Blow-up phenomena for scalar-flat metrics on manifolds with boundary

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Abstract

Let (M^n, g) be a compact Riemannian manifold with boundary ∂M. This article is concerned with the set of scalar-flat metrics on M which are in the conformal class of g and have ∂M as a constant mean curvature hypersurface. We construct examples of metrics on the unit ball B^n, in dimensions $n \geq 25$, for which this set is noncompact. These manifolds have umbilic boundary, but they are not conformally equivalent to B^n.

1 Introduction

Let (M^n, g) be a compact Riemannian manifold with boundary ∂M and dimension $n \geq 3$. In 1992, J. Escobar addressed the question of finding a scalar-flat conformal metric $\tilde{g} = u^{\frac{4}{n-2}} g$ which has ∂M as a constant mean curvature hypersurface. This problem was studied in [2], [9], [16], [17], [18], [27] and [28]. In analytical terms, it corresponds to the existence of a positive solution to the equations

$$\begin{cases} \Delta_g u - c_n R_g u = 0, & \text{in } M, \\ \frac{\partial u}{\partial \eta} - d_n \kappa_g u + K u^{\frac{n}{n-2}} = 0, & \text{on } \partial M, \end{cases} \tag{1.1}$$

for some constant K, where $c_n = \frac{n-2}{4(n-1)}$ and $d_n = \frac{n-2}{2}$. Here, Δ_g is the Laplace-Beltrami operator, R_g is the scalar curvature, κ_g is the mean curvature of ∂M and η is the inward unit normal vector to ∂M.

Escobar's question was motivated by the classical Yamabe problem, which consists of finding a conformal metric of constant scalar curvature on a given closed Riemannian manifold. This was completely solved after the works of H. Yamabe ([35]), N. Trudinger ([34]), T. Aubin ([4]) and R. Schoen ([30]). (See [22] and [32] for nice surveys on the issue.) Conformal metrics of constant scalar curvature and zero boundary mean curvature on the boundary were studied in [7], [15] (see also [3] and [20]).

The solutions to the equations (1.1) are the critical points of the functional

$$Q(u) = \frac{\int_M |du|_g^2 + c_n R_g u^2 dv_g + \int_{\partial M} d_n \kappa_g u^2 d\sigma_g}{\left(\int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} d\sigma_g \right)^{\frac{n-2}{n-1}}} ,$$

where dv_g and $d\sigma_g$ denote the volume forms of M and ∂M, respectively. In order to prove the existence of these solutions, Escobar introduced the conformally invariant Sobolev quotient

$$Q(M, \partial M) = \inf\{Q(u); \ u \in C^1(\bar{M}), u \not\equiv 0 \text{ on } \partial M\}.$$

In this work we are interested in the question of whether the full set of solutions to (1.1) is compact. A necessary condition is that M is not conformally equivalent to the standard ball B^n. We point out that if the equations (1.1) have a solution $u > 0$ with K positive (resp. zero and negative), then $Q(M, \partial M)$ has to be positive (resp. zero and negative). If $K < 0$, the solution to the equations (1.1) is unique. If $K = 0$, the equations (1.1) become linear and the solutions are unique up to a multiplication by a positive constant. Hence, the only interesting case is the one when $K > 0$.

The problem of compactness of solutions to the equations (1.1) was studied by V. Felli and M. Ould Ahmedou in the conformally flat case with umbilic boundary ([18]) and in the three-dimensional case with umbilic boundary ([19]). In [1], the author proved compactness for dimensions $n \geq 7$ under a generic condition. Other compactness results for similar equations were obtained by Z. Djadli, A. Malchiodi and M. Ould Ahmedou in [11, 12], by Z. Han and Y. Li in [20] and by M. Ould Ahmedou in [29].

In the case of manifolds without boundary, the question of compactness of the full set of solutions to the Yamabe equation was first raised by R. Schoen in a topics course at Stanford University in 1988. A necessary condition is that the manifold M^n is not conformally equivalent to the sphere S^n. This problem was studied in [13], [14], [23], [24], [25], [26], [31] and [33] and was completely solved in a series of three papers: [6], [8] and [21]. In [6], S. Brendle discovered the first smooth counterexamples for dimensions $n \geq 52$ (see [5] for nonsmooth examples). In [21], M. Khuri, F. Marques and R. Schoen proved compactness for dimensions $3 \leq n \leq 24$. Finally, in [8], Brendle and Marques extended the counterexamples of [6] to the remaining dimensions $25 \leq n \leq 51$.

It is expected that, as in the case of manifolds without boundary, there should be a critical dimension n_0 such that compactness in the case of manifolds with boundary holds for $n < n_0$ and fails for $n \geq n_0$. In this work we partially answer this question by showing that compactness fails for dimensions $n \geq 25$. More precisely we prove:

Main Theorem. *Let $n \geq 25$. Then there exists a smooth Riemannian metric g on B^n and a sequence of positive smooth functions $\{v_\nu\}_{\nu=1}^{\infty}$ with the following properties:*

(i) g is not conformally flat;

(ii) ∂B^n is umbilic with respect to the induced metric by g;

(iii) for all ν, v_ν is a solution to the equations (1.1) with a constant $K > 0$ and $M = B^n$;

(iv) $Q(v_\nu) < Q(B^n, \partial B)$ for all ν;

(v) $\sup_{\partial B^n} v_\nu \to \infty$ as $\nu \to \infty$.

In order to prove the Main Theorem, we follow the program adopted in [6] and [8]. In Section 2, we show that the problem can be reduced to finding critical points of a certain function $\mathcal{F}_g(\xi, \epsilon)$, where ξ is a vector in $\mathbb{R}^{n-1}$ and ϵ is a positive real number. In Section 3, we show that the function $\mathcal{F}_g(\xi, \epsilon)$ can be approximated by an auxiliary function $F(\xi, \epsilon)$. In Section 4, we prove that the function $F(\xi, \epsilon)$ has a strict local minimum point. The cases $n \geq 53$ and $25 \leq n \leq 52$ are handled separately in Subsections 4.1 and 4.2 respectively. Finally, in Section 5, we use a perturbation argument to construct critical points of the function $\mathcal{F}_g(\xi, \epsilon)$ and prove the non-compactness theorem.

Notation. Throughout this work we will make use of the index notation for tensors. We will adopt the summation convention whenever confusion is not possible and use indices $1 \leq i, i, j, k, l, m, p, q, r, s \leq n - 1$ and $1 \leq a, b, c, d \leq n$. We also define constants $c_n = \frac{n-2}{4(n-1)}$ and $d_n = \frac{n-2}{2}$.

We will denote by Δ_g the Laplace-Beltrami operator. The volume forms of M and ∂M will be denoted by dv_g and $d\sigma_g$, respectively. By η we will denote the inward unit normal vector to ∂M. The scalar curvature will be denoted by R_g, the second fundamental form of ∂M by π_{kl} and the mean curvature, $\frac{1}{n-1} tr(\pi_{kl})$, by κ_g.

By $\mathbb{R}^n_+$ we will denote the half-space $\{x = (x_1, ..., x_n) \in \mathbb{R}^n; x_n \geq 0\}$. If $x \in \mathbb{R}^n_+$ we set $\bar{x} = (x_1, ..., x_{n-1}, 0) \in \partial \mathbb{R}^n_+ \cong \mathbb{R}^{n-1}$. For any $x_0 \in \mathbb{R}^n_+$ we set $B_r^+(x_0) = \{x \in \mathbb{R}^n_+ ; |x - x_0| < r\}$. The n-dimensional sphere of radius r in $\mathbb{R}^{n+1}$ will be denoted by S_r^n and σ_n will denote the area of the n-dimensional unit sphere S_1^n.

2   Lyapunov-Schmidt reduction

Given a pair $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$ we set

$$u_{(\xi, \epsilon)}(x) = \left(\frac{\epsilon}{(\epsilon + x_n)^2 + |\bar{x} - \xi|^2} \right)^{\frac{n-2}{2}}, \quad \text{for } x \in \mathbb{R}^n_+ .$$

Observe that $u_{(\xi, \epsilon)}$ satisfies

$$\begin{cases} \Delta u_{(\xi, \epsilon)} = 0, & \text{in } \mathbb{R}^n_+, \\ \frac{\partial}{\partial x_n} u_{(\xi, \epsilon)} + (n - 2) u_{(\xi, \epsilon)}^{\frac{n}{n-2}} = 0, & \text{on } \partial \mathbb{R}^n_+, \end{cases} \tag{2.1}$$

and

$$\int_{\partial \mathbb{R}^n_+} u_{(\xi, \epsilon)}^{\frac{2(n-1)}{n-2}} = \left(\frac{Q(B^n, \partial B)}{n - 2} \right)^{n-1} . \tag{2.2}$$

Let us define

$$\phi_{(\xi,\epsilon,n)}(x) = \left(\frac{\epsilon}{(\epsilon + x_n)^2 + |\bar{x} - \xi|^2}\right)^{\frac{n}{2}} \frac{\epsilon^2 - x_n^2 - |\bar{x} - \xi|^2}{(\epsilon + x_n)^2 + |\bar{x} - \xi|^2}$$

and

$$\phi_{(\xi,\epsilon,k)}(x) = \left(\frac{\epsilon}{(\epsilon + x_n)^2 + |\bar{x} - \xi|^2}\right)^{\frac{n}{2}} \frac{2\epsilon(x_k - \xi_k)}{(\epsilon + x_n)^2 + |\bar{x} - \xi|^2}$$

for $x \in \mathbb{R}^n_+$ and $k = 1, ..., n-1$. Observe that

$$\phi_{(\xi,\epsilon,n)}(x) \cdot ((\epsilon + x_n)^2 + |\bar{x} - \xi|^2) = -\frac{2\epsilon^2}{n-2} \frac{\partial}{\partial \epsilon} u_{(\xi,\epsilon)}(x),$$

$$\phi_{(\xi,\epsilon,k)}(x) \cdot ((\epsilon + x_n)^2 + |\bar{x} - \xi|^2) = \frac{2\epsilon^2}{n-2} \frac{\partial}{\partial \xi_k} u_{(\xi,\epsilon)}(x),$$

for $k = 1, ..., n-1$, and that $\|\phi_{(\xi,\epsilon,a)}\|_{L^{\frac{2(n-1)}{n}}(\partial \mathbb{R}^n_+)}$ is independent of $(\epsilon, \xi) \in \mathbb{R}^{n-1} \times (0, \infty)$, for any $a = 1, ..., n$.

We also set

$$\Sigma = \left\{w \in L^{\frac{2n}{n-2}}(\mathbb{R}^n_+) \cap L^{\frac{2(n-1)}{n-2}}(\partial \mathbb{R}^n_+) \cap H^1_{loc}(\mathbb{R}^n_+)\,;\, \int_{\mathbb{R}^n_+} |dw|^2 < \infty\right\},$$

$$\Sigma_{(\xi,\epsilon)} = \left\{w \in \Sigma\,;\, \int_{\partial \mathbb{R}^n_+} \phi_{(\xi,\epsilon,a)} w = 0\,,\, a = 1, ..., n\right\}$$

and $\|w\|_\Sigma = \left(\int_{\mathbb{R}^n_+} |dw|^2\right)^{\frac{1}{2}}$ for $w \in \Sigma$. Observe that $u_{(\xi,\epsilon)} \in \Sigma_{(\xi,\epsilon)}$ for each $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$. By Sobolev's inequality, there exists $K = K(n) > 0$ such that

$$\left(\int_{\mathbb{R}^n_+} |w|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} + \left(\int_{\partial \mathbb{R}^n_+} |w|^{\frac{2(n-1)}{n-2}}\right)^{\frac{n-2}{n-1}} \leq K \int_{\mathbb{R}^n_+} |dw|^2 \tag{2.3}$$

for all $w \in \Sigma$.

In what follows in this section we are going to find, for each pair $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$, a function $v_{(\xi,\epsilon)} \in \Sigma$ which is an approximate weak solution to a Yamabe-type problem (1.1) on $\mathbb{R}^n_+$. Then we will show that $v_{(\xi,\epsilon)}$ is in fact a classical solution to this problem whenever (ξ, ϵ) is a critical point of a certain energy function defined on $\mathbb{R}^{n-1} \times (0, \infty)$.

The following result is Proposition 26 of [6] and will be used throughout this work:

Lemma 2.1. *Suppose that we express the Riemannian metric g as $g = \exp(h)$, where h is a trace-free symmetric two-tensor defined on $\mathbb{R}^n_+$ and satisfying $|h(x)| \leq 1$ for any $x \in \mathbb{R}^n_+$. Then there exists $C = C(n) > 0$ such that*

$$\left|R_g - \left\{\partial_a \partial_b h_{ab} - \partial_a(h_{ac}\partial_b h_{bc}) + \frac{1}{2}\partial_a h_{ac}\partial_b h_{bc} - \frac{1}{4}\partial_c h_{ab}\partial_c h_{ab}\right\}\right| \leq C|h|^2|\partial^2 h| + C|h||\partial h|^2.$$

Notation. In this section we suppose that g is a Riemannian metric on $\mathbb{R}^n_+$ expressed as $g = exp(h)$, where h is a trace-free symmetric two-tensor satisfying $h(x) = 0$ for any $|x| \geq 1$.

Proposition 2.2. *If $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha \leq 1$ for any $x \in \mathbb{R}^n_+$, then there exists $C = C(n) > 0$ such that*

$$\left\| \Delta_g u_{(\xi,\epsilon)} - c_n R_g u_{(\xi,\epsilon)} \right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} + \left\| d_n \kappa_g u_{(\xi,\epsilon)} \right\|_{L^{\frac{2(n-1)}{n}}(\partial\mathbb{R}^n_+)} \leq C\alpha$$

for all pairs $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$.

Proof. It follows from the pointwise estimates

$$\left| \Delta_g u_{(\xi,\epsilon)} - c_n R_g u_{(\xi,\epsilon)} \right| \leq C \left\{ |h||\partial^2 u_{(\xi,\epsilon)}| + |\partial h||\partial u_{(\xi,\epsilon)}| + (|\partial^2 h| + |\partial h|^2)|u_{(\xi,\epsilon)}| \right\}$$

and $|d_n \kappa_g u_{(\xi,\epsilon)}| \leq C|\partial h||u_{(\xi,\epsilon)}|$ that

$$\left\| \Delta_g u_{(\xi,\epsilon)} - c_n R_g u_{(\xi,\epsilon)} \right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)}$$

$$\leq C \left\{ \|h\|_{L^\infty(\mathbb{R}^n_+)} \|\partial^2 u_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} + \|\partial h\|_{L^n(\mathbb{R}^n_+)} \|\partial u_{(\xi,\epsilon)}\|_{L^2(\mathbb{R}^n_+)} \right\}$$

$$+ C \left\{ \|\partial^2 h\|_{L^{\frac{n}{2}}(\mathbb{R}^n_+)} + \|\partial h\|^2_{L^n(\mathbb{R}^n_+)} \right\} \|u\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)}$$

and

$$\|d_n \kappa_g u_{(\xi,\epsilon)}\|_{L^{\frac{2(n-1)}{n}}(\partial\mathbb{R}^n_+)} \leq C\|\partial h\|_{L^{n-1}(\partial\mathbb{R}^n_+)} \|u_{(\xi,\epsilon)}\|_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}^n_+)} .$$

From this the result follows. $\qquad\qquad\square$

Lemma 2.3. *Let $B^n = B^n_{1/2}(0, ..., 0, -\frac{1}{2}) \subset \mathbb{R}^n$ be the ball with radius $\frac{1}{2}$ and center $(0, ..., 0, -\frac{1}{2})$. Let $z_1, ..., z_n$ be the coordinates of B^n taken with center $(0, ..., 0, -\frac{1}{2})$. For each pair $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$, the expression*

$$C_{(\xi,\epsilon)}(x) = \frac{\epsilon(x_1 - \xi_1, ..., x_{n-1} - \xi_{n-1}, x_n + \epsilon)}{|\bar{x} - \xi|^2 + (x_n + \epsilon)^2} + (0, ..., 0, -1)$$

defines a conformal equivalence

$$C_{(\xi,\epsilon)} : \mathbb{R}^n_+ \to B^n \backslash \{(0, ..., 0, -1)\}$$

that satisfies $C^*_{(\xi,\epsilon)} \delta_{B^n} = u_{(\xi,\epsilon)}^{\frac{4}{n-2}} \delta$, where δ_{B^n} is the Euclidean metric on B^n and δ is the Euclidean metric on $\mathbb{R}^n_+$. For any smooth function f on $\mathbb{R}^n_+$, we have

$$\Delta_{B^n} \tilde{u}_{(\xi,\epsilon)} = u_{(\xi,\epsilon)}^{-\frac{n+2}{n-2}} \Delta f \qquad\qquad (2.4)$$

and

$$\frac{\partial}{\partial\eta} \tilde{u}_{(\xi,\epsilon)} - (n-2)\tilde{u}_{(\xi,\epsilon)} = u_{(\xi,\epsilon)}^{-\frac{n}{n-2}} \frac{\partial f}{\partial x_n} , \qquad\qquad (2.5)$$

where $\tilde{u}_{(\xi,\epsilon)} = (fu_{(\xi,\epsilon)}^{-1}) \circ C_{(\xi,\epsilon)}^{-1}$. Moreover,

$$z_n \circ C_{(\xi,\epsilon)} = -\frac{\epsilon}{n-2} u_{(\xi,\epsilon)}^{-1} \frac{\partial}{\partial \epsilon} u_{(\xi,\epsilon)} = \frac{1}{2} u_{(\xi,\epsilon)}^{-\frac{n}{n-2}} \phi_{(\xi,\epsilon,n)} \tag{2.6}$$

and

$$z_k \circ C_{(\xi,\epsilon)} = \frac{\epsilon}{n-2} u_{(\xi,\epsilon)}^{-1} \frac{\partial}{\partial \xi_k} u_{(\xi,\epsilon)} = \frac{1}{2} u_{(\xi,\epsilon)}^{-\frac{n}{n-2}} \phi_{(\xi,\epsilon,k)} , \quad k = 1, ..., n-1 . \tag{2.7}$$

Proof. These are direct computations. The assertions (2.4) and (2.5) follow from the following properties of the conformal operators $L_g = \Delta_g - c_n R_g$ and $B_g = \frac{\partial}{\partial \eta} - d_n \kappa_g$:

$$L_{u^{\frac{4}{n-2}}g}(fu^{-1}) = u^{-\frac{n+2}{n-2}} L_g f \quad \text{and} \quad B_{u^{\frac{4}{n-2}}g}(fu^{-1}) = u^{-\frac{n}{n-2}} B_g f . \tag{2.8}$$

$\square$

Lemma 2.4. *There exists $\theta = \theta(n) > 0$ such that*

$$\int_{B^n} |dw|^2 - 2 \int_{\partial B^n} w^2 - 2\theta \left(\int_{B^n} |dw|^2 + (n-2) \int_{\partial B^n} w^2 \right) + \frac{4}{\theta} \left(\int_{\partial B^n} w \right)^2 \geq 0$$

for any $w \in H^1(B^n)$ such that $w \perp_{L^2(\partial B^n)} \{z_1, ..., z_n\}$. Here, we are following the notations of Lemma 2.3.

Proof. First we fix $0 \not\equiv w \in H^1(B^n)$ such that $w \perp_{L^2(\partial B^n)} \{1, z_1, ..., z_n\}$. Since

$$\inf \left\{ \frac{\int_{B^n} |d\psi|^2}{\int_{\partial B^n} \psi^2} , \text{ such that } \psi \in H^1(B^n), \ \psi \not\equiv 0 \text{ on } \partial B^n \text{ and } \psi \perp_{L^2(\partial B^n)} 1 \right\} = 2$$

and this infimum is realized only by the functions $z_1, ..., z_n$, we see that

$$\int_{B^n} |dw|^2 - 2 \int_{\partial B^n} w^2 > 0 .$$

Hence,

$$\int_{B^n} |dw|^2 - 2 \int_{\partial B^n} w^2 \geq 2\theta \left(\int_{B^n} |dw|^2 + (n-2) \int_{\partial B^n} w^2 \right) \tag{2.9}$$

holds for any $\theta > 0$ satisfying

$$\theta \leq \theta(w) = \frac{1}{2} \frac{\int_{B^n} |dw|^2 - 2 \int_{\partial B^n} w^2}{\int_{B^n} |dw|^2 + (n-2) \int_{\partial B^n} w^2}$$

and the equality is realized by $\theta = \theta(w)$.

We claim that there exists $\theta_0 > 0$ such that $\theta(w) \geq \theta_0$ for any $w \in H^1(B^n)$ satisfying $w \perp_{L^2(\partial B^n)} \{1, z_1, ..., z_n\}$. Suppose by contradiction this is not true. Thus

we can choose a sequence $\{w_j\}_{j=1}^{\infty} \subset H^1(B^n)$ such that $w_j \perp_{L^2(\partial B^n)} \{1, z_1, ..., z_n\}$ and $\theta(w_j) \to 0$ as $j \to \infty$. Hence

$$\int_{B^n} |dw_j|^2 - 2\int_{\partial B^n} w_j^2 = 2\theta(w_j)\left(\int_{B^n} |dw_j|^2 + (n-2)\int_{\partial B^n} w_j^2\right)$$

holds and we can assume that $\int_{B^n} |dw_j|^2 = 1$ for any j. Thus, $\int_{\partial B^n} w_j^2 \leq \frac{1}{2}$ for all j and we can suppose that $w_j \rightharpoonup w_0$ in $H^1(B^n)$ for some w_0. Since $H^1(B^n)$ is compactly imbedded in $L^2(\partial B^n)$, we know that $w_0 \perp_{L^2(\partial B^n)} \{1, z_1, ..., z_n\}$. Let us first assume that $w_0 \not\equiv 0$. We set

$$\beta = \int_{B^n} |dw_0|^2 - 2\int_{\partial B^n} w_0^2 > 0.$$

Since $\liminf_{i \to \infty} \int_{B^n} |dw_j|^2 \geq \int_{B^n} |dw_0|^2$ and $\lim_{i \to \infty} \int_{\partial B^n} w_j^2 = \int_{\partial B^n} w^2$, we can assume that

$$\int_{B^n} |dw_j|^2 - 2\int_{\partial B^n} w_j^2 \geq \frac{\beta}{2} \quad \text{for all } j.$$

On the other hand,

$$\frac{\beta}{n}\left\{\int_{B^n} |dw_j|^2 + (n-2)\int_{\partial B^n} w_j^2\right\} \leq \frac{\beta}{2},$$

since $\int_{B^n} |dw_j|^2 = 1$ and $\int_{\partial B^n} w_j^2 \leq \frac{1}{2}$. Hence,

$$2\theta(w_j)\left(\int_{B^n} |dw_j|^2 + (n-2)\int_{\partial B^n} w_j^2\right) = \int_{B^n} |dw_j|^2 - 2\int_{\partial B^n} w_j^2$$

$$\geq \frac{\beta}{n}\left(\int_{B^n} |dw_j|^2 + (n-2)\int_{\partial B^n} w_j^2\right).$$

which implies that $2\theta(w_j) \geq \frac{\beta}{n}$ for all j and contradicts the fact that $\theta(w_j) \to 0$.

Thus we must have $w_0 \equiv 0$, which implies that $\int_{\partial B^n} w_j^2 \to 0$ as $j \to \infty$. Then, if we set $\tilde{w}_j = \left(\int_{\partial B^n} w_j^2\right)^{-\frac{1}{2}} w_j$, we have $\tilde{w}_j \rightharpoonup \tilde{w}_0$ in $H^1(B^n)$, for some $\tilde{w}_0$. Moreover,

$$0 = \lim_{j \to \infty} \int_{B^n} |d\tilde{w}_j|^2 \geq \int_{B^n} |d\tilde{w}_0|^2$$

and

$$\int_{\partial B^n} \tilde{w}_j^2 = 1 = \int_{\partial B^n} \tilde{w}_0^2.$$

From this we conclude that $\tilde{w}_0 \equiv \text{const} \neq 0$, which contradicts the fact that $\tilde{w}_0 \perp_{L^2(\partial B^n)} 1$. This proves that there exists $\theta_0 > 0$ such that $\theta(w) \geq \theta_0$ for any $w \in H^1(B^n)$ satisfying $w \perp_{L^2(\partial B^n)} \{1, z_1, ..., z_n\}$. In particular, (2.9) holds, with $\theta = \theta_0$, for any $w \in H^1(B^n)$ satisfying $w \perp_{L^2(\partial B^n)} \{1, z_1, ..., z_n\}$.

Now, let $w \in H^1(B^n)$ satisfy $w \perp_{L^2(\partial B^n)} \{z_1, ..., z_n\}$. We write $w = w_1 + b$ where b is a constant and $w_1 \perp_{L^2(\partial B^n)} 1$. Then we have

$$\int_{B^n} |dw|^2 - 2 \int_{\partial B^n} w^2 - 2\theta_0 \left(\int_{B^n} |dw|^2 + (n-2) \int_{\partial B^n} w^2 \right) + \frac{4}{\theta_0} \left(\int_{\partial B^n} w \right)^2$$

$$= \int_{B^n} |dw_1|^2 - 2 \int_{\partial B^n} w_1^2 - 2\theta_0 \left(\int_{B^n} |dw_1|^2 + (n-2) \int_{\partial B^n} w_1^2 \right)$$

$$- 2(1 + (n-2)\theta_0) \int_{\partial B^n} b^2 + \frac{4}{\theta_0} \left(\int_{\partial B^n} b \right)^2$$

$$\geq \left(\frac{4}{\theta_0} - 2 - 2(n-2)\theta_0 \right) \int_{\partial B^n} b^2$$

Choosing θ_0 smaller if necessary, we can suppose that $\frac{4}{\theta_0} - 2 - 2(n-2)\theta_0 > 0$ and the result follows. □

Proposition 2.5. *There exists $\theta = \theta(n) > 0$ such that*

$$\int_{\mathbb{R}^n_+} |dw|^2 - n \int_{\partial \mathbb{R}^n_+} u_{(\xi,\epsilon)}^{\frac{2}{n-2}} w^2 \geq 2\theta |w|_{\Sigma}^2 - \frac{4}{\theta} \left(\int_{\partial \mathbb{R}^n_+} u_{(\xi,\epsilon)}^{\frac{n}{n-2}} w \right)^2$$

for all $w \in \Sigma_{(\xi,\epsilon)}$ and any pair $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$.

Proof. Let $w \in \Sigma_{(\xi,\epsilon)}$ and set $\bar{w} = (wu_{(\xi,\epsilon)}^{-1}) \circ C_{(\xi,\epsilon)}^{-1}$. Using the fact that $C_0^\infty(\mathbb{R}^n_+)$ is dense in Σ with respect to the norms $\| \cdot \|_{\Sigma}$, $\| \cdot \|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)}$ and $\| \cdot \|_{L^{\frac{2(n-1)}{n-2}}(\partial \mathbb{R}^n_+)}$, it is easy to see that we can assume that $\bar{w} \in H^1(B^n)$. It follows from the expressions (2.6) and (2.7) that

$$\int_{\partial B^n} \bar{w} z_n = \frac{1}{2} \int_{\partial \mathbb{R}^n_+} w \, \phi_{(\xi,\epsilon,n)} = 0$$

and

$$\int_{\partial B^n} \bar{w} z_k = \frac{1}{2} \int_{\partial \mathbb{R}^n_+} w \, \phi_{(\xi,\epsilon,k)} = 0, \quad k = 1, ..., n-1.$$

Then, according to Lemma 2.4, we have

$$\int_{B^n} |d\bar{w}|^2 - 2 \int_{\partial B^n} \bar{w}^2 - 2\theta \left(\int_{B^n} |d\bar{w}|^2 + (n-2) \int_{\partial B^n} \bar{w}^2 \right) + \frac{4}{\theta} \left(\int_{\partial B^n} \bar{w} \right)^2 \geq 0. \quad (2.10)$$

Hence, using the formulas (2.4) and (2.5), we ealisy see that

$$\int_{B^n} |d\bar{w}|^2 - 2 \int_{\partial B^n} \bar{w}^2 = \int_{\mathbb{R}^n_+} |dw|^2 - n \int_{\partial \mathbb{R}^n_+} u_{(\xi,\epsilon)}^{\frac{2}{n-2}} w^2,$$

$$\int_{B^n} |d\bar{w}|^2 + (n-2) \int_{\partial B^n} \bar{w}^2 = \int_{\mathbb{R}^n_+} |dw|^2 \quad \text{and} \quad \int_{\partial B^n} \bar{w} = \int_{\partial \mathbb{R}^n_+} u_{(\xi,\epsilon)}^{\frac{n}{n-2}} w.$$

Now the result follows from substituting these last three equations in (2.10). □

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Corollary 2.6. *Let K be as in (2.3) and θ be as in Proposition 2.5. Then there exists $0 < \alpha_0 = \alpha_0(n) \le 1$ such that, whenever $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \le \alpha_0$ for all $x \in \mathbb{R}^n_+$, we have*

$$\left(\int_{\mathbb{R}^n_+} |w|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} + \left(\int_{\partial\mathbb{R}^n_+} |w|^{\frac{2(n-1)}{n-2}} \right)^{\frac{n-2}{n-1}} \le 2K \int_{\mathbb{R}^n_+} \left(|dw|^2_g + c_n R_g w^2 \right) + 2K \int_{\partial\mathbb{R}^n_+} d_n \kappa_g w^2$$

(2.11)

for all $w \in \Sigma$ and

$$\int_{\mathbb{R}^n_+} (|dw|^2_g + c_n R_g w^2) + \int_{\partial\mathbb{R}^n_+} \left(d_n \kappa_g w^2 - nu^{\frac{2}{n-2}}_{(\xi,\epsilon)} w^2 \right) \ge \frac{\theta}{2} \|w\|^2_\Sigma - \frac{1}{\theta} A(w)^2 \quad (2.12)$$

for all $w \in \Sigma_{(\xi,\epsilon)}$ and any pair $(\xi,\epsilon) \in \mathbb{R}^{n-1} \times (0,\infty)$. Here,

$$A(w) = \int_{\mathbb{R}^n_+} \left(\Delta_g u_{(\xi,\epsilon)} - c_n R_g u_{(\xi,\epsilon)} \right) w + \int_{\partial\mathbb{R}^n_+} \left(-d_n \kappa_g u_{(\xi,\epsilon)} + 2u^{\frac{n}{n-2}} \right) w \,.$$

Proof. Let us first prove the estimate (2.12). Observe that

$$\int_{\mathbb{R}^n_+} \left(\Delta_g u_{(\xi,\epsilon)} - c_n R_g u_{(\xi,\epsilon)} \right) w \ge - \left\| \Delta_g u_{(\xi,\epsilon)} - c_n R_g u_{(\xi,\epsilon)} \right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} \|w\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)}$$

and

$$\int_{\partial\mathbb{R}^n_+} \left(-d_n \kappa_g u_{(\xi,\epsilon)} + 2u^{\frac{n}{n-2}} \right) w$$

$$\ge - \left\| d_n \kappa_g u_{(\xi,\epsilon)} \right\|_{L^{\frac{2(n-1)}{n}}(\partial\mathbb{R}^n_+)} \|w\|_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}^n_+)} + 2 \int_{\partial\mathbb{R}^n_+} u^{\frac{n}{n-2}}_{(\xi,\epsilon)} w$$

Hence, by Proposition 2.2 and inequality (2.3) we have

$$A(w) \ge -C\alpha_0 \|w\|_\Sigma + 2 \int_{\partial\mathbb{R}^n_+} u^{\frac{n}{n-2}}_{(\xi,\epsilon)} w \,.$$

Choosing α_0 small this implies

$$A(w)^2 \ge 4 \left(\int_{\partial\mathbb{R}^n_+} u^{\frac{n}{n-2}}_{(\xi,\epsilon)} w \right)^2 - \theta^2 \|w\|^2_\Sigma$$

which, together with Proposition 2.5, gives

$$\int_{\mathbb{R}^n_+} |dw|^2 - \int_{\partial\mathbb{R}^n_+} nu^{\frac{2}{n-2}}_{(\xi,\epsilon)} w^2 \ge \theta \|w\|^2_\Sigma - \frac{1}{\theta} A(w)^2 \,. \quad (2.13)$$

On the other hand,

$$\int_{\mathbb{R}^n_+} (|dw|^2_g + c_n R_g w^2) + \int_{\partial\mathbb{R}^n_+} (d_n \kappa_g w^2 - nu^{\frac{2}{n-2}}_{(\xi,\epsilon)} w^2)$$

$$= \int_{\mathbb{R}^n_+} |dw|^2 - \int_{\partial\mathbb{R}^n_+} nu^{\frac{2}{n-2}}_{(\xi,\epsilon)} w^2 + \int_{\mathbb{R}^n_+} \left\{ (g^{ij} - \delta^{ij}) \partial_i w \partial_j w + c_n R_g w^2 \right\} + \int_{\partial\mathbb{R}^n_+} d_n \kappa_g w^2 \,.$$

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The fact that $h(x) = 0$ for $|x| \geq 1$ and (2.3) imply that

$$\int_{\mathbb{R}^n_+} \left\{ (g^{ij} - \delta^{ij})\partial_i w \partial_j w + c_n R_g w^2 \right\} + \int_{\partial\mathbb{R}^n_+} d_n \kappa_g w^2 \tag{2.14}$$

$$\leq C\alpha_0 \|w\|^2_\Sigma + C\alpha_0 \|w\|^2_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)} + C\alpha_0 \|w\|^2_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}^n_+)}$$

$$\leq C\alpha_0(1 + K)\|w\|^2_\Sigma .$$

Hence,

$$\int_{\mathbb{R}^n_+} (|dw|^2_g + c_n R_g w^2) + \int_{\partial\mathbb{R}^n_+} (d_n \kappa_g w^2 - nu^{\frac{2}{n-2}}_{(\xi,\epsilon)} w^2) \tag{2.15}$$

$$\geq \int_{\mathbb{R}^n_+} |dw|^2 - \int_{\partial\mathbb{R}^n_+} nu^{\frac{2}{n-2}}_{(\xi,\epsilon)} w^2 - C\alpha_0(1 + K)\|w\|^2_\Sigma .$$

Now the result follows from the inequalities (2.13) and (2.15), choosing α_0 small. The estimate (2.11) follows easily from the inequalities (2.3) and (2.14). □

Proposition 2.7. *Suppose that $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_0$ for all $x \in \mathbb{R}^n_+$, where α_0 is the constant obtained in Corollary 2.6. Given any pair $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$ and functions $f \in L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)$ and $\bar{f} \in L^{\frac{2(n-1)}{n}}(\partial\mathbb{R}^n_+)$ there exists a unique $w \in \Sigma_{(\xi,\epsilon)}$ such that*

$$\int_{\mathbb{R}^n_+} \left(< dw, d\psi >_g + c_n R_g w\psi \right) + \int_{\partial\mathbb{R}^n_+} \left(d_n \kappa_g w\psi - nu^{\frac{2}{n-2}}_{(\xi,\epsilon)} w\psi \right) = \int_{\mathbb{R}^n_+} f\psi + \int_{\partial\mathbb{R}^n_+} \bar{f}\psi$$
(2.16)

for all $\psi \in \Sigma_{(\xi,\epsilon)}$. Moreover, there exists $C = C(n) > 0$ such that

$$\|w\|_\Sigma \leq C\|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} + C\|\bar{f}\|_{L^{\frac{2(n-1)}{n}}(\partial\mathbb{R}^n_+)} .$$

Proof. Let us first prove the existence part. Following the notations of Corollary 2.6, we define the funcional

$$T(w) = \int_{\mathbb{R}^n_+} (|dw|^2_g + c_n R_g w^2 - 2fw) + \int_{\partial\mathbb{R}^n_+} (d_n \kappa_g w^2 - nu^{\frac{2}{n-2}}_{(\xi,\epsilon)} w^2 - 2\bar{f}w) + \frac{1}{\theta} A(w)^2$$

for $w \in \Sigma_{(\xi,\epsilon)}$. Hence

$$dT_w(\psi) = 2\int_{\mathbb{R}^n_+} \left(< dw, d\psi >_g + c_n R_g w\psi - f\psi \right) + 2\int_{\partial\mathbb{R}^n_+} \left(d_n \kappa_g w\psi - nu^{\frac{2}{n-2}}_{(\xi,\epsilon)} w\psi - \bar{f}\psi \right)$$

$$+ \frac{2}{\theta} A(w)A(\psi) .$$

It follows from the identity (2.12) that

$$T(w) \geq \frac{\theta}{2}\|w\|^2_\Sigma - 2\int_{\mathbb{R}^n_+} fw - 2\int_{\partial\mathbb{R}^n_+} \bar{f}w$$

$$\geq \frac{\theta}{2}\|w\|^2_\Sigma - 2\|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)}\|w\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)} - 2\|\bar{f}\|_{L^{\frac{2(n-1)}{n}}(\partial\mathbb{R}^n_+)}\|w\|_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}^n_+)}$$

$$\geq \frac{\theta}{4}\|w\|^2_\Sigma - C\left(\|f\|^2_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} + \|\bar{f}\|^2_{L^{\frac{2(n-1)}{n}}(\partial\mathbb{R}^n_+)} \right)$$

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where in the last inequality we used the estimate (2.3). So, T is bounded below and by a standard argument we can find a minimizer w_0 for T over all functions in $\Sigma_{(\xi,\epsilon)}$. Now, integrating by parts we see that

$$\int_{\mathbb{R}^n_+} \left(<du_{(\xi,\epsilon)}, d\psi>_g + c_n R_g u_{(\xi,\epsilon)} \psi \right) + \int_{\partial\mathbb{R}^n_+} \left(d_n \kappa_g u_{(\xi,\epsilon)} \psi - n u_{(\xi,\epsilon)}^{\frac{n}{n-2}} \psi \right) = -A(\psi),$$

holds for all $\psi \in C_0^\infty(\mathbb{R}^n_+)$. Since this space is dense in Σ with respect to the norms $\|\cdot\|_\Sigma$, $\|\cdot\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)}$ and $\|\cdot\|_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}^n_+)}$, this identity holds for all $\psi \in \Sigma$. Hence, the function $w = w_0 - \frac{1}{\theta} A(w_0) u_{(\xi,\epsilon)}$ satisfies (2.16) for all $\psi \in \Sigma_{(\xi,\epsilon)}$, proving the existence part.

In order to prove the uniqueness part, suppose that $w \in \Sigma_{(\xi,\epsilon)}$ satisfies (2.16) for all $\psi \in \Sigma_{(\xi,\epsilon)}$. In particular,

$$\int_{\mathbb{R}^n_+} \left(|dw|_g^2 + c_n R_g w^2 \right) + \int_{\partial\mathbb{R}^n_+} \left(d_n \kappa_g w^2 - n u_{(\xi,\epsilon)}^{\frac{2}{n-2}} w^2 \right) = \int_{\mathbb{R}^n_+} fw + \int_{\partial\mathbb{R}^n_+} \bar{f}w$$

and

$$-A(w) = \int_{\mathbb{R}^n_+} \left(<dw, du_{(\xi,\epsilon)}>_g + c_n R_g w u_{(\xi,\epsilon)} \right) + \int_{\partial\mathbb{R}^n_+} \left(d_n \kappa_g w u_{(\xi,\epsilon)} - n u_{(\xi,\epsilon)}^{\frac{n}{n-2}} w \right)$$

$$= \int_{\mathbb{R}^n_+} f u_{(\xi,\epsilon)} + \int_{\partial\mathbb{R}^n_+} \bar{f} u_{(\xi,\epsilon)},$$

since $u_{(\xi,\epsilon)} \in \Sigma_{(\xi,\epsilon)}$. Then (2.12) implies

$$\frac{\theta}{2}\|w\|_\Sigma^2 \le \int_{\mathbb{R}^n_+} (|dw|_g^2 + c_n R_g w^2) + \int_{\partial\mathbb{R}^n_+} \left(d_n \kappa_g w^2 - n u_{(\xi,\epsilon)}^{\frac{2}{n-2}} w^2 \right) + \frac{1}{\theta} A(w)^2$$

$$= \int_{\mathbb{R}^n_+} fw + \int_{\partial\mathbb{R}^n_+} \bar{f}w + \frac{1}{\theta}\left(\int_{\mathbb{R}^n_+} f u_{(\xi,\epsilon)} + \int_{\partial\mathbb{R}^n_+} \bar{f} u_{(\xi,\epsilon)} \right)^2$$

$$\le \left\{ \|w\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)} + \frac{2}{\theta}\|u_{(\xi,\epsilon)}\|^2_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)} \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} \right\} \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)}$$

$$+ \left\{ \|w\|_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}^n_+)} + \frac{2}{\theta}\|u_{(\xi,\epsilon)}\|^2_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}^n_+)} \|\bar{f}\|_{L^{\frac{2(n-1)}{n}}(\partial\mathbb{R}^n_+)} \right\} \|\bar{f}\|_{L^{\frac{2(n-1)}{n}}(\partial\mathbb{R}^n_+)}$$

$$\le \left\{ K^{\frac{1}{2}}\|w\|_\Sigma + \frac{2}{\theta}\|u_{(\xi,\epsilon)}\|^2_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)} \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} \right\} \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)}$$

$$+ \left\{ K^{\frac{1}{2}}\|w\|_\Sigma + \frac{2}{\theta}\|u_{(\xi,\epsilon)}\|^2_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}^n_+)} \|\bar{f}\|_{L^{\frac{2(n-1)}{n}}(\partial\mathbb{R}^n_+)} \right\} \|\bar{f}\|_{L^{\frac{2(n-1)}{n}}(\partial\mathbb{R}^n_+)}.$$

Hence,

$$\frac{\theta}{4}\|w\|_\Sigma^2 \le \left\{ \frac{K}{\theta} + \frac{2}{\theta}\|u_{(\xi,\epsilon)}\|^2_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)} \right\} \|f\|^2_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)}$$

$$+ \left\{ \frac{K}{\theta} + \frac{2}{\theta}\|u_{(\xi,\epsilon)}\|^2_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}^n_+)} \right\} \|\bar{f}\|^2_{L^{\frac{2(n-1)}{n}}(\partial\mathbb{R}^n_+)}$$

and the result follows. $\qquad\qquad\square$

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Proposition 2.8. *Let α_0 be the constant obtained in Corollary 2.6. There is a constant $\alpha_1 = \alpha_1(n)$, $0 < \alpha_1 \leq \alpha_0$, with the following property: if $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_1$ for all $x \in \mathbb{R}^n_+$, given any pair $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$ there exists a unique $v_{(\xi,\epsilon)} \in \Sigma$ such that $v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} \in \Sigma_{(\xi,\epsilon)}$ and*

$$\int_{\mathbb{R}^n_+} \left(< dv_{(\xi,\epsilon)}, d\psi >_g + c_n R_g v_{(\xi,\epsilon)} \psi\right) + \int_{\partial \mathbb{R}^n_+} \left(d_n \kappa_g v_{(\xi,\epsilon)} \psi - (n-2)|v_{(\xi,\epsilon)}|^{\frac{2}{n-2}} v_{(\xi,\epsilon)} \psi\right) = 0$$

for all $\psi \in \Sigma_{(\xi,\epsilon)}$. Moreover, there exists $C = C(n) > 0$ such that

$$\|v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}\|_\Sigma \leq C\|\Delta_g u_{(\xi,\epsilon)} - c_n R_g u_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} + C\|d_n \kappa_g u_{(\xi,\epsilon)}\|_{L^{\frac{2(n-1)}{n}}(\partial \mathbb{R}^n_+)} \quad (2.17)$$

In particular, $v_{(\xi,\epsilon)} \not\equiv 0$.

Proof. Using Proposition 2.7 we can define

$$\mathcal{G}_{(\xi,\epsilon)} : L^{\frac{2n}{n+2}}(\mathbb{R}^n_+) \times L^{\frac{2(n-1)}{n}}(\partial \mathbb{R}^n_+) \longrightarrow \Sigma_{(\xi,\epsilon)}$$

by $\mathcal{G}_{(\xi,\epsilon)}(f, \bar{f}) = w$, where $w \in \Sigma_{(\xi,\epsilon)}$ satisfies (2.16) for all $\psi \in \Sigma_{(\xi,\epsilon)}$. Hence, there exists $C = C(n)$ such that

$$\|\mathcal{G}_{(\xi,\epsilon)}(f, \bar{f})\|_\Sigma \leq C\|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} + C\|\bar{f}\|_{L^{\frac{2(n-1)}{n}}(\partial \mathbb{R}^n_+)}. \quad (2.18)$$

We define a nonlinear mapping $\Phi_{(\xi,\epsilon)}(w) : \Sigma_{(\xi,\epsilon)} \to \Sigma_{(\xi,\epsilon)}$ by

$$\Phi_{(\xi,\epsilon)}(w) = \mathcal{G}_{(\xi,\epsilon)}(f_{(\xi,\epsilon)}, \bar{f}_{(\xi,\epsilon,w)})$$

where

$$f_{(\xi,\epsilon)} = \Delta_g u_{(\xi,\epsilon)} - c_n R_g u_{(\xi,\epsilon)}$$

and

$$\bar{f}_{(\xi,\epsilon,w)} = -d_n \kappa_g u_{(\xi,\epsilon)} + (n-2)\left\{|u_{(\xi,\epsilon)} + w|^{\frac{2}{n-2}}(u_{(\xi,\epsilon)} + w) - u_{(\xi,\epsilon)}^{\frac{n}{n-2}} - \frac{n}{n-2}u_{(\xi,\epsilon)}^{\frac{2}{n-2}}w\right\}.$$

It follows from Proposition 2.2 and the inequality (2.18) that $\|\Phi_{(\xi,\epsilon)}(0)\|_\Sigma \leq C\alpha_1$. Since

$$\left||u_{(\xi,\epsilon)} + w|^{\frac{2}{n-2}}(u_{(\xi,\epsilon)} + w) - |u_{(\xi,\epsilon)} + \tilde{w}|^{\frac{2}{n-2}}(u_{(\xi,\epsilon)} + \tilde{w}) - \frac{n}{n-2}u_{(\xi,\epsilon)}^{\frac{2}{n-2}}(w - \tilde{w})\right|$$

$$\leq C\left(|w|^{\frac{2}{n-2}} + |\tilde{w}|^{\frac{2}{n-2}}\right)|w - \tilde{w}|,$$

we have

$$\|\Phi_{(\xi,\epsilon)}(w) - \Phi_{(\xi,\epsilon)}(\tilde{w})\|_\Sigma$$

$$\leq C\left\|\left(|w|^{\frac{2}{n-2}} + |\tilde{w}|^{\frac{2}{n-2}}\right)(w - \tilde{w})\right\|_{L^{\frac{2(n-1)}{n}}(\partial \mathbb{R}^n_+)}$$

$$\leq C\left\{\|w\|_{L^{\frac{2(n-1)}{n-2}}(\partial \mathbb{R}^n_+)}^{\frac{2}{n-2}} + \|\tilde{w}\|_{L^{\frac{2(n-1)}{n-2}}(\partial \mathbb{R}^n_+)}^{\frac{2}{n-2}}\right\}\|w - \tilde{w}\|_{L^{\frac{2(n-1)}{n-2}}(\partial \mathbb{R}^n_+)}$$

for all $w, \tilde{w} \in \Sigma_{(\xi, \epsilon)}$. Hence, it follows from the estimate (2.3) that

$$\|\Phi_{(\xi, \epsilon)}(w) - \Phi_{(\xi, \epsilon)}(\tilde{w})\|_\Sigma \leq C \left(\|w\|_\Sigma^{\frac{2}{n-2}} + \|\tilde{w}\|_\Sigma^{\frac{2}{n-2}} \right) \|w - \tilde{w}\|_\Sigma$$

for any $w, \tilde{w} \in \Sigma_{(\xi, \epsilon)}$. Thus, for α_1 small, the contraction maximum principle implies that the mapping $\Phi_{(\xi, \epsilon)}$ has a fixed point $w_{(\xi, \epsilon)}$. Now the result follows from choosing $v_{(\xi, \epsilon)} = u_{(\xi, \epsilon)} + w_{(\xi, \epsilon)}$. Observe that $v_{(\xi, \epsilon)}$ cannot be identically zero because of (2.17) and Proposition 2.2 with $\alpha = \alpha_1$ small. $\qquad\square$

Given a pair $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$ we define

$$\mathcal{F}_g(\xi, \epsilon) = \int_{\mathbb{R}_+^n} (|dv_{(\xi, \epsilon)}|_g^2 + c_n R_g v_{(\xi, \epsilon)}^2) + \int_{\partial \mathbb{R}_+^n} d_n \kappa_g v_{(\xi, \epsilon)}^2 \qquad (2.19)$$

$$- \frac{(n-2)^2}{n-1} \int_{\partial \mathbb{R}_+^n} |v_{(\xi, \epsilon)}|^{\frac{2(n-1)}{n-2}} - \frac{n-2}{n-1} \int_{\partial \mathbb{R}_+^n} u_{(\xi, \epsilon)}^{\frac{2(n-1)}{n-2}} .$$

Proposition 2.9. *Suppose that* $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_1$ *for all* $x \in \mathbb{R}_+^n$, *where* α_1 *is the constant obtained in Proposition 2.8. Choosing* α_1 *smaller if necessary, the function* $\mathcal{F}_g$ *is continuously differentiable and, if* $(\bar{\xi}, \bar{\epsilon})$ *is a critical point of* $\mathcal{F}_g$, *then* $v_{(\bar{\xi}, \bar{\epsilon})}$ *is a positive smooth solution of*

$$\begin{cases} \Delta_g v_{(\bar{\xi}, \bar{\epsilon})} - c_n R_g v_{(\bar{\xi}, \bar{\epsilon})} = 0, & in \ \mathbb{R}_+^n, \\ \frac{\partial}{\partial x_n} v_{(\bar{\xi}, \bar{\epsilon})} - d_n \kappa_g v_{(\bar{\xi}, \bar{\epsilon})} + (n-2) v_{(\bar{\xi}, \bar{\epsilon})}^{\frac{n}{n-2}} = 0, & on \ \partial \mathbb{R}_+^n. \end{cases} \qquad (2.20)$$

In the proof of Proposition 2.9 we will use the following removable singularities theorem, which is a slight modification of Proposition 2.7 of [22]:

Lemma 2.10. *Let* (M^n, g) *be a Riemannian manifold with boundary* ∂M. *Let* $x \in \partial M$ *be a boundary point and* $\mathcal{U} \subset M$ *an open set containing* x. *Let* u *be a weak solution to*

$$\begin{cases} \Delta_g u + \phi u = 0, & in \ \mathcal{U} \setminus \{x\} \\ \frac{\partial u}{\partial \eta} + \psi u = 0, & on \ \mathcal{U} \cap \partial M \setminus \{x\}, \end{cases}$$

where $\phi \in L^{\frac{n}{2}}(\mathcal{U})$ *and* $\psi \in L^{n-1}(\mathcal{U} \cap \partial M)$. *Suppose that* $u \in L^q(\mathcal{U}) \cap L^p(\mathcal{U} \cap \partial M)$ *for some* $q > \frac{n}{n-2}$ *and* $p > \frac{n-1}{n-2}$. *Then* u *is a weak solution to*

$$\begin{cases} \Delta_g u + \phi u = 0, & in \ \mathcal{U}, \\ \frac{\partial u}{\partial \eta} + \psi u = 0, & on \ \mathcal{U} \cap \partial M. \end{cases}$$

Proof of Proposition 2.9. Given a pair $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$, by the definition of $v_{(\xi, \epsilon)}$, there exist $b_a(\xi, \epsilon) \in \mathbb{R}$, $a = 1, ..., n$, such that

$$\int_{\mathbb{R}_+^n} \left(< dv_{(\xi, \epsilon)}, d\psi >_g + c_n R_g v_{(\xi, \epsilon)} \psi \right) + \int_{\partial \mathbb{R}_+^n} \left(d_n \kappa_g v_{(\xi, \epsilon)} \psi - (n-2) |v_{(\xi, \epsilon)}|^{\frac{2}{n-2}} v_{(\xi, \epsilon)} \psi \right)$$

$$= \sum_{a=1}^n b_a(\xi, \epsilon) \cdot \int_{\partial \mathbb{R}_+^n} \phi_{(\xi, \epsilon, a)} \psi$$

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for any $\psi \in \Sigma$. Hence, derivating the expression (2.19) and observing the identity (2.2), we obtain

$$\frac{\partial \mathcal{F}_g}{\partial \epsilon}(\xi, \epsilon) = 2 \sum_{a=1}^{n} b_a(\xi, \epsilon) \cdot \int_{\partial \mathbb{R}_+^n} \phi_{(\xi, \epsilon, b)} \frac{\partial}{\partial \epsilon} v_{(\xi, \epsilon)}$$

and

$$\frac{\partial \mathcal{F}_g}{\partial \xi_k}(\xi, \epsilon) = 2 \sum_{a=1}^{n} b_a(\xi, \epsilon) \cdot \int_{\partial \mathbb{R}_+^n} \phi_{(\xi, \epsilon, a)} \frac{\partial}{\partial \xi_k} v_{(\xi, \epsilon)}, \quad k = 1, ..., n-1.$$

On the other hand,

$$\int_{\partial \mathbb{R}_+^n} \phi_{(\xi, \epsilon, a)} (v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)}) = 0, \quad a = 1, ..., n,$$

since $v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)} \in \Sigma_{(\xi, \epsilon)}$. This implies

$$0 = \int_{\partial \mathbb{R}_+^n} \frac{\partial}{\partial \epsilon} \phi_{(\xi, \epsilon, a)} (v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)}) + \int_{\partial \mathbb{R}_+^n} \phi_{(\xi, \epsilon, a)} \frac{\partial}{\partial \epsilon} (v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)})$$

$$= \int_{\partial \mathbb{R}_+^n} \frac{\partial}{\partial \epsilon} \phi_{(\xi, \epsilon, a)} (v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)}) + \int_{\partial \mathbb{R}_+^n} \phi_{(\xi, \epsilon, a)} \frac{\partial}{\partial \epsilon} v_{(\xi, \epsilon)} + \beta(n) \delta_{an} \epsilon^{-1}$$

and

$$0 = \int_{\partial \mathbb{R}_+^n} \frac{\partial}{\partial \xi_k} \phi_{(\xi, \epsilon, a)} (v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)}) + \int_{\partial \mathbb{R}_+^n} \phi_{(\xi, \epsilon, a)} \frac{\partial}{\partial \xi_k} v_{(\xi, \epsilon)} - \beta(n) \delta_{ak} \epsilon^{-1},$$

where

$$\beta(n) = -\epsilon \int_{\partial \mathbb{R}_+^n} \phi_{(\xi, \epsilon, n)} \frac{\partial}{\partial \epsilon} u_{(\xi, \epsilon)} = \epsilon \int_{\partial \mathbb{R}_+^n} \phi_{(\xi, \epsilon, k)} \frac{\partial}{\partial \xi_k} u_{(\xi, \epsilon)} > 0, \quad k = 1, ..., n-1.$$

Thus

$$-b_n(\xi, \epsilon) \beta(n) = \frac{\epsilon}{2} \frac{\partial \mathcal{F}_g}{\partial \epsilon}(\xi, \epsilon) + \epsilon \sum_{a=1}^{n} b_a(\xi, \epsilon) \cdot \int_{\partial \mathbb{R}_+^n} \frac{\partial}{\partial \epsilon} \phi_{(\xi, \epsilon, a)} (v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)}).$$

Similarly,

$$b_k(\xi, \epsilon) \beta(n) = \frac{\epsilon}{2} \frac{\partial \mathcal{F}_g}{\partial \xi_k}(\xi, \epsilon) + \epsilon \sum_{a=1}^{n} b_a(\xi, \epsilon) \cdot \int_{\partial \mathbb{R}_+^n} \frac{\partial}{\partial \xi_k} \phi_{(\xi, \epsilon, a)} (v_{(\xi, \epsilon)} - u_{(\xi, \epsilon)})$$

for $k = 1, ..., n-1$. Hence, if $(\bar{\xi}, \bar{\epsilon})$ is a critical point of $\mathcal{F}_g$, then there exists $C = C(n)$ such that

$$\sum_{a=1}^{n} |b_a(\bar{\xi}, \bar{\epsilon})| \le C \|v_{(\bar{\xi}, \bar{\epsilon})} - u_{(\bar{\xi}, \bar{\epsilon})}\|_{L^{\frac{2(n-1)}{n-2}}(\partial \mathbb{R}_+^n)} \sum_{a=1}^{n} |b_a(\bar{\xi}, \bar{\epsilon})|.$$

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By the estimate (2.3) and Propositions 2.2 and 2.8, $\|v_{(\bar{\xi},\bar{\epsilon})} - u_{(\bar{\xi},\bar{\epsilon})}\|_{L^{\frac{2(n-1)}{n-2}}(\partial \mathbb{R}^n_+)} \leq CK^{\frac{1}{2}}\alpha_1$. Thus, choosing α_1 small, we must have $b_a(\bar{\xi}, \bar{\epsilon}) = 0$ for $a = 1, ..., n$. Hence,

$$\int_{\mathbb{R}^n_+} \left(< dv_{(\bar{\xi},\bar{\epsilon})}, d\psi >_g + c_n R_g v_{(\bar{\xi},\bar{\epsilon})}\psi \right) \tag{2.21}$$

$$+ \int_{\partial \mathbb{R}^n_+} \left(d_n \kappa_g v_{(\bar{\xi},\bar{\epsilon})}\psi - (n-2)|v_{(\bar{\xi},\bar{\epsilon})}|^{\frac{2}{n-2}}v_{(\bar{\xi},\bar{\epsilon})}\psi \right) = 0$$

for any $\psi \in \Sigma$.

Now we are going to show that $v_{(\bar{\xi},\bar{\epsilon})} \geq 0$ on $\partial \mathbb{R}^n_+$. To that end, we set $\psi = \min\{v_{(\bar{\xi},\bar{\epsilon})}, 0\}$ and use the equation (2.21) to conclude that

$$\int_{\mathbb{R}^n_+ \cap \{v_{(\bar{\xi},\bar{\epsilon})}<0\}} \left(|dv_{(\bar{\xi},\bar{\epsilon})}|^2_g + c_n R_g v^2_{(\bar{\xi},\bar{\epsilon})} \right) \tag{2.22}$$

$$+ \int_{\partial \mathbb{R}^n_+ \cap \{v_{(\bar{\xi},\bar{\epsilon})}<0\}} d_n \kappa_g v^2_{(\bar{\xi},\bar{\epsilon})} = (n-2) \int_{\partial \mathbb{R}^n_+ \cap \{v_{(\bar{\xi},\bar{\epsilon})}<0\}} |v_{(\bar{\xi},\bar{\epsilon})}|^{\frac{2(n-1)}{n-2}} .$$

Using (2.11) with $w = \psi$ we see that

$$\left(\int_{\partial \mathbb{R}^n_+ \cap \{v_{(\bar{\xi},\bar{\epsilon})}<0\}} |v_{(\bar{\xi},\bar{\epsilon})}|^{\frac{2(n-1)}{n-2}} \right)^{\frac{n-2}{n-1}} \leq 2K \int_{\mathbb{R}^n_+ \cap \{v_{(\bar{\xi},\bar{\epsilon})}<0\}} \left(|dv_{(\bar{\xi},\bar{\epsilon})}|^2_g + c_n R_g v^2_{(\bar{\xi},\bar{\epsilon})} \right)$$

$$+ 2K \int_{\partial \mathbb{R}^n_+ \cap \{v_{(\bar{\xi},\bar{\epsilon})}<0\}} d_n \kappa_g v^2_{(\bar{\xi},\bar{\epsilon})} .$$

From this, together with (2.22), we deduce that $v_{(\bar{\xi},\bar{\epsilon})} \geq 0$ almost everywhere on $\partial \mathbb{R}^n_+$ or

$$\left(\int_{\partial \mathbb{R}^n_+ \cap \{v_{(\bar{\xi},\bar{\epsilon})}<0\}} |v_{(\bar{\xi},\bar{\epsilon})}|^{\frac{2(n-1)}{n-2}} \right)^{\frac{1}{n-1}} \geq \frac{1}{2K(n-2)} .$$

On the other hand,

$$\left(\int_{\partial \mathbb{R}^n_+ \cap \{v_{(\bar{\xi},\bar{\epsilon})}<0\}} |v_{(\bar{\xi},\bar{\epsilon})}|^{\frac{2(n-1)}{n-2}} \right)^{\frac{n-2}{2(n-1)}} \leq \left(\int_{\partial \mathbb{R}^n_+} |v_{(\bar{\xi},\bar{\epsilon})} - u_{(\bar{\xi},\bar{\epsilon})}|^{\frac{2(n-1)}{n-2}} \right)^{\frac{n-2}{2(n-1)}} \leq CK^{\frac{1}{2}}\alpha_1 .$$

Hence, choosing α_1 sufficiently small we have $v_{(\bar{\xi},\bar{\epsilon})} \geq 0$ on $\partial \mathbb{R}^n_+$. In particular, the equation (2.21) can be written as

$$\int_{\mathbb{R}^n_+} \left(< dv_{(\bar{\xi},\bar{\epsilon})}, d\psi >_g + c_n R_g v_{(\bar{\xi},\bar{\epsilon})}\psi \right) + \int_{\partial \mathbb{R}^n_+} \left(d_n \kappa_g v_{(\bar{\xi},\bar{\epsilon})}\psi - (n-2)v^{\frac{n}{n-2}}_{(\bar{\xi},\bar{\epsilon})}\psi \right) = 0$$

for any $\psi \in \Sigma$. By a result of Cherrier in [10], $v_{(\bar{\xi},\bar{\epsilon})}$ is smooth.

The fact that $v_{(\xi,\bar{\epsilon})} > 0$ in $\mathbb{R}^n_+$ is just a consequence of the maximum principle, as follows. We set $\tilde{g} = \tilde{u}^{\frac{4}{n-2}} g$, where $\tilde{u}(x) = (1 + |x|^2)^{\frac{2-n}{2}}$. Observe that $\tilde{u}$ satisfies $\Delta \tilde{u} + n(n-2)\tilde{u}^{\frac{n+2}{n-2}} = 0$ in $\mathbb{R}^n_+$ and we have

$$
\begin{aligned}
c_n R_{\tilde{g}} &= -\tilde{u}^{-\frac{n+2}{n-2}} \Delta \tilde{u} - \tilde{u}^{-\frac{n+2}{n-2}}(\Delta_g \tilde{u} - \Delta \tilde{u} - c_n R_g \tilde{u}) \\
&\geq n(n-2) - C\tilde{u}^{-\frac{n+2}{n-2}} \left\{ |h||\partial^2 \tilde{u}| + |\partial h||\partial \tilde{u}| + (|\partial^2 h| + |\partial h|^2)|\tilde{u}| \right\}.
\end{aligned}
$$

Using the facts that $h(x) = 0$ for $|x| \geq 1$ and $|h| + |\partial h| + |\partial^2 h| \leq C\alpha_1$ we can assume that $R_{\tilde{g}} > 0$, by choosing α_1 small .

Let S^n_+ be a hemisphere of $S^n_{1/2}$. We will use the well known conformal equivalence between $S^n_+ \backslash \{x_0\}$ and $\mathbb{R}^n_+$ realized by the stereographic projection, where $x_0 \in \partial S^n_+$. Under this equivalence, the standard metric on S^n_+ is written on $\mathbb{R}^n_+$ as $\tilde{u}^{\frac{4}{n-2}}\delta$, where δ is the Euclidean metric on $\mathbb{R}^n_+$. We set $\tilde{v} = \tilde{u}^{-1} v_{(\xi,\bar{\epsilon})}$. By the properties (2.8) of the operators $L_g = \Delta_g - c_n R_g$ and $B_g = \frac{\partial}{\partial \eta} - d_n \kappa_g$, we have

$$
L_{\tilde{g}}(\tilde{v}) = \tilde{u}^{-\frac{n+2}{n-2}} L_g v_{(\xi,\bar{\epsilon})} = 0, \quad \text{in } S^n_+,
$$

and

$$
B_{\tilde{g}}(\tilde{v}) + (n-2)\tilde{v}^{\frac{n}{n-2}} = \tilde{u}^{-\frac{n}{n-2}} B_g v_{(\xi,\bar{\epsilon})} + (n-2)(\tilde{u}^{-1} v_{(\xi,\bar{\epsilon})})^{\frac{n}{n-2}} = 0, \quad \text{on } \partial S^n_+.
$$

To establish the last two equations, we also used Lemma 2.10.

Since $R_{\tilde{g}} > 0$, it follows from the maximum principle in S^n_+ and the Hopf Lemma that if $\tilde{v} \geq 0$ on ∂S^n_+ then we have either $\tilde{v} > 0$ or $\tilde{v} \equiv 0$ in S^n_+. The latter contradicts the last assertion of Proposition 2.8. Hence, $\tilde{v} \geq 0$ on ∂S^n_+ implies that $\tilde{v} > 0$ in S^n_+. Since we have proved that $v_{(\xi,\bar{\epsilon})} \geq 0$ on $\partial \mathbb{R}^n_+$, we conclude that $v_{(\xi,\bar{\epsilon})} > 0$ in $\mathbb{R}^n_+$. $\qquad \square$

3   An estimate for the energy of a bubble

In this section we will show that the energy function $\mathcal{F}_g$ can be approximated by a certain auxiliary function.

We fix a multi-linear form $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying the algebraic properties of the Weyl tensor. We set

$$
|W|^2 = \sum_{a,b,c,d=1}^{n} (W_{acbd} + W_{adbc})^2
$$

and assume that $|W|^2 > 0$. Recall that throughout this article we work with indices $1 \leq i, j, k, l \leq n-1$ and $1 \leq a, b, c, d \leq n$ and set $\bar{x} = (x_1, ..., x_{n-1}, 0) \in \partial \mathbb{R}^n_+$ whenever $x = (x_1, ..., x_{n-1}, x_n) \in \mathbb{R}^n_+$. For $x \in \mathbb{R}^n_+$ we set

$$
H_{ij}(x) = H_{ij}(\bar{x}) = W_{ikjl} x^k x^l \quad \text{and} \quad H_{nb}(x) = 0
$$

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and define $\bar{H}_{ab}(x) = f(|\bar{x}|^2)H_{ab}(x)$, where

$$f(s) = \sum_{j=0}^{d} a_j s^j. \tag{3.1}$$

The integer $0 < d < \frac{n-6}{4}$ and the coefficients $a_0, ..., a_d \in \mathbb{R}$ will be chosen later. Observe that H is symmetric, trace-free, independent of the coordinate x_n and satisfies

$$x^a H_{ab}(x) = x^i H_{ib}(x) = 0 = \partial_a H_{ab}(x) = \partial_i H_{ib}(x), \quad \text{for any } x \in \mathbb{R}^n_+ .$$

We define a Riemannian metric $g = \exp(h)$ on $\mathbb{R}^n_+$ where h is a trace-free symmetric two tensor on $\mathbb{R}^n_+$ satisfying

$$\begin{cases} h_{ab}(x) = \mu \lambda^{2d} f(\lambda^{-2}|\bar{x}|^2)H_{ab}(x), & \text{for } |x| \leq \rho, \\ h_{ab}(x) = 0, & \text{for } |x| \geq 1. \end{cases}$$

Here, $\mu \leq 1$, $\lambda \leq \rho \leq 1$ and we suppose that $h_{nb}(x) = 0$ for any $x \in \mathbb{R}^n_+$ and $\partial_n h_{ab}(x) = 0$ for any $x \in \partial \mathbb{R}^n_+$. We also assume that $|h| + |\partial h| + |\partial^2 h| \leq \alpha_1$ where α_1 is the constant obtained in Proposition 2.8. Observe that

$$x^a h_{ab}(x) = x^i h_{ib}(x) = 0 = \partial_a h_{ab}(x) = \partial_i h_{ib}(x), \quad \text{for } |x| \leq \rho .$$

and $h_{ab}(x) = O(\mu(\lambda + |x|)^{2d+2})$. The second fundamental form of $\partial \mathbb{R}^n_+$ satisfies

$$\pi_{ij} = \Gamma^n_{ij} = \frac{1}{2}(g_{in,j} + g_{jn,i} - g_{ij,n}) = 0 .$$

In particular, the mean curvature of $\partial \mathbb{R}^n_+$ is given by $\kappa_g = \frac{1}{n-1}g^{ij}\pi_{ij} = 0$.

Using Proposition 2.8, for each pair $(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty)$ we choose $v_{(\xi,\epsilon)}$ to be the unique element of Σ such that $v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} \in \Sigma_{(\xi,\epsilon)}$ and

$$\int_{\mathbb{R}^n_+} \left(< dv_{(\xi,\epsilon)}, d\psi >_g + c_n R_g v_{(\xi,\epsilon)}\psi \right) - (n-2) \int_{\partial \mathbb{R}^n_+} |v_{(\xi,\epsilon)}|^{\frac{2}{n-2}} v_{(\xi,\epsilon)}\psi = 0$$

for all $\psi \in \Sigma_{(\xi,\epsilon)}$.

Finally, we define $\Omega = \{(\xi, \epsilon) \in \mathbb{R}^{n-1} \times (0, \infty); |\xi| < 1, \frac{1}{2} < \epsilon < 2\}$.

Proposition 3.1. *For any pair $(\xi, \epsilon) \in \lambda\Omega$ we have the estimates*

$$\left\|\Delta_g u_{(\xi,\epsilon)} - c_n R_g u_{(\xi,\epsilon)}\right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} \leq C\mu\lambda^{2d+2} + C\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}$$

and

$$\left\|\Delta_g u_{(\xi,\epsilon)} - c_n R_g u_{(\xi,\epsilon)} + \mu\lambda^{2d} f(\lambda^{-2}|\bar{x}|^2)H_{ij}\partial_i\partial_j u_{(\xi,\epsilon)}\right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} \leq C\mu^2\lambda^{4d+4} + C\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}.$$

Proof. We just observe that

$$|\Delta_g u_{(\xi,\epsilon)}(x) - c_n R_g(x) u_{(\xi,\epsilon)}(x)| \le C\mu\lambda^{\frac{n-2}{2}}(\lambda + |x|)^{2d+2-n}$$

and

$$|\Delta_g u_{(\xi,\epsilon)}(x) - c_n R_g(x) u_{(\xi,\epsilon)}(x) + \mu\lambda^{2d} f(\lambda^{-2}|\bar{x}|^2) H_{ij}(x)\partial_i\partial_j u_{(\xi,\epsilon)}(x)| \le C\mu^2\lambda^{\frac{n-2}{2}}(\lambda + |x|)^{4d+4-n}$$

for $|x| \le \rho$. In the last inequality we used the fact that, since $\partial_a h_{ab}(x) = 0$ for $|x| \le \rho$, Lemma 2.1 implies that $|R_g(x)| \le |\partial h(x)|^2 + |h(x)||\partial^2 h(x)|$ for $|x| \le \rho$. $\qquad\square$

Corollary 3.2. *For any pair $(\xi, \epsilon) \in \lambda\Omega$ we have the estimate*

$$\|v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)} + \|v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}\|_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}^n_+)} \le C\mu\lambda^{2d+2} + C\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}.$$

Proof. It follows from Proposition 2.8 and the estimate (2.3) that

$$\|v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)} + \|v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}\|_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}^n_+)} \le C\left\|\Delta_g u_{(\xi,\epsilon)} - c_n R_g u_{(\xi,\epsilon)}\right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)}$$

$$\le C\mu\lambda^{2d+2} + C\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}},$$

where we used Proposition 3.1 in the last inequality. $\qquad\square$

In order to refine the estimate of Corollary 3.2, using Proposition 2.7 with $h_{ab} = 0$ we choose the function $w_{(\xi,\epsilon)}$ to be the unique element of $\Sigma_{(\xi,\epsilon)}$ satisfying

$$\int_{\mathbb{R}^n_+} <dw_{(\xi,\epsilon)}, d\psi> - \int_{\partial\mathbb{R}^n_+} nu_{(\xi,\epsilon)}^{\frac{2}{n-2}} w_{(\xi,\epsilon)}\psi = -\int_{\mathbb{R}^n_+} \mu\lambda^{2d} f(\lambda^{-2}|\bar{x}|^2) H_{ij}\partial_i\partial_j u_{(\xi,\epsilon)}\psi$$

$$(3.2)$$

for all $\psi \in \Sigma_{(\xi,\epsilon)}$. Observe that, since $x^i H_{ij}(x) = 0$ for any $x \in \mathbb{R}^n_+$, we have $w_{(0,\epsilon)} = 0$.

Proposition 3.3. *The function $w_{(\xi,\epsilon)}$ is smooth and satisfies, for any pair $(\xi, \epsilon) \in \lambda\Omega$,*

$$|\partial^k w_{(\xi,\epsilon)}(x)| \le C\lambda^{\frac{n-2}{2}}\mu(\lambda + |x|)^{2d+4-k-n}, \quad \text{for all } x \in \mathbb{R}^n_+, \ k = 0, 1, 2.$$

Proof. First observe that there exist real numbers $b_a(\xi, \epsilon)$, $1 \le a \le n$, such that $w_{(\xi,\epsilon)}$ satisfies

$$\int_{\mathbb{R}^n_+} <dw_{(\xi,\epsilon)}, d\psi> - \int_{\partial\mathbb{R}^n_+} nu_{(\xi,\epsilon)}^{\frac{2}{n-2}} w_{(\xi,\epsilon)}\psi \qquad\qquad (3.3)$$

$$= -\int_{\mathbb{R}^n_+} \mu\lambda^{2d} f(\lambda^{-2}|\bar{x}|^2) H_{ij}\partial_i\partial_j u_{(\xi,\epsilon)}\psi + \sum_{a=1}^{n} b_a(\xi, \epsilon) \int_{\partial\mathbb{R}^n_+} \phi_{(a,\xi,\epsilon)}\psi$$

for all $\psi \in \Sigma$. Hence, it follows from standard elliptic theory that $w_{(\xi,\epsilon)}$ is smooth.

Now we are going to prove the pointwise estimates. Observe that

$$\left\| \mu\lambda^{2d} f(\lambda^{-2}|\bar{x}|^2) H_{ij}(x)\partial_i\partial_j u_{(\xi,\epsilon)}(x) \right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} \leq C\mu\lambda^{2d+2}. \qquad (3.4)$$

Then we apply Proposition 2.7 with $h_{ab} = 0$ and use the estimates (2.3) and (3.4) to conclude that

$$\|w_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)} + \|w_{(\xi,\epsilon)}\|_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}^n_+)} \leq K^{\frac{1}{2}}\|w_{(\xi,\epsilon)}\|_\Sigma \leq C\mu\lambda^{2d+2}.$$

Moreover, we can use the equation (3.3) with $\psi = \phi_{(\xi,\epsilon,a)}$ to conclude that

$$\sum_{a=0}^n |b_a(\xi,\epsilon)| \leq C\mu\lambda^{2d+2}.$$

Hence,

$$|\Delta w_{(\xi,\epsilon)}(x)| = \left|\mu\lambda^{2d} f(\lambda^{-2}|\bar{x}|^2) H_{ij}(x)\partial_i\partial_j u_{(\xi,\epsilon)}(x)\right| \leq \mu\lambda^{\frac{n-2}{2}}(\lambda + |x|)^{2d+2-n},$$

for all $x \in \mathbb{R}^n_+$, and

$$\left|\frac{\partial}{\partial x_n}w_{(\xi,\epsilon)}(x) + n u_{(\xi,\epsilon)}^{\frac{2}{n-2}}w_{(\xi,\epsilon)}(x)\right| = \left|-\sum_{a=1}^n b_a(\xi,\epsilon)\phi_{(a,\xi,\epsilon)}(x)\right| \leq \mu\lambda^{\frac{n}{2}}(\lambda + |x|)^{2d+2-n}$$

for all $x \in \partial\mathbb{R}^n_+$.

Claim. $\sup_{x\in\mathbb{R}^n_+}(\lambda + |x|)^{\frac{n-2}{2}}|w_{(\xi,\epsilon)}(x)| \leq C\mu\lambda^{2d+2}$

We fix $x_0 \in \mathbb{R}^n_+$ and set $r = \frac{1}{2}(\lambda + |x_0|)$. Then we see that

$$u_{(\xi,\epsilon)}^{\frac{2}{n-2}}(x) \leq Cr^{-1}, \quad \text{for all } x \in B_r^+(x_0),$$

$$\left|\frac{\partial}{\partial x_n}w_{(\xi,\epsilon)}(x) + n u_{(\xi,\epsilon)}^{\frac{2}{n-2}}w_{(\xi,\epsilon)}(x)\right| \leq C\mu\lambda^{\frac{n}{2}}r^{2d+2-n}, \quad \text{for all } x \in B_r^+(x_0) \cap \partial\mathbb{R}^n_+$$

and

$$|\Delta w_{(\xi,\epsilon)}(x)| \leq C\mu\lambda^{\frac{n-2}{2}}r^{2d+2-n}, \quad \text{for all } x \in B_r^+(x_0).$$

It follows from standard interior estimates that

$$r^{\frac{n-2}{2}}|w_{(\xi,\epsilon)}(x_0)| \leq C\|w_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n-2}}(B_r^+(x_0))} + Cr^{\frac{n+2}{2}}\|\Delta w_{(\xi,\epsilon)}\|_{L^\infty(B_r^+(x_0))}$$

$$+ Cr^{\frac{n}{2}}\left\|\frac{\partial}{\partial x_n}w_{(\xi,\epsilon)} + n u_{(\xi,\epsilon)}^{\frac{2}{n-2}}w_{(\xi,\epsilon)}\right\|_{L^\infty(B_r^+(x_0)\cap\partial\mathbb{R}^n_+)}$$

$$\leq C\mu\lambda^{2d+2} + C\mu\lambda^{\frac{n-2}{2}}r^{2d+2+\frac{2-n}{2}} + C\mu\lambda^{\frac{n}{2}}r^{2d+2-\frac{n}{2}}$$

$$\leq C\mu\lambda^{2d+2},$$

since we are assuming that $d < \frac{n-6}{4}$. This proves the Claim.

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Since $\sup_{x\in\mathbb{R}^n_+}|x|^{\frac{n-2}{2}}|w_{(\xi,\epsilon)}(x)| < \infty$, for all $x = (x_1,...,x_{n-1},x_n) \in \mathbb{R}^n_+$ we have

$$w_{(\xi,\epsilon)}(x) = -\frac{1}{(n-2)\sigma_{n-2}} \int_{\mathbb{R}^n_+} \left(|x-y|^{2-n} + |\tilde{x}-y|^{2-n}\right) \Delta w_{(\xi,\epsilon)}(y)dy$$

$$-\frac{1}{(n-2)\sigma_{n-2}} \int_{\partial\mathbb{R}^n_+} \left(|x-y|^{2-n} + |\tilde{x}-y|^{2-n}\right) \frac{\partial}{\partial y_n} w_{(\xi,\epsilon)}(y)dy,$$

where $\tilde{x} = (x_1,...,x_{n-1},-x_n)$. Now we use a bootstrap argument to prove the pointwise estimates. It follows from the last two inequalities that

$$\sup_{x\in\mathbb{R}^n_+}(\lambda + |x|)^\beta|w_{(\xi,\epsilon)}(x)| \leq C\sup_{x\in\mathbb{R}^n_+}(\lambda + |x|)^{\beta+2}|\Delta w_{(\xi,\epsilon)}(x)|$$

$$+ C\sup_{x\in\partial\mathbb{R}^n_+}(\lambda + |x|)^{\beta+1}\left|\frac{\partial}{\partial x_n}w_{(\xi,\epsilon)}(x)\right|$$

for all $0 < \beta < n-2$. Since

$$|\Delta w_{(\xi,\epsilon)}(x)| \leq \mu\lambda^{\frac{n-2}{2}}(\lambda + |x|)^{2d+2-n}, \quad \text{for all } x \in \mathbb{R}^n_+,$$

and

$$\left|\frac{\partial}{\partial x_n}w_{(\xi,\epsilon)}(x)\right| \leq nu_{(\xi,\epsilon)}^{\frac{2}{n-2}}(x)|w_{(\xi,\epsilon)}(x)| + \mu\lambda^{\frac{n}{2}}(\lambda + |x|)^{2d+2-n}, \quad \text{for all } x \in \partial\mathbb{R}^n_+,$$

we see that

$$\sup_{x\in\mathbb{R}^n_+}(\lambda + |x|)^\beta|w_{(\xi,\epsilon)}(x)| \leq C\lambda\sup_{x\in\partial\mathbb{R}^n_+}(\lambda + |x|)^{\beta-1}|w_{(\xi,\epsilon)}(x)| + C\mu\lambda^{\beta+2d+3-\frac{n}{2}}$$

for all $0 < \beta \leq n-4-2d$. Interating we obtain

$$\sup_{x\in\mathbb{R}^n_+}(\lambda + |x|)^{n-2d-4}|w_{(\xi,\epsilon)}(x)| \leq C\mu\lambda^{\frac{n-2}{2}}.$$

The derivative estimates follow from elliptic theory, finishing the proof. $\quad\square$

Corollary 3.4. *For any* $(\xi,\epsilon) \in \lambda\Omega$, *the function* $v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} - w_{(\xi,\epsilon)}$ *satisfies*

$$\|v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} - w_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)} + \|v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} - w_{(\xi,\epsilon)}\|_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}^n_+)}$$

$$\leq C\mu^{\frac{n}{n-2}}\lambda^{\frac{(2d+2)\cdot n}{n-2}} + C\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}.$$

Proof. It follows from the definition of $w_{(\xi,\epsilon)}$ that

$$\int_{\mathbb{R}^n_+} \left(< dw_{(\xi,\epsilon)}, d\psi >_g + c_n R_g w_{(\xi,\epsilon)}\psi\right) - \int_{\partial\mathbb{R}^n_+} nu_{(\xi,\epsilon)}^{\frac{2}{n-2}}w_{(\xi,\epsilon)}\psi$$

$$= -\int_{\mathbb{R}^n_+} \left\{\partial_j\left((g^{ij} - \delta_{ij})\partial_i w_{(\xi,\epsilon)}\right)\psi - c_n R_g w_{(\xi,\epsilon)}\psi\right\}$$

$$-\int_{\mathbb{R}^n_+} \mu\lambda^{2d}f(\lambda^{-2}|\tilde{x}|^2)H_{ij}\partial_i\partial_j u_{(\xi,\epsilon)}\psi,$$

for any $\psi \in \Sigma_{(\xi,\epsilon)}$. Hence we can write $w_{(\xi,\epsilon)} = -\mathcal{G}_{(\xi,\epsilon)}(B_1 + B_2, 0)$, where

$$B_1 = \partial_j\left((g^{ij} - \delta_{ij})\partial_i w_{(\xi,\epsilon)}\right) - c_n R_g w_{(\xi,\epsilon)},$$

$$B_2 = \mu \lambda^{2d} f(\lambda^{-2}|\bar{x}|^2) H_{ij} \partial_i \partial_j u_{(\xi,\epsilon)}$$

and $\mathcal{G}_{(\xi,\epsilon)}$ is the operator defined in the proof of Proposition 2.8.

On the other hand,

$$\int_{\mathbb{R}^n_+} \left\{ < d(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}), d\psi >_g + c_n R_g (v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)})\psi \right\}$$

$$- \int_{\partial \mathbb{R}^n_+} n u_{(\xi,\epsilon)}^{\frac{2}{n-2}} (v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)})\psi$$

$$= - \int_{\mathbb{R}^n_+} \left\{ < du_{(\xi,\epsilon)}, d\psi >_g + c_n R_g u_{(\xi,\epsilon)}\psi \right\}$$

$$+ \int_{\partial \mathbb{R}^n_+} \left\{ (n-2)|v_{(\xi,\epsilon)}|^{\frac{2}{n-2}} v_{(\xi,\epsilon)}\psi - n u_{(\xi,\epsilon)}^{\frac{2}{n-2}}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)})\psi \right\}$$

$$= \int_{\mathbb{R}^n_+} (\Delta_g u_{(\xi,\epsilon)} - c_n R_g u_{(\xi,\epsilon)})\psi$$

$$+ (n-2)\int_{\partial \mathbb{R}^n_+} \left\{ |v_{(\xi,\epsilon)}|^{\frac{2}{n-2}} v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}^{\frac{n}{n-2}} - \frac{n}{n-2} u_{(\xi,\epsilon)}^{\frac{2}{n-2}}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}) \right\} \psi.$$

Hence we can write $v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} = \mathcal{G}_{(\xi,\epsilon)}(B_3, (n-2)B_4)$, where

$$B_3 = \Delta_g u_{(\xi,\epsilon)} - c_n R_g u_{(\xi,\epsilon)},$$

$$B_4 = (|v_{(\xi,\epsilon)}|^{\frac{2}{n-2}} v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}^{\frac{n}{n-2}}) - \frac{n}{n-2} u_{(\xi,\epsilon)}^{\frac{2}{n-2}}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}).$$

Puting this facts together we conclude that

$$v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} - w_{(\xi,\epsilon)} = \mathcal{G}_{(\xi,\epsilon)}(B_1 + B_2 + B_3, (n-2)B_4).$$

Now we are going to estimate the terms B_1, B_2, B_3, B_4. Since

$$|B_1(x)| \leq C\partial(|h||\partial w_{(\xi,\epsilon)}|)(x) + C(|\partial^2 h||h| + |\partial h|^2)|w_{(\xi,\epsilon)}|(x)$$

$$\leq C\mu^2 \lambda^{\frac{n-2}{2}}(\lambda + |x|)^{4d+4-n}, \quad \text{for } |x| \leq \rho,$$

we have

$$\|B_1\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} \leq C\mu^2 \lambda^{4d+4} + C\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}.$$

It follows from Proposition 3.1 that

$$\|B_2 + B_3\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} \leq C\mu^2 \lambda^{4d+4} + C\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}.$$

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Since $|B_4(x)| \leq C|v_{(\xi,\epsilon)}(x) - u_{(\xi,\epsilon)}(x)|^{\frac{n}{n-2}}$ for any $x \in \partial\mathbb{R}_+^n$, we have

$$\|B_4\|_{L^{\frac{2(n-1)}{n}}(\partial\mathbb{R}_+^n)} \leq C\|v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}\|_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}_+^n)}^{\frac{n}{n-2}} \leq C\mu^{\frac{n}{n-2}}\lambda^{\frac{(2d+2)n}{n-2}} + C\left(\frac{\lambda}{\rho}\right)^{\frac{n}{2}},$$

where in the last inequality we used Corollary 3.2.

Using the estimates above we see that

$$\|B_1 + B_2 + B_3\|_{L^{\frac{2n}{n+2}}(\mathbb{R}_+^n)} + \|(n-2)B_4\|_{L^{\frac{2(n-1)}{n}}(\partial\mathbb{R}_+^n)} \leq C\mu^{\frac{n}{n-2}}\lambda^{\frac{(2d+2)n}{n-2}} + C\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}.$$

Hence,

$$\|v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} - w_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}_+^n)} + \|v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} - w_{(\xi,\epsilon)}\|_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}_+^n)} \leq C\mu^{\frac{n}{n-2}}\lambda^{\frac{(2d+2)n}{n-2}} + C\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}.$$

$\square$

Lemma 3.5. *For any $(\xi,\epsilon) \in \lambda\Omega$ we have the estimate*

$$\left|\int_{\partial\mathbb{R}_+^n}(|v_{(\xi,\epsilon)}|^{\frac{2}{n-2}} - u_{(\xi,\epsilon)}^{\frac{2}{n-2}})u_{(\xi,\epsilon)}v_{(\xi,\epsilon)} - \frac{1}{n-1}\int_{\partial\mathbb{R}_+^n}(|v_{(\xi,\epsilon)}|^{\frac{2(n-1)}{n-2}} - u_{(\xi,\epsilon)}^{\frac{2(n-1)}{n-2}})\right|$$

$$\leq C\mu^{\frac{2(n-1)}{n-2}}\lambda^{\frac{(4d+4)(n-1)}{n-2}} + C\left(\frac{\lambda}{\rho}\right)^{n-1}.$$

Proof. It follows from the pointwise estimate

$$\left|\left(|v_{(\xi,\epsilon)}|^{\frac{2}{n-2}} - u_{(\xi,\epsilon)}^{\frac{2}{n-2}}\right)u_{(\xi,\epsilon)}v_{(\xi,\epsilon)} - \frac{1}{n-1}\left(|v_{(\xi,\epsilon)}|^{\frac{2(n-1)}{n-2}} - u_{(\xi,\epsilon)}^{\frac{2(n-1)}{n-2}}\right)\right| \leq C|v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}|^{\frac{2(n-1)}{n-2}}$$

that

$$\left|\int_{\partial\mathbb{R}_+^n}(|v_{(\xi,\epsilon)}|^{\frac{2}{n-2}} - u_{(\xi,\epsilon)}^{\frac{2}{n-2}})u_{(\xi,\epsilon)}v_{(\xi,\epsilon)} - \frac{1}{n-1}\int_{\partial\mathbb{R}_+^n}(|v_{(\xi,\epsilon)}|^{\frac{2(n-1)}{n-2}} - u_{(\xi,\epsilon)}^{\frac{2(n-1)}{n-2}})\right|$$

$$\leq C\|v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}\|_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}_+^n)}^{\frac{2(n-1)}{n-2}} \leq C\left(\mu\lambda^{2d+2} + \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}\right)^{\frac{2(n-1)}{n-2}},$$

where in the last inequality we used Corollary 3.2. Now the result follows. $\square$

Proposition 3.6. *Let $\mathcal{F}_g$ be the function defined by the formula (2.19). For any pair $(\xi,\epsilon) \in \lambda\Omega$ we have the estimate*

$$\left|\mathcal{F}_g(\xi,\epsilon) - \frac{1}{2}\int_{B_\rho^+(0)}h_{il}h_{jl}\partial_i u_{(\xi,\epsilon)}\partial_j u_{(\xi,\epsilon)} + \frac{c_n}{4}\int_{B_\rho^+(0)}(\partial_l h_{ij})^2 u_{(\xi,\epsilon)}^2\right.$$

$$\left. - \int_{\mathbb{R}_+^n}\mu\lambda^{2d}f(\lambda^{-2}|\bar{x}|^2)H_{ij}\partial_i\partial_j u_{(\xi,\epsilon)}\,w_{(\xi,\epsilon)}\right|$$

$$\leq C\mu^{\frac{2(n-1)}{n-2}}\lambda^{\frac{(4d+4)(n-1)}{n-2}} + C\mu\lambda^{2d+2}\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}} + C\left(\frac{\lambda}{\rho}\right)^{n-2}.$$

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Proof. It follows from the definition of $v_{(\xi,\epsilon)}$ that

$$\int_{\mathbb{R}^n_+} \left\{ < dv_{(\xi,\epsilon)}, d(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}) >_g + c_n R_g v_{(\xi,\epsilon)}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}) \right\}$$

$$- (n-2) \int_{\partial\mathbb{R}^n_+} |v_{(\xi,\epsilon)}|^{\frac{2}{n-2}} v_{(\xi,\epsilon)}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}) = 0$$

Thus,

$$\int_{\mathbb{R}^n_+} \left\{ |dv_{(\xi,\epsilon)}|^2_g - < dv_{(\xi,\epsilon)}, du_{(\xi,\epsilon)} >_g + c_n R_g (v^2_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} v_{(\xi,\epsilon)}) \right\} \tag{3.5}$$

$$- (n-2) \int_{\partial\mathbb{R}^n_+} \left\{ |v_{(\xi,\epsilon)}|^{\frac{2(n-1)}{n-2}} - |v_{(\xi,\epsilon)}|^{\frac{2}{n-2}} v_{(\xi,\epsilon)} u_{(\xi,\epsilon)} \right\} = 0 \,.$$

We set

$$\varrho = \int_{\mathbb{R}^n_+} \left\{ < du_{(\xi,\epsilon)}, d(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}) >_g + c_n R_g u_{(\xi,\epsilon)}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}) \right\} \tag{3.6}$$

$$- \int_{\mathbb{R}^n_+} h_{ij} \partial_i \partial_j u_{(\xi,\epsilon)}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}) - (n-2) \int_{\partial\mathbb{R}^n_+} u^{\frac{n}{n-2}}_{(\xi,\epsilon)}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)})$$

Thus,

$$\varrho = \int_{\mathbb{R}^n_+} \left\{ -|du_{(\xi,\epsilon)}|^2_g + < du_{(\xi,\epsilon)}, dv_{(\xi,\epsilon)} >_g + c_n R_g (u_{(\xi,\epsilon)} v_{(\xi,\epsilon)} - u^2_{(\xi,\epsilon)}) \right\}$$

$$- \int_{\mathbb{R}^n_+} h_{ij} \partial_i \partial_j u_{(\xi,\epsilon)}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}) - (n-2) \int_{\partial\mathbb{R}^n_+} \left\{ u^{\frac{n}{n-2}}_{(\xi,\epsilon)} v_{(\xi,\epsilon)} - u^{\frac{2(n-1)}{n-2}}_{(\xi,\epsilon)} \right\}$$

Hence, summing (3.5) and (3.7),

$$\varrho = \int_{\mathbb{R}^n_+} \left\{ |dv_{(\xi,\epsilon)}|^2_g - |du_{(\xi,\epsilon)}|^2_g + c_n R_g (v^2_{(\xi,\epsilon)} - u^2_{(\xi,\epsilon)}) \right\} \tag{3.7}$$

$$- \int_{\mathbb{R}^n_+} h_{ij} \partial_i \partial_j u_{(\xi,\epsilon)}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)})$$

$$- (n-2) \int_{\partial\mathbb{R}^n_+} \left\{ (|v_{(\xi,\epsilon)}|^{\frac{2(n-1)}{n-2}} - u^{\frac{2(n-1)}{n-2}}_{(\xi,\epsilon)}) + (u^{\frac{2}{n-2}}_{(\xi,\epsilon)} - |v_{(\xi,\epsilon)}|^{\frac{2}{n-2}}) u_{(\xi,\epsilon)} v_{(\xi,\epsilon)} \right\} \,.$$

Then

$$\varrho = \int_{\mathbb{R}^n_+} \left\{ |dv_{(\xi,\epsilon)}|^2_g + c_n R_g v^2_{(\xi,\epsilon)} \right\} - \int_{\partial\mathbb{R}^n_+} \left\{ \frac{(n-2)^2}{n-1} |v_{(\xi,\epsilon)}|^{\frac{2(n-1)}{n-2}} + \frac{n-2}{n-1} u^{\frac{2(n-1)}{n-2}}_{(\xi,\epsilon)} \right\} \tag{3.8}$$

$$- \int_{\partial\mathbb{R}^n_+} \left\{ \frac{n-2}{n-1} |v_{(\xi,\epsilon)}|^{\frac{2(n-1)}{n-2}} + \frac{(n-2)^2}{n-1} u^{\frac{2(n-1)}{n-2}}_{(\xi,\epsilon)} \right\}$$

$$- (n-2) \int_{\partial\mathbb{R}^n_+} (u^{\frac{2}{n-2}}_{(\xi,\epsilon)} - |v_{(\xi,\epsilon)}|^{\frac{2}{n-2}}) u_{(\xi,\epsilon)} v_{(\xi,\epsilon)} + 2(n-2) \int_{\partial\mathbb{R}^n_+} u^{\frac{2(n-1)}{n-2}}_{(\xi,\epsilon)}$$

$$- \int_{\mathbb{R}^n_+} \left\{ |du_{(\xi,\epsilon)}|^2_g + c_n R_g u^2_{(\xi,\epsilon)} + h_{ij} \partial_i \partial_j u_{(\xi,\epsilon)}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}) \right\} \,.$$

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We set

$$B = \int_{\mathbb{R}^n_+} \left\{ |du_{(\xi,\epsilon)}|_g^2 - |du_{(\xi,\epsilon)}|^2 + c_n R_g u_{(\xi,\epsilon)}^2 + h_{ij}\partial_i\partial_j u_{(\xi,\epsilon)}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}) \right\}$$

and observe that $\int_{\mathbb{R}^n_+} |du_{(\xi,\epsilon)}|^2 = (n-2)\int_{\partial\mathbb{R}^n_+} u_{(\xi,\epsilon)}^{\frac{2(n-1)}{n-2}}$. Hence,

$$\mathcal{F}_g(\xi,\epsilon) - B = \frac{n-2}{n-1}\int_{\partial\mathbb{R}^n_+} \left\{ |v_{(\xi,\epsilon)}|^{\frac{2(n-1)}{n-2}} - u_{(\xi,\epsilon)}^{\frac{2(n-1)}{n-2}} \right\} \tag{3.9}$$

$$- (n-2)\int_{\partial\mathbb{R}^n_+} (|v_{(\xi,\epsilon)}|^{\frac{2}{n-2}} - u_{(\xi,\epsilon)}^{\frac{2}{n-2}})\, u_{(\xi,\epsilon)} v_{(\xi,\epsilon)} + \varrho$$

$$= O\left\{ \lambda^{\frac{(4d+4)(n-1)}{n-2}} \mu^{\frac{2(n-1)}{n-2}} + \left(\frac{\lambda}{\rho}\right)^{n-1} \right\} + \varrho$$

where in the last inequality we used Lemma 3.5.

On the other hand,

$$B = \frac{1}{2}\int_{B_\rho^+(0)} h_{il}h_{jl}\partial_i u_{(\xi,\epsilon)}\partial_j u_{(\xi,\epsilon)} - \frac{c_n}{4}\int_{B_\rho^+(0)} (\partial_l h_{ij})^2 u_{(\xi,\epsilon)}^2 \tag{3.10}$$

$$+ \int_{\mathbb{R}^n_+} \mu\lambda^{2d} f(\lambda^{-2}|\bar{x}|^2) H_{ij}\partial_i\partial_j u_{(\xi,\epsilon)} w_{(\xi,\epsilon)} + e_1 + e_2 + e_3 + e_4 + e_5$$

where

$$e_1 = -\int_{\mathbb{R}^n_+} h_{ij}\partial_i u_{(\xi,\epsilon)}\partial_j u_{(\xi,\epsilon)} + c_n\int_{\mathbb{R}^n_+} \partial_i\partial_j h_{ij}\, u_{(\xi,\epsilon)}^2\,,$$

$$e_2 = \int_{\mathbb{R}^n_+} (g^{ij} - \delta_{ij} + h_{ij})\partial_i u_{(\xi,\epsilon)}\partial_j u_{(\xi,\epsilon)} - \int_{B_\rho^+(0)} \frac{1}{2}h_{il}h_{jl}\partial_i u_{(\xi,\epsilon)}\partial_j u_{(\xi,\epsilon)}$$

$$= \int_{\mathbb{R}^n_+\setminus B_\rho^+(0)} (g^{ij} - \delta_{ij} + h_{ij})\partial_i u_{(\xi,\epsilon)}\partial_j u_{(\xi,\epsilon)}$$

$$+ \int_{B_\rho^+(0)} \left\{ g^{ij} - \delta_{ij} + h_{ij} - \frac{1}{2}h_{il}h_{jl} \right\} \partial_i u_{(\xi,\epsilon)}\partial_j u_{(\xi,\epsilon)}\,,$$

$$e_3 = c_n\int_{\mathbb{R}^n_+} (R_g - \partial_i\partial_j h_{ij})\, u_{(\xi,\epsilon)}^2 + c_n\int_{B_\rho^+(0)} \frac{1}{4}(\partial_l h_{ij})^2 u_{(\xi,\epsilon)}^2$$

$$= c_n\int_{\mathbb{R}^n_+\setminus B_\rho^+(0)} (R_g - \partial_i\partial_j h_{ij})\, u_{(\xi,\epsilon)}^2 + c_n\int_{B_\rho^+(0)} \left\{ R_g + \frac{1}{4}(\partial_l h_{ij})^2 \right\} u_{(\xi,\epsilon)}^2\,,$$

$$e_4 = \int_{\mathbb{R}^n_+} h_{ij}\partial_i\partial_j u_{(\xi,\epsilon)}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} - w_{(\xi,\epsilon)})\,,$$

$$e_5 = -\int_{\mathbb{R}^n_+\setminus B_\rho^+(0)} \mu\lambda^{2d} f(\lambda^{-2}|\bar{x}|^2) H_{ij}\partial_i\partial_j u_{(\xi,\epsilon)} w_{(\xi,\epsilon)} + \int_{\mathbb{R}^n_+\setminus B_\rho^+(0)} h_{ij}\partial_i\partial_j u_{(\xi,\epsilon)}\, w_{(\xi,\epsilon)}\,.$$

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For the expression of e_3 we used the fact that $\partial_j h_{ij}(x) = 0$ for $|x| \le \rho$. We are going to use this same fact in the rest of this proof.

Now we are going to estimate the terms $e_1, ..., e_5$. First observe that for $|x| \le \rho$ we have

$$\left| g^{ij}(x) - \delta_{ij} + h_{ij}(x) - \frac{1}{2} h_{ij} h_{jl}(x) \right| \le C|h(x)|^3 \le C\mu^3 (\lambda + |x|)^{6d+6} \qquad (3.11)$$

$$\le C\mu^3 (\lambda + |x|)^{\frac{n-1}{n-2}(4d+4)}$$

and

$$\left| R_g(x) + \frac{1}{4} (\partial_l h_{ij})^2(x) \right| \le C|h(x)|^2 |\partial^2 h(x)| + C|h(x)||\partial h(x)|^2 \qquad (3.12)$$

$$\le C\mu^3 (\lambda + |x|)^{6d+4} \le C\mu^3 (\lambda + |x|)^{\frac{n-1}{n-2}(4d+4)-2}.$$

Here, we used Lemma 2.1.

From the identity $u_{(\xi,\epsilon)} \partial_i \partial_j u_{(\xi,\epsilon)} - \frac{n}{n-2} \partial_i u_{(\xi,\epsilon)} \partial_j u_{(\xi,\epsilon)} = -\frac{1}{n-2} |du_{(\xi,\epsilon)}|^2 \delta_{ij}$ and the fact that $\sum_{j=1}^{n-1} h_{jj} = 0$ we see that

$$\frac{n}{n-2} \int_{\mathbb{R}^n_+} h_{ij} \partial_i u_{(\xi,\epsilon)} \partial_j u_{(\xi,\epsilon)} = \int_{\mathbb{R}^n_+} h_{ij} u_{(\xi,\epsilon)} \partial_i \partial_j u_{(\xi,\epsilon)}$$

$$= -\int_{\mathbb{R}^n_+} \partial_i h_{ij} \, u_{(\xi,\epsilon)} \partial_j u_{(\xi,\epsilon)} - \int_{\mathbb{R}^n_+} h_{ij} \partial_i u_{(\xi,\epsilon)} \partial_j u_{(\xi,\epsilon)},$$

where in the last equality we integrated by parts. Thus,

$$e_1 = \frac{n-2}{2(n-1)} \int_{\mathbb{R}^n_+ \backslash B^+_\rho(0)} \partial_i h_{ij} \, u_{(\xi,\epsilon)} \partial_j u_{(\xi,\epsilon)} + c_n \int_{\mathbb{R}^n_+ \backslash B^+_\rho(0)} \partial_i \partial_j h_{ij} \, u^2_{(\xi,\epsilon)}. \qquad (3.13)$$

Then we use the identities (3.13), (3.11) and (3.12) to estimate e_1, e_2 and e_3

respectively and conclude that

$$|e_1| \leq C\rho \left(\frac{\lambda}{\rho}\right)^{n-2}, \tag{3.14}$$

$$|e_2| \leq C\left(\frac{\lambda}{\rho}\right)^{n-2} + C\mu^3 \lambda^{\frac{n-1}{n-2}(4d+4)},$$

$$|e_3| \leq C\rho^2\left(\frac{\lambda}{\rho}\right)^{n-2} + C\mu^3 \lambda^{\frac{n-1}{n-2}(4d+4)}$$

$$|e_4| \leq C\int_{\mathbb{R}^n_+} |h||\partial^2 u_{(\xi,\epsilon)}||v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} - w_{(\xi,\epsilon)}|$$

$$\leq C\|h\partial^2 u_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} \|v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)} - w_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)}$$

$$\leq C\left\{\mu\lambda^{2d+2} + \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}\right\} \cdot \left\{\mu^{\frac{n}{n-2}}\lambda^{\frac{(2d+2)n}{n-2}} + \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}\right\}$$

$$\leq C\mu^{\frac{2(n-1)}{n-2}}\lambda^{\frac{(4d+4)(n-1)}{n-2}} + \mu\lambda^{2d+2}\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}} + \mu^{\frac{n}{n-2}}\lambda^{\frac{(2d+2)n}{n-2}}\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}} + \left(\frac{\lambda}{\rho}\right)^{n-2},$$

$$|e_5| \leq \rho^{2d+2}\left(\frac{\lambda}{\rho}\right)^{n-2}.$$

Now we are going to estimate ϱ using its definition (equation (3.6)). Integrating by parts and using the second equation of (2.1), we obtain

$$|\varrho| \leq \int_{\mathbb{R}^n_+} \left| -\Delta_g u_{(\xi,\epsilon)}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}) + c_n R_g u_{(\xi,\epsilon)}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}) \right. \tag{3.15}$$

$$\left. - h_{ij}\partial_i\partial_j u_{(\xi,\epsilon)}(v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}) \right|$$

$$\leq \left\|\Delta_g u_{(\xi,\epsilon)} - c_n R_g u_{(\xi,\epsilon)} + h_{ij}\partial_i\partial_j u_{(\xi,\epsilon)}\right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n_+)} \|v_{(\xi,\epsilon)} - u_{(\xi,\epsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)}$$

$$\leq C\left\{\mu^2\lambda^{4d+4} + \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}\right\} \cdot \left\{\mu\lambda^{2d+2} + \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}\right\}$$

$$\leq C\mu^3\lambda^{6d+6} + C\mu\lambda^{2d+2}\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}} + C\left(\frac{\lambda}{\rho}\right)^{n-2}.$$

Here, we used Proposition 3.1 and Corollary 3.2 in the second inequality.

The result now follows from (3.9), (3.10), (3.14) and (3.15). □

4 Finding a critical point of an auxiliary function

Let us follow the notations of the last section. We define

$$F(\xi,\epsilon) = \frac{1}{2}\int_{\mathbb{R}^n_+} \bar{H}_{il}\bar{H}_{jl}\partial_i u_{(\xi,\epsilon)}\partial_j u_{(\xi,\epsilon)} - \frac{c_n}{4}\int_{\mathbb{R}^n_+} (\partial_l\bar{H}_{ij})^2 u_{(\xi,\epsilon)}^2 + \int_{\mathbb{R}^n_+} \bar{H}_{ij}\partial_i\partial_j u_{(\xi,\epsilon)}\, z_{(\xi,\epsilon)}$$

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where $z_{(\xi,\epsilon)}$ is the unique element of $\Sigma_{(\xi,\epsilon)}$ that satisfies

$$\int_{\mathbb{R}^n_+} <dz_{(\xi,\epsilon)}, d\psi> - \int_{\partial\mathbb{R}^n_+} nu^{\frac{2}{n-2}}_{(\xi,\epsilon)} z_{(\xi,\epsilon)}\psi = -\int_{\mathbb{R}^n_+} \bar{H}_{ij}\partial_i\partial_j u_{(\xi,\epsilon)}\psi \qquad (4.1)$$

for any $\psi \in \Sigma_{(\xi,\epsilon)}$. The function $z_{(\xi,\epsilon)}$ is obtained in Proposition 2.7 with $h_{ab} = 0$.

In this section we will show that the function $F(\xi,\epsilon)$ has a critical point, which is a strict local minimum. Recall that throughout this article we use indices $1 \le i, i, j, k, l, m, p, q, r, s \le n - 1$.

Since $\bar{H}_{ab}(-x) = \bar{H}_{ab}(x)$ for any $x \in \mathbb{R}^n_+$, the function $F(\xi,\epsilon)$ satisfies $F(\xi,\epsilon) = F(-\xi,\epsilon)$ for all $(\xi,\epsilon) \in \mathbb{R}^{n-1} \times (0,\infty)$. In particular,

$$\frac{\partial}{\partial\xi_p}F(0,\epsilon) = \frac{\partial^2}{\partial\epsilon\partial\xi_p}F(0,\epsilon) = 0, \quad \text{for all } \epsilon > 0. \qquad (4.2)$$

Proposition 4.1. *We have*

$$\int_{S^{n-2}_r} (\partial_l H_{ij})^2(x)x^p x^q = \frac{2\sigma_{n-2}r^{n+2}}{(n-1)(n+1)}(W_{ipjl} + W_{iljp})(W_{iqjl} + W_{iljq})$$

$$+ \frac{\sigma_{n-2}r^{n+2}}{(n-1)(n+1)}(W_{ikjl} + W_{iljk})^2\delta_{pq}$$

and

$$\int_{S^{n-2}_r} (H_{ij})^2(x)x^p x^q = \frac{2\sigma_{n-2}r^{n+4}}{(n-1)(n+1)(n+3)}(W_{ipjl} + W_{iljp})(W_{iqjl} + W_{iljq})$$

$$+ \frac{\sigma_{n-2}r^{n+4}}{2(n-1)(n+1)(n+3)}(W_{ikjl} + W_{iljk})^2\delta_{pq}.$$

Proof. Observe that

$$\int_{S^{n-2}_r} (\partial_l H_{ij})^2(x)x^p x^q = \int_{S^{n-2}_r} (W_{iljr} + W_{irjl})(W_{iljm} + W_{imjl})x^r x^m x^p x^q$$

and

$$\int_{S^{n-2}_r} (H_{ij})^2(x)x^p x^q = \int_{S^{n-2}_r} W_{ikjl}W_{irjm}x^k x^l x^r x^m x^p x^q.$$

Now we just need to apply Corollary A-3 in the Appendix. $\qquad\square$

Proposition 4.2. *We have*

$$\int_{S^{n-2}_r} (\partial_l\bar{H}_{ij})^2(x)x^p x^q = \frac{2\sigma_{n-2}r^{n+2}}{(n-1)(n+1)(n+3)}(W_{ipjl} + W_{iljp})(W_{iqjl} + W_{iljq})$$

$$\cdot \left\{(n+3)f(r^2)^2 + 8r^2 f(r^2)f'(r^2) + 4r^4 f'(r^2)^2\right\}$$

$$+ \frac{\sigma_{n-2}r^{n+2}}{(n-1)(n+1)(n+3)}(W_{ikjl} + W_{iljk})^2\delta_{pq}$$

$$\cdot \left\{(n+3)f(r^2)^2 + 4r^2 f(r^2)f'(r^2) + 2r^4 f'(r^2)^2\right\}.$$

Proof. Since

$$\partial_l \bar{H}_{ij}(x) = f(|\bar{x}|^2)\partial_l H_{ij}(x) + 2f'(|\bar{x}|^2)x^l H_{ij}(x)$$

we obtain

$$(\partial_l \bar{H}_{ij})^2(x) = f(|\bar{x}|^2)^2(\partial_l H_{ij})^2(x) + 4f(|\bar{x}|^2)f'(|\bar{x}|^2)x^l\partial_l H_{ij}H_{ij}(x) + 4|\bar{x}|^2 f'(|\bar{x}|^2)^2(H_{ij})^2$$
$$= f(|\bar{x}|^2)^2(\partial_l H_{ij})^2(x) + 8f(|\bar{x}|^2)f'(|\bar{x}|^2)(H_{ij})^2(x) + 4|\bar{x}|^2 f'(|\bar{x}|^2)^2(H_{ij})^2(x)$$

Hence,

$$\int_{S_r^{n-2}} (\partial_l \bar{H}_{ij})^2(x)x^p x^q = f(|\bar{x}|^2)^2 \int_{S_r^{n-2}} (\partial_l H_{ij})^2(x)x^p x^q$$
$$+ \left(8f(|\bar{x}|^2)f'(|\bar{x}|^2) + 4r^2 f'(|\bar{x}|^2)^2\right) \int_{S_r^{n-2}} (H_{ij})^2(x)x^p x^q$$

and the result follows from Proposition 4.1. $\qquad\square$

Corollary 4.3. *We have*

$$\int_{S_r^{n-2}} (\partial_l \bar{H}_{ij})^2(x) =$$
$$\frac{\sigma_{n-2}r^n}{(n-1)(n+1)}(W_{ikjl} + W_{iljk})^2 \left\{(n+1)f(r^2)^2 + 4r^2 f(r^2)f'(r^2) + 2r^4 f'(r^2)^2\right\}.$$

Proof. By Proposition 4.2,

$$r^2 \int_{S_r^{n-2}} (\partial_l \bar{H}_{ij})^2(x) = \sum_{p=1}^{n-1} \int_{S_r^{n-2}} (\partial_l \bar{H}_{ij})^2(x)(x^p)^2$$

$$= \frac{2\sigma_{n-2}r^{n+2}}{(n-1)(n+1)(n+3)}(W_{ikjl} + W_{iljk})^2 \left\{(n+3)f(r^2)^2 + 8r^2 f(r^2)f'(r^2) + 4r^4 f'(r^2)^2\right\}$$

$$+ \frac{\sigma_{n-2}r^{n+2}}{(n-1)(n+1)(n+3)}(n-1)(W_{ikjl} + W_{iljk})^2$$
$$\cdot \left\{(n+3)f(r^2)^2 + 4r^2 f(r^2)f'(r^2) + 2r^4 f'(r^2)^2\right\}$$

$$= \frac{\sigma_{n-2}r^{n+2}}{(n-1)(n+1)}(W_{ikjl} + W_{iljk})^2 \left\{(n+1)f(r^2)^2 + 4r^2 f(r^2)f'(r^2) + 2r^4 f'(r^2)^2\right\}.$$

$\qquad\square$

Proposition 4.4. *We have*

$$F(0,\epsilon) = -\frac{c_n \cdot \sigma_{n-2}}{4(n-1)(n+1)}(W_{ikjl} + W_{iljk})^2$$
$$\cdot \int_0^\infty \int_0^\infty r^n \left\{(n+1)f(r^2)^2 + 4r^2 f(r^2)f'(r^2) + 2r^4 f'(r^2)^2\right\} \epsilon^{n-2}((\epsilon+t)^2 + r^2)^{2-n} dr dt.$$

Proof. It follows from symmetry arguments that $z_{(0,\epsilon)} = 0$ and

$$\int_{S_r^{n-2}} \bar{H}_{il}\bar{H}_{jl}\partial_i u_{(0,\epsilon)}\partial_j u_{(0,\epsilon)}(x)$$

$$= \int_{S_r^{n-2}} \frac{(n-2)^2\epsilon^{n-2}}{((\epsilon+x_n)^2+|\bar{x}|^2)^n} f(|\bar{x}|^2)^2 W_{iplq}W_{jrlm}x^i x^j x^p x^q x^r x^m = 0.$$

Hence, we have

$$F(0,\epsilon) = -\frac{c_n}{4}\int_{\mathbb{R}_+^n}(\partial_l\bar{H}_{ij})^2(x)\, u_{(0,\epsilon)}^2(x)$$

$$= -\frac{c_n}{4}\int_0^\infty\int_0^\infty\int_{S_r^{n-2}}(\partial_l\bar{H}_{ij})^2(x)\, u_{(0,\epsilon)}^2(x)\, d\sigma_r(x)\, dr\, dx_n .$$

The result now follows from Corollary 4.3. $\qquad\qquad\square$

We write

$$F(0,\epsilon) = -\beta_n \cdot \sum_{q=0}^{2d} \alpha_q \int_0^\infty\int_0^\infty r^{2q+n}\epsilon^{n-2}((\epsilon+t)^2+r^2)^{2-n}dr\, dt,$$

where
$$\beta_n = \frac{c_n\cdot\sigma_{n-2}}{4(n-1)(n+1)}(W_{ikjl}+W_{iljk})^2,$$

and define the coefficients $\alpha_q \in \mathbb{R}$ by the formula

$$\sum_{q=0}^{2d}\alpha_q s^q = (n+1)f(s)^2 + 4sf(s)f'(s) + 2s^2 f'(s)^2 . \qquad (4.3)$$

Here, d is the integer in the formula (3.1). Changing variables $t' = t/\epsilon$ and $r' = r/\epsilon$ we obtain

$$F(0,\epsilon) = -\beta_n \cdot \sum_{q=0}^{2d}\alpha_q\epsilon^{2q+4}\int_0^\infty\int_0^\infty\frac{r^{2q+n}}{((1+t)^2+r^2)^{n-2}}dr\, dt$$

and, changing variables $r' = r/(1+t)$,

$$F(0,\epsilon) = -\beta_n \cdot \sum_{q=0}^{2d}\alpha_q\epsilon^{2q+4}\int_0^\infty\frac{1}{(1+t)^{n-5-2q}}dt \cdot \int_0^\infty\frac{r^{2q+n}}{(1+r^2)^{n-2}}dr$$

Now, we have
$$\int_0^\infty\frac{1}{(1+t)^{n-5-2q}}dt = \frac{1}{n-6-2q}$$

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and

$$\int_0^\infty \frac{r^{2q+n}}{(1+r^2)^{n-2}} dr = \left\{ \prod_{j=0}^q \frac{n-1+2j}{n-5-2j} \right\} \cdot \int_0^\infty \frac{r^{n-2}}{(1+r^2)^{n-2}} dr,$$

where we used Lemma A-1. Hence, we can write

$$F(0,\epsilon) = -\beta_n \cdot I(\epsilon^2) \cdot \int_{r=0}^\infty \frac{r^{n-2}}{(1+r^2)^{n-2}} dr \tag{4.4}$$

where

$$I(s) = \sum_{q=0}^{2d} \frac{\alpha_q}{n-6-2q} \left\{ \prod_{j=0}^q \frac{n-1+2j}{n-5-2j} \right\} s^{q+2}. \tag{4.5}$$

We will now turn our attention to the second order derivatives of the function $F(\xi, \epsilon)$.

Proposition 4.5. *We have*

$$\frac{\partial^2}{\partial \xi_p \partial \xi_q} F(0,\epsilon) = (n-2)^2 \int_{\mathbb{R}_+^n} \frac{\epsilon^{n-2}}{((\epsilon+x_n)^2 + |\bar{x}|^2)^n} \bar{H}_{pl}(x)\bar{H}_{ql}(x)$$

$$- \frac{(n-2)^2}{4} \int_{\mathbb{R}_+^n} \frac{\epsilon^{n-2}}{((\epsilon+x_n)^2 + |\bar{x}|^2)^n} (\partial_l \bar{H}_{ij}(x))^2 x^p x^q$$

$$+ \frac{(n-2)^2}{8(n-1)} \int_{\mathbb{R}_+^n} \frac{\epsilon^{n-2}}{((\epsilon+x_n)^2 + |\bar{x}|^2)^{n-1}} (\partial_l \bar{H}_{ij}(x))^2 \delta_{pq}.$$

Proof. The proof is the same of Proposition 21 of [6]. $\qquad\square$

Proposition 4.6. *We have*

$$\frac{\partial^2}{\partial \xi_p \partial \xi_q} F(0,\epsilon) \tag{4.6}$$

$$= -\frac{2(n-2)^2 \sigma_{n-2}}{(n-1)(n+1)(n+3)} (W_{ipjl} + W_{iljp})(W_{iqjl} + W_{iljq})$$

$$\cdot \int_0^\infty \int_0^\infty \frac{\epsilon^{n-2}}{((\epsilon+x_n)^2 + r^2)^n} r^{n+4} \left(2f(r^2)f'(r^2) + r^2 f'(r^2)^2 \right) dr dt$$

$$- \frac{(n-2)^2 \sigma_{n-2}}{2(n-1)(n+1)(n+3)} (W_{ikjl} + W_{iljk})^2 \delta_{pq}$$

$$\cdot \int_0^\infty \int_0^\infty \frac{\epsilon^{n-2}}{((\epsilon+x_n)^2 + r^2)^n} r^{n+4} \left\{ 2f(r^2)f'(r^2) + r^2 f'(r^2)^2 \right\} dr dt$$

$$+ \frac{(n-2)^2 \sigma_{n-2}}{4(n-1)^2(n+1)} (W_{ikjl} + W_{iljk})^2 \delta_{pq}$$

$$\cdot \int_0^\infty \int_0^\infty \frac{\epsilon^{n-2}}{((\epsilon+x_n)^2 + r^2)^{n-1}} r^{n+4} f'(r^2)^2 dr dt.$$

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Proof. It follows from Corollary A-3 in the Appendix that

$$\int_{S_r^{n-2}} \bar{H}_{pl}\bar{H}_{ql}(x) = \int_{S_r^{n-2}} f(r^2)^2 H_{pl}H_{ql}(x) = f(r^2)^2 \int_{S_r^{n-2}} W_{ipkl}W_{jqml}x^i x^j x^k x^m$$

$$= \frac{\sigma_{n-2}}{2(n-1)(n+1)}(W_{ipkl} + W_{ilkp})(W_{iqkl} + W_{ilkq})r^{n+2}f(r^2)^2 \, .$$

Hence,

$$\int_{\mathbb{R}_+^n} \frac{\epsilon^{n-2}\bar{H}_{pl}\bar{H}_{ql}(x)}{((\epsilon+x_n)^2 + |\bar{x}|^2)^n} = \frac{\sigma_{n-2}}{2(n-1)(n+1)}(W_{ipkl} + W_{ilkp})(W_{iqkl} + W_{ilkq}) \tag{4.7}$$

$$\cdot \int_0^\infty \int_0^\infty \frac{\epsilon^{n-2}r^{n+2}}{((\epsilon+x_n)^2 + r^2)^n}f(r^2)^2 dt dr \, .$$

It follows from Proposition 4.2 that

$$\int_{\mathbb{R}_+^n} \frac{\epsilon^{n-2}(\partial_l \bar{H}_{ij})^2(x)x^p x^q}{((\epsilon+x_n)^2 + |\bar{x}|^2)^n} \tag{4.8}$$

$$= \frac{2\sigma_{n-2}}{(n-1)(n+1)(n+3)}(W_{ipjl} + W_{iljp})(W_{iqjl} + W_{iljq})$$

$$\cdot \int_0^\infty \int_0^\infty \frac{\epsilon^{n-2}r^{n+2}}{((\epsilon+x_n)^2 + r^2)^n} \left\{ (n+3)f(r^2)^2 + 8r^2 f(r^2)f'(r^2) + 4r^4 f'(r^2)^2 \right\} dt dr$$

$$+ \frac{\sigma_{n-2}}{(n-1)(n+1)(n+3)}(W_{ikjl} + W_{iljk})^2 \delta_{pq}$$

$$\cdot \int_0^\infty \int_0^\infty \frac{\epsilon^{n-2}r^{n+2}}{((\epsilon+x_n)^2 + r^2)^n} \left\{ (n+3)f(r^2)^2 + 4r^2 f(r^2)f'(r^2) + 2r^4 f'(r^2)^2 \right\} dt dr \, .$$

and from Corollary 4.3 that

$$\int_{\mathbb{R}_+^n} \frac{\epsilon^{n-2}(\partial_l \bar{H}_{ij})^2(x)\delta_{pq}}{((\epsilon+x_n)^2 + |\bar{x}|^2)^{n-1}} = \frac{\sigma_{n-2}}{(n-1)(n+1)}(W_{ikjl} + W_{iljk})^2 \delta_{pq} \tag{4.9}$$

$$\cdot \int_0^\infty \int_0^\infty \frac{\epsilon^{n-2}r^n}{((\epsilon+x_n)^2 + r^2)^{n-1}} \left\{ (n+1)f(r^2)^2 + 4r^2 f(r^2)f'(r^2) + 2r^4 f'(r^2)^2 \right\} dt dr \, .$$

Observe that

$$\frac{r^n}{((\epsilon+x_n)^2 + r^2)^{n-1}} \left\{ (n+1)f(r^2)^2 + 4r^2 f(r^2)f'(r^2) \right\} \tag{4.10}$$

$$= \frac{2(n-1)r^{n+2}f(r^2)^2}{((\epsilon+x_n)^2 + r^2)^n} + \frac{d}{dr}\left\{ \frac{r^{n+1}f(r^2)^2}{((\epsilon+x_n)^2 + r^2)^{n-1}} \right\} \, .$$

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Substituting the equation (4.10) in the equation (4.9) we obtain

$$\int_{\mathbb{R}^n_+} \frac{\epsilon^{n-2}(\partial_l \bar{H}_{ij})^2(x)\delta_{pq}}{((\epsilon + x_n)^2 + |\bar{x}|^2)^{n-1}} \tag{4.11}$$

$$= \frac{2\sigma_{n-2}}{n+1}(W_{ikjl} + W_{iljk})^2\delta_{pq} \cdot \int_0^\infty \int_0^\infty \frac{\epsilon^{n-2}r^{n+2}f(r^2)^2}{((\epsilon + x_n)^2 + r^2)^n}dtdr$$

$$+ \frac{2\sigma_{n-2}}{(n-1)(n+1)}(W_{ikjl} + W_{iljk})^2\delta_{pq} \cdot \int_0^\infty \int_0^\infty \frac{\epsilon^{n-2}r^{n+4}f'(r^2)^2}{((\epsilon + x_n)^2 + r^2)^{n-1}}dtdr\,,$$

since we are assuming that $n > 4d + 6$. Now, using the equations (4.7), (4.8) and (4.11) in Proposition 4.5, we obtain

$$\frac{\partial^2}{\partial\xi_p\partial\xi_q}F(0,\epsilon)$$

$$= \frac{(n-2)^2\sigma_{n-2}}{2(n-1)(n+1)}(W_{ipjl} + W_{iljp})(W_{iqjl} + W_{iljq})$$

$$\cdot \int_0^\infty \int_0^\infty \frac{\epsilon^{n-2}}{((\epsilon + x_n)^2 + r^2)^n}r^{n+2}f(r^2)^2drdt$$

$$- \frac{(n-2)^2\sigma_{n-2}}{2(n-1)(n+1)(n+3)}(W_{ipjl} + W_{iljp})(W_{iqjl} + W_{iljq})$$

$$\cdot \int_0^\infty \int_0^\infty \frac{\epsilon^{n-2}}{((\epsilon + x_n)^2 + r^2)^n}r^{n+2}\left\{(n+3)f(r^2)^2 + 8r^2f(r^2)f'(r^2) + 4r^4f'(r^2)^2\right\}drdt$$

$$- \frac{(n-2)^2\sigma_{n-2}}{4(n-1)(n+1)(n+3)}(W_{ikjl} + W_{iljk})^2\delta_{pq}$$

$$\cdot \int_0^\infty \int_0^\infty \frac{\epsilon^{n-2}}{((\epsilon + x_n)^2 + r^2)^n}r^{n+2}\left\{(n+3)f(r^2)^2 + 4r^2f(r^2)f'(r^2) + 2r^4f'(r^2)^2\right\}drdt$$

$$+ \frac{(n-2)^2\sigma_{n-2}}{4(n-1)(n+1)}(W_{ikjl} + W_{iljk})^2\delta_{pq}$$

$$\cdot \int_0^\infty \int_0^\infty \frac{\epsilon^{n-2}}{((\epsilon + x_n)^2 + r^2)^n}r^{n+2}f(r^2)^2drdt$$

$$+ \frac{(n-2)^2\sigma_{n-2}}{4(n-1)^2(n+1)}(W_{ikjl} + W_{iljk})^2\delta_{pq}$$

$$\cdot \int_0^\infty \int_0^\infty \frac{\epsilon^{n-2}}{((\epsilon + x_n)^2 + r^2)^{n-1}}r^{n+4}f'(r^2)^2drdt$$

and the result follows after we cancel out some terms in the above equation. $\quad\square$

Let us define constants β_q, for $q = 0, ..., 2d - 1$, by the following expression:

$$\sum_{q=0}^{2d-1} \beta_q s^q = 2f(s)f'(s) + sf'(s)^2\,.$$

Proposition 4.7. *We have*

$$\int_0^\infty \int_0^\infty \frac{\epsilon^{n-2}}{((\epsilon + x_n)^2 + r^2)^n} r^{n+4} \left(2f(r^2)f'(r^2) + r^2 f'(r^2)^2\right) dr dt \qquad (4.12)$$

$$= J(\epsilon^2) \cdot \int_0^\infty \frac{r^{n+2}}{(1 + r^2)^n} dr,$$

where

$$J(s) = \sum_{q=0}^{2d-1} \frac{\beta_q s^{q+2}}{n - 6 - 2q} \cdot \left\{ \prod_{j=0}^q \frac{n + 3 + 2j}{n - 5 - 2j} \right\}.$$

Proof.

$$\int_0^\infty \int_0^\infty \frac{\epsilon^{n-2}}{((\epsilon + x_n)^2 + r^2)^n} r^{n+4} \left(2f(r^2)f'(r^2) + r^2 f'(r^2)^2\right) dr dt$$

$$= \sum_{q=0}^{2d-1} \beta_q \int_0^\infty \int_0^\infty \frac{\epsilon^{n-2} r^{n+4+2q}}{((\epsilon + x_n)^2 + r^2)^n} dr dt$$

$$= \sum_{q=0}^{2d-1} \beta_q \epsilon^{2q+4} \int_0^\infty \int_0^\infty \frac{r^{n+4+2q}}{((1 + x_n)^2 + r^2)^n} dr dt$$

$$= \sum_{q=0}^{2d-1} \beta_q \epsilon^{2q+4} \int_0^\infty \frac{1}{(1 + t)^{n-5-2q}} dt \int_0^\infty \frac{r^{n+4+2q}}{(1 + r^2)^n} dr$$

Now we observe that

$$\int_0^\infty \frac{1}{(1 + t)^{n-5-2q}} dt = \frac{1}{n - 6 - 2q}$$

and apply Lemma A-1 to see that

$$\int_0^\infty \frac{r^{n+4+2q}}{(1 + r^2)^n} dr = \left\{ \prod_{j=0}^q \frac{n + 3 + 2j}{n - 5 - 2j} \right\} \cdot \int_0^\infty \frac{r^{n+2}}{(1 + r^2)^n} dr.$$

$\square$

4.1   The case $n \geq 53$

In this case we choose $d = 1$ in the equation (3.1). Then the coeficients α_q in the equation (4.3) are given by

$$\alpha_0 = (n + 1) a_0^2, \quad \alpha_1 = 2(n + 3) a_0 a_1, \quad \alpha_2 = (n + 7) a_1^2.$$

Thus, derivating $I(s)$ in the expression (4.5) we obtain

$$
I'(s) = \sum_{q=0}^{2} \frac{(q+2)\,\alpha_q}{n-6-2q} \left\{ \prod_{j=0}^{q} \frac{n-1+2j}{n-5-2j} \right\} s^{q+1}
$$

$$
= \frac{2\alpha_0(n-1)}{(n-6)(n-5)} \cdot s + \frac{3\alpha_1(n-1)(n+1)}{(n-8)(n-5)(n-7)} \cdot s^2 + \frac{4\alpha_2(n-1)(n+1)(n+3)}{(n-10)(n-5)(n-7)(n-9)} \cdot s^3
$$

$$
= \frac{2(n+1)(n-1)}{n-5} \left\{ \frac{1}{n-6} a_0^2 s + \frac{3(n+3)}{(n-8)(n-7)} a_0 a_1 s^2 + \frac{2(n+3)(n+7)}{(n-10)(n-7)(n-9)} a_1^2 s^3 \right\}.
$$

Now we choose $a_1 = -1$ and define the polynomial p_n by

$$
p_n(a_0) = \frac{a_0^2}{n-6} - \frac{3(n+3)\,a_0}{(n-8)(n-7)} + \frac{2(n+3)(n+7)}{(n-10)(n-7)(n-9)}.
$$

Hence,

$$
I'(1) = \frac{2(n+1)(n-1)}{n-5} p_n(a_0).
$$

The discriminant of p_n is then given by

$$
\mathrm{discrim}(p_n) = \frac{(n+3)^2}{(n-7)^2(n-8)^2} \left\{ 9 - \frac{8(n-7)(n-8)^2(n+7)}{(n+3)(n-6)(n-9)(n-10)} \right\}
$$

$$
= \frac{(n+3)^2}{(n-7)^2(n-8)^2} \frac{q(n)}{(n+3)(n-6)(n-9)(n-10)},
$$

where

$$
q(n) = 9(n+3)(n-6)(n-9)(n-10) - 8(n-7)(n-8)^2(n+7).
$$

Observe that

$$
q'(n) = 4n^3 - 210n^2 + 2082n - 5624
$$

and

$$
q''(n) = 6(2n^2 - 70n + 347).
$$

Since the roots $\frac{70 \pm \sqrt{2124}}{4}$ of q'' are less than 53, we see that $q''(n) > 0$ for $n \geq 53$. Since $q(53) = 105696$ and $q'(53) = 110340$, we conclude that $\mathrm{discrim}(p_n) > 0$ for $n \geq 53$. Hence, if we set

$$
a_0 = \frac{(n+3)(n-6)}{2(n-7)(n-8)} \left\{ 3 + \sqrt{9 - \frac{8(n-7)(n-8)^2(n+7)}{(n+3)(n-6)(n-9)(n-10)}} \right\},
$$

then $s = 1$ is critical point of $I(s)$. According to Proposition B-1 in the Appendix, $I''(1) < 0$ for $n \geq 53$.

Now we will handle $J(s)$, as defined in Proposition 4.7. We have

$$
J(s) = \frac{(n+3)\,\beta_0\, s^2}{(n-6)(n-5)} + \frac{(n+3)(n+5)\,\beta_1\, s^3}{(n-8)(n-5)(n-7)}
$$

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where
$$\beta_0 = 2\,a_0\,a_1 \quad \text{and} \quad \beta_1 = 3\,a_1^2.$$

Hence,
$$J(s) = \frac{(n+3)\,a_1}{n-5}\left\{\frac{2\,a_0\,s^2}{n-6} + \frac{3(n+5)\,a_1\,s^3}{(n-8)(n-7)}\right\}.$$

If we set a_0 and a_1 as above we have

$$J(1) = \frac{n+3}{(n-8)(n-5)(n-7)}$$

$$\cdot\left\{6 - (n+3)\sqrt{9 - \frac{8(n-7)(n-8)^2(n+7)}{(n+3)(n-6)(n-9)(n-10)}}\right\}$$

According to Proposition B-2 in the Appendix, $J(1) < 0$ for $n \geq 53$.

From the equations (4.2), (4.4), (4.6) and (4.12) and the above results we can conclude the following:

Proposition 4.8. *Suppose that $n \geq 53$. If we set $a_1 = -1$ and*

$$a_0 = \frac{(n+3)(n-6)}{2(n-7)(n-8)}\left\{3 + \sqrt{9 - \frac{8(n-7)(n-8)^2(n+7)}{(n+3)(n-6)(n-9)(n-10)}}\right\},$$

then $I'(1) = 0$, $I''(1) < 0$ and $J(1) < 0$. In particular, the function $F(\xi, \epsilon)$ has a strict local minimum at the point $(0, 1)$.

4.2   The case $25 \leq n \leq 52$

In this case we choose $d = 4$ in the equation (3.1). The coeficients α_q in the equation (4.3) are then given by

$$\alpha_0 = (n+1)\,a_0^2,$$
$$\alpha_1 = 2(n+3)\,a_0\,a_1,$$
$$\alpha_2 = 2(n+5)\,a_0\,a_2 + (n+7)\,a_1^2,$$
$$\alpha_3 = 2(n+11)\,a_1\,a_2 + 2(n+7)\,a_0\,a_3,$$
$$\alpha_4 = 2(n+15)\,a_1\,a_3 + (n+17)\,a_2^2 + 2(n+9)\,a_0\,a_4,$$
$$\alpha_5 = 2(n+23)\,a_2\,a_3 + 2(n+19)\,a_1\,a_4,$$
$$\alpha_6 = (n+31)\,a_3^2 + 2(n+29)\,a_2\,a_4,$$
$$\alpha_7 = 2(n+39)\,a_3\,a_4,$$
$$\alpha_8 = (n+49)\,a_4^2.$$

Thus, derivating $I(s)$ in the expression (4.5) we obtain

$$I'(s) = \sum_{q=0}^{8}\frac{(q+2)\,\alpha_q}{n-6-2q}\left\{\prod_{j=0}^{q}\frac{n-1+2j}{n-5-2j}\right\}s^{q+1}.$$

Now we choose $a_1 = -3/5$, $a_2 = 1/8$, $a_3 = -1/125$, $a_4 = 10^{-4}$ and define the polynomial r_n by $r_n(a_0) = I'(1)$. Hence,

$$r_n(a_0) = \frac{2(n-1)(n+1)}{(n-6)(n-5)} \cdot a_0^2 + \left\{ \sum_{q=1}^{4} \gamma_q(n) \frac{q+2}{n-6-2q} \prod_{j=0}^{q} \frac{n-1+2j}{n-5-2j} \right\} \cdot a_0$$

$$+ \sum_{q=2}^{8} \delta_q(n) \frac{q+2}{n-6-2q} \prod_{j=0}^{q} \frac{n-1+2j}{n-5-2j},$$

where

$$\gamma_1(n) = -\frac{6}{5}(n+3), \quad \gamma_2(n) = \frac{n+5}{4}, \quad \gamma_3(n) = -\frac{2}{125}(n+7), \quad \gamma_4(n) = \frac{n+9}{5000},$$

$$\delta_2(n) = \frac{9(n+7)}{25}, \delta_3(n) = -\frac{3(n+11)}{20}, \delta_4(n) = \frac{1009n+16385}{40000}, \delta_5(n) = -\frac{53n+1207}{25000},$$

$$\delta_6(n) = \frac{89n+2709}{10^6}, \quad \delta_7(n) = -\frac{n+39}{625000}, \quad \delta_8(n) = \frac{n+49}{10^8}.$$

Direct computations show that $\mathrm{discrim}(r_n) > 0$ for $25 \leq n \leq 52$.

If we choose

$$a_0 = \frac{(n-6)(n-5)}{4(n-1)(n+1)} \cdot \left\{ -\sum_{q=1}^{4} \gamma_q(n) \frac{q+2}{n-6-2q} \prod_{j=0}^{q} \frac{n-1+2j}{n-5-2j} + \sqrt{\mathrm{discrim}(r_n)} \right\}$$

then $s = 1$ is critical point of $I(s)$. For $25 \leq n \leq 52$, direct computations show that $I''(1)$ is of the form $-e_1 - e_2 \sqrt{e_3}$, where e_1, e_2, e_3 are positive rational numbers.

The function $J(s)$, defined in Proposition 4.7, is written as

$$J(s) = \sum_{q=0}^{7} \frac{\beta_q s^{q+2}}{n-6-2q} \cdot \left\{ \prod_{j=0}^{q} \frac{n+3+2j}{n-5-2j} \right\}.$$

where

$$\beta_0 = 2\,a_0\,a_1\,, \quad \beta_1 = 4\,a_0\,a_2 + 3\,a_1^2\,, \quad \beta_2 = 6\,a_0\,a_3 + 10\,a_1\,a_2\,, \quad \beta_3 = 8\,a_0\,a_4 + 14\,a_1\,a_3 + 8\,a_2^2\,,$$

$$\beta_4 = 18\,a_1\,a_4 + 22\,a_2\,a_3\,, \quad \beta_5 = 28\,a_2\,a_4 + 15\,a_3^2\,, \quad \beta_6 = 38\,a_3\,a_4\,, \quad \beta_7 = 24\,a_4^2\,.$$

For $25 \leq n \leq 52$, direct computations show that $J(1)$ is of the form $-e_1 - e_2 \sqrt{e_3}$, where e_1, e_2, e_3 are positive rational numbers. From the equations (4.2), (4.4), (4.6) and (4.12) and the above results we can conclude the following:

Proposition 4.9. *Suppose that $25 \leq n \leq 52$. If $a_1 = -3/5$, $a_2 = 1/8$, $a_3 = -1/125$, $a_4 = 10^{-4}$ and*

$$a_0 = \frac{(n-6)(n-5)}{4(n-1)(n+1)} \cdot \left\{ -\sum_{q=1}^{4} \gamma_q(n) \frac{q+2}{n-6-2q} \prod_{j=0}^{q} \frac{n-1+2j}{n-5-2j} + \sqrt{\mathrm{discrim}(r_n)} \right\}$$

then $I'(1) = 0$, $I''(1) < 0$ and $J(1) < 0$. In particular, the function $F(\xi, \epsilon)$ has a strict local minimum at the point $(0, 1)$.

5 Proof of the main theorem

In this section we will make use of the two-tensor H, defined on $\mathbb{R}_+^n$, the polynomial f and the open set $\Omega \subset \mathbb{R}^{n-1} \times (0, \infty)$, which were defined in Section 3. As in Sections 4.1 and 4.2, we fix $d = 1$ if $n \geq 53$ and $d = 4$ if $25 \leq n \leq 52$. We set $D_r(0) = \{x \in \partial \mathbb{R}_+^n \, ; \, |x| < r\}$.

The basic ingredient in the proof of the Main Theorem is the following result:

Proposition 5.1. *Assume that $n \geq 25$. Let g be a smooth Riemannan metric on $\mathbb{R}_+^n$ expressed as $g = \exp(h)$, where h is a symmetric trace-free two-tensor on $\mathbb{R}_+^n$ satisfying the following properties:*

$$
\begin{cases}
h_{ab}(x) = \mu \lambda^{2d} f(\lambda^{-2}|\bar{x}|) H_{ab}(x) \, , & \text{for } |x| \leq \rho \, , \\
h_{ab}(x) = 0 \, , & \text{for } |x| \geq 1 \, , \\
h_{nb}(x) = 0 \, , & \text{for } x \in \mathbb{R}_+^n \, , \\
\partial_n h_{ab}(x) = 0 \, , & \text{for } x \in \partial \mathbb{R}_+^n \, ,
\end{cases}
\tag{5.1}
$$

where $a, b = 1, ..., n$. We also assume that

$$
|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha \leq \alpha_1 \, , \quad \text{for all } x \in \mathbb{R}_+^n \, ,
$$

where α_1 is the constant obtained in Proposition 2.8.

If α and $\mu^{-2} \lambda^{n-4d-6} \rho^{2-n}$ are sufficiently small, then there exists a positive smooth function v satisfying

$$
\begin{cases}
\Delta_g v - c_n R_g v = 0 \, , & \text{in } \mathbb{R}_+^n \, , \\
\frac{\partial}{\partial x_n} v - d_n \kappa_g v + (n-2) v^{\frac{n}{n-2}} = 0 \, , & \text{on } \partial \mathbb{R}_+^n
\end{cases}
\tag{5.2}
$$

and

$$
\int_{\partial \mathbb{R}_+^n} v^{\frac{2(n-1)}{n-2}} < \left(\frac{Q(B^n, \partial B)}{n-2} \right)^{n-1} \, .
\tag{5.3}
$$

Moreover, there exists $c = c(n) > 0$ such that

$$
\sup_{D_\lambda(0)} v \geq c \lambda^{\frac{2-n}{2}} \, .
\tag{5.4}
$$

Proof. It follows from the fact that

$$
(n+1) f(s)^2 + 4s f(s) f'(s) + 2s^2 f'(s)^2 = (n-1) f(s)^2 + 2(f(s) + s f'(s))^2
$$

and Propostion 4.4 that $F(0, 1) < 0$. According to Propositions 4.8 and 4.9, we can choose the coefficients $a_0, ..., a_d$ in the formula (3.1) such that the point $(0, 1)$ is a strict local minimum of F. Hence, we can find an open set $\Omega' \subset \Omega$ such that $(0, 1) \in \Omega'$ and

$$
F(0, 1) < \inf_{(\xi, \epsilon) \in \partial \Omega'} F(\xi, \epsilon) < 0 \, .
$$

Observe that $u_{(\lambda\xi,\lambda\epsilon)}(\lambda x) = \lambda^{-\frac{n-2}{2}} u_{(\xi,\epsilon)}(x)$ and $w_{(\lambda\xi,\lambda\epsilon)}(\lambda x) = \mu\lambda^{2d+2-\frac{n-2}{2}} z_{(\xi,\epsilon)}(x)$ for all $x \in \mathbb{R}^n_+$. Here, $w_{(\xi,\epsilon)}$ and $z_{(\xi,\epsilon)}$ are the functions defined by the formulas (3.2) and (4.1) respectively. Thus, it follows from Proposition 3.6 that

$$\left| \mathcal{F}_g(\lambda\xi, \lambda\epsilon) - \mu^2\lambda^{4d+4} F(\xi,\epsilon) \right| \leq C\mu^{\frac{2(n-1)}{n-2}} \lambda^{\frac{(4d+4)(n-1)}{n-2}} + C\mu\lambda^{2d+2} \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}} + C\left(\frac{\lambda}{\rho}\right)^{n-2}$$

for all $(\xi,\epsilon) \in \Omega$. Hence,

$$\left| \mu^{-2}\lambda^{-4d-4} \mathcal{F}_g(\lambda\xi, \lambda\epsilon) - F(\xi,\epsilon) \right| \leq C\mu^{\frac{2}{n-2}} \lambda^{\frac{4d+4}{n-2}}$$
$$+ C\mu^{-1}\lambda^{\frac{n-4d-6}{2}} \rho^{\frac{2-n}{2}} + C\mu^{-2}\lambda^{n-4d-6}\rho^{2-n}$$

for all $(\xi,\epsilon) \in \Omega$. If $\mu^{-2}\lambda^{n-4d-6}\rho^{2-n}$ is sufficiently small then we have

$$\mathcal{F}_g(0,\lambda) < \inf_{(\xi,\epsilon)\in\partial\Omega'} \mathcal{F}_g(\lambda\xi, \lambda\epsilon) < 0 .$$

Thus we conclude that there exists a point $(\bar\xi, \bar\epsilon) \in \Omega'$ such that

$$\mathcal{F}_g(\lambda\bar\xi, \lambda\bar\epsilon) = \inf_{(\xi,\epsilon)\in\Omega'} \mathcal{F}_g(\lambda\xi, \lambda\epsilon) < 0 .$$

By Proposition 2.9, the function $v = v_{(\lambda\bar\xi, \lambda\bar\epsilon)}$ obtained in Proposition 2.8 is a positive smooth solution to the equations (5.2). Hence, by the definition of $\mathcal{F}_g$ (see the formula (2.19)) and the formula (2.2), we have

$$\frac{n-2}{n-1} \int_{\partial\mathbb{R}^n_+} v^{\frac{2(n-1)}{n-2}} = \frac{n-2}{n-1} \left(\frac{Q(B^n,\partial B)}{n-2}\right)^{n-1} + \mathcal{F}(\lambda\bar\xi, \lambda\bar\epsilon) .$$

This implies the inequality (5.3).

In order to prove the inequality (5.4), observe that

$$\|v - u_{(\lambda\bar\xi, \lambda\bar\epsilon)}\|_{L^{\frac{2(n-1)}{n-2}}(D_\lambda(0))} \leq \|v - u_{(\lambda\bar\xi, \lambda\bar\epsilon)}\|_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}^n_+)} \leq C\alpha$$

by Propositions 2.2 and 2.8. Hence,

$$|D_\lambda(0)|^{\frac{n-2}{2(n-1)}} \sup_{D_\lambda(0)} v \geq \|v\|_{L^{\frac{2(n-1)}{n-2}}(D_\lambda(0))} \geq -C\alpha + \|u_{(\lambda\bar\xi, \lambda\bar\epsilon)}\|_{L^{\frac{2(n-1)}{n-2}}(D_\lambda(0))} .$$

Now, the inequality (5.4) follows from choosing α sufficiently small. $\qquad\square$

Now the Main Theorem follows from the next theorem, using the conformal equivalence between $B^n \setminus \{(0,...,0,-1)\}$ and $\mathbb{R}^n_+$ (see Lemma 2.3), the properties (2.8) and Lemma 2.10.

Theorem 5.2. *Assume that $n \geq 25$. Then there exists a smooth Riemannian metric g on $\mathbb{R}^n_+$ with the following properties:*

(a) $g_{ab}(x) = \delta_{ab}$ for $|x| \geq 1/2$;

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(b) g is not conformally flat;

(c) $\partial\mathbb{R}^n_+$ is totally geodesic with respect to the induced metric by g;

(d) there exists a sequence of positive smooth functions $\{v_\nu\}^\infty_{\nu=1}$ satisfying

$$\begin{cases} \Delta_g v_\nu - c_n R_g v_\nu = 0, & \text{in } \mathbb{R}^n_+, \\ \frac{\partial}{\partial x_n} v_\nu - d_n \kappa_g v_\nu + (n-2)v_\nu^{\frac{n}{n-2}} = 0, & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{5.5}$$

for all ν,

$$\int_{\partial\mathbb{R}^n_+} v_\nu^{\frac{2(n-1)}{n-2}} < \left(\frac{Q(B^n, \partial B)}{n-2}\right)^{n-1},$$

for all ν, and $\sup_{D_1(0)} v_\nu \to \infty$ as $\nu \to \infty$.

Proof. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth cutoff function such that $\chi(t) = 1$ for $t \leq 1$ and $\chi(t) = 0$ for $t \geq 2$. We define the trace-free symmetric two-tensor h on $\mathbb{R}^n_+$ by

$$h_{ab}(x) = \sum_{N=N_0}^\infty \chi(4N^2|x - x_N|)2^{-dN} f(2^N|\bar{x} - x_N|)H_{ab}(x - x_N)$$

where $x_N = (\frac{1}{N}, 0, ..., 0) \in \partial\mathbb{R}^n_+$. Observe that h is smooth and satisfies $h_{an}(x) = 0$ for $x \in \mathbb{R}^n_+$ and $\partial_n h_{ab}(x) = 0$ for $x \in \partial\mathbb{R}^n_+$. If N_0 is sufficiently large, then $h_{ab}(x) = 0$ for $|x| \geq \frac{1}{2}$ and $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha$ for $x \in \mathbb{R}^n_+$, with α sufficiently small as in Proposition 5.1. Then we define the metric $g(x) = \exp(h(x))$ for $x \in \mathbb{R}^n_+$ and the result follows from Proposition 5.1. $\qquad\square$

Appendix A

In this section we establish some useful identities used in Section 4. They are simple computations which are performed in the Appendix of [6].

Lemma A-1. *We have $\int_0^\infty \frac{s^\alpha ds}{(1+s^2)^m} = \frac{2m-\alpha-3}{\alpha+1} \int_0^\infty \frac{s^{\alpha+2}ds}{(1+s^2)^m}$, for $\alpha + 3 < 2m$.*

Proposition A-2. *We have*

$$\int_{S^{n-2}_r} p_k = \frac{r^2}{k(k + n - 3)} \int_{S^{n-2}_r} \Delta p_k$$

for every homogeneous polynomial p_k of degree k.

Corollary A-3. *We have*

$$\int_{S^{n-2}} x_i x_j = \frac{\sigma_{n-2}}{n - 1},$$

$$\int_{S^{n-2}} x_i x_j x_k x_l = \frac{\sigma_{n-2}}{(n-1)(n+1)}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

and

$$\int_{S^{n-2}} x_i x_j x_k x_l x_p x_q = \frac{\sigma_{n-2}}{(n-1)(n+1)(n+3)} (\delta_{ij}\delta_{kl}\delta_{pq} + \delta_{ij}\delta_{kp}\delta_{lq} + \delta_{ij}\delta_{kq}\delta_{lp}$$
$$+\delta_{ik}\delta_{jl}\delta_{pq} + \delta_{ik}\delta_{jp}\delta_{lq} + \delta_{ik}\delta_{jq}\delta_{lp}$$
$$+\delta_{il}\delta_{jk}\delta_{pq} + \delta_{il}\delta_{jp}\delta_{kq} + \delta_{il}\delta_{jq}\delta_{kp}$$
$$+\delta_{ip}\delta_{jk}\delta_{lq} + \delta_{ip}\delta_{jl}\delta_{kq} + \delta_{ip}\delta_{jq}\delta_{kl}$$
$$+\delta_{iq}\delta_{jk}\delta_{lp} + \delta_{iq}\delta_{jl}\delta_{kp} + \delta_{iq}\delta_{jp}\delta_{kl}).$$

Appendix B

In this section we establish some results used in Section 4.1. The notations here are the same of that section. In particular, we fix $a_1 = -1$ and

$$a_0 = \frac{(n+3)(n-6)}{2(n-7)(n-8)} \left\{ 3 + \sqrt{9 - \frac{8(n-7)(n-8)^2(n+7)}{(n+3)(n-6)(n-9)(n-10)}} \right\}.$$

Proposition B-1. *We have $I''(1) < 0$ for $n \geq 53$.*

Proof. We are going to prove that $I''(1) < 0$ for $n \geq 70$. If $25 \leq n \leq 69$ the result follows from direct computations. We write

$$a_0 = \frac{(n+3)(n-6)}{2(n-7)(n-8)} \left\{ 3 + \sqrt{9 - \frac{8p_A(n)}{p_B(n)}} \right\},$$

where $p_A(n) = (n-7)(n-8)^2(n+7)$, $p_B(n) = (n+3)(n-6)(n-9)(n-10)$ and define

$$q_L(n) = p_A(n) - p_B(n) \quad \text{and} \quad q_U(n) = \alpha p_B(n) - p_A(n),$$

where $\alpha = \frac{31439}{28800}$.

Claim. $q_L(n) > 0$ for $n \geq 9$ and $q_U(n) > 0$ for $n \geq 70$.

In order to prove the Claim, first observe that the forth order terms of q_L cancel out and we have $q_L(n) = 6n^3 - 114n^2 + 712n - 1516$. Hence, $q_L''(n) = 36n - 228 > 0$ for $n \geq 7$, $q_L(9) = 32$ and $q_L'(9) = 118$. Thus, $q_L(n) > 0$ for $n \geq 9$.

Now we observe that

$$q_U(n) = \frac{2639}{28800}n^4 - \frac{115429}{14400}n^3 + \frac{1207877}{9600}n^2 - \frac{282161}{400}n + \frac{218809}{160}.$$

Hence, $q_U'''(n) = \frac{2639}{1200}n - \frac{115439}{2400} > 0$ for $n \geq 70$, $q_U(70) = \frac{287074}{15}$, $q_U'(70) = \frac{178522037}{7200}$ and $q_U''(70) = \frac{10910017}{4800}$. Thus, $q_U(n) > 0$ for $n \geq 70$, proving the Claim.

We assume that $n \geq 70$. In particular, we conclude from the Claim that $\alpha > \frac{p_A(n)}{p_B(n)} > 1$, which implies

$$\frac{2(n+3)(n-6)}{(n-7)(n-8)} > a_0 > \frac{(n+3)(n-6)}{2(n-7)(n-8)}(3 + \sqrt{9 - 8\alpha}).$$

Now we use this estimate in

$$I''(1) = \frac{2(n+1)(n-1)}{n-5}\left\{\frac{a_0^2}{n-6} - \frac{6(n+3)a_0}{(n-8)(n-7)} + \frac{6(n+3)(n+7)}{(n-10)(n-7)(n-9)}\right\}$$

to see that

$$\frac{(n-5)I''(1)}{2(n+1)(n-1)} < \frac{4(n+3)^3(n-6)}{(n-7)^2(n-8)^2} - \frac{3(3 + \sqrt{9 - 8\alpha})(n+3)^2(n-6)}{(n-7)^2(n-8)^2}$$
$$+ \frac{6(n+3)(n+7)}{(n-10)(n-7)(n-9)}.$$

This can be written as

$$I''(1) < \frac{2(n+3)(n+1)(n-1)\,\gamma(n)}{(n-8)^2(n-10)(n-5)(n-7)^2(n-9)},$$

where

$$\gamma(n) = -(5 + 3\sqrt{9 - 8\alpha})(n+3)(n-6)(n-10)(n-9) + 6(n+7)(n-7)(n-8)^2.$$

In order to complete our proof, we will show that $\gamma(n) < 0$ under our assumption on the dimension. Observe that $\gamma(n) = -\frac{11}{20}n^4 + \frac{481}{10}n^3 - \frac{15099}{20}n^2 + \frac{21162}{5}n - 8205$. Hence $\gamma'''(n) = -\frac{66}{5}n + \frac{1443}{5} < 0$ for $n \geq 70$, $\gamma(70) = -118392$, $\gamma'(70) = -\frac{744953}{5}$ and $\gamma''(70) = -\frac{136479}{10}$. Now the result follows. $\qquad\square$

Proposition B-2. *We have* $J(1) < 0$ *for* $n \geq 53$.

Proof. Let us assume that $n \geq 53$. We want to show that $(n+3)\sqrt{9 - \frac{8p_A(n)}{p_B(n)}} - 6 > 0$, where we are using the polynomials p_A and p_B as in the proof of Proposition B-1. We set again $q_U(n) = \alpha p_B(n) - p_A(n)$ and choose $\alpha = \frac{7047}{6272}$.

Claim. $q_U(n) > 0$.

In order to prove the Claim, first observe that

$$q_U(n) = \frac{775}{6272}n^4 - \frac{27341}{3136}n^3 + \frac{814983}{6272}n^2 - \frac{551233}{784}n + \frac{2063213}{1568}.$$

Hence, $q_U'''(n) = \frac{2325}{784}n - \frac{82023}{1568} > 0$ for $n \geq 53$, $q_U(53) = \frac{169857}{28}$, $q_U'(53) = \frac{20672955}{1568}$ and $q_U''(53) = \frac{5182395}{3136}$. Thus, $q_U(n) > 0$ for $n \geq 53$, proving the Claim.

The Claim implies that $(n + 3) \sqrt{9 - \frac{8 p_A(n)}{p_B(n)}} > (n + 3) \sqrt{9 - 8\alpha}$, which reduces the problem to prove that

$$(n + 3) \sqrt{9 - 8\alpha} - 6 \geq 0 . \tag{B-1}$$

On the other hand, the fact that $\alpha = \frac{1}{8} \left\{ 9 - \frac{36}{56^2} \right\}$ implies $\frac{1}{8} \left\{ 9 - \frac{36}{(n+3)^2} \right\} \geq \alpha$, which is equivalent to the inequality (B-1). $\qquad \square$

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