The asymptotic behavior of Palais-Smale sequences on manifolds with boundary

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Abstract

We describe the asymptotic behavior of Palais-Smale sequences associated to certain Yamabe-type equations on manifolds with boundary. We prove that each of those sequences converges to a solution of the limit equation plus a finite number of "bubbles" which are obtained by rescaling fundamental solutions of the corresponding Euclidean equations.

1 Introduction

Let (M^n, g) be a compact Riemannian manifold with boundary ∂M and dimension $n \geq 3$. For $u \in H^1(M)$, we consider the following family of equations, indexed by $\nu \in \mathbb{N}$:

$$\begin{cases} \Delta_g u = 0\,, & \text{in } M\,, \\ \frac{\partial}{\partial \eta_g} u - h_\nu u + u^{\frac{n}{n-2}} = 0\,, & \text{on } \partial M\,, \end{cases} \tag{1.1}$$

and their associated functionals

$$I_g^\nu(u) = \frac{1}{2}\int_M |du|_g^2 dv_g + \frac{1}{2}\int_{\partial M} h_\nu u^2 d\sigma_g - \frac{n-2}{2(n-1)}\int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} d\sigma_g\,. \tag{1.2}$$

Here, $\{h_\nu\}_{\nu\in\mathbb{N}}$ is a sequence of functions in $C^\infty(\partial M)$, Δ_g is the Laplace-Beltrami operator, and η_g is the inward unit normal vector to ∂M. Moreover, dv_g and $d\sigma_g$ are the volume forms of M and ∂M respectively and $H^1(M)$ is the Sobolev space $H^1(M) = \{u \in L^2(M)\,;\, du \in L^2(M)\}$.

Definition 1.1. We say that $\{u_\nu\}_{\nu\in\mathbb{N}} \subset H^1(M)$ is a *Palais-Smale* sequence for $\{I_g^\nu\}$ if
(i) $\{I_g^\nu(u_\nu)\}_{\nu\in\mathbb{N}}$ is bounded, and
(ii) $dI_g^\nu(u_\nu) \to 0$ strongly in $H^1(M)'$ as $\nu \to \infty$.

In this paper we establish a result describing the asymptotic behavior of those Palais-Smale sequences. This work is inspired by Struwe's theorem in

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[18] for equations $\Delta u + \lambda u + |u|^{\frac{4}{n-2}}u = 0$ on Euclidean domains. We refer the reader to [11, Chapter 3] for a version of Struwe's theorem on closed Riemannian manifolds, and to [7, 8, 17] for similar equations with boundary conditions.

Roughly speaking, as $\nu \to \infty$ and $h_\nu \to h_\infty$ we prove that each Palais-Smale sequence $\{u_\nu \geq 0\}_{\nu\in\mathbb{N}}$ is $H^1(M)$-asymptotic to a nonnegative solution of the limit equations

$$\begin{cases} \Delta_g u = 0\,, & \text{in } M\,,\\ \frac{\partial}{\partial\eta_g}u - h_\infty u + u^{\frac{n}{n-2}} = 0\,, & \text{on } \partial M\,, \end{cases} \tag{1.3}$$

plus a finite number of "bubbles" obtained by rescaling fundamental positive solutions of the Euclidean equations

$$\begin{cases} \Delta u = 0\,, & \text{in } \mathbb{R}^n_+\,,\\ \frac{\partial}{\partial y_n}u + u^{\frac{n}{n-2}} = 0\,, & \text{on } \partial\mathbb{R}^n_+\,, \end{cases} \tag{1.4}$$

where $\mathbb{R}^n_+ = \{(y_1, ..., y_n) \in \mathbb{R}^n\,;\ y_n \geq 0\}$.

Palais-Smale sequences frequently appear in the blow-up analysis of geometric problems. In the particular case when h_∞ is $\frac{n-2}{2}$ times the boundary mean curvature, the equations (1.3) are satisfied by a positive smooth function u representing a conformal scalar-flat Riemannian metric $u^{\frac{4}{n-2}}g$ with positive constant boundary mean curvature. The existence of those metrics is the Yamabe-type problem for manifolds with boundary introduced by Escobar in [14].

An application of our result is the blow-up analysis performed by the author in [2] for the proof of a convergence theorem for a Yamabe-type flow introduced by Brendle in [5].

We now begin to state our theorem more precisely.

Convention. We assume that there is some $h_\infty \in C^\infty(\partial M)$ and $C > 0$ such that $h_\nu \to h_\infty$ in $L^2(\partial M)$ as $\nu \to \infty$ and $|h_\nu(x)| \leq C$ for all $x \in \partial M$, $\nu \in \mathbb{N}$. This obviously implies that $h_\nu \to h_\infty$ in $L^p(\partial M)$ as $\nu \to \infty$, for any $p \geq 1$.

Notation. *If (M, g) is a Riemannian manifold with boundary ∂M, we will denote by $D_r(x)$ the metric ball in ∂M with center at $x \in \partial M$ and radius r.*

If $z_0 \in \mathbb{R}^n_+$, we set $B^+_r(z_0) = \{z \in \mathbb{R}^n_+\,;\ |z - z_0| < r\}$. We define

$$\partial^+ B^+_r(z_0) = \partial B^+_r(z_0) \cap \mathbb{R}^n_+\,, \quad \textit{and} \quad \partial' B^+_r(z_0) = B^+_r(z_0) \cap \partial\mathbb{R}^n_+\,.$$

Thus, $\partial' B^+_r(z_0) = \emptyset$ if $z_0 = (z_0^1, ..., z_0^n)$ satisfies $z_0^n > r$.

We define the Sobolev space $D^1(\mathbb{R}^n_+)$ as the completion of $C^\infty_0(\mathbb{R}^n_+)$ with respect to the norm

$$||u||_{D^1(\mathbb{R}^n_+)} = \sqrt{\int_{\mathbb{R}^n_+} |du(y)|^2 dy}\,.$$

It follows from a Liouville-type theorem established by Li and Zhu in [15] (see also [13] and [10]) that any nonnegative solution in $D^1(\mathbb{R}^n_+)$ to the equations

(1.4) is of the form

$$U_{\epsilon,a}(y) = \left(\frac{\epsilon}{(y_n + \frac{\epsilon}{n-2})^2 + |\bar{y} - a|^2}\right)^{\frac{n-2}{2}}, \quad a \in \mathbb{R}^{n-1}, \epsilon > 0, \tag{1.5}$$

or is identically zero (see Remark 2.5). By Escobar ([12]) or Beckner ([4]) we have the sharp Euclidean Sobolev inequality

$$\left(\int_{\partial\mathbb{R}^n_+} |u(y)|^{\frac{2(n-1)}{n-2}} dy\right)^{\frac{n-2}{n-1}} \leq K_n^2 \int_{\mathbb{R}^n_+} |du(y)|^2 dy, \tag{1.6}$$

for $u \in D^1(\mathbb{R}^n_+)$, which has the family of functions (1.5) as extremal functions. Here,

$$K_n = \left(\frac{n-2}{2}\right)^{-\frac{1}{2}} \sigma_{n-1}^{-\frac{1}{2(n-1)}},$$

where σ_{n-1} is the area of the unit $(n-1)$-sphere in $\mathbb{R}^n$. Up to a multiplicative constant, the functions defined by (1.5) are the only nontrivial extremal ones for the inequality (1.6).

Definition 1.2. Fix $x_0 \in \partial M$ and geodesic normal coordinates for ∂M centered at x_0. Let $(x_1, ..., x_{n-1})$ be the coordinates of $x \in \partial M$ and $\eta_g(x)$ be the inward unit vector normal to ∂M at x. For small $x_n \geq 0$, the point $\exp_x(x_n\eta_g(x)) \in M$ is said to have *Fermi coordinates* $(x_1, ..., x_n)$ (centered at x_0).

For small $\rho > 0$ the Fermi coordinates centered at $x_0 \in \partial M$ define a smooth map $\psi_{x_0} : B^+_\rho(0) \subset \mathbb{R}^n_+ \to M$.

We define the functional I^∞_g by the same expression as I^ν_g with $h_\nu = h_\infty$ for all ν, and state our main theorem as follows:

Theorem 1.3. *Let (M^n, g) be a compact Riemannian manifold with boundary ∂M and dimension $n \geq 3$. Suppose $\{u_\nu \geq 0\}_{\nu\in\mathbb{N}}$ is a Palais-Smale sequence for $\{I^\nu_g\}$. Then there exist $m \in \{0, 1, 2, ...\}$, a nonnegative solution $u^0 \in H^1(M)$ of (1.3), and m nontrivial nonegative solutions $U^j = U_{\epsilon_j,a_j} \in D^1(\mathbb{R}^n_+)$ of (1.4), sequences $\{R^j_\nu > 0\}_{\nu\in\mathbb{N}}$, and sequences $\{x^j_\nu\}_{\nu\in\mathbb{N}} \subset \partial M$, $1 \leq j \leq m$, the whole satisfying the following conditions for $1 \leq j \leq m$, possibly after taking subsequences:*
(i) $R^j_\nu \to \infty$ as $\nu \to \infty$.
(ii) x^j_ν converges as $\nu \to \infty$.
(iii) $\left\|u_\nu - u^0 - \sum_{j=1}^m \eta^j_\nu u^j_\nu\right\|_{H^1(M)} \to 0$ as $\nu \to \infty$, where

$$u^j_\nu(x) = (R^j_\nu)^{\frac{n-2}{2}} U^j(R^j_\nu \psi^{-1}_{x^j_\nu}(x)) \quad \textit{for } x \in \psi_{x^j_\nu}(B^+_{2r_0}(0)).$$

Here, $r_0 > 0$ is small, the

$$\psi_{x^j_\nu} : B^+_{2r_0}(0) \subset \mathbb{R}^n_+ \to M$$

are Fermi coordinates centered at $x^j_\nu \in \partial M$, and the η^j_ν are smooth cutoff functions such that $\eta^j_\nu \equiv 1$ in $\psi_{x^j_\nu}(B^+_{r_0}(0))$ and $\eta^j_\nu \equiv 0$ in $M\backslash\psi_{x^j_\nu}(B^+_{2r_0}(0))$.

Moreover,

$$I_g^\nu(u_\nu) - I_g^\infty(u^0) - \frac{m}{2(n-1)} K_n^{-2(n-1)} \to 0 \quad \text{as } \nu \to \infty\,,$$

and we can assume that for all $i \neq j$

$$\frac{R_\nu^i}{R_\nu^j} + \frac{R_\nu^j}{R_\nu^i} + R_\nu^i R_\nu^j d_g(x_\nu^i, x_\nu^j)^2 \to \infty \quad \text{as } \nu \to \infty\,. \tag{1.7}$$

Remark 1.4. Relations of the type (1.7) were previously obtained in [3, 6].

2 Proof of the main theorem

The rest of this paper is devoted to the proof of Theorem 1.3 which will be carried out in several lemmas. Our presentation will follow the same steps as Chapter 3 of [11], with the necessary modifications.

Lemma 2.1. *Let $\{u_\nu\}$ be a Palais-Smale sequence for $\{I_g^\nu\}$. Then there exists $C > 0$ such that $||u_\nu||_{H^1(M)} \leq C$ for all ν.*

Proof. It suffices to prove that $||du_\nu||_{L^2(M)}$ and $||u_\nu||_{L^2(\partial M)}$ are uniformly bounded. The proof follows the same arguments as [11, p.27]. □

Define I_g as the functional I_g^ν when $h_\nu \equiv 0$ for all ν.

Lemma 2.2. *Let $\{u_\nu \geq 0\}$ be a Palais-Smale sequence for $\{I_g^\nu\}$ such that $u_\nu \rightharpoonup u^0 \geq 0$ in $H^1(M)$ and set $\hat{u}_\nu = u_\nu - u^0$. Then $\{\hat{u}_\nu\}$ is a Palais-Smale sequence for $\{I_g\}$ and satisfies*

$$I_g(\hat{u}_\nu) - I_g^\nu(u_\nu) + I_g^\infty(u^0) \to 0 \quad \text{as } \nu \to \infty\,. \tag{2.1}$$

Moreover, u^0 is a (weak) solution of (1.3).

Proof. First observe that $u_\nu \rightharpoonup u^0$ in $H^1(M)$ implies that $u_\nu \to u^0$ in $L^{\frac{n}{n-2}}(\partial M)$ and a.e. in ∂M. Using the facts that $dI_g^\nu(u_\nu)\phi \to 0$ for any $\phi \in C^\infty(\bar{M})$ and $h_\nu \to h_\infty$ in $L^p(\partial M)$ for any $p \geq 1$, it is not difficult to see that the last assertion of Lemma 2.2 follows.

In order to prove (2.1), we first observe that

$$I_g^\nu(u_\nu) = I_g(\hat{u}_\nu) + I_g^\infty(u^0) - \frac{(n-2)}{2(n-1)} \int_{\partial M} \Phi_\nu d\sigma_g + o(1)\,,$$

where $\Phi_\nu = |\hat{u}_\nu + u^0|^{\frac{2(n-1)}{n-2}} - |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}} - |u^0|^{\frac{2(n-1)}{n-2}}$, and $o(1) \to 0$ as $\nu \to \infty$. Then (2.1) follows from the fact that there exists $C > 0$ such that

$$\int_{\partial M} \Phi_\nu d\sigma_g \leq C \int_{\partial M} |\hat{u}_\nu|^{\frac{n}{n-2}} |u^0| d\sigma_g + C \int_{\partial M} |u^0|^{\frac{n}{n-2}} |\hat{u}_\nu| d\sigma_g\,, \quad \text{for all } \nu\,,$$

and, by basic integration theory, the right side of this last inequality goes to 0 as $\nu \to \infty$.

Now we prove that $\{\hat{u}_\nu\}$ is a Palais-Smale sequence for I_g. Let $\phi \in C^\infty(M)$. Observe that

$$\left|\int_{\partial M} h_\nu u_\nu \phi d\sigma_g - \int_{\partial M} h_\infty u_\nu \phi d\sigma_g\right| \leq \|u_\nu\|_{L^2(\partial M)} \|h_\nu - h_\infty\|_{L^{2(n-1)}(\partial M)} \|\phi\|_{L^{\frac{2(n-1)}{n-2}}(\partial M)}$$

by Hölder's inequality. Then, by the Sobolev embedding theorem,

$$\int_{\partial M} h_\nu u_\nu \phi d\sigma_g = \int_{\partial M} h_\infty u^0 \phi d\sigma_g + o(\|\phi\|_{H^1(M)})$$

from which follows that

$$dI_g^\nu(u_\nu)\phi = dI_g(\hat{u}_\nu)\phi - \int_{\partial M} \psi_\nu \phi d\sigma_g + o(\|\phi\|_{H^1(M)})\,, \tag{2.2}$$

where $\psi_\nu = |\hat{u}_\nu + u^0|^{\frac{2}{n-2}}(\hat{u}_\nu + u^0) - |\hat{u}_\nu|^{\frac{2}{n-2}}\hat{u}_\nu - |u^0|^{\frac{2}{n-2}}u^0$.

Next we observe that there exists $C > 0$ such that

$$\int_{\partial M} |\psi_\nu \phi| d\sigma_g \leq C \int_{\partial M} |\hat{u}_\nu|^{\frac{2}{n-2}} |u^0|\,|\phi| d\sigma_g + C \int_{\partial M} |u^0|^{\frac{2}{n-2}} |\hat{u}_\nu|\,|\phi| d\sigma_g\,,$$

for all ν, and use Hölder's inequality and basic integration theory to obtain

$$\begin{aligned}\int_{\partial M} |\psi_\nu \phi| d\sigma_g &\leq \left(\left\||\hat{u}_\nu|^{\frac{2}{n-2}} u^0\right\|_{L^{\frac{2(n-1)}{n}}(\partial M)} + \left\||u^0|^{\frac{2}{n-2}} \hat{u}_\nu\right\|_{L^{\frac{2(n-1)}{n}}(\partial M)}\right) \|\phi\|_{L^{\frac{2(n-1)}{n-2}}(\partial M)} \\ &= o\left(\|\phi\|_{L^{\frac{2(n-1)}{n-2}}(\partial M)}\right).\end{aligned}$$

Then we can use this and the Sobolev embedding theorem in (2.2) to conclude that

$$dI_g^\nu(u_\nu)\phi = dI_g(\hat{u}_\nu)\phi + o(\|\phi\|_{H^1(M)})\,,$$

finishing the proof. □

Lemma 2.3. *Let $\{\hat{u}_\nu\}_{\nu\in\mathbb{N}}$ be a Palais-Smale sequence for I_g such that $\hat{u}_\nu \rightharpoonup 0$ in $H^1(M)$ and $I_g(\hat{u}_\nu) \to \beta$ as $\nu \to \infty$ for some $\beta < \dfrac{K_n^{-2(n-1)}}{2(n-1)}$. Then $\hat{u}_\nu \to 0$ in $H^1(M)$ as $\nu \to \infty$.*

Proof. Since

$$\int_M |d\hat{u}_\nu|^2 dv_g - \int_{\partial M} |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_g = dI_g(\hat{u}_\nu) \cdot \hat{u}_\nu = o(\|\hat{u}_\nu\|_{H^1(M)})$$

and $\{\|\hat{u}_\nu\|_{H^1(M)}\}$ is uniformly bounded due to Lemma 2.1, we can see that

$$\begin{aligned}\beta + o(1) = I_g(\hat{u}_\nu) &= \frac{1}{2(n-1)} \int_{\partial M} |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_g + o(1) \\ &= \frac{1}{2(n-1)} \int_M |d\hat{u}_\nu|_g^2 dv_g + o(1)\end{aligned} \tag{2.3}$$

which already implies $\beta \geq 0$. At the same time, as proved by Li and Zhu in [16], there exists $B = B(M, g) > 0$ such that

$$\left(\int_{\partial M} |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_g\right)^{\frac{n-2}{n-1}} \leq K_n^2 \int_M |d\hat{u}_\nu|_g^2 dv_g + B \int_{\partial M} |\hat{u}_\nu|^2 d\sigma_g\,.$$

Since $H^1(M)$ is compactly embedded in $L^2(\partial M)$, we have $\|\hat{u}_\nu\|_{L^2(\partial M)} \to 0$. Then we obtain

$$(2(n-1)\beta + o(1))^{\frac{n-2}{n-1}} \leq 2(n-1)K_n^2\beta + o(1)$$

from which we conclude that either

$$\frac{K_n^{-2(n-1)}}{2(n-1)} \leq \beta + o(1)$$

or $\beta = 0$. Hence, our hypotheses imply $\beta = 0$. Using (2.3) finishes the proof. □

Define the functional

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^n_+} |du(y)|^2 dy - \frac{n-2}{2(n-1)}\int_{\partial\mathbb{R}^n_+} |u(y)|^{\frac{2(n-1)}{n-2}} dy$$

for $u \in D^1(\mathbb{R}^n_+)$ and observe that $E(U_{\epsilon,a}) = \dfrac{K_n^{-2(n-1)}}{2(n-1)}$ for any $a \in \mathbb{R}^{n-1}$, $\epsilon > 0$.

Lemma 2.4. *Let $\{\hat{u}_\nu\}_{\nu\in\mathbb{N}}$ be a Palais-Smale sequence for I_g. Suppose $\hat{u}_\nu \rightharpoonup 0$ in $H^1(M)$, but not strongly. Then there exist a sequence $\{R_\nu > 0\}_{\nu\in\mathbb{N}}$ with $R_\nu \to \infty$, a convergent sequence $\{x_\nu\}_{\nu\in\mathbb{N}} \subset \partial M$, and a nontrivial solution $u \in D^1(\mathbb{R}^n_+)$ of*

$$\begin{cases} \Delta u = 0\,, & \text{in } \mathbb{R}^n_+\,, \\ \frac{\partial}{\partial y_n}u - |u|^{\frac{2}{n-2}}u = 0\,, & \text{on } \partial\mathbb{R}^n_+\,, \end{cases} \tag{2.4}$$

the whole such that, up to a subsequence, the following holds: If

$$\hat{v}_\nu(x) = \hat{u}_\nu(x) - \eta_\nu(x)R_\nu^{\frac{n-2}{2}}u(R_\nu\psi_{x_\nu}^{-1}(x))\,,$$

then $\{\hat{v}_\nu\}_{\nu\in\mathbb{N}}$ is a Palais-Smale sequence for I_g satisfying $\hat{v}_\nu \rightharpoonup 0$ in $H^1(M)$ and

$$\lim_{\nu\to\infty}\Big(I_g(\hat{u}_\nu) - I_g(\hat{v}_\nu)\Big) = E(u)\,.$$

Here, the $\psi_{x_\nu} : B^+_{2r_0}(0) \subset \mathbb{R}^n_+ \to M$ are Fermi coordinates centered at x_ν and the $\eta_\nu(x)$ are smooth cutoff functions such that $\eta_\nu \equiv 1$ in $\psi_{x_\nu}(B^+_{r_0}(0))$ and $\eta_\nu \equiv 0$ in $M\backslash\psi_{x_\nu}(B^+_{2r_0}(0))$.

Proof. By the density of $C^\infty(M)$ in $H^1(M)$ we can assume that $\hat{u}_\nu \in C^\infty(M)$. We can also assume that $I_g(\hat{u}_\nu) \to \beta$ as $\nu \to \infty$ and, since $dI_g(\hat{u}_\nu) \to 0$ in $H^1(M)'$, we obtain

$$\lim_{\nu\to\infty}\int_{\partial M} |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_g = 2(n-1)\beta \geq K_n^{-2(n-1)}$$

as in the proof of Lemma 2.3. Hence, given $t_0 > 0$ small we can choose $x_0 \in \partial M$ and $\lambda_0 > 0$ such that

$$\int_{D_{t_0}(x_0)} |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_g \geq \lambda_0$$

up to a subsequence. Now we set

$$\mu_\nu(t) = \max_{x \in \partial M} \int_{D_t(x)} |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_g$$

for $t > 0$, and, for any $\lambda \in (0, \lambda_0)$, choose sequences $\{t_\nu\} \subset (0, t_0)$ and $\{x_\nu\} \subset \partial M$ such that

$$\lambda = \mu_\nu(t_\nu) = \int_{D_{t_\nu}(x_\nu)} |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_g \,. \tag{2.5}$$

We can also assume that x_ν converges. Now we choose $r_0 > 0$ small such that for any $x_0 \in \partial M$ the Fermi coordinates $\psi_{x_0}(z)$ centered at x_0 are defined for all $z \in B^+_{2r_0}(0) \subset \mathbb{R}^n_+$ and satisfy

$$\frac{1}{2}|z - z'| \leq d_g(\psi_{x_0}(z), \psi_{x_0}(z')) \leq 2|z - z'| \,, \quad \text{for any } z, z' \in B^+_{r_0}(0) \,.$$

For each ν we consider Fermi coordinates

$$\psi_\nu = \psi_{x_\nu} : B^+_{2r_0}(0) \to M \,.$$

For any $R_\nu \geq 1$ and $y \in B^+_{R_\nu r_0}(0)$ we set

$$\tilde{u}_\nu(y) = R_\nu^{-\frac{n-2}{2}} \hat{u}_\nu(\psi_\nu(R_\nu^{-1} y)) \quad \text{and} \quad \tilde{g}_\nu(y) = (\psi_\nu^* g)(R_\nu^{-1} y) \,.$$

Let us consider $z \in \mathbb{R}^n_+$ and $r > 0$ such that $|z| + r < R_\nu r_0$. Then we have

$$\int_{B^+_r(z)} |d\tilde{u}_\nu|^2_{\tilde{g}_\nu} dv_{\tilde{g}_\nu} = \int_{\psi_\nu(R_\nu^{-1} B^+_r(z))} |d\hat{u}_\nu|^2_g dv_g \,,$$

and, if in addition $z \in \partial \mathbb{R}^n_+$,

$$\begin{aligned} \int_{\partial' B^+_r(z)} |\tilde{u}_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} &= \int_{\psi_\nu(R_\nu^{-1} \partial' B^+_r(z))} |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_g \\ &\leq \int_{D_{2R_\nu^{-1} r}(\psi_\nu(R_\nu^{-1} z))} |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_g \,, \end{aligned} \tag{2.6}$$

where we have used the fact that

$$\psi_\nu(R_\nu^{-1} \partial' B^+_r(z)) = \psi_\nu(\partial' B^+_{R_\nu^{-1} r}(R_\nu^{-1} z)) \subset D_{2R_\nu^{-1} r}(\psi_\nu(R_\nu^{-1} z)) \,.$$

Given $r \in (0, r_0)$ we fix $t_0 \leq 2r$. Then, given a $\lambda \in (0, \lambda_0)$ to be fixed later, we set $R_\nu = 2r t_\nu^{-1} \geq 2r t_0^{-1} \geq 1$. Then it follows from (2.5) and (2.6) that

$$\int_{\partial' B^+_r(z)} |\tilde{u}_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} \leq \lambda \,. \tag{2.7}$$

Moreover, since $\psi_\nu(\partial' B^+_{2R_\nu^{-1}r}(0)) = D_{t_\nu}(x_\nu)$ we have

$$\int_{\partial' B^+_{2r}(0)} |\tilde{u}_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} = \int_{D_{t_\nu}(x_\nu)} |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_g = \lambda\,. \tag{2.8}$$

Choosing r_0 smaller if necessary, we can suppose that

$$\frac{1}{2}\int_{\mathbb{R}^n_+} |du|^2 dy \le \int_{\mathbb{R}^n_+} |du|^2_{\tilde{g}_{x_0,R}} dv_{\tilde{g}_{x_0,R}} \le 2\int_{\mathbb{R}^n_+} |du|^2 dy \tag{2.9}$$

for any $R \ge 1$ and any $u \in D^1(\mathbb{R}^n_+)$ such that $\mathrm{supp}(u) \subset B^+_{2r_0R}(0)$. Here, $\tilde{g}_{x_0,R}(y) = (\psi^*_{x_0} g)(R^{-1}y)$. We can also assume that

$$\frac{1}{2}\int_{\partial\mathbb{R}^n_+} |u| dy \le \int_{\partial\mathbb{R}^n_+} |u| d\sigma_{\tilde{g}_{x_0,R}} \le 2\int_{\partial\mathbb{R}^n_+} |u| dy \tag{2.10}$$

for all $u \in L^1(\partial\mathbb{R}^n_+)$ such that $\mathrm{supp}(u) \subset \partial' B^+_{2r_0R}(0)$.

Let $\tilde{\eta}$ be a smooth cutoff function on $\mathbb{R}^n$ such that $0 \le \tilde{\eta} \le 1$, $\tilde{\eta}(z) = 1$ for $|z| \le \frac{1}{4}$, and $\tilde{\eta}(z) = 0$ for $|z| \ge \frac{3}{4}$. We set $\tilde{\eta}_\nu(y) = \tilde{\eta}(r_0^{-1}R_\nu^{-1}y)$.

It is easy to check that $\left\{ \int_{\mathbb{R}^n_+} |d(\tilde{\eta}_\nu\tilde{u}_\nu)|^2_{\tilde{g}_\nu} dv_{\tilde{g}_\nu} \right\}$ is uniformly bounded. Then the inequality (2.9) implies that $\{\tilde{\eta}_\nu\tilde{u}_\nu\}$ is uniformly bounded in $D^1(\mathbb{R}^n_+)$ and we can assume that $\tilde{\eta}_\nu\tilde{u}_\nu \rightharpoonup u$ in $D^1(\mathbb{R}^n_+)$ for some u.

Claim 1. Let us set $r_1 = r_0/24$. There exists $\lambda_1 = \lambda_1(n)$ such that for any $0 < r < r_1$ and $0 < \lambda < \min\{\lambda_1, \lambda_0\}$ we have

$$\tilde{\eta}_\nu\tilde{u}_\nu \to u\,, \quad \text{in } H^1(B^+_{2Rr}(0))\,, \quad \text{as } \nu \to \infty\,,$$

for any $R \ge 1$ satisfying $R \le R_\nu$ for all ν large.

Proof of Claim 1. We consider $r \in (0, r_1)$, $\lambda \in (0, \lambda_0)$ and choose $z_0 \in \partial\mathbb{R}^n_+$ such that $|z_0| < 3(2R-1)r_1$. By Fatou's lemma,

$$\int_r^{2r} \liminf_{\nu\to\infty} \left\{ \int_{\partial^+ B^+_\rho(z_0)} \Big\{ |d(\tilde{\eta}_\nu\tilde{u}_\nu)|^2 + |\tilde{\eta}_\nu\tilde{u}_\nu|^2 \Big\} d\sigma_\rho \right\} d\rho$$
$$\le \liminf_{\nu\to\infty} \int_{B^+_{2r}(z_0)} \Big\{ |d(\tilde{\eta}_\nu\tilde{u}_\nu)|^2 + |\tilde{\eta}_\nu\tilde{u}_\nu|^2 \Big\} dy \le C\,,$$

where $d\sigma_\rho$ is the volume form on $\partial^+ B^+_\rho(z_0)$ induced by the Euclidean metric. Thus there exists $\rho \in [r, 2r]$ such that, up to a subsequence,

$$\int_{\partial^+ B^+_\rho(z_0)} \Big\{ |d(\tilde{\eta}_\nu\tilde{u}_\nu)|^2 + |\tilde{\eta}_\nu\tilde{u}_\nu|^2 \Big\} d\sigma_\rho \le C\,, \quad \text{for all } \nu\,.$$

Hence, $\{\|\tilde{\eta}_\nu\tilde{u}_\nu\|_{H^1(\partial^+ B^+_\rho(z_0))}\}$ is uniformly bounded and, since the embedding

$$H^1(\partial^+ B^+_\rho(z_0)) \subset H^{1/2}(\partial^+ B^+_\rho(z_0))$$

is compact, we can assume that

$$\tilde{\eta}_\nu \tilde{u}_\nu \to u \text{ in } H^{1/2}(\partial^+ B^+_\rho(z_0)), \quad \text{as } \nu \to \infty.$$

We set $\mathcal{A} = B^+_{3r}(z_0) - \overline{B^+_\rho(z_0)}$ and let $\{\phi_\nu\} \subset D^1(\mathbb{R}^n_+)$ be such that

$$\phi_\nu = \begin{cases} \tilde{\eta}_\nu \tilde{u}_\nu - u, & \text{in } B^+_{\rho+\epsilon}(z_0), \\ 0, & \text{in } \mathbb{R}^n_+ \backslash B^+_{3r-\epsilon}(z_0), \end{cases}$$

with $\epsilon > 0$ small. Then

$$\|\tilde{\eta}_\nu \tilde{u}_\nu - u\|_{H^{1/2}(\partial^+ B^+_\rho(z_0))} = \|\phi_\nu\|_{H^{1/2}(\partial^+ B^+_\rho(z_0))} \to 0, \quad \text{as } \nu \to \infty$$

and there exists $\{\phi^0_\nu\} \subset D^1(\mathcal{A})$ such that

$$\|\phi_\nu + \phi^0_\nu\|_{H^1(\mathcal{A})} \leq C\|\phi_\nu\|_{H^{1/2}(\partial^+\mathcal{A})} = C\|\phi_\nu\|_{H^{1/2}(\partial^+ B^+_\rho(z_0))}$$

for some $C > 0$ independent of ν. Here, $D^1(\mathcal{A})$ is the closure of $C^\infty_0(\mathcal{A})$ in $H^1(\mathcal{A})$ and we have set $\partial^+\mathcal{A} = \partial\mathcal{A} \cap (\mathbb{R}^n_+ \backslash \partial\mathbb{R}^n_+)$ and $\partial'\mathcal{A} = \partial\mathcal{A} \cap \partial\mathbb{R}^n_+$.

The sequence of functions $\{\zeta_\nu\} = \{\phi_\nu + \phi^0_\nu\} \subset D^1(\mathbb{R}^n_+)$ satisfies

$$\zeta_\nu = \begin{cases} \tilde{\eta}_\nu \tilde{u}_\nu - u, & \text{in } \overline{B^+_\rho(z_0)}, \\ \phi_\nu + \phi^0_\nu, & \text{in } B^+_{3r}(z_0) \backslash \overline{B^+_\rho(z_0)}, \\ 0, & \text{in } \mathbb{R}^n_+ \backslash B^+_{3r}(z_0). \end{cases}$$

In particular, $\zeta_\nu \to 0$ in $H^1(\mathcal{A})$. We set

$$\tilde{\zeta}_\nu(x) = R_\nu^{\frac{n-2}{2}} \zeta_\nu(R_\nu \psi_\nu^{-1}(x)), \quad \text{if } x \in \psi_\nu(B^+_{6r_1}(0)),$$

and $\tilde{\zeta}_\nu(x) = 0$ otherwise. Since we are assuming $|z_0| < 3(2R-1)r_1 \leq 3(2R_\nu - 1)r_1$ for all ν large, $B^+_{3r}(z_0) \subset B^+_{6r_1 R_\nu}(0)$. Hence,

$$\tilde{\zeta}_\nu(x) = \begin{cases} R_\nu^{\frac{n-2}{2}} (\tilde{\eta}_\nu \tilde{u}_\nu - u)(R_\nu \psi_\nu^{-1}(x)), & \text{if } x \in \psi_\nu(R_\nu^{-1} \overline{B^+_\rho(z_0)}), \\ R_\nu^{\frac{n-2}{2}} (\phi_\nu + \phi^0_\nu)(R_\nu \psi_\nu^{-1}(x)), & \text{if } x \in \psi_\nu\Big(R_\nu^{-1}(\overline{B^+_{3r}(z_0)} \backslash B^+_\rho(z_0))\Big), \end{cases}$$

and $\tilde{\zeta}_\nu(x) = 0$ otherwise, and

$$\begin{aligned} dI_g(\hat{u}_\nu) \cdot \tilde{\zeta}_\nu &= dI_g(\hat{\eta}_\nu \hat{u}_\nu) \cdot \tilde{\zeta}_\nu \qquad (2.11) \\ &= \int_{B^+_{3r}(z_0)} < d(\tilde{\eta}_\nu \tilde{u}_\nu), d\zeta_\nu >_{\tilde{g}_\nu} dv_{\tilde{g}_\nu} - \int_{\partial' B^+_{3r}(z_0)} |\tilde{\eta}_\nu \tilde{u}_\nu|^{\frac{2}{n-2}} (\tilde{\eta}_\nu \tilde{u}_\nu) \zeta_\nu d\sigma_{\tilde{g}_\nu}, \end{aligned}$$

where $\hat{\eta}_\nu(x) = \tilde{\eta}(r_0^{-1} \psi_\nu^{-1}(x))$.

Since there exists $C > 0$ such that $\|\tilde{\zeta}_\nu\|_{H^1(M)} \leq C\|\zeta_\nu\|_{D^1(\mathbb{R}^n_+)}$, the sequence $\{\tilde{\zeta}_\nu\}$ is uniformly bounded in $H^1(M)$. Hence,

$$dI_g(\hat{u}_\nu) \cdot \tilde{\zeta}_\nu \to 0 \quad \text{as } \nu \to \infty. \qquad (2.12)$$

Noting that $\zeta_\nu \to 0$ in $H^1(\mathcal{A})$ and $\zeta_\nu \rightharpoonup 0$ in $D^1(\mathbb{R}^n_+)$, we obtain

$$\int_{B^+_{3r}(z_0)} < d(\tilde{\eta}_\nu \tilde{u}_\nu), d\zeta_\nu >_{\tilde{g}_\nu} dv_{\tilde{g}_\nu} = \int_{B^+_\rho(z_0)} < d(\zeta_\nu + u), d\zeta_\nu >_{\tilde{g}_\nu} dv_{\tilde{g}_\nu} + o(1) \tag{2.13}$$
$$= \int_{\mathbb{R}^n_+} |d\zeta_\nu|^2_{\tilde{g}_\nu} dv_{\tilde{g}_\nu} + o(1)\,.$$

Similarly,

$$\int_{\partial' B^+_{3r}(z_0)} |\tilde{\eta}_\nu \tilde{u}_\nu|^{\frac{2}{n-2}} (\tilde{\eta}_\nu \tilde{u}_\nu)\zeta_\nu \, d\sigma_{\tilde{g}_\nu} = \int_{\partial\mathbb{R}^n_+} |\zeta_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} + o(1)\,. \tag{2.14}$$

Using (2.11), (2.12), (2.13) and (2.14) we conclude that

$$\int_{\mathbb{R}^n_+} |d\zeta_\nu|^2_{\tilde{g}_\nu} dv_{\tilde{g}_\nu} = \int_{\partial\mathbb{R}^n_+} |\zeta_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} + o(1)\,. \tag{2.15}$$

Using again the facts that $\zeta_\nu \to 0$ in $H^1(\mathcal{A})$ and $\zeta_\nu \rightharpoonup 0$ in $D^1(\mathbb{R}^n_+)$, we can apply the inequality

$$\left| |\tilde{\eta}_\nu \tilde{u}_\nu - u|^{\frac{2(n-1)}{n-2}} - |\tilde{\eta}_\nu \tilde{u}_\nu|^{\frac{2(n-1)}{n-2}} + |u|^{\frac{2(n-1)}{n-2}} \right| \leq C|u|^{\frac{n}{n-2}} |\tilde{\eta}_\nu \tilde{u}_\nu - u| + C|\tilde{\eta}_\nu \tilde{u}_\nu - u|^{\frac{n}{n-2}} |u|$$

to see that

$$\int_{\partial\mathbb{R}^n_+} |\zeta_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} = \int_{\partial' B^+_\rho(z_0)} |\tilde{\eta}_\nu \tilde{u}_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} - \int_{\partial' B^+_\rho(z_0)} |u|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} + o(1)\,.$$

This implies

$$\int_{\partial\mathbb{R}^n_+} |\zeta_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} \leq \int_{\partial' B^+_\rho(z_0)} |\tilde{\eta}_\nu \tilde{u}_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} + o(1) \tag{2.16}$$
$$= \int_{\partial' B^+_\rho(z_0)} |\tilde{u}_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} + o(1)\,,$$

where we have used the fact that $\tilde{\eta}_\nu(z) = 1$ for all $z \in B^+_\rho(z_0)$.

If $N = N(n) \in \mathbb{N}$ is such that $\partial' B^+_2(0)$ is covered by N discs in $\partial\mathbb{R}^n_+$ of radius 1 with center in $\partial' B^+_2(0)$, then we can choose points $z_i \in \partial' B^+_{2r}(z_0)$, $i = 1, ..., N$, satisfying

$$\partial' B^+_\rho(z_0) \subset \partial' B^+_{2r}(z_0) \subset \bigcup_{i=1}^{N} \partial' B^+_r(z_i)\,.$$

Hence, using (2.7), (2.15) and (2.16) we see that

$$\int_{\mathbb{R}^n_+} |d\zeta_\nu|^2_{\tilde{g}_\nu} dv_{\tilde{g}_\nu} + o(1) = \int_{\partial\mathbb{R}^n_+} |\zeta_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} \leq N\lambda + o(1)\,. \tag{2.17}$$

It follows from (2.9), (2.10) and the Sobolev inequality (1.6) that

$$\left(\int_{\partial\mathbb{R}^n_+} |\zeta_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu}\right)^{\frac{n-2}{n-1}} \leq 2^{\frac{n-2}{n-1}} \left(\int_{\partial\mathbb{R}^n_+} |\zeta_\nu|^{\frac{2(n-1)}{n-2}} dx\right)^{\frac{n-2}{n-1}}$$
$$\leq 2^{\frac{n-2}{n-1}} K_n^2 \int_{\mathbb{R}^n_+} |d\zeta_\nu|^2 dx \leq 2^{1+\frac{n-2}{n-1}} K_n^2 \int_{\mathbb{R}^n_+} |d\zeta_\nu|^2_{\tilde{g}_\nu} dv_{\tilde{g}_\nu}\,.$$

Then using (2.15) and (2.17) we obtain

$$\int_{\mathbb{R}^n_+} |d\zeta_\nu|^2_{\tilde{g}_\nu} dv_{\tilde{g}_\nu} = \int_{\partial\mathbb{R}^n_+} |\zeta_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} + o(1)$$
$$\leq \left(2^{1+\frac{n-2}{n-1}} K_n^2\right)^{\frac{n-1}{n-2}} \left(\int_{\mathbb{R}^n_+} |d\zeta_\nu|^2_{\tilde{g}_\nu} dv_{\tilde{g}_\nu}\right)^{\frac{n-1}{n-2}} + o(1)$$
$$\leq 2^{1+\frac{n-1}{n-2}} K_n^{\frac{2(n-1)}{n-2}} (N\lambda + o(1))^{\frac{1}{n-2}} \int_{\mathbb{R}^n_+} |d\zeta_\nu|^2_{\tilde{g}_\nu} dv_{\tilde{g}_\nu} + o(1)\,.$$

Now we set $\lambda_1 = \dfrac{K_n^{-2(n-1)}}{2^{2n-3}N}$ and assume that $\lambda < \lambda_1$. Then

$$2^{1+\frac{n-1}{n-2}} (N\lambda)^{\frac{1}{n-2}} K_n^{\frac{2(n-1)}{n-2}} < 1,$$

and we conclude that

$$\lim_{\nu\to\infty} \int_{\mathbb{R}^n_+} |d\zeta_\nu|^2_{\tilde{g}_\nu} dv_{\tilde{g}_\nu} = 0\,.$$

Hence, $\zeta_\nu \to 0$ in $D^1(\mathbb{R}^n_+)$. Since $r \leq \rho$, we have

$$\tilde{\eta}_\nu \tilde{u}_\nu \to u \quad \text{in } H^1(B^+_r(z_0))\,. \tag{2.18}$$

Now let us choose any $z_0 = ((z_0)^1, ..., (z_0)^n) \in \mathbb{R}^n_+$ satisfying $(z_0)^n > \dfrac{r}{2}$ and $|z_0| < 3(2R-1)r_1$. Using this choice of z_0 and $r' = \dfrac{r}{6}$ replacing r, the process above can be performed with some obvious modifications. In this case, we have $\partial' B^+_{3r'}(z_0) = \emptyset$ and the boundary integrals vanish. Hence, the equality (2.15) already implies that $\tilde{\eta}_\nu \tilde{u}_\nu \to u$ in $H^1(B^+_{r'}(z_0))$.

If $N_1 = N_1(R,n) \in \mathbb{N}$ and $N_2 = N_2(R,n) \in \mathbb{N}$ are such that the half-ball $B^+_{2R}(0)$ is covered by N_1 half-balls of radius 1 with center in $\partial' B^+_{2R}(0)$ plus N_2 balls of radius 1/6 with center in $\{z = (z^1, ..., z^n) \in' B^+_{2R}(0)\,;\, z^n > 1/2\}$, then the half-ball $B^+_{2Rr}(0)$ is covered by N_1 half-balls of radius r with center in $\partial' B^+_{2Rr}(0)$ plus N_2 balls of radius $r/6$ with center in $\{z = (z^1, ..., z^n) \in B^+_{2Rr}(0)\,;\, z^n > r/2\}$.

Hence, $\tilde{\eta}_\nu \tilde{u}_\nu \to u$ in $H^1(B^+_{2Rr}(0))$, finishing the proof of Claim 1.

Using (2.8), (2.10) and Claim 1 with $R = 1$ we see that

$$\lambda = \int_{\partial' B_r^+(0)} |\tilde{u}_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} = \int_{\partial' B_r^+(0)} |\tilde{\eta}_\nu \tilde{u}_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} \qquad (2.19)$$
$$\leq 2 \int_{\partial' B_r^+(0)} |u|^{\frac{2(n-1)}{n-2}} dx + o(1)\,.$$

It follows that $u \not\equiv 0$, due to (1.6).

Claim 2. We have $\lim_{\nu\to\infty} R_\nu = \infty$. In particular, Claim 1 can be stated for any $R \geq 1$.

Proof of Claim 2. Suppose by contradiction that, up to a subsequence, $R_\nu \to R'$ as $\nu \to \infty$, for some $1 \leq R' < \infty$. Then, since $\hat{u}_\nu \rightharpoonup 0$ in $H^1(M)$, we have $\tilde{u}_\nu \rightharpoonup 0$ in $H^1(B^+_{2r}(0))$. This contradicts the fact that

$$\tilde{u}_\nu \tilde{\eta}_\nu \to u \not\equiv 0\,, \quad \text{in } H^1(B^+_{2r}(0))\,,$$

which is obtained by applying Claim 1 with $R = 1$. This proves Claim 2.

That u is a (weak) solution of (2.4) follows easily from the fact that $\{\hat{u}_\nu\}$ is a Palais-Smale sequence for I_g and $\tilde{\eta}_\nu \tilde{u}_\nu \to u$ in $D^1(\mathbb{R}^n_+)$.

Now we set

$$V_\nu(x) = \eta_\nu(x) R_\nu^{\frac{n-2}{2}} u(R_\nu \psi_{x_\nu}^{-1}(x))$$

for $x \in \psi_{x_\nu}(B^+_{2r_0}(0))$ and 0 otherwise. The proof of the following claim is totally analogous to Step 3 on p.37 of [11] with some obvious modifications.

Claim 3. We have $\hat{u}_\nu - V_\nu \rightharpoonup 0$, as $\nu \to \infty$, in $H^1(M)$. Moreover, as $\nu \to \infty$,

$$dI_g(V_\nu) \to 0 \quad \text{and} \quad dI_g(\hat{u}_\nu - V_\nu) \to 0$$

strongly in $H^1(M)'$, and

$$I_g(\hat{u}_\nu) - I_g(\hat{u}_\nu - V_\nu) \to E(u).$$

We finally observe that if $r_0' > 0$ is also sufficiently small then $|(\eta_\nu - \eta_\nu')V_\nu| \to 0$ as $\nu \to \infty$, where η_ν' is a smooth cutoff function such that $\eta_\nu' \equiv 1$ in $\psi_{x_\nu}(B^+_{r_0'}(0))$ and $\eta_\nu' \equiv 0$ in $M \backslash \psi_{x_\nu}(B^+_{2r_0'}(0))$. Hence, the statement of Lemma 2.4 holds for any $r_0 > 0$ sufficiently small, finishing the proof. □

Proof of Theorem 1.3. According to Lemma 2.1, the Palais-Smale sequence $\{u_\nu\}$ for I_g^ν is uniformly bounded in $H^1(M)$. Hence, we can assume that $u_\nu \rightharpoonup u^0$ in $H^1(M)$, and $u_\nu \to u^0$ a.e in M, for some $0 \leq u^0 \in H^1(M)$. By Lemma 2.2, u^0 is a solution to the equations (1.3). Moreover, $\hat{u}_\nu = u_\nu - u^0$ is Palais-Smale for I_g and satisfies

$$I_g(\hat{u}_\nu) = I_g^\nu(u_\nu) - I_g^\infty(u^0) + o(1)\,.$$

If $\hat{u}_\nu \to 0$ in $H^1(M)$, then the theorem is proved. If $\hat{u}_\nu \rightharpoonup 0$ in $H^1(M)$ but not strongly, then we apply Lemma 2.4 to obtain a new Palais-Smale sequence $\{\hat{u}_\nu^1\}$ satisfying

$$I_g(\hat{u}_\nu^1) \le I_g(\hat{u}_\nu) - \beta^* + o(1) = I_g^\nu(u_\nu) - I_g^\infty(u^0) - \beta^* + o(1)\,,$$

where $\beta^* = \dfrac{K_n^{-2(n-1)}}{2(n-1)}$. The term β^* appears in the above inequality because $E(u) \ge \beta^*$ for any nontrivial solution $u \in D^1(\mathbb{R}^n_+)$ to the equations (1.1). This can be seen using the Sobolev inequality (1.6).

Now we again have either $\hat{u}_\nu^1 \to 0$ in $H^1(M)$, in which case the theorem is proved, or we apply Lemma 2.4 to obtain a new Palais-Smale sequence $\{\hat{u}_\nu^2\}$. The process follows by induction and stops by virtue of Lemma 2.3, once we obtain a Palais-Smale sequence $\{\hat{u}_\nu^m\}$ with $I_g(\hat{u}_\nu^m)$ converging to some $\beta < \beta^*$.

We are now left with the proof of (1.7) and the fact that the U^j' obtained by the process above are of the form (1.5). To that end, we can follow the proof of Lemma 3.3 in [11], with some simple changes, to obtain the relation (1.7) and to prove that the U^j are nonnegative. For the reader's convenience this is outlined below.

Claim. The functions u^0 and U^j obtained above are nonnegative. Moreover, the identity (1.7) holds.

Proof of the Claim. That u^0 is nonnegative is straightforward. In order to prove that the U^j are also nonnegative we set $\hat{u}_\nu = u_\nu - u^0$ and $\mu_\nu^j = 1/R_\nu^j$.

Given integers $N \in [1, m]$ and $s \in [0, N-1]$, we will prove that there exist an integer p and sequences $\{\tilde{x}_\nu^k\}_{\nu\in\mathbb{N}} \subset \partial M$ and $\{\lambda_\nu^k > 0\}_{\nu\in\mathbb{N}}$, for each $k = 1, ..., p$, such that $d_g(x_\nu^N, \tilde{x}_\nu^k)/\mu_\nu^N$ is bounded and $\lim_{\nu\to\infty} \lambda_\nu^k/\mu_\nu^N = 0$, and such that

$$\int_{\Omega_\nu^N(R)\setminus\bigcup_{k=1}^p \tilde{\Omega}_\nu^k(R')} \left|\hat{u}_\nu - \sum_{j=1}^{s} u_\nu^j - u_\nu^N\right|^{\frac{2n}{n-2}} dv_g = o(1) + \epsilon(R')\,, \tag{2.20}$$

for any $R, R' > 0$. Here, $\Omega_\nu^N(R) = \psi_{x_\nu^N}(B^+_{R\mu_\nu^N}(0))$, $\tilde{\Omega}_\nu^k(R') = \psi_{\tilde{x}_\nu^k}(B^+_{R'\lambda_\nu^k}(0))$ and $\epsilon(R') \to 0$ as $R' \to \infty$.

We prove (2.20) by reverse induction on s. It follows from Claim 2 in the proof of Lemma 2.4 that

$$\int_{\Omega_\nu^N(R)} \left|\hat{u}_\nu - \sum_{j=1}^{N-1} u_\nu^j - u_\nu^N\right|^{\frac{2n}{n-2}} dv_g = o(1)\,,$$

so that (2.20) holds for $s = N-1$.

Assuming (2.20) holds for some $s \in [1, N-1]$, let us prove it does for $s-1$.

We first consider the case when $d_g(x_\nu^s, x_\nu^N)$ does not converge to zero as $\nu \to \infty$. In this case, we can assume $\Omega_\nu^N(R) \cap \Omega_\nu^s(\tilde{R}) = \emptyset$ for any $\tilde{R} > 0$. Then after rescaling we have

$$\int_{\Omega_\nu^N(R) \setminus \bigcup_{k=1}^p \tilde{\Omega}_\nu^k(R')} |u_\nu^s|^{\frac{2n}{n-2}} dv_g \le C \int_{\mathbb{R}_+^n \setminus B_{\tilde{R}}^+(0)} |U^s|^{\frac{2n}{n-2}} dy. \tag{2.21}$$

Since $\tilde{R} > 0$ is arbitrary and $U^s \in L^{\frac{2n}{n-2}}(\mathbb{R}_+^n)$, the left side of (2.21) converges to zero as $\nu \to \infty$. Hence, (2.20) still holds replacing s by $s-1$.

Let us now consider the case when $d_g(x_\nu^s, x_\nu^N) \to 0$ as $\nu \to \infty$. According to Claim 2 in the proof of Lemma 2.4, given $\tilde{R} > 0$ we have

$$\int_{\Omega_\nu^s(\tilde{R})} \Big| \hat{u}_\nu - \sum_{j=1}^s u_\nu^j \Big|^{\frac{2n}{n-2}} dv_g = o(1)\,.$$

Using the induction hypothesis (2.20) we then conclude that

$$\int_{(\Omega_\nu^N(R) \setminus \bigcup_{k=1}^p \tilde{\Omega}_\nu^k(R')) \cap \Omega_\nu^s(\tilde{R})} |u_\nu^N|^{\frac{2n}{n-2}} dv_g = o(1) + \epsilon(R')\,.$$

First assume that $d_g(x_\nu^s, x_\nu^N)/\mu_\nu^N \to \infty$. Rescaling by μ_ν^N and using coordinates centered at x_ν^N, it's not difficult to see that $d_g(x_\nu^s, x_\nu^N)/\mu_\nu^s \to \infty$. Hence we can assume that $\Omega_\nu^N(R) \cap \Omega_\nu^s(\tilde{R}) = \emptyset$ for any $\tilde{R} > 0$ and we proceed as in the case when $d_g(x_\nu^s, x_\nu^N)$ does not converge to 0 to conclude that (2.20) holds for $s-1$.

If $d_g(x_\nu^s, x_\nu^N)/\mu_\nu^N$ does not go to infinity, we can assume that it converges. In this case one can check that $\mu_\nu^s/\mu_\nu^N \to 0$. We set $\tilde{x}_\nu^{p+1} = x_\nu^s$ and $\lambda_\nu^{p+1} = \mu_\nu^s$, so that $\lambda_\nu^{p+1}/\mu_\nu^N \to 0$ as $\nu \to \infty$. Observing that

$$\int_{\Omega_\nu^N(R) \setminus \bigcup_{k=1}^{p+1} \tilde{\Omega}_\nu^k(R')} |u_\nu^s|^{\frac{2n}{n-2}} dv_g \le \int_{M \setminus \Omega_\nu^s(R')} |u_\nu^s|^{\frac{2n}{n-2}} dv_g \le \epsilon(R')\,,$$

it follows that (2.20) holds when we replace p by $p+1$ and s by $s-1$.

This proves (2.20). The above also proves (1.7).

We fix an integer $N \in [1, m]$ and $s = 0$. Let $\tilde{y}_\nu^k \in \partial\mathbb{R}_+^n$ be such that $\tilde{x}_\nu^k = \psi_{x_\nu^N}^N(\mu_\nu^N \tilde{y}_\nu^k)$, for $k = 1, ..., p$. For each k, the sequence $\{\tilde{y}_\nu^k\}_{\nu \in \mathbb{N}}$ is bounded so there exists $\tilde{y}^k \in \partial\mathbb{R}_+^n$ such that $\lim_{\nu\to\infty} \tilde{y}_\nu^k = \tilde{y}^k$, possibly after taking a subsequence. Let us set $\tilde{X} = \bigcup_{k=1}^p \tilde{y}^k$ and

$$\tilde{u}_\nu^N(y) = (\mu_\nu^N)^{\frac{n-2}{2}} \hat{u}_\nu^N(\psi_{x_\nu^N}(\mu_\nu^N y))\,.$$

It follows from (2.20) that

$$\tilde{u}_\nu^N \to U^N\,, \quad \text{in } L_{loc}^{\frac{2n}{n-2}}(B_R^+(0) \setminus \tilde{X})\,, \ \text{ as } \nu \to \infty\,.$$

Therefore we can assume that $\tilde{u}_\nu \to U^N$ a.e. in $\mathbb{R}_+^n$ as $\nu \to \infty$.

If we set
$$\tilde{u}^{0,N}_\nu(y) = (\mu^N_\nu)^{\frac{n-2}{2}} u^0(\psi_{x^N_\nu}(\mu^N_\nu y)),$$
it's easy to prove that
$$\tilde{u}^{0,N}_\nu \to 0\,, \quad \text{in } L^{\frac{2n}{n-2}}_{loc}(B^+_R(0))\,, \ \text{as } \nu \to \infty\,.$$
Hence, $\tilde{u}^{0,N}_\nu \to 0$ a.e. in $\mathbb{R}^n_+$ as $\nu \to \infty$. Setting
$$v^N_\nu(y) = (\mu^N_\nu)^{\frac{n-2}{2}} u^N_\nu(\psi_{x^N_\nu}(\mu^N_\nu y)),$$
we see that $v^N_\nu \to U^N$ a.e. in $\mathbb{R}^n_+$ as $\nu \to \infty$. In particular, U^N is nonnegative. This proves the Claim.

Remark 2.5. For the regularity of the U^j we can use [9, Théorème 1]. Although this theorem is established for compact manifolds we can use the conformal equivalence between $\mathbb{R}^n_+$ and $B^n\backslash\{\text{point}\}$ and a removable singularities theorem (see Lemma 2.7 on p.1821 of [1]) to apply it in B^n.

Thus we are able to use the result in [15] to conclude that the U^j are of the form (1.5), so we can write $U^j = U_{\epsilon_j,a_j}$.

This finishes the proof of Theorem 1.3. □

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