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Wave train defocusing in the presence of highly disordered topography

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Introduction

Equation for the wave train envelope in a finite and constant depth were derived by Benney and Roskes (1969), Hashimoto and Ono (1972) and Davey and Stewartson (1974). In this work we derive a nonlinear Schrödinger equation for the envelope for slowly modulated waves over a highly disordered topography at intermediate depth using assintotic analysis following Mei and Hancock (JFM, 2003) but generalizing their results. The topography has an impact on the stability of Stokes waves.

Evolution equations of the wave envelope over a disordered bottom

We consider the two-dimensional irrotational flow of an incompressible and inviscid fluid governed according to the theory of nonlinear potential in (x,z)-plane, where x is the direction of propagation of surface waves in this flow and z is the vertical coordinate. The region of interest is bounded below by an impermeable and stationary topography defined by a random function $z=-h(x)$ with zero mean and the free surface displacement given by $z=\eta(x,t)$.

The governing equations and nonlinear boundary conditions for the velocity potential $\Phi(x,z,t)$ and the frre surface displacement $z=\eta(x,t)$ are

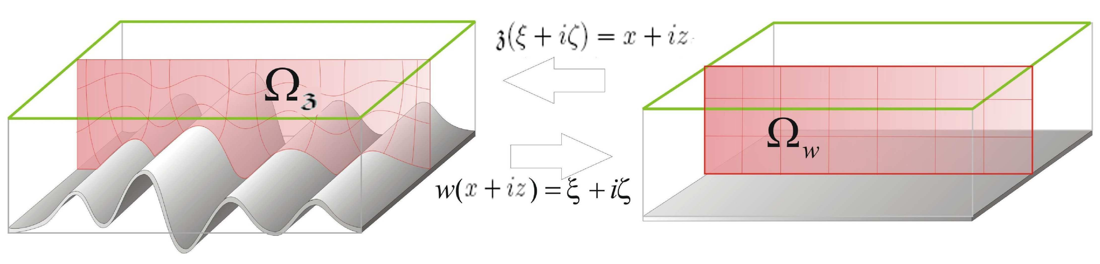
$$\begin{aligned} \Phi_{xx}+\Phi_{zz} &= 0, \quad -h(x)<z<\eta, \\ g\eta+\Phi_t+\frac{1}{2}|\nabla\Phi|^2 &= 0, \quad z=\eta(x,t) \\ \eta_t+\Phi_x\eta_x &= \Phi_z, \quad z=\eta(x,t) \\ h_x\Phi_x+\Phi_z &= 0, \quad z=-h(x). \end{aligned}$$

Hypotheses: $\eta=O(A)$ and $kA\ll 1$, ie we are considering the typical slope of the free surface height be small (small amplitude variation with respect to the wavelength). Also let us assume $k\,h=O(1)$ (a regime of intermediate depth).

Following Hamilton (1977) we define a conformal mapping:

$$\mathfrak{z}(\xi+i\zeta)=x+iz. \qquad (1)$$

Below we have a schematic figure showing the conformal mapping and its inverse.



The nonlinear potential theory equations in curvilinear (ξ,ζ) coordinates are

$$\begin{aligned} \Phi_{\xi\xi}+\Phi_{\zeta\zeta} &= 0, \quad -1<\zeta<N(\xi,t), \\ |J|N_t+\Phi_\xi N_\xi-\Phi_\zeta &= 0, \quad \zeta=N(\xi,t) \\ |J|\left(g\eta+\Phi_t\right)+\frac{1}{2}|\nabla_{\xi\zeta}\Phi|^2 &= 0, \quad \zeta=N(\xi,t) \\ \Phi_\zeta &= 0, \quad \zeta=-1. \end{aligned}$$

Here $N(\xi,t)$ is the function that describes the free surface profile in the new coordinate system and $|J|=|J|(\xi,\zeta)$ is the Jacobian of the transformation (1). Following Hamilton (1977), Nachbin (2003) e Artiles Roqueta (2004) the known results are

$$|J|(\xi,\zeta)=z_\xi^2+z_\zeta^2.$$

$$|J|(\xi,\zeta)=M(\xi)^2+\mathscr{R}_J(\xi,\zeta).$$

where $M(\xi)\equiv z_\zeta(\xi,0)$ (a variable free surface coefficient). Solving a boundary value problem for $z(\xi,\zeta)$ we obtain that:

$$M(\xi)\equiv z_\zeta(\xi,0)=\frac{\pi}{4}\int_{-\infty}^{\infty}\frac{h\left(x(\xi',-1)\right)}{cosh_2^{\pi}\left(\xi-\xi'\right)}d\xi'. \qquad (2)$$

Suppose that the function describing the topography of the seabed is given by

$$h(x)=\begin{cases} 1+n(x), & -L<x<L \\ 1, & x<-L \quad \text{e} \quad x>L. \end{cases} \qquad (3)$$

The topography can be of large amplitude and we do not need to assume that the fluctuations $n(x)$ are small, nor continuous, nor slowly varying.

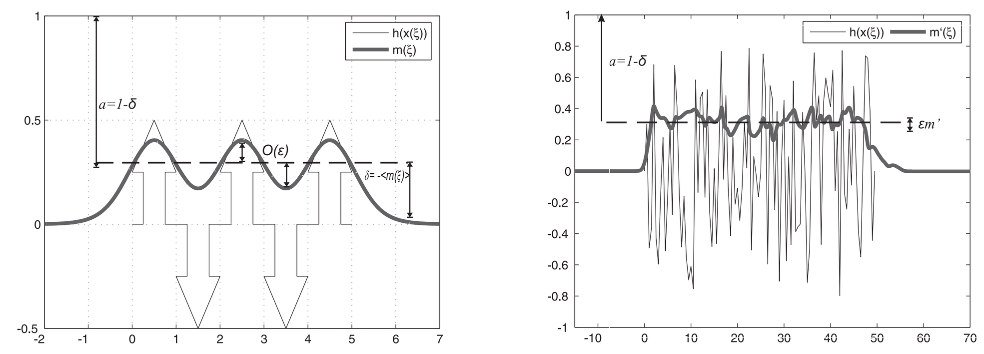
When $h(x)$ is written this way, the free surface coefficient (also known as metric coefficient) has the form:

$$M(\xi)=1+m(\xi) \text{ where } m(\xi)=\frac{\pi}{4}\int_{-\infty}^{\infty}\frac{n\left(x(\xi',-1)\right)}{cosh_2^{\pi}\left(\xi-\xi'\right)}d\xi'.$$

We can adopt the following expansion to $M(\xi)$,

$$M(\xi)=a+\varepsilon\, m'(\xi). \qquad (4)$$

where a represents the effective depth ("felt"at the free surface) while $m'(\xi)$ represents the fluctuation which has zero mean ($\langle m'\rangle=0$) and $\varepsilon=O(kA)$. Actually $a=1-\delta$, where $\delta=-\langle m(\xi)\rangle$ with $0<\delta<1$, ie $a<1$.



Replace $M(\xi)=a+\varepsilon\, m'(\xi)$ in the Taylor expansion of the free surface conditions. We introduce multiple scales

$$\xi,\ \xi_1=\varepsilon\xi,\ \xi_2=\varepsilon^2\xi\ldots,$$
$$t,\ t_1=\varepsilon t,\ t_2=\varepsilon^2 t\ldots,$$

where $\varepsilon=ka\ll 1$. Suppose Φ and η as power series in ε:

$$\Phi=\sum_{n=1}\varepsilon^n\phi_n, \qquad \eta=\sum_{n=1}\varepsilon^n\eta_n,$$

where

$$\phi_n=\phi_n\left(\xi,\xi_1,\xi_2,\ldots;\ldots;\zeta;t,t_1,t_2,\ldots\right),$$
$$\eta_n=\eta_n\left(\xi,\xi_1,\xi_2,\ldots;t,t_1,t_2,\ldots\right),$$

Equating like powers of ε yields

$$\begin{aligned} \left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\zeta^2}\right)\phi_n &= F_n, \quad -1<\zeta<0, \\ \mathcal{L}_a\phi_n\equiv\left(\frac{\partial^2}{\partial t^2}+\frac{g}{a}\frac{\partial}{\partial\zeta}\right)\phi_n &= G_n, \quad \zeta=0, \\ \frac{\partial\phi_n}{\partial\zeta} &= 0, \quad \zeta=-1. \end{aligned}$$

We get η_n from $-g\eta_n=H_n$, $\zeta=0$. For boundary value problems above, from $n=2$, the forcing terms will carry topography information through the a and $m'(\xi)$.

Let $\langle\ldots\rangle$ be the stochastic average and $(\ldots)'$ the random component. At all the orders, we express the solution as

$$\phi_n=\langle\phi_n\rangle+\phi_n', \quad \eta_n=\langle\eta_n\rangle+\eta_n', \quad n=1,2,3,\ldots$$

We also write

$$F_n=\langle F_n\rangle+F_n', \quad G_n=\langle G_n\rangle+G_n', \quad H_n=\langle H_n\rangle+H_n'.$$

If at all orders seek solutions (both for the deterministic part, as for random), as a series involving the Fourier modes as

$$\{\phi_n,F_n,G_n\}=\sum_{m=-n}^{n}e^{im\psi}\{\phi_{nm},F_{nm},G_{nm}\}. \qquad (5)$$

We obtain the following family of subproblems

$$\left(\frac{\partial^2}{\partial z^2}-m^2k^2\right)\phi_{nm}=F_{nm}, \quad -h<z<0, \qquad (6)$$

$$\left(\frac{g}{a}\frac{\partial}{\partial z}-m^2\omega^2\right)\phi_{nm}=G_{nm}, \quad z=0, \qquad (7)$$

$$\frac{\partial}{\partial z}\phi_{nm}=0, \quad z=-h. \qquad (8)$$

The Schrödinger equation arises from the condition of compatibility of boundary value problems in third order for the deterministic part.

$$-i\frac{\partial B}{\partial\tau}+\alpha_1\frac{\partial^2 B}{\partial\chi^2}+\alpha_2|B|^2B-i\Theta B=0,$$

where $\Theta=\frac{\widehat{\beta_{ai}}}{\omega}\left(\frac{\sigma_0}{A_0}\right)^2$, σ_0 is the root-mean-square of the free surface coeficient and $\widehat{\beta_{ai}}$ depends of $m'(\xi)$ correlation.

Stokes Waves: stability versus topography

$$-i\frac{\partial B}{\partial\tau}+\alpha_1\frac{\partial^2 B}{\partial\chi^2}+\alpha_2|B|^2B=0, \qquad (9)$$

with α_1 and α_2 given by

$$\alpha_1=-\frac{k^2}{2\omega}\frac{\partial^2\omega}{\partial k^2}, \quad \text{onde} \quad \omega^2=\frac{gk}{a}\tanh q$$

$$\alpha_2=\frac{\omega\,k^2}{16\,\sinh^4 q}\Big\{(6a^2+2)\cosh^4 q+(37-45a^2)\cosh^2 q+1+(36a^2-28)-2\tanh^2 q\Big\}-\frac{\left(a\,\widetilde{C_g}/c_p+2\cosh^2 q\right)^2}{2\,a\sinh^2 2q\left(\frac{q}{\tanh q}-\frac{\widetilde{C_g}^2}{c_p}\right)}-\frac{\left(a\,\widetilde{C_g}/c_p+2\cosh^2 q\right)(\Upsilon)}{2\,a^2\sinh^2 2q\left(\frac{q}{\tanh q}-\frac{\widetilde{C_g}^2}{c_p}\right)}.$$

Consider a solution of the form of a perturbed Stokes wave

$$B=B_0\left(1+\Delta b\right)e^{i(-\Omega\tau+\Delta\theta)}, \qquad (10)$$

where $b=b(\chi,\tau)$, $\theta(\chi,\tau)$ (both are real functions) and whose dispersion relation is given by $\Omega=\alpha_2|B_0|^2$. Replacing this solution in equation (9) we obtain

$$-(b_\tau+i\theta_\tau-ib\Omega)+\alpha_1(b_{\chi\chi}+i\theta_{\chi\chi})+3\alpha_2|B_0|^2b=0.$$

Since b and θ are real functions and using the dispersion relation above to eliminate Ω, we have:

$$\theta_\tau+\alpha_1 b_{\chi\chi}+2\alpha_2|B_0|^2b=0; \qquad (11)$$
$$-b_\tau+\alpha_1\theta_{\chi\chi}=0. \qquad (12)$$

As the equations (11) - (12) are linear with constant coefficients we look for a solution of the form:

$$\begin{pmatrix} b \\ \theta \end{pmatrix}=\begin{pmatrix} b_o \\ \theta_o \end{pmatrix}e^{i(\kappa\chi-\mu\tau)}+*, \qquad (13)$$

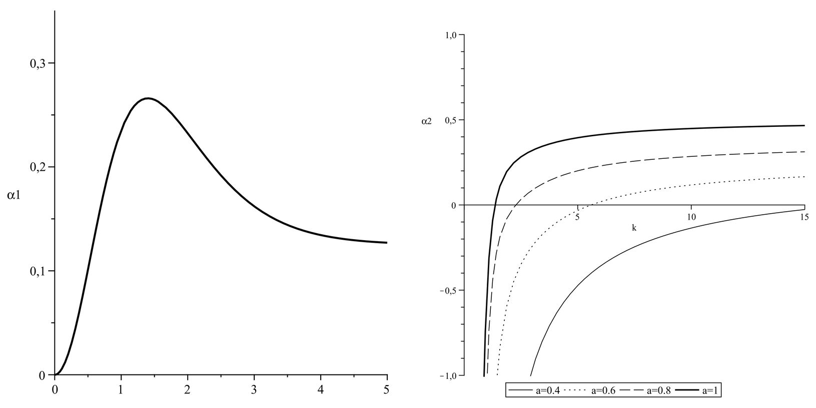
where b_o, θ_o, κ (>0) and μ are constant. This solution exists since

$$\begin{vmatrix} i\mu & -\kappa^2\alpha_1 \\ -\alpha_1\kappa^2+2|B_0|^2\alpha_2 & -i\mu \end{vmatrix}=0\Leftrightarrow\mu^2=(\alpha_1\kappa)^2(\kappa^2-2\frac{\alpha_2}{\alpha_1}|B_0|^2).$$

This equation define the following stability criteria:

$\frac{\alpha_2}{\alpha_1}<0$	(μ is real) stable	defocusing
$\frac{\alpha_2}{\alpha_1}>0$	(μ is a imaginary) unstable	focusing (NLS+)

The graphs below show the behavior of α_1 and α_2 as a function of k.



$\alpha_1\times k$ $\qquad$ $\alpha_2\times k$

The coefficient α_1 is positive for all $k>0$ independent of a. However, for a value of a fixed, α_2 changes sign for a given value of k. We calculate that this critical value k_0 where the coefficient α_2 changes sign when $a=1$ is ≈ 1.363 (the classical value to flat bottom); when a is decreasing we have k_0 increasing.

We can conclude that decreasing the value of the parameter a, ie increasing the amplitude of the fluctuations in topography, we regularize the solution: wave numbers that were unstable become stable. In other words, as the topography floats, the equation becomes defocusing in regions where it was focusing.

References

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