A correspondence for isometric immersions into product spaces and its applications

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Abstract

We present a correspondence for isometric immersions that are graphs in Riemannian or semi-Riemannian warped product spaces. We use this correspondence to give several existence and non-existence theorems for hypersurfaces in Riemannian or Lorentzian spaces. In the case of surfaces, we obtain further applications regarding height estimates, harmonic representation of surfaces or existence of holomorphic quadratic differentials in homogeneous and non-homogeneous spaces.

1 Introduction

In this paper we present a correspondence for isometric immersions into product spaces endowed with warped product metrics, and we explore its applications in topics such as non-immersion theorems, existence and rigidity problems, height or area estimates, representation formulas for surfaces in terms of harmonic maps, or existence of holomorphic quadratic differentials.

This basic correspondence works, roughly, as follows. Assume that there exists an isometric immersion ψ of a semi-Riemannian manifold (M^n, g_M) into a product space $N^n \times P^r$ endowed with a semi-Riemannian warped product metric

$$g = g_N \otimes_{\mathcal{F}} g_P := (\mathcal{F}^2 g_N) \otimes g_P, \qquad \mathcal{F} \in C^\infty(P),\ \mathcal{F} > 0.$$

Assume also that ψ is a *graph*, i.e. we can write $\psi = (\pi, h \circ \pi)$, where π is a diffeomorphism of M^n into some open domain $\Omega \subset N^n$, and $h : \Omega \to P$ is smooth. Then $\widehat{\psi} = (\pi^{-1}, h)$ describes an isometric immersion of $\Omega \subset N^n$ as a graph into the product space $M^n \times P^r$ endowed with the semi-Riemannian metric

$$g' = \frac{1}{\mathcal{F}^2} \left(g_M \otimes (-g_P) \right).$$

For example, in the most simple case of surfaces in $\mathbb{R}^3$, the basic correspondence implies that if $\psi : (M^2, g_M) \to \mathbb{R}^3$ is a local isometric immersion as a graph in $\mathbb{R}^3$, then there is an associated isometric immersion of a planar Euclidean domain $\Omega \subset \mathbb{R}^2$ as a graph into the Lorentzian product space $M^2 \times \mathbb{L} \equiv (M^2 \times \mathbb{R}, g := g_M - dt^2)$. This particular case is one of the classical key tools in the study of the local isometric embedding problem in $\mathbb{R}^3$ (see the books [8, 15, 27] and references therein), and our basic correspondence can be seen as a wide generalization of this fact.

We shall see that this extended correspondence has as well interesting geometric consequences for hypersurfaces in Lorentzian product manifolds. In this sense, it should be observed that the theory of (hyper)surfaces in Riemannian product spaces is currently experiencing a great activity, and that its Lorentzian counterpart is also starting to develop accordingly. In this sense, our basic correspondence will give very simple proofs of previously known results, but it will also yield new theorems that, in some cases, seem quite hard to obtain directly, i.e. without using in some way this correspondence.

Some of the main results that we will obtain are:

Theorem. *Let (M^n, g_M), $n \geq 2$, be a complete Riemannian manifold with negative Ricci curvature, and whose scalar curvature S_g satisfies $S_g \leq c < 0$. Then the Euclidean space $\mathbb{R}^n$ cannot be isometrically immersed into the Lorentzian product space $M^n \times \mathbb{L}$.*

Theorem. *Let (M^2, g_M) be a complete surface with constant Gauss curvature $K_M < 0$. Then no complete Riemannian surface $(\Sigma, \langle , \rangle_\Sigma)$ of constant curvature $c > K_M$ can be isometrically immersed into $M^2 \times \mathbb{L}$.*

Theorem. *Let (M^2, g_M) be a topological sphere endowed with a Riemannian metric g_M of positive curvature. Then there is a smooth 3-parameter family of isometric embeddings of the unit sphere $\mathbb{S}^2$ into the steady state type spacetime $M^2 \times_{e^t} \mathbb{L}$.*

Theorem. *Let (M^2, g_M) be a Riemannian surface of constant curvature $K_M > 0$, and let $\psi : \Omega \subset \mathbb{R}^2 \to M^2 \times \mathbb{L}$ be an isometric immersion of a planar domain Ω as a graph into $M^2 \times \mathbb{L}$, whose boundary lies on the slice $M^2 \times \{0\}$. Then the maximum height that ψ can rise over $M^2 \times \{0\}$ is $1/\sqrt{K_M}$, with equality holding at some point if and only if the projection of ψ over $M^2 \times \{0\}$ is isometric to a hemisphere of $\mathbb{S}^2(K_M)$.*

Theorem. *Any spacelike graph of constant curvature* $K = 1$ *in the steady state type spacetime* $\mathbb{S}^2(c) \times_{e^t} \mathbb{L}$ *has an associated holomorphic quadratic differential. The same is true for spacelike surfaces of constant curvature* $K_M < c$ *in the homogeneous Lorentzian product space* $\mathbb{S}^2(c) \times \mathbb{L}$.

Theorem. *There exists a correspondence between the space of harmonic diffeomorphisms from* $\Sigma = \mathbb{D}$ *or* $\mathbb{C}$ *onto* $\mathbb{H}^2$ *and the space of entire flat graphs in* $\mathbb{H}^2 \times \mathbb{R}$.

Specifically, any such harmonic diffeomorphism can be realized as the projection onto $\mathbb{H}^2$ *of a 2-parameter family of (generically non-congruent) entire flat graphs, for the conformal structure of the second fundamental form.*

As a general (but somehow inexact) rule of thought, the basic correspondence we expose here indicates that any result on (semi)-Riemannian submanifold theory which only has intrinsic hypotheses, and for which the ambient space admits a warped product structure, admits a *dual* result of the same nature in a different ambient manifold. In the simplest case of surfaces in product 3-manifolds, this duality produces results for surfaces in Lorentzian spaces from results in Riemannian spaces (and viceversa).

The paper is outlined as follows. In Section 2 we shall explain the correspondence in detail, and we will analyze how it changes the second fundamental form for the case of surfaces in product 3-manifolds. This will be useful for proving existence of holomorphic quadratic differentials later on.

In Section 3 we will describe how the correspondence works in the most simple cases, i.e. those associated with surfaces in 3-dimensional space forms. Section 4 will be devoted to obtain both existence and non-existence theorems for isometric immersions into Lorentzian warped product spaces. In Section 5 we will derive height and area estimates for surfaces of constant curvature in Lorentzian product 3-spaces. In Section 6 we will construct a holomorphic quadratic differential for constant curvature surfaces in Lorentzian product spaces $M^2 \times \mathbb{L}$, where M^2 also has constant curvature. Finally, also in Section 6, we will analyze the structure of the space of entire flat graphs in the Riemannian space $\mathbb{H}^2 \times \mathbb{R}$ in terms of harmonic maps into the hyperbolic space $\mathbb{H}^2$.

2 The basic correspondence

All along this paper (M^n, g_M), (N^n, g_N) and (P^r, g_P) will denote semi-Riemannian manifolds of dimensions n, n and r, respectively. These manifolds will actually be Riemannian (resp. Lorentzian) if the index of their metric is 0 (resp. 1).

Let us consider now an isometric immersion

$$\psi : (M, g_M) \to (N \times P, g),$$

where $N^n \times P^r$ is endowed with a warped product metric

$$g = g_N \otimes_{\mathcal{F}} g_P = (\mathcal{F}^2 g_N) \otimes g_P, \tag{2.1}$$

where $\mathcal{F}$ is a smooth positive function on P. When $\mathcal{F} = 1$ we get the usual product metric.

We shall assume furthermore that this isometric immersion is a *graph* over N, i.e. we can write

$$\psi(x) = (\pi(x), h(\pi(x)), \tag{2.2}$$

where $\pi : M \to \pi(M) \subset N$ is a diffeomorphism and $h : \pi(M) \to P$ is smooth. Then, associated to ψ we can build a new immersion, namely:

$$\widehat{\psi} : (\pi(M), g_N) \to M \times P$$

$$\widehat{\psi}(y) = (\pi^{-1}(y), h(y)), \quad \forall y \in \pi(M). \tag{2.3}$$

Observe that π^{-1} is well defined because π is a diffeomorphism. Hence, $\widehat{\psi}$ is a graph over M. Let us endow $M \times P$ with the metric

$$g' = \frac{1}{\mathcal{F}^2}(g_M \otimes (-g_P)). \tag{2.4}$$

Then we have

Lemma 1 *The map* $\widehat{\psi} : (\pi(M), g_N) \to (M \times P, g')$ *defined in* (2.3) *is an isometric immersion.*

Proof: Let $x \in M$, $y = \pi(x) \in N$, and consider $v, w \in T_yN$ as well as the vectors $\widetilde{v}, \widetilde{w} \in T_xM$ such that $v = d\pi_x(\widetilde{v})$, $w = d\pi_x(\widetilde{w})$. In what follows, we will omit for clarity the point at which we are working. Note that, as ψ is isometric, by (2.1),

$$\begin{aligned} g_M(\widetilde{v}, \widetilde{w}) &= \mathcal{F}^2 g_N(v, w) + g_P\left(d(h \circ \pi)(\widetilde{v}), d(h \circ \pi)(\widetilde{w})\right) \\ &= \mathcal{F}^2 g_N(v, w) + g_P(dh(v), dh(w)). \end{aligned} \tag{2.5}$$

So, since $d\pi^{-1}(v) = \widetilde{v}$ and $d\pi^{-1}(w) = \widetilde{w}$, we have using (2.4) and (2.5) that

$$\begin{aligned} g'(d\widehat{\psi}(v), d\widehat{\psi}(w)) &= g'((d\pi^{-1}(v), dh(v)), (d\pi^{-1}(w), dh(w))) \\ &= \frac{1}{\mathcal{F}^2}(g_M(\widetilde{v}, \widetilde{w}) - g_P(dh(v), dh(w))) \\ &= g_N(v, w), \end{aligned}$$

as claimed.

□

Lemma 1 provides a duality for isometric immersions into product spaces that can be formulated as follows:

The basic correspondence: If we have an isometric immersion of a semi-Riemannian manifold (M, g_M) as a graph in a product space $N \times P$ endowed with a warped metric g as in (2.1), then we can obtain an associated isometric immersion of a domain $\pi(M) \subset N$ in the ambient space $M \times P$ endowed with the semi-Riemannian metric g' in (2.4), which is a graph over M. This process can obviously be reversed. Moreover, in the case that all M, N, P are Riemannian, then the warped metric g' on $M \times P$ is semi-Riemannian of index $r = \dim P$.

Let us observe that g' can be seen as a warped product metric

$$g' = g_M \otimes_{1/\mathcal{F}} \left(\frac{-g_P}{\mathcal{F}^2} \right).$$

So, we can apply again the basic correspondence to $\widehat{\psi}$, from which we recover ψ, i.e. the basic correspondence is involutive.

It is also important to understand how the basic correspondence behaves with respect to ambient isometries. For this, let us consider the diffeomorphisms

$$\Phi_N : N \to N, \qquad \Phi_P : P \to P$$

of (N, g_N) and (P, g_P), respectively, and assume that Φ_N is an isometry. Then the product map $\Phi_N \times \Phi_P : N \times P \to N \times P$ will be an isometry of $(N \times P, g)$ (g as in (2.1)) if and only if Φ_P is an isometry of P and $\mathcal{F} \circ \Phi_P = \mathcal{F}$. We shall call to these isometries of a warped product space *split-isometries*. We have then:

Lemma 2 *Let $\psi_i : (M, g_M) \to (N \times P, g)$, $\psi_i = (\pi_i, h_i \circ \pi_i)$, $i = 1, 2$ be two isometric immersions as graphs that differ by a split-isometry $\Phi = \Phi_N \times \Phi_P$ of the ambient space $(N \times P, g)$ (g as in (2.1)). Assume without loss of generality that $\pi_1(M) = \pi_2(M) =: \Omega$. Then, the isometric immersions*

$$\widehat{\psi_1}, \widehat{\psi_2} : (\Omega \subset N, g_N) \to (M \times P, g'), \qquad (g' \text{ as in } (2.4))$$

differ, up to a reparametrization, by the split-isometry $\mathrm{Id}_M \times \Phi_P$ of $(M \times P, g')$.

Proof: By hypothesis, $\psi_2 = \Phi \circ \psi_1$, and thus $\Phi_N \circ \pi_1 = \pi_2$ and $\Phi_P \circ h_1 \circ \pi_1 = h_2 \circ \pi_2$, i.e. $\Phi_P \circ h_1 = h_2 \circ \Phi_N$.

Consider now $\widehat{\psi_i} = (\pi_i^{-1}, h_i)$, $i = 1, 2$. From the above relations we see that

$$\widehat{\Phi} \circ \widehat{\psi_1} = \widehat{\psi_2} \circ \Phi_N, \qquad \widehat{\Phi} := \mathrm{Id}_M \times \Phi_P,$$

which proves the result.

□

The above Lemma tells that the basic correspondence preserves the property of differing by a split-isometry of the ambient space. However, this is not true for general isometries. That is, if we consider two isometric immersions

$$\psi_1, \psi_2 : (M, g_M) \to (N \times P, g)$$

as graphs over N that are *congruent*, then their associated immersions $\widehat{\psi_1}, \widehat{\psi_2}$ will not be congruent in $(M \times P, g')$ in general. This situation can be used to deform submanifolds in an isometric but non-congruent way in product spaces, see Example 5 or Theorem 11, for instance.

Next, let us analyze in more detail this correspondence for the special case of product 3-spaces $M^2 \times \mathbb{R}$ with the usual product metric. That is, we will assume in the previous duality that $P = \mathbb{R}$, and that both M, N are 2-dimensional manifolds.

Let $p \in M$, $q \in N$ and $\pi : M \to \pi(M) \subset N$ be a diffeomorphism. Observe that, using that π is a diffeomorphism around p, it is possible to parametrize neighbourhoods of p in M and q in N in a way such that $\pi = \mathrm{Id}$. In other words, we can regard locally M, N and π as $M \equiv (\Omega, g_M)$, $N \equiv (\Omega, g_N)$ and $\pi = \mathrm{Id}_\Omega$, where $\Omega \subset \mathbb{R}^2$ is a planar domain.

With this, the isometric immersion (2.2) is written in these coordinates as

$$\psi(u,v) = (u, v, h(u,v)) : (\Omega, g_M) \to \left(\Omega \times \mathbb{R}, g := g_N \otimes dt^2\right).$$

Hence, its associated isometric immersion is

$$\widehat{\psi}(u,v) = (u, v, h(u,v)) : \left(\Omega, g_N\right) \to \left(\Omega \times \mathbb{R}, g' := g_M \otimes (-dt^2)\right).$$

If we write now the metric g_N in these local coordinates as

$$g_N = Edu^2 + 2Fdudv + Gdv^2,$$

then the first fundamental form of ψ is

$$I_\psi = (E + h_u^2)du^2 + 2(F + h_u h_v)dudv + (G + h_v^2)dv^2.$$

Hence, the unit normal vectors of ψ and $\widehat{\psi}$, denoted respectively in coordinates by $\mathcal{N}_\psi = (\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3)$ and $N_{\widehat{\psi}} = (n_1, n_2, n_3)$, can be computed by solving the following systems:

$$\left\{\begin{array}{l} 0 = g(\psi_u, \mathcal{N}_\psi) = E\mathcal{N}_1 + F\mathcal{N}_2 + h_u\mathcal{N}_3 \\ 0 = g(\psi_v, \mathcal{N}_\psi) = F\mathcal{N}_1 + G\mathcal{N}_2 + h_v\mathcal{N}_3 \\ 1 = g(\mathcal{N}_\psi, \mathcal{N}_\psi) = E\mathcal{N}_1^2 + 2F\mathcal{N}_1\mathcal{N}_2 + G\mathcal{N}_2^2 + \mathcal{N}_3^2 \end{array}\right. \tag{2.6}$$

$$\left\{\begin{array}{l} 0 = g'(\widehat{\psi}_u, N_{\widehat{\psi}}) = (E + h_u^2)n_1 + (F + h_u h_v)n_2 - h_u n_3 \\ 0 = g'(\widehat{\psi}_v, N_{\widehat{\psi}}) = (F + h_u h_v)n_1 + (G + h_v^2)n_2 - h_v n_3 \\ -1 = g'(N_{\widehat{\psi}}, N_{\widehat{\psi}}) = (E + h_u^2)n_1^2 + 2(F + h_u h_v)n_1 n_2 + (G + h_v^2)n_2^2 - n_3^2 \end{array}\right. \tag{2.7}$$

With this, a straightforward computation provides the following relations (up to a sign, i.e. up to a change of orientation in one of the surfaces):

$$n_1 = \mathcal{N}_1, \quad n_2 = \mathcal{N}_2, \quad n_3\mathcal{N}_3 = -1. \tag{2.8}$$

Let us consider now the respective second fundamental forms of ψ and $\widehat{\psi}$, written with respect to the (u,v) coordinates as

$$II_\psi = e_\psi du^2 + 2f_\psi dudv + g_\psi dv^2, \quad II_{\widehat{\psi}} = e_{\widehat{\psi}}du^2 + 2f_{\widehat{\psi}}dudv + g_{\widehat{\psi}}dv^2.$$

Denote by ∇, ∇' the metric connections associated to g and g' and by ∇^M, ∇^N the connections of M and N. Also, denote by Γ_{ij}^{kM} and Γ_{ij}^{kN} the Christoffel symbols of M and N in the (u,v) coordinates, respectively. Then,

$$\begin{aligned}
e_\psi = g(\nabla_{\psi_u}\psi_u, \mathcal{N}_\psi) &= g_N(\nabla^N_{\partial_u}\partial_u, \mathcal{N}_1\partial_u + \mathcal{N}_2\partial_v) + h_{uu}\mathcal{N}_3 \\
&= \Gamma_{11}^{1N}(\mathcal{N}_1 E + \mathcal{N}_2 F) + \Gamma_{11}^{2N}(\mathcal{N}_1 F + \mathcal{N}_2 G) + h_{uu}\mathcal{N}_3 \\
f_\psi = g(\nabla_{\psi_v}\psi_u, \mathcal{N}_\psi) &= g_N(\nabla^N_{\partial_v}\partial_u, \mathcal{N}_1\partial_u + \mathcal{N}_2\partial_v) + h_{uv}\mathcal{N}_3 \\
&= \Gamma_{12}^{1N}(\mathcal{N}_1 E + \mathcal{N}_2 F) + \Gamma_{12}^{2N}(\mathcal{N}_1 F + \mathcal{N}_2 G) + h_{uv}\mathcal{N}_3 \\
g_\psi = g(\nabla_{\psi_v}\psi_v, \mathcal{N}_\psi) &= g_N(\nabla^N_{\partial_v}\partial_v, \mathcal{N}_1\partial_u + \mathcal{N}_2\partial_v) + h_{vv}\mathcal{N}_3 \\
&= \Gamma_{22}^{1N}(\mathcal{N}_1 E + \mathcal{N}_2 F) + \Gamma_{22}^{2N}(\mathcal{N}_1 F + \mathcal{N}_2 G) + h_{vv}\mathcal{N}_3
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
e_{\widehat{\psi}} = g'(\nabla'_{\widehat{\psi}_u}\widehat{\psi}_u, n_{\widehat{\psi}}) &= g_M(\nabla^M_{\partial_u}\partial_u, n_1\partial_u + n_2\partial_v) - h_{uu}n_3 \\
&= \Gamma_{11}^{1M}(n_1(E + h_u^2) + n_2(F + h_u h_v)) \\
&\quad +\Gamma_{11}^{2M}(n_1(F + h_u h_v) + n_2(G + h_v^2)) - h_{uu}n_3 \\
f_{\widehat{\psi}} = g'(\nabla'_{\widehat{\psi}_v}\widehat{\psi}_u, n_{\widehat{\psi}}) &= g_M(\nabla^M_{\partial_v}\partial_u, n_1\partial_u + n_2\partial_v) - h_{uv}n_3 \\
&= \Gamma_{12}^{1M}(n_1(E + h_u^2) + n_2(F + h_u h_v)) \\
&\quad +\Gamma_{12}^{2M}(n_1(F + h_u h_v) + n_2(G + h_v^2)) - h_{uv}n_3 \\
g_{\widehat{\psi}} = g'(\nabla'_{\widehat{\psi}_v}\widehat{\psi}_v, n_{\widehat{\psi}}) &= g_M(\nabla^M_{\partial_v}\partial_v, n_1\partial_u + n_2\partial_v) - h_{vv}n_3 \\
&= \Gamma_{22}^{1M}(n_1(E + h_u^2) + n_2(F + h_u h_v)) \\
&\quad +\Gamma_{22}^{2M}(n_1(F + h_u h_v) + n_2(G + h_v^2)) - h_{vv}n_3
\end{aligned} \tag{2.10}$$

If we compute the Christoffel symbols and solve (2.6) and (2.7), then from (2.8), (2.9) and (2.10), a long but direct computation ensures that

$$e_\psi = e_{\widehat{\psi}}, \quad f_\psi = f_{\widehat{\psi}}, \quad g_\psi = g_{\widehat{\psi}}. \tag{2.11}$$

Summarizing, we have

Proposition 3 *Let*

$$\psi = (\pi, h \circ \pi) : (M^2, g_M) \to (N^2 \times \mathbb{R}, g := g_N \otimes dt^2)$$

be an isometric immersion as a graph, and, denoting $\Omega := \pi(M^2) \subset N^2$, let

$$\widehat{\psi} = (\pi^{-1}, h) : (\Omega, g_N) \to \left(M^2 \times \mathbb{R}, g' := g_M \otimes (-dt^2)\right)$$

be its associated isometric immersion. Then for any $p \in M^2$ and $q = \pi(p) \in N^2$, the second fundamental forms II_ψ and $II_{\widehat{\psi}}$ verify

$$II_{\psi(p)}(v_1, v_2) = II_{\widehat{\psi}(q)}(d\pi(v_1), d\pi(v_2)) \qquad \forall\, v_1, v_2 \in T_pM. \tag{2.12}$$

In particular:

1. *ψ has vanishing extrinsic curvature if and only if so does $\widehat{\psi}$.*

2. *II_ψ is a Riemannian metric on M^2 if and only if so is $II_{\widehat{\psi}}$.*

Let us recall that any Riemannian surface carries an associated conformal structure. Thus, if II_ψ is positive definite, it induces a conformal structure on the surface. In particular, from (2.12) we see that, in the conditions of Proposition 3, z is a local conformal parameter for II_ψ on M if and only if $\zeta = z \circ \pi^{-1}$ is a local conformal parameter for $II_{\widehat{\psi}}$ on Ω.

The following corollary is an immediate consequence of Proposition 3 and the above fact. For stating it, we introduce the following notation:

Notation: let $\psi, \widehat{\psi}$ denote the isometric immersions (2.2), (2.3) associated by the basic correspondence, and let α denote a symmetric $(0,2)$ tensor on M^n. Then, if $\Omega := \pi(M^n) \subset N^n$, we will denote by $\widehat{\alpha}$ the symmetric $(0,2)$ tensor on Ω given by $\widehat{\alpha} = \pi_*(\alpha)$, i.e.

$$\widehat{\alpha}_q(\widehat{v}, \widehat{w}) = \alpha_p(v, w) \tag{2.13}$$

for $p \in M^n$, $v, w \in T_pM$, $q = \pi(p)$ and $\widehat{v} = d\pi_p(v)$, $\widehat{w} = d\pi_p(w)$.

Notation: let α denote a real-valued symmetric $(0,2)$ tensor on a Riemann surface Σ. We define its associated quadratic differential $\alpha^{(2,0)}$ as the $(2,0)$-part of the complexification of α, i.e.

$$\alpha^{(2,0)} := \alpha(\partial_z, \partial_z)\, dz^2 = \frac{1}{4}\left(\alpha(\partial_u, \partial_u) - \alpha(\partial_v, \partial_v) - 2i\, \alpha(\partial_u, \partial_v)\right) dz^2,$$

where $z = u + iv$ is a complex parameter on Σ. This definition is independent of z and produces a globally defined complex-valued quadratic differential $\alpha^{(2,0)}$ on Σ, that is holomorphic whenever $(\alpha(\partial_z, \partial_z))_{\bar{z}} = 0$.

Corollary 4 *Let $\psi : M^2 \to \Omega \times \mathbb{R}$ and $\widehat{\psi} : \Omega \to M^2 \times \mathbb{R}$ be as in Proposition 3, and assume that II_ψ (and hence $II_{\widehat{\psi}}$) is positive definite. Then:*

1. *$f : M^2 \to \bar{\mathbb{C}}$ is a meromorphic function with respect to the conformal structure induced by II_ψ if and only if $f \circ \pi^{-1} : \Omega \to \bar{\mathbb{C}}$ is meromorphic for the conformal structure induced by $II_{\widehat{\psi}}$.*

2. *For any symmetric $(0,2)$ tensor α on M^2, $\alpha^{(2,0)}$ is a holomorphic quadratic differential for the conformal structure induced by II_ψ if and only if $\widehat{\alpha}^{(2,0)}$ is a holomorphic quadratic differential on Ω for the conformal structure induced by $II_{\widehat{\psi}}$.*

This result will have important geometric consequences in Section 6.

3 Examples

In this section we will explain how the correspondence described in Section 2 works for the most elementary cases of isometric immersions. Specifically, we will analyze the basic information provided by this correspondence for the case of surfaces in 3-dimensional Riemannian and Lorentzian spaces of constant curvature. It is convenient to remark that, in order to apply the correspondence, we need to express the ambient manifold as a warped product. As this can be done for a 3-dimensional space form in several different ways, it turns out that the correspondence of Section 2 can also be applied in different forms.

Let us remark that some of the classes of surfaces that we are going to connect via the basic correspondence are already connected by other (much more concrete) classical dualities. This is the case, for instance, of flat surfaces in $\mathbb{R}^3$ and spacelike flat surfaces in Minkowski 3-space, or spacelike surfaces with $K = 1$ in de Sitter 3-space. It is also the case of flat surfaces in $\mathbb{H}^3$ and spacelike flat surfaces in de Sitter 3-space. However, the correspondence that we introduce here is different from the previous ones.

Notation: We shall denote by $\mathbb{R}^n_s$ the pseudo-Euclidean space of dimension n and index s, endowed with the metric

$$\langle , \rangle = \sum_{i=1}^{n-s} dx_i^2 - \sum_{i=n-s+1}^{n} dx_i^2,$$

and by $\mathbb{L}^n, \mathbb{S}^n_1$ and $\mathbb{H}^n_1$, respectively, the *Minkowski space*

$$\mathbb{L}^n = \mathbb{R}^n_1 \equiv (\mathbb{R}^n, \langle , \rangle_{\mathbb{L}^n} = dx_1^2 + \cdots + dx_{n-1}^2 - dx_n^2),$$

the *de Sitter space*

$$\mathbb{S}^n_1 = \{x \in \mathbb{L}^{n+1} : \langle x, x \rangle = 1\},$$

and the *anti de-Sitter space*

$$\mathbb{H}^n_1 = \{x \in \mathbb{R}^{n+2}_2 : \langle x, x \rangle = -1\}.$$

These will be regarded as the canonical Lorentzian spaces of constant curvature.

Notation: From now on, if (M^n, g_M) is a Riemannian manifold and $\mathcal{F} : \mathbb{R} \to (0, \infty)$ is a smooth positive function, we will denote by

$$M^n \times_{\mathcal{F}} \mathbb{L}$$

the product space $M^n \times \mathbb{R}$ endowed with the Lorentzian warped product metric

$$\langle , \rangle := \mathcal{F}(t)^2 g_M - dt^2.$$

In particular, $M^2 \times \mathbb{L}$ will stand for $M^2 \times \mathbb{R}$ with its associated Lorentzian product metric.

3.1 Surfaces in $\mathbb{R}^3$

This is the simplest and most known case. Let

$$\psi : (M^2, g_M) \to \mathbb{R}^3 \equiv (\mathbb{R}^2 \times \mathbb{R}, dx^2 + dy^2 + dz^2),$$

$$\psi(x) = (\pi(x), h(\pi(x))),$$

be an isometric immersion of a surface as a graph in the Euclidean 3-dimensional space. Then, its associated immersion $\widehat{\psi}$ is a flat immersion into a Lorentzian product 3-manifold:

$$\widehat{\psi} : (\pi(M^2), dx^2 + dy^2) \to M^2 \times \mathbb{L}.$$

Here, we observe that the induced metric of $\widehat{\psi}$ is the flat metric referred to in the introduction that is commonly used for studying the local isometric embedding problem in $\mathbb{R}^3$, see [8, 15, 27] and references therein.

Even more specifically, we get that flat graphs in $\mathbb{R}^3$ are associated with spacelike flat surfaces in $\mathbb{L}^3$. In this sense, let us recall that any spacelike surface in a Lorentzian warped product 3-space $M^2 \times_{\mathcal{F}} \mathbb{L}$ is automatically a local graph over M^2.

Consider now $\mathbb{R}^3 \setminus \{0\} =: \mathbb{R}^3_*$ foliated by round spheres centered at the origin. Then $\mathbb{R}^3_* = \mathbb{S}^2 \times_r \mathbb{R}_+$, i.e. the usual Euclidean metric of $\mathbb{R}^3$ is expressed as the warped metric

$$\langle , \rangle_{\mathbb{R}^3} = r^2 \, g_{\mathbb{S}^2} + dr^2,$$

where $g_{\mathbb{S}^2}$ denotes the usual metric of the unit sphere and r denotes the distance to the origin.

Now, taking a flat graph over some domain of $\mathbb{S}^2$,

$$\psi : (\mathbb{R}^2, dx^2 + dy^2) \to \mathbb{S}^2 \times_r \mathbb{R}^+ \equiv \mathbb{R}^3_*,$$

$$\psi(x) = (\pi(x), h(\pi(x))),$$

we obtain an immersion of $\pi(\mathbb{R}^2) \subset \mathbb{S}^2$ into the Lorentzian space

$$\mathbb{S}^3_{1,+} \equiv (\mathbb{R}^2 \times \mathbb{R}^+, \frac{1}{r^2}(dx^2 + dy^2 - dr^2)).$$

It is known that this space $\mathbb{S}^3_{1,+}$ is one half of the usual de Sitter 3-space $\mathbb{S}^3_1$. Actually, $\mathbb{S}^3_{1,+}$ can be seen as the usual *steady state space* (see [7, 18, 16, 28, 2]), i.e. the open piece of $\mathbb{S}^3_1$ given by

$$\mathbb{S}^3_{1,+} = \{x \in \mathbb{S}^3_1 \subset \mathbb{L}^4 : \langle x, a \rangle > 0\},$$

where $a \in \mathbb{L}^4$ satisfies $\langle a, a \rangle = 0$ and $\langle a, (0, 0, 0, 1) \rangle > 0$. An alternative model for $\mathbb{S}^3_1$, which can be obtained by making the change of coordinates $r = e^{-t}$ in the previous half-space type model, is

$$\mathbb{S}^3_1 \equiv \mathbb{R}^2 \times_{e^t} \mathbb{L}.$$

More generally, consider now an isometric immersion of (M^2, g_M) into $\mathbb{R}^3$ as a graph over $\mathbb{S}^2$. Then, by the basic correspondence and making again the change $r = e^{-t}$, we obtain the existence of an associated isometric immersion of a spherical domain $\pi(M^2) \subset \mathbb{S}^2$ into the Lorentzian 3-space

$$M^2 \times_{e^t} \mathbb{L}.$$

These type of spaces appear in [2] as a generalization of the steady state space, and were called there *steady state type spacetimes*.

3.2 Surfaces in $\mathbb{S}^3$

As in the Euclidean case, the usual metric in the 3-sphere can be written as a warped product metric in several ways. For instance, by stereographic projection we can regard $\mathbb{S}^3 \setminus \{\text{north}\}$ as $\mathbb{R}^3$ endowed with the metric

$$\frac{4}{(1+r^2)^2}\langle , \rangle_{\mathbb{R}^3}, \qquad r := ||p||, \qquad p \in \mathbb{R}^3,$$

and so, by using the change of coordinates $r = e^t$, this metric can be seen as

$$\frac{1}{\cosh^2 t}(g_{\mathbb{S}^2} \otimes dt^2). \tag{3.1}$$

Then, if

$$\psi : (M^2, g_M) \to \mathbb{S}^3 \setminus \{\text{north}, \text{south}\},$$
$$\psi(x) = (\pi(x), h(\pi(x))),$$

is an isometric immersion as a graph over a domain of $\mathbb{S}^2$, the basic correspondence shows the existence of an associated isometric immersion of $\pi(M) \subset \mathbb{S}^2$ into the Lorentzian space

$$\bar{M}^3 \equiv M^2 \times_{\mathcal{F}} \mathbb{L}, \qquad \mathcal{F}(t) = \cosh t. \tag{3.2}$$

Let us point out that this type of Lorentzian 3-spaces include again the de Sitter space $\mathbb{S}^3_1$, in the case $M^2 = \mathbb{S}^2$ (a complete list of the Lorentzian warped product structures with base $\mathbb{S}^2$, $\mathbb{H}^2$ or $\mathbb{R}^2$ that have constant curvature can be found in [5]).

3.3 Surfaces in $\mathbb{H}^3$

The hyperbolic 3-space admits again different warped product structures, that will provide different geometric information.

First, let us consider the half-space model of $\mathbb{H}^3$, that is

$$\mathbb{H}^3 \equiv (\mathbb{R}^2 \times \mathbb{R}^+, \frac{1}{z^2}(dx^2 + dy^2 + dz^2)).$$

Take now an immersed surface in $\mathbb{H}^3$ that is a graph over a region of a horosphere. Then, up to an isometry of $\mathbb{H}^3$, we can view this surface as a vertical graph in the above model, and apply the basic correspondence. In this way, from a graph

$$\psi : (M^2, g_M) \to (\mathbb{R}^2 \times \mathbb{R}^+, \frac{1}{z^2}(dx^2 + dy^2 + dz^2)),$$

$$\psi(p) = (\pi(p), h(\pi(p))),$$

we obtain the associated isometric immersion

$$\widehat{\psi} : (\pi(M^2), dx^2 + dy^2) \rightarrow M^2 \times_z \mathbb{L}^+,$$

$$\widehat{\psi}(q) = (\pi^{-1}(q), h(q)).$$

This is a flat surface in the Lorentzian 3-space $M^2 \times_z \mathbb{L}^+$, which can be seen as a *generalized Lorentzian cone.* Indeed, the usual timelike cone in $\mathbb{L}^4$ is obtained when we choose $M^2 = \mathbb{S}^2$. By the usual substitution $z = e^{-t}$, this generalized timelike cone can also be seen as the Lorentzian space

$$\bar{M}^3 \equiv \left(M^2 \times \mathbb{R}, g := e^{-2t} \left(g_M \otimes (-dt^2)\right)\right).$$

In particular, we obtain a quite explicit link between the well-developed theory of flat surfaces in $\mathbb{H}^3$ (see for instance [12, 14, 19, 20]) and the theory of flat surface in the cone-type Lorentzian warped product space $\mathbb{R}^2 \times_z \mathbb{L}$. This seems to indicate that the existing holomorphic representation formula for flat surfaces in $\mathbb{H}^3$ could possibly be extended to the case of spacelike flat surfaces in this special Lorentzian 3-manifold.

On the other hand, let us consider the Poincaré ball model for $\mathbb{H}^3$, i.e. the unit ball $\mathbb{B}^3$ of $\mathbb{R}^3$ endowed with the Poincaré metric

$$g_P = \frac{4}{(1 - ||p||^2)^2} \langle , \rangle_{\mathbb{R}^3}, \qquad ||p|| < 1.$$

If we view now $\mathbb{R}^3_*$ as the warped space $\mathbb{S}^2 \times_r \mathbb{R}_+$ as explained in Subsection 3.1, and make the usual change $r = e^{-t}$, we can regard $\mathbb{H}^3_* := \mathbb{H}^3 \setminus \{0\}$ as

$$\mathbb{H}^3_* \equiv \left(\mathbb{S}^2 \times \mathbb{R}^+, g = \frac{1}{\sinh^2(t)} (g_{\mathbb{S}^2} \otimes (dt^2))\right).$$

Therefore, by the basic correspondence, from an immersed surface in $\mathbb{H}^3$ that can be viewed as a graph over some domain of a totally umbilical round sphere,

$$\psi : (M^2, g_M) \rightarrow \mathbb{H}^3_* \equiv (\mathbb{S}^2 \times \mathbb{R}^+, g),$$

$$\psi(x) = (\pi(x), h(\pi(x))),$$

we obtain an associated isometric immersion of a spherical domain $\pi(M^2) \subset \mathbb{S}^2$ into the Lorentzian ambient space

$$M^2 \times_{\mathcal{F}} \mathbb{L}^+, \qquad \mathcal{F}(t) := \sinh(t). \tag{3.3}$$

In particular, if we start with $(M^2, g_M) \equiv \mathbb{H}^2$, we obtain a spacelike graph of constant curvature 1 in de Sitter 3-space $\mathbb{S}^3_1$, since a model for this space is exactly

$$\mathbb{S}^3_1 \equiv \mathbb{H}^2 \times_{\mathcal{F}} \mathbb{L}^+, \qquad \mathcal{F}(t) := \sinh(t).$$

In other words, isometric immersions of $\mathbb{H}^2$ into $\mathbb{H}^3$ as local graphs provide isometric immersions of domains $\pi(\mathbb{H}^2) \subset \mathbb{S}^2$ into $\mathbb{S}^3_1$. Again, the Lorentzian warped product spaces (3.3) can be seen as *generalized de Sitter spaces* in a natural way.

3.4 Surfaces in Lorentzian space forms

The above usages of the basic correspondence starting with a surface in a Riemannian space form $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$ work similarly when considering the Lorentzian model spaces $\mathbb{L}^3$, $\mathbb{S}^3_1$ and $\mathbb{H}^3_1$. For instance:

1. Any spacelike surface (M^2, g_M) in $\mathbb{L}^3 \equiv \mathbb{R}^2 \times \mathbb{L}$ produces a flat surface in the Riemannian product space $M^2 \times \mathbb{R}$.

2. Consider the warped product structure for de Sitter 3-space

$$\mathbb{S}^3_1 \equiv \mathbb{S}^2 \times_{\mathcal{F}} \mathbb{L}, \qquad \mathcal{F}(t) := \cosh t.$$

Then, if

$$\psi : (M^2, g_M) \to \mathbb{S}^3_1,$$
$$\psi(p) = (\pi(p), h(\pi(p))),$$

is a spacelike isometric immersion which is a graph over $\mathbb{S}^2$, then we obtain an associated isometric immersion of curvature 1 of the spherical domain $\pi(M^2) \subset \mathbb{S}^2$ into the Riemannian product space

$$(M^2 \times \mathbb{R}, g), \qquad g := \frac{1}{\cosh^2 t}\left(g_M \otimes (dt^2)\right).$$

Let us remark that this Riemannian warped product metric is the canonical one in $\mathbb{S}^3$ minus two antipodal points when $M^2 \equiv \mathbb{S}^2$ (see (3.1)).

Several other examples can be obtained similarly if we express the Lorentzian space $\mathbb{L}^3$, $\mathbb{S}^3_1$, $\mathbb{H}^3_1$ as a warped product in a different way. We omit further details, since the process was already explained in the Riemannian case.

Let us conclude this section exposing a more specific example, that will be studied in detail in Section 6.

Example 5 *The class of isometric immersions of $\mathbb{H}^2$ into $\mathbb{L}^3$ corresponds by the basic correspondence to the class of entire flat graphs in $\mathbb{H}^2 \times \mathbb{R}$. Moreover, by Lemma 2, congruent entire flat graphs in $\mathbb{H}^2 \times \mathbb{R}$ correspond exactly to isometric immersions of $\mathbb{H}^2$ into $\mathbb{L}^3$ differing by a split-isometry of $\mathbb{L}^3$, since all isometries of $\mathbb{H}^2 \times \mathbb{R}$ are actually split-isometries.*

But now, observe that in $\mathbb{L}^3$ there exist isometries that are not split-isometries, and that the quotient space $\{\text{isometries}\}/\{\text{split-isometries}\}$ *is* 2*-dimensional. Thus, associated to a congruence class of isometric immersions of $\mathbb{H}^2$ into $\mathbb{L}^3$ we get, generically, a* 2*-parameter family of congruency classes of entire flat graphs in $\mathbb{H}^2 \times \mathbb{R}$.*

4 Existence and non-existence theorems

In this section, some results about existence and non-existence of isometric immersions into Lorentzian product manifolds are obtained. For that, we apply the basic correspondence introduced in Section 2 to some non-immersion theorems in Riemannian space forms [9, 25, 26, 10, 17] and homogeneous spaces [3], as well as to some existence theorems regarding Weyl's embedding problem [22]. It is remarkable that some of these results would be extremely difficult to prove in a direct way, i.e. without using the basic correspondence.

One of the most famous non-immersion theorems is Efimov's theorem [9], which states that no complete surface (M^2, g) with $K_M \leq c < 0$ can be isometrically immersed into $\mathbb{R}^3$. This result had been previously obtained by Hilbert when K_M is constant, and by Heinz [17] for the case of entire graphs. It was subsequently generalized by Smyth and Xavier [26] to the case of hypersurfaces in $\mathbb{R}^{n+1}$, under some additional assumptions in the case $n > 3$.

Our first non-immersion theorem uses the basic correspondence together with Heinz's result and its recent extension to arbitrary dimension by Fontenele:

Theorem 6 *Let (M^n, g), $n \geq 2$, be a complete Riemannian manifold with negative Ricci curvature, and whose scalar curvature S_g satisfies $S_g \leq c < 0$.*

Then the Euclidean space $(\mathbb{R}^n, \langle,\rangle_{\mathbb{R}^n})$ cannot be isometrically immersed into the Lorentzian product space $M^n \times \mathbb{L}$.

Proof: Let $(\widetilde{M}, g_{\widetilde{M}})$ denote the universal Riemannian covering of (M^n, g_M), and suppose that

$$\widehat{\psi} : (\mathbb{R}^n, \langle,\rangle_{\mathbb{R}^n}) \to \widetilde{M} \times \mathbb{L}$$
$$x \mapsto (\pi(x), h(x))$$

is an isometric immersion. It is clear that the map $\pi : \mathbb{R}^n \to \widetilde{M}$ is a local diffeomorphism, since $\widehat{\psi}$ is spacelike. Moreover, the projection π increases distances. Then, since $\mathbb{R}^n$ is complete, π is a covering map. Moreover, as $\widetilde{M}$ is simply connected, π is actually a diffeomorphism. Hence, by the basic correspondence, we get an isometric immersion of $\widetilde{M}$ as a graph in $\mathbb{R}^{n+1}$:

$$\psi : (\widetilde{M}, g_{\widetilde{M}}) \to (\mathbb{R}^n \times \mathbb{R}, \langle,\rangle_{\mathbb{R}^n} \otimes dt^2) \equiv \mathbb{R}^{n+1}$$
$$y \mapsto (\pi^{-1}(y), h(\pi^{-1}(y))).$$

However, for $n \geq 3$ this contradicts a recent theorem by Fontenele [10], which states that any entire graph in $\mathbb{R}^{n+1}$ with negative Ricci curvature verifies that $\inf_M \|A\|^2 = 0$, where here $||A||$ stands for the norm of the second fundamental form. As $|S_g| \leq \delta \|A\|^2$ for a positive constant $\delta > 0$, this implies the desired result. In the case $n = 2$, Heinz proved in [17] that $\inf_{M^2} |K_M| = 0$ for the Gauss curvature K_M of any entire graph M^2 in $\mathbb{R}^3$. Thus, in any case, we conclude that $\mathbb{R}^n$ cannot be isometrically immersed into $M^n \times \mathbb{L}$.

□

In particular, we obtain the following consequences:

1. The Euclidean plane $\mathbb{R}^2$ cannot be isometrically immersed into a Lorentzian product space $M^2 \times \mathbb{L}$ in which (M, g_M) is a complete Riemannian surface with $K_M \leq \text{const} < 0$.

2. The Euclidean space $\mathbb{R}^n$ cannot be isometrically immersed into the Lorentzian product space $\mathbb{H}^n(\kappa) \times \mathbb{L}$, for any $n \geq 2$.

The classical Efimov non-immersion theorem was extended by Schlenker [25] for surfaces in the remaining space forms $\mathbb{S}^3, \mathbb{H}^3$. Specifically, he proved that: (a) no complete Riemannian surface whose curvature satisfies $K \leq -1-\varepsilon < -1$ and $\sup_M |\nabla(\frac{1}{\sqrt{K}})| < \infty$ can be isometrically immersed into $\mathbb{H}^3$, and (b) no complete Riemannian surface with $K \leq -\varepsilon < 0$ and $\sup_M |\nabla(\frac{1}{\sqrt{K}})| < \infty$ can be isometrically immersed into $\mathbb{S}^3$.

Theorems 7 and 8 are a direct consequence of applying the basic correspondence of Section 2 to these results. We can easily deduce them by using the expression of $\mathbb{H}^3$ and $\mathbb{S}^3$ as warped product spaces given in Section 3.

Theorem 7 *Let (M^2, g_M) be a complete surface whose Gauss curvature K_M satisfies $K_M \leq -1 - \varepsilon < -1$ and $\sup_M |\nabla(\frac{1}{\sqrt{K}})| < \infty$. Then there is no isometric immersion of the Euclidean plane $\mathbb{R}^2$ into the generalized Lorentzian cone $M^2 \times_z \mathbb{L}^+$. Neither there are isometric immersions of the unit sphere $\mathbb{S}^2$ into*

$$M^2 \times_{\mathcal{F}} \mathbb{L}^+, \qquad \mathcal{F}(t) := \sinh(t).$$

Theorem 8 *Let (M, g_M) be a complete surface whose Gauss curvature satisfies $K_M \leq -\varepsilon < 0$ and $\sup_M |\nabla(\frac{1}{\sqrt{K}})| < \infty$. Then there is no isometric immersion of the unit sphere $\mathbb{S}^2$ into the Lorentzian space $\bar{M}^3$ given by (3.2).*

Next, we will present some non-immersion theorems for constant curvature surfaces in Lorentzian product spaces $M^2 \times \mathbb{L}$, where M^2 also has constant curvature.

Theorem 9 *Let (M^2, g_M) be a complete surface with constant Gauss curvature $K_M < 0$. Then no complete Riemannian surface $(\Sigma, \langle , \rangle_\Sigma)$ of constant curvature $c > K_M$ can be isometrically immersed into $M^2 \times \mathbb{L}$.*

Proof: If $c > 0$, the proof follows from a simple topological argument as in the proof of Theorem 6, bearing in mind that in that case Σ would be compact while the Riemannian universal covering $\widetilde{M}$ would be diffeomorphic to the plane. The case $c = 0$ is covered by Theorem 6.

Now, assume that $c < 0$, and consider an isometric immersion

$$\widehat{\psi} : (\Sigma, \langle , \rangle_\Sigma) \to \widetilde{M} \times \mathbb{L}.$$

Arguing as in the proof of Theorem 6, and noting that $\widetilde{\Sigma} = \mathbb{H}^2(c)$, we get the existence of an associated isometric immersion $\psi : (\widetilde{M^2}, g_M) \to \mathbb{H}^2(c) \times \mathbb{R}$. This is impossible if $K_M < c$, as follows from the Hilbert-type theorem [3, Theorem 3].

□

The same argument together with [3, Theorem 2] and [4, Proposition 2] provide a direct proof of the following theorem in [1]:

Theorem 10 *Let (M^2, g_M) be a complete surface with constant curvature $K_M > 0$. Then, if $0 < c \neq K_M$, the unit sphere $\mathbb{S}^2(c)$ cannot be isometrically immersed into the Lorentzian product space $M^2 \times \mathbb{L}$.*

Specifically, the result follows from the basic correspondence and [3], [4], since there are no complete surfaces with $K_M < c$ in $\mathbb{S}^2(c) \times \mathbb{R}$, and the only complete surfaces with $K_M > c$ in $\mathbb{S}^2(c) \times \mathbb{R}$ are the rotational ones, which are not graphs over $\mathbb{S}^2$.

Besides all these non-existence results, we can also use the basic correspondence to prove existence results, as the following one:

Theorem 11 *Let (M^2, g_M) be a topological sphere endowed with a Riemannian metric g_M of positive curvature $K_M > 0$. Then there exists an isometric embedding $\psi : (\mathbb{S}^2, g_{\mathbb{S}^2}) \to M^2 \times_{e^t} \mathbb{L}$ of the unit sphere $\mathbb{S}^2$ into the steady state type spacetime $M^2 \times_{e^t} \mathbb{L}$.*

Moreover this embedding is unique in the following sense: if ψ_1, ψ_2 are two isometric immersions of $\mathbb{S}^2$ into $M^2 \times_{e^t} \mathbb{L}$, then their associated isometric immersions via the basic correspondence $\widehat{\psi_1}, \widehat{\psi_2} : M^2 \to \mathbb{R}^3_*$ differ at most by an isometry of $\mathbb{R}^3$.

Proof: By Pogorelov's solution to the classical Weyl embedding problem (see [22, 21]), there exists an isometric embedding $\widehat{\psi} : (M^2, g_M) \to \mathbb{R}^3$, which is unique up to isometries in $\mathbb{R}^3$.

Hence, we can assume that the origin lies in the interior of the domain of $\mathbb{R}^3$ bounded by the ovaloid $\widehat{\psi}(M)$, and thus $\widehat{\psi}$ is a global graph over the unit sphere $\mathbb{S}^2$. In this way, by applying the basic correspondence with respect to the warped product $\mathbb{R}^3_* \equiv \mathbb{S}^2 \times_r \mathbb{R}^+$, as explained in Section 2, we obtain an isometric immersion $\psi : (\mathbb{S}^2, g_{\mathbb{S}^2}) \to M^2 \times_{e^t} \mathbb{L}$, as wished.

Conversely, if $\psi_1, \psi_2 : (\mathbb{S}^2, g_{\mathbb{S}^2}) \to M^2 \times_{e^t} \mathbb{L}$ are two isometric immersions, then ψ_1 and ψ_2 are entire graphs over M^2. In this way, using again the basic correspondence we obtain that the isometric immersions $\widehat{\psi_1}, \widehat{\psi_2} : (M^2, g_M) \to \mathbb{R}^3$ must agree up to isometries, by the uniqueness of the solution to Weyl's problem.

□

Let us observe that, under the hypothesis of Theorem 11, if $\psi_1, \psi_2 : (\mathbb{S}^2, g_{\mathbb{S}^2}) \to M^2 \times_{e^t} \mathbb{L}$ are two isometric immersions, then $\widehat{\psi_2} = \varphi \circ \widehat{\psi_1}$, where φ is an isometry of $\mathbb{R}^3$. If φ is an isometry of $\mathbb{R}^3$ that leaves $\mathbb{S}^2$ invariant, then by Lemma 2 we see that ψ_1, ψ_2 are congruent immersions in $M^2 \times_{e^t} \mathbb{L}$.

Now, as the isometry group of $\mathbb{R}^3$ is 6-dimensional, and the isometry subgroup leaving $\mathbb{S}^2$ invariant is 3-dimensional, it is easy to observe that, generically, the family of non-congruent isometric immersions of $\mathbb{S}^2$ into $M^2 \times_{e^t} \mathbb{L}$ is 3-dimensional.

Remark 12 *Pogorelov's solution to Weyl's embedding problem also holds in the 3-sphere $\mathbb{S}^3$ and the hyperbolic space $\mathbb{H}^3$ for surfaces with positive extrinsic curvature. So, using the basic correspondence, the previous theorem also occurs when the ambient space $\mathbb{M} \times_{e^t} \mathbb{L}$ is replaced by the Lorentzian 3-spaces given by* (3.2) *and* (3.3).

A similar argument to the previous one, using this time the existence and uniqueness theorem in [25] for isometric immersions of topological spheres with curvature smaller than one into the de Sitter space $\mathbb{S}^3_1$, yields the following result via the basic correspondence:

Theorem 13 *Let M^2 be a topological sphere endowed with a Riemannian metric g_M of curvature $K_M < 1$. Assume that every closed geodesic of M^2 has length greater than or equal to 2π. Then there exists an isometric embedding $\psi : (\mathbb{S}^2, g_{\mathbb{S}^2}) \to (M^2 \times \mathbb{R}, g)$ with $g = \frac{1}{\cosh^2 t}(g_M \otimes dt^2)$.*

Moreover, this isometric embedding is unique in the following sense: if ψ_1, ψ_2 are two isometric immersions of $\mathbb{S}^2$ into $M^2 \times \mathbb{R}$, then the associated isometric immersions via the basic correspondence $\widehat{\psi_1}, \widehat{\psi_2} : M^2 \to \mathbb{S}^3_1$ differ at most by an isometry of $\mathbb{S}^3_1$.

As in the case of Theorem 11, it is easy to observe that the family of non-congruent isometric immersions of $\mathbb{S}^2$ into the previous warped product space $(M^2 \times \mathbb{R}, g)$ is, generically, 3-dimensional.

5 Height and area estimates

Our aim in this section is to obtain optimal height and area estimates for constant curvature surfaces in 3-dimensional Lorentzian product spaces, using our construction of associated immersions.

Notation: From now on, $\mathbb{Q}^2(c)$, will denote the 2-dimensional Riemannian space form of constant curvature c. That is, $\mathbb{Q}^2(0) = \mathbb{R}^2$, $\mathbb{Q}^2(c) = \mathbb{H}^2(c)$ if $c < 0$, and $\mathbb{Q}^2(c) = \mathbb{S}^2(c)$ if $c > 0$. We will also let $\varepsilon \in \{-1, 0, 1\}$.

Theorem 14 *Let (M^2, g_M) be a Riemannian surface of constant Gauss curvature $K_M > 0$, and let $\Omega \subset \mathbb{Q}^2(\varepsilon)$ be a compact domain of $\mathbb{Q}^2(\varepsilon)$. Assume moreover that $K_M > 1$ if $\varepsilon = 1$. Consider*

$$\widehat{\psi} : \Omega \subset \mathbb{M}^2(\varepsilon) \to M^2 \times \mathbb{L},$$
$$\widehat{\psi}(p) = (\pi(p), h(\pi(p)))$$

an isometric immersion of Ω as a graph in $M^2 \times \mathbb{L}$, whose boundary lies on the slice $M^2 \times \{0\}$ (i.e. $h \circ \pi(\partial\Omega) = 0$).

Then, the height estimate

$$h(\pi(p)) \leq C(K_M) \tag{5.1}$$

holds for every $p \in \Omega$, where

$$C(K_M) := \begin{cases} \dfrac{1}{\sqrt{K_M}} & \text{if } \varepsilon = 0, \\ \sqrt{\dfrac{K_M+1}{K_M}} \arctan\left(\dfrac{1}{\sqrt{K_M}}\right) & \text{if } \varepsilon = -1, \\ \sqrt{\dfrac{K_M-1}{K_M}} \ln\left(\dfrac{\sqrt{K_M}+1}{\sqrt{K_M}-1}\right) & \text{if } \varepsilon = 1. \end{cases} \tag{5.2}$$

Moreover, equality in (5.1) *holds for some $p \in \Omega$ if and only if $\pi(\Omega) \subset M^2$ is isometric to a hemisphere of the* 2*-dimensional sphere $\mathbb{S}^2(K_M)$.*

Proof: The associated isometric immersion to $\widehat{\psi}$ via the basic correspondence is

$$\psi : (\pi(\Omega) \subset M^2, g_M) \to \mathbb{M}^2(\varepsilon) \times \mathbb{R},$$

$$\psi(q) = (\pi^{-1}(q), h(q)).$$

Thus, ψ is a graph of constant curvature $K_M > 0$ ($K_M > 1$ if $\varepsilon = 1$), and whose boundary is contained in the slice $\mathbb{M}^2(\varepsilon) \times \{0\}$. In these conditions, we can apply the height estimates of constant curvature graphs obtained in [23] if $\varepsilon = 0$, and in [4] if $\varepsilon = \pm 1$, to conclude the inequalities (5.2),(5.1). Moreover, the claimed uniqueness when equality is attained follows from the corresponding property of the Riemannian case, proved in [11, 4].

□

By using the area estimates in [11] and [6] for constant Gauss curvature surfaces in $\mathbb{R}^3$, we can obtain the following result, whose proof is omitted since it is very similar to the one of Theorem 14.

Theorem 15 *Let (M^2, g_M) be a Riemannian surface of constant curvature $K_M > 0$, and let $\Omega \subset \mathbb{R}^2$ be a compact simply connected planar domain. Assume that*

$$\widehat{\psi} : \Omega \subset \mathbb{R}^2 \to M^2 \times \mathbb{L}$$

$$\widehat{\psi}(p) = (\Pi(p), h(\Pi(p)))$$

is an isometric immersion of Ω as a graph in $M^2 \times \mathbb{L}$, whose boundary lies on a slice $M^2 \times \{h_0\}$ (i.e. $h(\Pi(\partial\Omega)) = h_0 \in \mathbb{R}$).

Then if we denote by A_Ω the area of Ω, and by $\overline{A}$ the area of $\Pi(\Omega) \subset M$, it holds that

$$\frac{2\pi - 2\sqrt{\pi^2 - \pi K_M A_\Omega}}{K_M} \leq \overline{A} \leq \frac{2\pi + 2\sqrt{\pi^2 - \pi K_M A_\Omega}}{K_M}.$$

Besides, if the length L of $\Gamma := \partial\Omega \subset \mathbb{R}^2$ satisfies $4\pi^2 - K_M L^2 \geq 0$, then

$$\frac{2\pi - 2\sqrt{\pi^2 - \pi K_M A_\Omega}}{K_M} \leq \overline{A} \leq \frac{2\pi - \sqrt{4\pi^2 - K_M L^2}}{K_M},$$

or

$$\frac{2\pi + \sqrt{4\pi^2 - K_M L^2}}{K_M} \leq \overline{A} \leq \frac{2\pi + 2\sqrt{\pi^2 - \pi K_M A_\Omega}}{K_M}.$$

Moreover, the equality holds if and only if $\Pi(\Omega) \subset M$ is isometric to a hemisphere of the 2-dimensional sphere $\mathbb{S}^2(K_M)$.

6 Holomorphic differentials and harmonic maps

A classical result of surface theory establishes that if $\psi : \Sigma \to \mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$ is a surface of positive constant curvature K (with $K > 1$ in $\mathbb{S}^3$), and if z is a conformal parameter for the second fundamental form II_ψ, then $\langle \psi_z, \psi_z \rangle dz^2$ is a holomorphic quadratic differential. In other words, if I denotes the first fundamental form of ψ, then $I^{(2,0)}$ is holomorphic for the conformal structure induced by II_ψ.

This classical result was extended by Aledo, Espinar and Gálvez in [3] to the case where the ambient space is one of the homogeneous product spaces $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$. Next, we extend this result to the Lorentzian product space $\mathbb{S}^2(c) \times \mathbb{L}$ by means of the basic correspondence and Corollary 4. Remarkably, we will obtain this result for all values $K_M < c$ of the curvature, thus including negative values.

Corollary 16 *Let $\psi : (M^2, g_M) \to \mathbb{S}^2(c) \times \mathbb{L}$, $\psi = (\pi, h \circ \pi)$, be an immersed spacelike surface of constant curvature $K_M < c$ as a graph in the Lorentzian product space $\mathbb{S}^2(c) \times \mathbb{L}$. Let us consider on M^2 the quadratic form*

$$\alpha := (c - K_M)\pi^*(g_{\mathbb{S}^2(c)}) + K_M \, \mathrm{d}(h \circ \pi)^2.$$

Then, $\alpha^{(2,0)}$ is a holomorphic quadratic differential on the surface for the conformal structure of the second fundamental form II_ψ.

Proof: Let $\widehat{\psi} : \Omega \subset \mathbb{S}^2(c) \to M^2 \times \mathbb{R}$, $\Omega := \pi(M^2)$, denote the isometric immersion $\widehat{\psi} = (\pi^{-1}, h)$ associated to ψ by the basic correspondence, and let $\widehat{\alpha}$ be given in terms of α by (2.13). Thus,

$$\widehat{\alpha} = (c - K_M) g_{\mathbb{S}^2(c)} + K_M (dh)^2.$$

The condition $K_M < c$ implies that the second fundamental forms II_ψ and $II_{\widehat{\psi}}$ are definite, and so they induce conformal structures on M^2 and Ω. Moreover, $\widehat{\psi}$ can be seen locally as an isometric immersion into $\mathbb{Q}^2(K_M) \times \mathbb{R}$, as (M^2, g_M) is locally isometric to $\mathbb{Q}^2(K_M)$. In this conditions, and by applying an appropriate dilation to the surface, by [3, Corollary 1] we see that $\widehat{\alpha}^{(2,0)}$ (and hence $\alpha^{(2,0)}$, by Corollary 4) is a holomorphic quadratic differential, as wished.

□

In the limit case $c = 0$ it is well-known that, for spacelike surfaces of negative constant curvature in $\mathbb{L}^3$, the quantity $I^{(2,0)}$ is a holomorphic quadratic differential with respect to the second fundamental form. A similar result is also true for all spacelike locally convex surfaces of constant curvature in $\mathbb{S}^3_1$ and $\mathbb{H}^3_1$.

It is remarkable that the same construction actually provides the existence of holomorphic quadratic differentials for surfaces of constant curvature in non-homogeneous (Riemannian or Lorentzian) warped product spaces. For instance, we have

Theorem 17 *Let $\psi : (M^2, g_M) \to \mathbb{S}^2(c) \times_{e^t} \mathbb{L}$, $\psi = (\pi, h \circ \pi)$, be an immersed spacelike surface of constant curvature $K_M = 1$ as a graph in the steady state spacetime $\mathbb{S}^2(c) \times_{e^t} \mathbb{L}$. Let α denote the pullback metric $\alpha := \pi^*(g_{\mathbb{S}^2(c)})$. Similarly, let us consider the conformal structure on M^2 induced by the Riemannian metric $\pi^*(II_{\widehat{\psi}})$, where here $II_{\widehat{\psi}}$ stands for the second fundamental form of the associated immersion $\widehat{\psi} : \Omega \subset \mathbb{S}^2(c) \to \mathbb{R}^3_*$, $\widehat{\psi} = (\pi^{-1}, h)$.*

Then $\alpha^{(2,0)}$ is a holomorphic quadratic differential for this conformal structure.

Proof: The argument is basically the one used in Corollary 16. This time, we simply have to write $\mathbb{R}^3_* \equiv \mathbb{S}^2 \times_r \mathbb{R}$, and use the basic correspondence as explained in Subsection 3.1, along with the existence of a holomorphic quadratic differential for surfaces of positive constant curvature in $\mathbb{R}^3$ explained above.

It must be remarked that, as we are using a non-trivial warping function, this time Proposition 3 does not apply, i.e. $II_\psi \neq \pi^*(II_{\widehat{\psi}})$ at first. However, α and $\pi^*(II_{\widehat{\psi}})$ are independent in general, since so are $I_{\widehat{\psi}}$ and $II_{\widehat{\psi}}$ in $\mathbb{R}^3$ (observe that $\alpha = \pi^*(I_{\widehat{\psi}})$). This tells that we are indeed constructing a non-trivial holomorphic quadratic differential.

□

Similar results can be proved for graphs of constant curvature in some Riemannian warped product 3-spaces, by applying the basic correspondence as above to the existing holomorphic quadratic differential for spacelike surfaces of constant curvature in Lorentzian space forms. In this line, let us consider one specific example in which more information can be obtained.

It is known (see [13]) that the Gauss map of a spacelike surface of constant negative curvature in $\mathbb{L}^3$ is a harmonic map into $\mathbb{H}^2$ for the conformal structure of the second fundamental form, and that the surface can be recovered in terms of the Gauss map. The next result shows that a similar situation holds for flat graphs in $\mathbb{H}^2 \times \mathbb{R}$.

Recall that a smooth map $G : \Sigma \to \mathbb{H}^2 \subset \mathbb{L}^3$ from a Riemann surface Σ is harmonic if and only if $Q_G := \langle G_z, G_z \rangle dz^2$ is holomorphic, and that two harmonic maps $G, G^* : \Sigma \to \mathbb{H}^2 \subset \mathbb{L}^3$ are *conjugate* if $Q_G = -Q_{G^*}$ and $\langle G_z, G_{\bar{z}} \rangle = \langle G^*_z, G^*_{\bar{z}} \rangle$. The harmonic conjugate of a given harmonic map $G : \Sigma \to \mathbb{H}^2$ is defined up to isometries of $\mathbb{H}^2$, and always exists if Σ is simply connected.

Theorem 18 *Let $S \equiv (x, y, h(x, y))$ denote a complete spacelike graph in $\mathbb{L}^3$ of constant curvature $K = -1$, and let $N(x, y) : \mathbb{R}^2 \to \mathbb{H}^2$ denote its Gauss map. If $N^*(x, y)$:*

$\mathbb{R}^2 \to \mathbb{H}^2$ *is a conjugate harmonic map of* N *(for the conformal structure of the second fundamental form), then the map*

$$\phi(x,y) = (N^*(x,y), h(x,y)) : \mathbb{R}^2 \to \mathbb{H}^2 \times \mathbb{R}$$

is an isometric immersion of the Euclidean plane $(\mathbb{R}^2, dx^2 + dy^2)$ *into the homogeneous product space* $\mathbb{H}^2 \times \mathbb{R}$*. This isometric immersion is moreover an entire graph over* $\mathbb{H}^2$*.*

Proof: We shall use (x,y) as global coordinates for S in the obvious way. Let $\Phi : S \to \mathbb{H}^2$ be an isometry, and $N : S \to \mathbb{H}^2$ be the Gauss map of S. Let also $z : S \to U \subset \mathbb{C}$ be a global conformal parameter on S for the second fundamental form. Then it follows from [13] that $\Phi \circ z^{-1} : U \to \mathbb{H}^2$ and $N \circ z^{-1} : U \to \mathbb{H}^2$ are harmonic maps into $\mathbb{H}^2$ that are *conjugate*. Now, by the basic correspondence we can associate to S the isometric immersion

$$\widehat{\psi}(x,y) = (x, y, h(x,y)) : (\mathbb{R}^2, dx^2 + dy^2) \to S \times \mathbb{R}.$$

Finally, $\phi(x,y) = (\Phi(x,y), h(x,y)) : \mathbb{R}^2 \to \mathbb{H}^2 \times \mathbb{R}$ gives an isometric immersion of $\mathbb{R}^2$ into $\mathbb{H}^2 \times \mathbb{R}$ as an entire graph, which has the desired form.

□

In particular, we see that the vertical projection on $\mathbb{H}^2$ of a flat graph in $\mathbb{H}^2 \times \mathbb{R}$ is harmonic for the conformal structure of the second fundamental form (see also [4]).

It is clear that Theorem 18 can also be applied locally, and implies that any flat graph in $\mathbb{H}^2 \times \mathbb{R}$ can be recovered by means of two conjugate harmonic maps G, G^* into $\mathbb{H}^2$. Specifically, let $\pi : \mathbb{H}^2 \to \mathbb{D}$ denote the stereographic projection, let $g := \pi \circ G$, $g^* = \pi \circ G^*$, and assume that $G^* : \Sigma \to G^*(\Sigma) \subset \mathbb{H}^2$ is a diffeomorphism. Then, by means of the Weierstrass representation formula in [13] and the proof of Theorem 18 we see that

$$\psi : \Sigma \to \mathbb{H}^2 \times \mathbb{R}, \qquad \psi = \left(g^*, \mathrm{Re} \int 4 \frac{-\bar{g} g_z + g \bar{g}_z}{(1 - |g|^2)^2} \, dz \right), \tag{6.1}$$

is a flat graph in $\mathbb{H}^2 \times \mathbb{R}$, conformally parametrized with respect to the second fundamental form. And conversely, it is also clear by the basic correspondence that any flat graph in $\mathbb{H}^2 \times \mathbb{R}$ can be expressed in this way for a suitable pair of conjugate harmonic maps G, G^* into $\mathbb{H}^2$.

As a direct consequence of this, it seems possible to develop an integrable systems theory for flat graphs in $\mathbb{H}^2 \times \mathbb{R}$, and to derive a Sym-type formula using loop groups.

Theorem 18 has the following global consequence:

Corollary 19 *There exists a correspondence between the space of harmonic diffeomorphism from* $\Sigma = \mathbb{D}$ *or* $\mathbb{C}$ *onto* $\mathbb{H}^2$ *and the space of entire flat graphs in* $\mathbb{H}^2 \times \mathbb{R}$*.*

Specifically, for each harmonic diffeomorphism $G : \Sigma \to \mathbb{H}^2$ *there is exactly a* 2*-parameter family of (generically non-congruent) entire flat graphs in* $\mathbb{H}^2 \times \mathbb{R}$*, whose vertical projection onto* $\mathbb{H}^2$ *coincides with* G*, for the conformal structure of the second fundamental form.*

Proof: The above development shows that any harmonic diffeomorphism G of $\Sigma = \mathbb{C}$ or $\mathbb{D}$ into $\mathbb{H}^2$ is the vertical projection of an entire flat graph in $\mathbb{H}^2 \times \mathbb{R}$, where the conformal structure is the one induced by the second fundamental form of the graph (by Proposition 3). And clearly, if we substitute G by $\Psi \circ G$, where Ψ is an isometry of $\mathbb{H}^2$, then the entire flat graph $\psi = (\pi, h \circ \pi)$ transforms into $\phi = (\Psi \circ \pi, h \circ \pi)$, which is congruent to ψ in $\mathbb{H}^2 \times \mathbb{R}$.

Finally, the fact that the class of surfaces with the same vertical projection is 2-dimensional follows from Example 5. It could be also deduced from (6.1), taking into account that the harmonic conjugate of a harmonic map is defined up to isometries of $\mathbb{H}^2$ (thus, a 3-parameter family), and that rotations of this harmonic conjugate do not change the height function in (6.1). So, we are left again with a 2-parameter family of entire graphs.

□

This Corollary indicates that the class of entire flat graphs in $\mathbb{H}^2 \times \mathbb{R}$ is extremely large, as so is the class of harmonic diffeomorphisms onto $\mathbb{H}^2$. Apart from these entire graphs, there also exist isometric immersions of $\mathbb{R}^2$ into $\mathbb{H}^2 \times \mathbb{R}$ in the form of a vertical cylinder $\gamma \times \mathbb{R} \subset \mathbb{H}^2 \times \mathbb{R}$, where γ is a regular curve in $\mathbb{H}^2$. Related to this, we would like to mention the following open problem: *are all isometric immersions of* $\mathbb{R}^2$ *into* $\mathbb{H}^2 \times \mathbb{R}$ *either vertical cylinders or entire flat graphs?*

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