The geometric Neumann problem for the Liouville equation

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Abstract

Let Ω denote the upper half-plane $\mathbb{R}^2_+$ or the upper half-disk $D_\varepsilon^+ \subset \mathbb{R}^2_+$ of center 0 and radius ε. In this paper we classify the solutions $v \in C^2(\overline{\Omega} \setminus \{0\})$ to the Neumann problem

$$\begin{cases} \Delta v + 2Ke^v = 0 & \text{in } \Omega \subseteq \mathbb{R}^2_+ = \{(s,t) \in \mathbb{R}^2 : \ t > 0\}, \\ \dfrac{\partial v}{\partial t} = c_1 e^{v/2} & \text{on } \partial\Omega \cap \{s > 0\}, \\ \dfrac{\partial v}{\partial t} = c_2 e^{v/2} & \text{on } \partial\Omega \cap \{s < 0\}, \end{cases}$$

where $K, c_1, c_2 \in \mathbb{R}$, with the finite energy condition $\int_\Omega e^v < \infty$. As a result, we classify the conformal Riemannian metrics of constant curvature and finite area on a half-plane that have a finite number of boundary singularities, not assumed a priori to be conical, and constant geodesic curvature along each boundary arc.

1 Introduction

The Liouville equation $\Delta v + 2Ke^v = 0$ has a natural geometric Neumann problem attached to it, that comes from the following question:

Let $\Omega \subset \mathbb{R}^2$ be a domain with smooth boundary $\partial\Omega$. *What are the conformal Riemannian metrics on Ω having constant curvature K, and constant geodesic curvature along each boundary component of $\partial\Omega$?* Here, we assume that the metric extends smoothly to the boundary $\partial\Omega$.

An important property of the Liouville equation is that it is conformally invariant. Thus, it is not very restrictive to consider only simple symmetric domains Ω, such as disks, half-planes or annuli. The most studied case is when $\Omega = \mathbb{R}^2_+$. In that situation, we are led to the Neumann problem

$$\begin{cases} \Delta v + 2Ke^v = 0 & \text{in } \mathbb{R}^2_+ = \{(s,t) \in \mathbb{R}^2 : \ t > 0\}, \\ \dfrac{\partial v}{\partial t} = c\, e^{v/2} & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{1}$$

where the first equation tells that the conformal metric $e^v|dz|^2$ has constant curvature K, and the free boundary condition gives that $\partial\mathbb{R}^2_+$ has constant geodesic curvature $-c/2$ for that metric. The above problem was fully solved by Zhang [Zha] (in the finite energy case) and Gálvez-Mira [GaMi] (in general), as an extension of previous results in [LiZh, Ou] (see also [ChLi, ChWa, HaWa]).

Recently, there has been some work on the geometric Neumann problem for Liouville's equation in $\mathbb{R}^2_+$ in the presence of a boundary singularity, i.e. the problem

$$\begin{cases} \Delta v + 2Ke^v = 0 & \text{in } \mathbb{R}^2_+ = \{(s,t) \in \mathbb{R}^2 : \ t > 0\}, \\ \dfrac{\partial v}{\partial t} = c_1 e^{v/2} & \text{on } \partial\mathbb{R}^2_+ \cap \{s > 0\}, \\ \dfrac{\partial v}{\partial t} = c_2 e^{v/2} & \text{on } \partial\mathbb{R}^2_+ \cap \{s < 0\}, \end{cases} \tag{P}$$

with $K \in \{-1, 0, 1\}$ and $c_1, c_2 \in \mathbb{R}$. In [JWZ], Jost, Wang and Zhou gave a complete classification of the solutions to the above problem under the following assumptions:

1. The metric $e^v|dz|^2$ has finite area in $\mathbb{R}^2_+$, i.e.

$$\int_{\mathbb{R}^2_+} e^v < \infty. \tag{2}$$

2. The boundary $\partial\mathbb{R}^2_+$ has finite length for the metric $e^v|dz|^2$, i.e.

$$\int_{\mathbb{R}_-} e^{v/2} + \int_{\mathbb{R}_+} e^{v/2} < \infty. \tag{3}$$

3. The metric $e^v|dz|^2$ has a *boundary conical singularity* at the origin, i.e. there exists $\lim_{z\to 0}|z|^{-2\alpha}e^v \neq 0$ for some $\alpha > -1$.

4. $K = 1$.

With these hypotheses, they showed that any solution to (P) corresponds to the conformal metric associated to the sector of a sphere of radius one limited by two circles that intersect at exactly two points, or to the complement of a closed arc of circle in the sphere, possibly composed with an adequate branched covering of the Riemann sphere $\overline{\mathbb{C}}$. In particular, they provided explicit analytic expressions for all these solutions.

Our main objective in this paper is to provide several improvements of the Jost-Wang-Zhou theorem. These are included in Theorem 2, Theorem 3, Theorem 4 and Corollary 4.

In Theorem 4 we will remove the last three hypotheses of the above list in the Jost-Wang-Zhou result, and prove that any solution to (P) of finite area is a *canonical solution*. These canonical solutions have explicit analytic expressions and a simple geometric interpretation as the conformal factor associated to basic regions of 2-dimensional space forms, up to composition with suitable branched coverings of $\overline{\mathbb{C}}$ if $K = 1$ (see Section 2). For the case $K = 1$ we recover the solutions obtained in [JWZ], together with some new solutions corresponding to the case that the boundary singularity at the origin is not conical; we do not prescribe here any asymptotic behavior at the origin, nor the finite length condition (3).

In Theorem 2 we give a general classification of all the solutions to (P), without any integral finiteness assumptions, in the spirit of [GaMi]. We show that the class of solutions to (P) is extremely large, but still it can be described in terms of entire holomorphic functions satisfying some adequate properties. As a matter of fact, we give such a result not only in $\mathbb{R}^2_+$ but also in an arbitrary half-disk $D^+_\varepsilon \subset \mathbb{R}^2_+$. That is, we also give a general classification result for the solutions $v \in C^2(\overline{D^+_\varepsilon} \setminus \{0\})$ to the local problem

$$\begin{cases} \Delta v + 2Ke^v = 0 & \text{in } D^+_\varepsilon = \{(s,t) \in \mathbb{R}^2 :\ s^2 + t^2 < \varepsilon^2,\ t > 0\}, \\ \dfrac{\partial v}{\partial t} = c_1 e^{v/2} & \text{on } I^+_\varepsilon = \{(s,0) \in \mathbb{R}^2 :\ 0 < s < \varepsilon\}, \\ \dfrac{\partial v}{\partial t} = c_2 e^{v/2} & \text{on } I^-_\varepsilon = \{(s,0) \in \mathbb{R}^2 :\ -\varepsilon < s < 0\}. \end{cases} \qquad (L)$$

In Theorem 3 we classify the solutions to the local problem (L) that satisfy the finite area condition

$$\int_{D_\varepsilon^+} e^v < \infty, \tag{4}$$

and give a general procedure to construct all of them. In particular, we describe the asymptotic behaviour at the origin of any solution to (L) that satisfies (4). This is a generalization to the case of boundary singularities of the well-known results in [Bry, ChWa, Hei, Nit, War] which describe the asymptotic behaviour of metrics of constant curvature and finite area in the punctured disk $\mathbb{D}^*$.

In Corollary 4 we extend Theorem 4 to the case of an arbitrary number of boundary singularities. This solves a problem posed in [JWZ], under milder hypotheses. The basic examples of conformal metrics of constant curvature with boundary singularities and constant geodesic curvature along each boundary component are the ones determined by circular polygons in $\overline{\mathbb{C}}$, but there are many others. To obtain this larger family, we consider *immersed* circular polygons for which we allow self-intersections, and give a differential-topological criterion (Alexandrov embeddedness) for them to generate such metrics, see Definition 5. With this, Corollary 4 proves the converse: any conformal metric of finite area and constant curvature on $\mathbb{R}^2_+$ (or equivalently on the unit disk $\mathbb{D}$), with finitely many boundary singularities and constant geodesic curvature along each boundary component, is one of those *circular polygonal metrics* constructed from Alexandrov-embedded, possibly self-intersecting, circular polygons. Analytically, those metrics will not have simple explicit expressions; yet, one can still give some analytic information about them. In Corollary 5 we will describe for $K = 1$ the moduli space of these metrics, by parametrizing it in terms of their associated Schwarzian maps, which have simple explicit expressions.

We have organized the paper as follows. In Section 2 we will present the *canonical solutions*, together with their geometric interpretation and their basic properties. Section 3 contains some preliminaries. In Section 4 we will study the local problem (L) at a boundary singularity, and prove Theorem 2. In Section 5 we will prove Theorem 3, which describes all the solutions to (L) that satisfy the finite energy condition (4). In Section 6 we will prove Theorem 4, which states that any finite area solution to (P) is a canonical solution. In Section 7 we will prove Corollary 4 and Corollary 5 on the classification of conformal metrics of constant curvature with a finite number of boundary singularities.

2 The canonical solutions

Our objective in this section is to describe, both analytically and geometrically, an explicit family of solutions to (P) satisfying the finite energy condition (2). We will prove in Theorem 4 that these are actually *all* the finite energy solutions to (P).

In all that follows, we assume that $K \in \{-1, 0, 1\}$, without loss of generality.

2.1 Analytic description

Definition 1. *A* canonical solution *is a function of one of the following types:*

1. $v_1 : \mathbb{R}^2_+ \to \mathbb{R}$ *given by*

$$v_1 = \log \frac{4\lambda^2\gamma^2|z|^{2(\gamma-1)}}{(K\lambda^2 + |z^\gamma - z_0|^2)^2} \tag{5}$$

where $\gamma, \lambda > 0$ *and* $z_0 \in \mathbb{C}$ *satisfy* $K\lambda^2 + |z^\gamma - z_0|^2 \neq 0$ *for all* $z \in \overline{\mathbb{C}^+} \equiv \overline{\mathbb{R}^2_+}$.

2. $v_2 : \mathbb{R}^2_+ \to \mathbb{R}$ *given by*

$$v_2 = \log \frac{4\,\lambda^2}{|z|^2(K\lambda^2 + |\log z - z_0|^2)^2}, \tag{6}$$

where $\gamma, \lambda > 0$ *and* $z_0 \in \mathbb{C}$, *satisfy* $K\lambda^2 + |\log z - z_0|^2 \neq 0$ *for all* $z \in \overline{\mathbb{C}^+} \equiv \overline{\mathbb{R}^2_+}$. *Here,* $\log z = \ln |z| + i \arg(z)$, *where* $\arg(z) \in [0, \pi]$.

Let us observe some elementary properties of these *canonical solutions*, and explain for what choices of the constants γ, λ, z_0 and K they exist.

The function v_1 given by (5) is well defined in $\overline{\mathbb{R}^2_+} \backslash \{0\}$ if $K = 1$ for all $\gamma, \lambda > 0$, $z_0 \in \mathbb{C}$. However, if $K = 0, -1$, v_1 is well defined if and only if $K\lambda^2 + |z^\gamma - z_0|^2 \neq 0$. In other words, if and only if the distance from the point z_0 to the sector $\{z^\gamma : z \in \overline{\mathbb{R}^2_+} \backslash \{0\}\}$ is bigger than $-K\lambda^2$. A simple analysis shows that this happens:

- for $K = 0$ if and only if $z_0 = 0$, or $z_0 \neq 0$ and $\pi\gamma < \theta_0$ with $\theta_0 = \arg(z_0) \in [0, 2\pi)$.
- for $K = -1$ if and only if $\lambda \leq |z_0|$, $\pi\gamma < \theta_0 - \alpha_0$, and $|\mathrm{Im}(z)| > \lambda$ when $\mathrm{Re}(z) > 0$. Here, $\theta_0 = \arg(z_0) \in [0, 2\pi)$ and $\alpha_0 \in (0, \pi/2)$ with $\sin \alpha_0 = \lambda/|z_0|$.

Besides, if $K=1$ the function v_1 satisfies the finite area condition

$$\int_{\mathbb{R}^2_+} e^{v_1} = \int_{\mathbb{R}^2_+} \frac{4\lambda^2\gamma^2|z|^{2(\gamma-1)}}{(K\lambda^2+|z^\gamma - z_0|^2)^2} < \infty$$

for every γ, λ, z_0.

In the other cases, if it holds $K\lambda^2 + |z^\gamma - z_0|^2 \neq 0$ for all $z \in \overline{\mathbb{C}^+} \equiv \overline{\mathbb{R}^2_+}$ (and not just for all $z \in \overline{\mathbb{R}^2_+} \setminus \{0\}$), then the metric trivially has finite area. Otherwise, it means that $z_0 = 0$ if $K = 0$, or $|z_0| = \lambda$ when $K = -1$. But in these cases we clearly have infinite area at the origin.

As a consequence, v_1 is a well defined function in $\overline{\mathbb{R}^2_+}\backslash\{0\}$ with finite area if and only if $K\lambda^2 + |z^\gamma - z_0|^2 \neq 0$ for all $z \in \overline{\mathbb{C}^+} \equiv \overline{\mathbb{R}^2_+}$, which is the condition of Definition 1. Observe that $\gamma < 2$ when $K = 0, -1$.

Analogously the function v_2 given in (6) is well defined in $\overline{\mathbb{R}^2_+}\backslash\{0\}$ and has finite area if and only if $K\lambda^2 + |\log z - z_0|^2 \neq 0$ for all $z \in \overline{\mathbb{C}^+} \equiv \overline{\mathbb{R}^2_+}$.

In particular, if $K = 1$ the condition $K\lambda^2 + |\log z - z_0|^2 \neq 0$ for all $z \in \overline{\mathbb{C}^+}$ is satisfied for every λ, z_0. However, in the other cases, we need to impose that the distance from the point z_0 to the strip $\{\log z : \ z \in \overline{\mathbb{R}^2_+}\backslash\{0\}\} = \{\zeta \in \mathbb{C} : 0 < \mathrm{Im}\zeta < \pi\}$ is bigger that $-K\lambda^2$. This condition happens

- for $K = 0$ if and only if $\mathrm{Im}(z_0) < 0$ or $\mathrm{Im}(z_0) > \pi$.
- for $K = -1$ if and only if $\mathrm{Im}(z_0) < -\lambda$ or $\mathrm{Im}(z_0) > \pi + \lambda$.

This analysis together with a simple computation shows that these canonical solutions are indeed finite area solutions to problem (P).

Lemma 1. *Any canonical solution $v : \overline{\mathbb{R}^2_+} \setminus \{0\} \to \mathbb{R}$ is a solution to the geometric Neumann problem (P) satisfying*

$$\int_{\mathbb{R}^2_+} e^v < \infty,$$

where the constants c_1, c_2 associated to the problem are given by the following expressions in terms of γ, λ and $z_0 := r_0 e^{i\theta_0}$:

1. *For v_1 as in* (5),
$$c_1 = 2\,\frac{r_0}{\lambda}\,\sin\theta_0, \qquad c_2 = -2\,\frac{r_0}{\lambda}\,\sin(\theta_0 - \pi\gamma). \tag{7}$$
2. *For v_2 as in* (6),
$$c_1 = \frac{2}{\lambda}\,\mathrm{Im}(z_0), \qquad c_2 = \frac{2}{\lambda}\,(\pi - \mathrm{Im}(z_0)). \tag{8}$$

2.2 Geometric description

Let $\mathcal{Q}^2(K)$ denote the 2-dimensional space form of constant curvature $K \in \{-1, 0, 1\}$, which will be viewed as (Σ_K, ds_K^2) where

$$\Sigma_K = \begin{cases} \overline{\mathbb{C}} & \text{if } K = 1, \\ \mathbb{C} & \text{if } K = 0, \\ \mathbb{D} \subset \mathbb{C} & \text{if } K = -1, \end{cases}$$

and ds_K^2 is the Riemannian metric on Σ_K given by

$$ds_K^2 = \frac{4|d\zeta|^2}{(1 + K|\zeta|^2)^2}. \tag{9}$$

So, a regular curve in Σ_K has constant curvature if and only if its image is a piece of a circle in $\bar{\mathbb{C}}$.

Definition 2. *Let $\mathcal{C}_1, \mathcal{C}_2$ be two different circles in $\overline{\mathbb{C}}$ such that $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$, and let $\mathcal{U} \subset \overline{\mathbb{C}}$ be any of the regions in which $\mathcal{C}_1$ and $\mathcal{C}_2$ divide $\overline{\mathbb{C}}$. Assume that $\overline{\mathcal{U}}$ is contained in Σ_K. Then $\mathcal{U}$ is called a* basic domain *of $\mathcal{Q}^2(K)$.*

Let now $\mathcal{U} \subset \Sigma_K$ be a basic domain equipped with the metric ds_K^2 in (9). Note that one can conformally parametrize $\mathcal{U}$ by a biholomorphism $g : \mathbb{C}^+ \to \mathcal{U}$ such that $g(\infty)$ is a point $p \in \mathcal{C}_1 \cap \mathcal{C}_2$, and in the case that $\mathcal{C}_1 \cap \mathcal{C}_2$ is not a single point $g(0)$ is also some $q \in \mathcal{C}_1 \cap \mathcal{C}_2$.

It is then clear from this process that the pull-back metric $g^*(ds_K^2)$ produces a conformal metric of constant curvature K in $\overline{\mathbb{C}^+} \equiv \mathbb{R}^2_+$, which has constant geodesic curvature along $\mathbb{R}_-$ and $\mathbb{R}_+$, and a singularity at the origin. Also, this metric trivially has finite area, so we have a solution to (P) that satisfies (2).

A similar process can be done if $K = 1$, by considering $\mathcal{U}$ to be the complement of an arc of a circle in $\overline{\mathbb{C}}$. This would correspond in some sense to taking $\mathcal{C}_1 = \mathcal{C}_2$ in the above process.

Furthermore, if $K = 1$ and $\mathcal{C}_1 \cap \mathcal{C}_2 = \{p, q\}$ consists of two points, we can easily create other finite area solutions to (P), starting from the basic region $\mathcal{U} \subset \overline{\mathbb{C}}$. For that, it suffices to consider a finite-folded branched holomorphic covering of $\overline{\mathbb{C}}$, with branching points at p and q. If we denote this branched covering by Φ, and consider $\widehat{g} := \Phi \circ g$, the pullback metric of ds_K^2 via $\widehat{g}$ again describes as before a finite area solution to (P).

This construction provides a geometric interpretation of the canonical solutions. Indeed, we have:

Fact: Let $v \in C^2(\overline{\mathbb{R}^2_+} \setminus \{0\})$ be a canonical solution. Then $e^v|dz|^2$ is the pullback metric on $\mathbb{R}^2_+$ of either:

a) some basic region $\mathcal{U}$ in $\mathcal{Q}^2(K)$, or

b) the complement in $\mathcal{Q}^2(1) \equiv \overline{\mathbb{C}}$ of a closed arc of a circle,

possibly composed with a suitable branched covering of $\overline{\mathbb{C}}$ in case $K = 1$.

We do not give a direct proof of this fact, since it will become evident from the proof of our main results. See Section 7.

3 Preliminaries

Let us start by explaining the classical relationship between the Liouville equation and complex analysis. From now on we will identify $\mathbb{R}^2$ and $\mathbb{C}$, and write $z = s+it \equiv (s,t)$ for points in the domain of a solution to the Liouville equation.

Theorem 1. *Let $v : \Omega \subset \mathbb{R}^2 \equiv \mathbb{C} \to \mathbb{R}$ denote a solution to $\Delta u + 2Ke^v = 0$ in a simply connected domain Ω. Then there exists a locally univalent meromorphic function g (holomorphic with $1 + K|g|^2 > 0$ if $K \leq 0$) in Ω such that*

$$v = \log \frac{4|g'|^2}{(1+K|g|^2)^2}. \tag{10}$$

Conversely, if g is a locally univalent meromorphic function (holomorphic with $1 + K|g|^2 > 0$ if $K \leq 0$) in Ω, then (10) is a solution to $\Delta v + 2Ke^v = 0$ in Ω.

Up to a dilation, we will assume from now on that $K \in \{-1, 0, 1\}$. Also, observe that the function g in the above theorem, which is called the *developing map* of the solution, is unique up to a Möbius transformation of the form

$$g \mapsto \frac{\alpha g - \bar{\beta}}{K\beta g + \bar{\alpha}}, \qquad |\alpha|^2 + K|\beta|^2 = 1. \tag{11}$$

Remark 1. *The developing map g has a natural geometric interpretation: if $v \in C^2(\Omega)$ is a solution to $\Delta v + 2Ke^v = 0$, then its developing map $g : \Omega \subseteq \mathbb{C} \to \Sigma \subseteq \bar{\mathbb{C}}$ provides a local isometry from $(\Omega, e^v|dz|^2)$ to $\mathcal{Q}^2(K) \equiv (\Sigma_K, ds_K^2)$, where ds_K^2 is given by (9).*

There is another relevant holomorphic function attached to any solution v of the Liouville equation. We will denote it by Q, and it is given by the formulas below, where g is the developing map of v:

$$Q := v_{zz} - \frac{1}{2} v_z^2 = \{g, z\} := \left(\frac{g_{zz}}{g_z}\right)_z - \frac{1}{2}\left(\frac{g_{zz}}{g_z}\right)^2. \tag{12}$$

Here, by definition $v_z = (v_s - iv_t)/2$ (and $g_z = g'$), and $\{g, z\}$ is the classical *Schwarzian derivative* of the meromorphic function g with respect to z. Observe that Q is holomorphic, i.e. it does not have poles, and it does not depend on the choice of the developing map g. We will call it the *Schwarzian map* associated to the solution v.

The following lemma gives some basic local properties of a solution to the geometric Neumann problem for the Liouville equation along the boundary. It is a consequence of some arguments in [GaMi], but we give a brief proof here for the convenience of the reader.

Lemma 2. *Let $D_\varepsilon^+ = \{z \in \mathbb{C} : |z| < \varepsilon, \mathrm{Im} z > 0\}$, and let $v \in C^2(\overline{D_\varepsilon^+})$ be a solution to*

$$\begin{cases} \Delta v + 2Kv = 0 & \text{in} \quad D_\varepsilon^+ \\ \dfrac{\partial v}{\partial t} = ce^{v/2} & \text{on} \quad I_\varepsilon = (-\varepsilon, \varepsilon) \subset \mathbb{R} \end{cases}$$

Then:

(i) The Schwarzian derivative map Q of v, defined by (12)*, takes real values along I_ε, and extends holomorphically to the whole disk D_ε by $Q(\bar{z}) = \overline{Q(z)}$.*

(ii) The developing map g of v can be extended to D_ε as a locally univalent meromorphic function.

(iii) $g(s, 0) : I_\varepsilon \to \overline{\mathbb{C}}$ is a regular parametrization of a piece of a circle $\mathcal{C}$ in $\overline{\mathbb{C}}$.

Proof. By the Neumann condition $v_t = ce^{v/2}$ along I_ε, we have

$$\mathrm{Im}\, Q(s, 0) = -\frac{1}{2}\left(\frac{c}{2}v'(s)e^{v(s)/2} - \frac{c}{2}v'(s)e^{v(s)/2}\right) = 0,$$

for every $s \in I_\varepsilon$. Thus, (i) holds immediately by Schwarzian reflection.

For (ii), we only need to recall that if $q(z)$ is a holomorphic function in a simply connected domain, then the equation $\{g, z\} = q(z)$ always has a locally univalent meromorphic solution g, which is unique up to linear fractional transformations. In our case, we have $\{g, z\} = Q$ on D_ε^+, and so (ii) follows from (i).

Finally, (iii) is clear from the fact that the developing map g defines a local isometry from $(\overline{D_\varepsilon^+}, e^v|dz|^2)$ into $\mathcal{Q}^2(K)$, and I_ε has constant curvature $-c/2$ for the metric $e^v|dz|^2$, by the Neumann condition $v_t = ce^{v/2}$. $\square$

For the proof of Theorem 2, we will also need the following elementary lemma.

Lemma 3. *Let* $\widetilde{\Omega} = \{w \in \mathbb{C} : \ a < Re(w) < b\}$*, with* $-\infty \leq a < b \leq +\infty$*, and let* $h : \widetilde{\Omega} \longrightarrow \overline{\mathbb{C}}$ *be a function such that* $h(w + 2\pi i) = h(w)$*. Then, there exists a well defined function* $f : \Omega \longrightarrow \mathbb{C}$ *on the topological annulus* $\Omega = \{z \in \mathbb{C} : \ a < \log|z| < b\}$ *such that* $h(w) = f(e^w)$ *for all* $w \in \widetilde{\Omega}$.

Moreover, if h *is a meromorphic function then so it is* f.

4 The local problem: proof of Theorem 2.

A general description of all solutions to (L), in the spirit of the main result in [GaMi], is given by the following theorem. We let D_ε^* denote $\{z \in \mathbb{C} : 0 < |z| < \varepsilon\}$.

Theorem 2. *Let* v *be a solution of* (L)*. Then there exists a meromorphic function* $F : D_\varepsilon^* \longrightarrow \overline{\mathbb{C}}$ *such that* v *can be computed from* (10) *for a locally univalent meromorphic function* $g : \overline{D_\varepsilon^+} \setminus \{0\} \to \overline{\mathbb{C}}$ *given by one of the following expressions:*

(i) $g(z) = \psi(z^\gamma F(z))$*, with* $\gamma \in [0, 1)$ *and* $F(r) \in \mathbb{R} \cup \{\infty\}$ *for any* $r \in \mathbb{R} \cap D_\varepsilon^*$,

(ii) $g(z) = \psi(F(z) + \log(z))$*, with* $F(r) \in \mathbb{R} \cup \{\infty\}$ *for any* $r \in \mathbb{R} \cap D_\varepsilon^*$,

(iii) $g(z) = \psi(z^{i\gamma} F(z))$*, with* $\gamma < 0$ *and* $|F(r)| = 1$ *for any* $r \in \mathbb{R} \cap D_\varepsilon^*$.

Here, ψ *is a Möbius transformation and* g *is holomorphic with* $1 + K|g|^2 > 0$ *if* $K \leq 0$.

Conversely, let $g : \overline{D_\varepsilon^+} \setminus \{0\} \to \overline{\mathbb{C}}$ *be a locally univalent meromorphic function, holomorphic with* $1 + K|g|^2 > 0$ *if* $K \leq 0$*, constructed from a meromorphic function* $F : D_\varepsilon^* \to \overline{\mathbb{C}}$ *as in* $(i) - (iii)$ *above. Then, the function* v *given by* (10) *is a solution of problem* (L).

Remark 2. *Theorem 2 also provides all the solutions of the global problem* (P)*. For that, it is enough to consider* $\varepsilon = \infty$ *in the previous theorem, that is, to change* D_ε^* *by* $\mathbb{C}^*$.

Proof. Let $v \in C^2(\overline{D_\varepsilon^+} \setminus \{0\})$ be a solution of problem (L), and consider an associated developing map g. As explained in Lemma 2, the Schwarzian map Q of v, given by (12), extends holomorphically to the punctured disk D_ε^*. Consider now the covering map $w \mapsto e^w$, from $\widetilde{D_\varepsilon^*} = \{z \in \mathbb{C} : \ Re(z) < \log \varepsilon\}$ to D_ε^*, which is a local biholomorphism. Then, in the region of $\widetilde{D_\varepsilon^*}$ such that $0 < \mathrm{Im} w < \pi$ we can take the meromorphic map $\widetilde{g}$ given by

$$\widetilde{g}(w) = g(e^w). \tag{13}$$

Moreover, the Schwarzian of $\widetilde{g}(w)$ satisfies

$$\{\widetilde{g}, w\} = e^{2w} Q(e^w) - \frac{1}{2}. \tag{14}$$

As $Q(e^w)$ is globally defined and holomorphic in $\widetilde{D_\varepsilon^*}$, we see by the existence of solutions to the Schwarzian equation that $\widetilde{g}(w)$ can be extended to a locally univalent meromorphic function globally defined on $\widetilde{D_\varepsilon^*}$. In addition, since the right hand side of (14) is $2\pi i$-periodic, and since solutions to the Schwarzian equation $\{y, w\} = q(w)$ are unique up to Möbius transformations, we see that the meromorphic function $\widetilde{g} : \widetilde{D_\varepsilon^*} \to \overline{\mathbb{C}}$ satisfies

$$\widetilde{g}(w + 2\pi i) = \psi(\widetilde{g}(w)) \tag{15}$$

for a certain Möbius transformation ψ.

As explained in Lemma 2, $\widetilde{g}(w)$ lies on a circle $\mathcal{C}_1 \subset \overline{\mathbb{C}}$ for $\{w \in \widetilde{D_\varepsilon} : \text{Im}(w) = 0\}$, and $\widetilde{g}(w)$ lies on another circle $\mathcal{C}_2 \subset \overline{\mathbb{C}}$ for $\{w \in \widetilde{D_\varepsilon} : \text{Im}(w) = \pi\}$. We will study the behavior of g in terms of the relative position of both circles.

Case 1: $\mathcal{C}_1$ intersects $\mathcal{C}_2$ in two points or they coincide.

If $\mathcal{C}_1$ and $\mathcal{C}_2$ share at least two points, then we can consider a Möbius transformation φ such that $\varphi(\mathcal{C}_1)$ is the circle $\mathbb{R} \cup \{\infty\} \subseteq \overline{\mathbb{C}}$ and $\varphi(\mathcal{C}_2)$ is the circle given by a straight line passing through the origin and $\infty \in \overline{\mathbb{C}}$. For that, observe that φ is the composition of a Möbius transformation which maps the previous two points of $\mathcal{C}_1 \cap \mathcal{C}_2$ into $\{0, \infty\}$, and a rotation with respect to the origin.

From (13) and (15), the new locally univalent meromorphic maps $G = \varphi \circ g$ and $\widetilde{G} = \varphi \circ \widetilde{g}$ satisfy

$$\widetilde{G}(w) = G(e^w) \tag{16}$$

and

$$\widetilde{G}(w + 2\pi i) = \Psi(\widetilde{G}(w)) \tag{17}$$

for a certain Möbius transformation Ψ.

For any real number $r \in (-\infty, \log \varepsilon)$ we have $\widetilde{G}(r) \in \varphi(\mathcal{C}_1) \subseteq \mathbb{R} \cup \{\infty\}$. Hence, by the Schwarz reflection principle,

$$\widetilde{G}(w) = \overline{\widetilde{G}(\overline{w})}, \qquad \text{for all } w \in \widetilde{D_\varepsilon}. \tag{18}$$

Thus, from (17) and (18),

$$\widetilde{G}(r + \pi i) = \Psi(\widetilde{G}(r - \pi i)) = \Psi\left(\overline{\widetilde{G}(r + \pi i)}\right), \qquad \text{for all } r \in (-\infty, \log \varepsilon). \tag{19}$$

And, since the set $\{\widetilde{G}(r+\pi i):\ r\in(-\infty,\log\varepsilon)\}$ lies on the circle $\varphi(\mathcal{C}_2)$ and has no empty interior in $\varphi(\mathcal{C}_2)$, then

$$\zeta=\Psi(\overline{\zeta}),\qquad \text{for all } \zeta\in\varphi(\mathcal{C}_2).$$

But a Möbius transformation is determined by the image of three points, and $\varphi(\mathcal{C}_2)$ passes through 0 and ∞. So, if we take an arbitrary point $\zeta_0\in\varphi(\mathcal{C}_2)\backslash\{0,\infty\}$ we easily obtain that

$$\Psi(\zeta)=\frac{\zeta_0}{\overline{\zeta_0}}\,\zeta,\qquad \text{for all } \zeta\in\overline{\mathbb{C}}.$$

Therefore, from (17), we get

$$\widetilde{G}(w+2\pi i)=e^{i\theta_0}\ \widetilde{G}(w),\qquad w\in\widetilde{D_\varepsilon},$$

where $e^{i\theta_0}=\zeta_0/\overline{\zeta_0}$ for a real constant $\theta_0\in[0,2\pi)$. Finally, in order to obtain $\widetilde{G}$ we observe that the new meromorphic function

$$H(w)=e^{-\frac{\theta_0}{2\pi}\,w}\ \widetilde{G}(w) \tag{20}$$

satisfies

$$H(w+2\pi i)=H(w),\qquad w\in\widetilde{D_\varepsilon}.$$

So, from Lemma 3, there exists a well defined meromorphic function $F(z)$ in the punctured disk D_ε^* such that

$$H(w)=F(e^w),\qquad w\in\widetilde{D_\varepsilon}.$$

Hence, (18) and (20) give

$$\widetilde{G}(w)=e^{\gamma w}\ F(e^w),\qquad w\in\widetilde{D_\varepsilon}, \tag{21}$$

with $\gamma=\theta_0/(2\pi)\in[0,1)$ and $F(z)=\overline{F(\overline{z})}$, $z\in D_\varepsilon^*$.

In particular, the developing map g of any solution of the local problem (L) when $\mathcal{C}_1$ and $\mathcal{C}_2$ have at least two common points is given, from (16) and (21), by

$$g(z)=\frac{A\,z^\gamma\,F(z)+B}{C\,z^\gamma\,F(z)+D}, \tag{22}$$

for certain complex constants A,B,C,D, with $AD-BC=1$, which determine the Möbius transformation φ^{-1}.

Remark 3. *If $\mathcal{C}_1=\mathcal{C}_2$, then $\zeta_0\in\mathbb{R}$ and so $\gamma=0$.*

Case 2: $\mathcal{C}_1$ intersects $\mathcal{C}_2$ in a unique point.

Let p_0 be the common point of the circles $\mathcal{C}_1$ and $\mathcal{C}_2$. Then we consider a Möbius transformation φ that maps $\mathcal{C}_1$ to the circle $\mathbb{R}\cup\{\infty\}\subseteq\overline{\mathbb{C}}$ and maps $\mathcal{C}_2$ to the circle $\{z\in\mathbb{C}:\operatorname{Im}(z)=\pi\}\cup\{\infty\}$. For that, observe that φ can be seen as a Möbius transformation mapping $\mathcal{C}_1$ into $\mathbb{R}\cup\{\infty\}$ which maps p_0 to ∞, composed with a homothety.

As in the previous case, we define the new locally univalent meromorphic maps $G=\varphi\circ g$ and $\widetilde{G}=\varphi\circ\widetilde{g}$ which satisfy (16), (17), (18) and (19).

Since the set $\{\widetilde{G}(r+\pi i):\ r\in(-\infty,\log\varepsilon)\}$ lies on the circle $\{z\in\mathbb{C}:\ Im(z)=\pi\}\cup\{\infty\}$ and has no empty interior there, then

$$\zeta=\Psi(\overline{\zeta}),\qquad \text{for all } \zeta\in\{z\in\mathbb{C}:\operatorname{Im}(z)=\pi\}\cup\{\infty\}.$$

Therefore, $\Psi(\zeta)=\zeta+2\pi i$, and so, from (17),

$$\widetilde{G}(w+2\pi i)=\widetilde{G}(w)+2\pi i,\qquad w\in\widetilde{D_\varepsilon}.$$

Now, the new meromorphic function $H(w)=\widetilde{G}(w)-w$ satisfies $H(w+2\pi i)=H(w)$ for all $w\in\widetilde{D_\varepsilon}$. Hence, using Lemma 3 for the meromorphic function $H(w)$, there exists a well defined meromorphic function $F(z)$ in the punctured disk D_ε^* such that

$$\widetilde{G}(w)=F(e^w)+w,\qquad w\in\widetilde{D_\varepsilon}. \tag{23}$$

Moreover, from (18), $F(z)=\overline{F(\overline{z})}$, $z\in D_\varepsilon^*$.

With all of this, the developing map g of any solution of the local problem (L) when $\mathcal{C}_1$ and $\mathcal{C}_2$ have only one common point is given, from (16) and (23), by

$$g(z)=\frac{A\left(F(z)+\log z\right)+B}{C\left(F(z)+\log z\right)+D}, \tag{24}$$

for certain complex constants A,B,C,D, with $AD-BC=1$, which determine the Möbius transformation φ^{-1}.

Case 3: $\mathcal{C}_1$ does not intersect $\mathcal{C}_2$.

In this case, it is well known that there exists a Möbius transformation φ such that the image of the circles $\mathcal{C}_1$ and $\mathcal{C}_2$ are the circles centered at the origin with radii 1 and $R>1$, respectively.

We start by considering the locally univalent meromorphic maps $G=\varphi\circ g$ and $\widetilde{G}=\varphi\circ\widetilde{g}$, which satisfy again (16) and (17) for a certain Möbius transformation Ψ.

Given a real number $r \in (-\infty, \log \varepsilon)$ we have $|\widetilde{G}(r)| = 1$. So, from the Schwarz reflection principle

$$\widetilde{G}(w) = \frac{1}{\overline{\widetilde{G}(\overline{w})}}, \qquad w \in \widetilde{D_\varepsilon}. \tag{25}$$

In addition, from (17),

$$\widetilde{G}(r + \pi i) = \Psi(\widetilde{G}(r - \pi i)) = \Psi\left(\frac{1}{\overline{\widetilde{G}(r + \pi i)}}\right), \qquad r \in (-\infty, \log \varepsilon).$$

Thus, proceeding as in the previous cases, we have

$$\zeta = \Psi(\frac{1}{\overline{\zeta}}), \qquad \text{for } |\zeta| = R,$$

that is, $\Psi(\frac{1}{R}\, e^{i\theta}) = R\, e^{i\theta}$ for any $\theta \in \mathbb{R}$.

Therefore, $\Psi(\zeta) = R^2 \zeta$, and so, from (17),

$$\widetilde{G}(w + 2\pi i) = R^2\, \widetilde{G}(w), \qquad w \in \widetilde{D_\varepsilon}.$$

Then we can apply Lemma 3 to the meromorphic function

$$H(w) = e^{-\frac{\log(R^2)}{2\pi i}\, w}\; \widetilde{G}(w).$$

Hence, there exists a well defined meromorphic function $F(z)$ in D_ε^* such that

$$\widetilde{G}(w) = e^{i\gamma w} F(e^w), \qquad w \in \widetilde{D_\varepsilon}$$

for the negative real constant $\gamma = -\frac{\log(R^2)}{2\,\pi}$. Moreover, from (25), $F(z)\,\overline{F(\overline{z})} = 1$, for any $z \in D_\varepsilon^*$.

As a consequence, the developing map g of any solution of the local problem (L) when $\mathcal{C}_1$ and $\mathcal{C}_2$ have no common points is given, from (16), by

$$g(z) = \frac{A\, z^{\gamma\, i} F(z) + B}{C\, z^{\gamma\, i} F(z) + D}, \tag{26}$$

for certain complex constants A, B, C, D, with $AD - BC = 1$, which determine the Möbius transformation φ^{-1}.

This completes the proof of the first part of Theorem 2. Also, the converse part of the theorem is just a straightforward computation, so we are done. $\square$

5 Finite area: proof of Theorem 3

The following result describes the solution to problem (L) under the additional assumption (4) of finite area.

Theorem 3. *Let v be a solution of (L) that satisfies the finite energy condition (4). Then, its developing map g is given by the cases (i) or (ii) in Theorem 2, and F does not have an essential singularity at the origin.*

In particular, g can be continuously extended to the origin, and the Schwarzian map $Q : D_\varepsilon^* \longrightarrow \mathbb{C}$ of v has at most a pole of order two at 0.

Proof. Let us start by explaining that it suffices to prove the result for the case $K = 1$. Indeed, let v be a solution of $(L)-(4)$ for a constant $K = K_0 = -1, 0$ and g an associated developing map. Now, let us consider the function v_1 given by (10) for the map g and $K = 1$. Then, v_1 is also a solution of (L), but in this case for $K = 1$, and it also satisfies (4) since

$$\int e^{v_1}\,|dz| = \int \frac{4|g'(z)|^2}{(1+|g(z)|^2)^2}\,|dz| \leq \int \frac{4|g'(z)|^2}{(1+K_0\,|g(z)|^2)^2}\,|dz| = \int e^{v}\,|dz| < \infty.$$

In other words, if the result is true for $K = 1$, it will automatically be true for $K = -1, 0$, as claimed.

Thus, let g be the developing map of a solution to (L) for $K = 1$. First, let us prove that the lengths of the semicircles

$$C_r = \{z \in D_\varepsilon^* : \ |z| = r, \ \mathrm{Im}(z) \geq 0\},$$

for the metric $e^v|dz|^2$ of constant curvature $K = 1$, tend to zero when r goes to zero.

If we denote by $L(r)$ the length of the semicircle C_r for $e^v|dz|^2$, and write $z = r\,e^{i\theta}$, we have from (10)

$$L(r) = r\int_0^\pi e^{v/2}d\theta = r\int_0^\pi \frac{2|g'|}{1+|g|^2}d\theta \leq 2\pi \sup_{0\leq\theta\leq\pi}(r\,g^\sharp(r\,e^{i\theta}))$$

where $g^\sharp(z)$ is the spherical derivative of g with respect to z, that is,

$$g^\sharp(z) := \frac{|g'(z)|}{1+|g(z)|^2}.$$

Hence,

$$\limsup_{|z|\to 0} L(|z|) \leq 2\pi \limsup_{z\to 0}\left(|z|\,g^\sharp(z)\right). \tag{27}$$

Claim: in the above conditions, we have $\limsup_{z\to 0}\left(|z|\, g^\sharp(z)\right) = 0$.

Let us prove the claim above. Assume that $\limsup_{z\to 0}\left(|z|\, g^\sharp(z)\right) \neq 0$, and so $\limsup_{z\to 0} g^\sharp(z) = \infty$. Consider for some $r \in (\varepsilon/4, \varepsilon/2)$ the fixed domain

$$\Omega = \{z \in \mathbb{R}^2_+ :\ r \leq |z| \leq \varepsilon\},$$

and the family of functions

$$g_n(z) = g\left(\frac{z}{2^n}\right), \quad z \in \Omega.$$

That is, the functions of the family $\mathfrak{G} = \{g_n\}_{n\in\mathbb{N}}$ are nothing but the function g evaluated over a domain $\Omega_n \subset \mathbb{R}^2_+$ that gets smaller and closer to the origin as n increases. Moreover, by the choice of r, it holds $\Omega_n \cap \Omega_{n+1} \neq \emptyset$ and $\Omega_n \cap \Omega_{n+2} = \emptyset$. The Theorem of Marty (see [Ma]) characterizes a family of meromorphic functions $\mathfrak{G}$ as a normal family if and only if for every compact $K \subset \Omega$ there exists a constant $M(K)$ such that $g_n^\sharp(z) \leq M(K)$ on K for every function $g_n \in \mathfrak{G}$. Thus, as $\limsup_{z\to 0} g^\sharp(z) = \infty$, $\mathfrak{G}$ is not normal on Ω. On the other hand, the Theorem of Montel (see [Mo]) asserts that as $\mathfrak{G}$ is not normal there will be at least one function $g_{n_0} \in \mathfrak{G}$ that takes every complex value with at most two exceptions on Ω. Then, as $\mathfrak{G} \setminus \{g_{n_0}\}$ is not a normal family, we can iterate this argument and conclude that g assumes every complex value infinitely many times with at most to exceptions in a neighborhood of the origin in $\mathbb{R}^2_+$. This means that $g(z)$ covers an infinite area on $\{z \in D^*_\varepsilon :\ \mathrm{Im}(z) \geq 0\}$, which is a contradiction.

Thus, the claim is proved.

Observe that g is a local isometry from $\{z \in D^*_\varepsilon :\ \mathrm{Im}(z) \geq 0\}$ with the metric $e^v|dz|^2$ into the unit sphere $\overline{\mathbb{C}}$ with its standard metric. Also, we know from Lemma 2 that $g(D_\varepsilon \cap \mathbb{R}^+)$ lies on a circle $\mathcal{C}_1 \subseteq \overline{\mathbb{C}}$ and $g(D_\varepsilon \cap \mathbb{R}^-)$ lies on a circle $\mathcal{C}_2 \subseteq \overline{\mathbb{C}}$. So, $g(C_r)$ is a curve in the sphere $\overline{\mathbb{C}}$ with length $L(r)$, joining the point $g(r)$ of $\mathcal{C}_1$ and the point $g(-r)$ of $\mathcal{C}_2$.

Thus, the case (iii) in Theorem 2 cannot happen because this kind of solutions only occur when $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, which contradicts $\lim_{r\to 0} L(r) = 0$.

The solutions of type (ii) in Theorem 2 happen when $\mathcal{C}_1$ intersects $\mathcal{C}_2$ at a unique point $p_0 \in \overline{\mathbb{C}}$. Let us see that, in this case, $\lim_{z\to 0} g(z) = p_0$, what shows that the function $F(z)$ such that $g(z) = \psi(F(z) + \log(z))$ cannot have an essential singularity at the origin as we wanted to show.

Let $\{z_n\}$ be a sequence converging to 0, and $\delta > 0$ small enough. Now, let us prove that there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $d(g(z_n), p_0) < \delta$, where $d(,)$ denotes distance in $\overline{\mathbb{C}} \equiv \mathcal{Q}^2(1)$.

Let D be the open disk of radius $\delta/2$ centered at p_0 and

$$\widehat{\delta} = \min\{d(\mathcal{C}_1 \backslash D, \mathcal{C}_2), d(\mathcal{C}_1, \mathcal{C}_2 \backslash D)\}.$$

Since $\lim_{r\to 0} L(r) = 0$ there exists $r_0 > 0$ such that if $r < r_0$ then $L(r) < \min\{\delta, \widehat{\delta}\}$. Now, we choose n_0 such that $|z_n| < r_0$ for all $n \geq n_0$. Thus, $L(|z_n|) < \widehat{\delta}$ and so $g(|z_n|) \in \mathcal{C}_1 \cap D$ and $g(-|z_n|) \in \mathcal{C}_2 \cap D$. Hence, if $d(g(z_n), p_0) \geq \delta$ we would have

$$L(|z_n|) \geq d(g(|z_n|), g(z_n)) + d(g(z_n), g(-|z_n|)) > \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

which is a contradiction. This proves that $\lim_{z\to 0} g(z) = p_0$.

Let us show now that the Schwarzian derivative map $Q(z)$, which is well defined in D_ε^*, has at most a pole of order two at the origin. Since the Schwarzian derivative is invariant under Möbius transformations, in this case we only need to do the computation for $g(z) = F(z) + \log z$. As

$$\frac{g''(z)}{g'(z)} = \frac{z^2 F''(z) - 1}{z\,(zF'(z) + 1)}$$

has a pole of order one at the origin, we get from (12) that $Q(z)$ has at most a pole of order two there.

Finally, we analyze the solutions of the case (i) in Theorem 2. They correspond to the situation in which $\mathcal{C}_1 \cap \mathcal{C}_2$ has exactly two points, or $\mathcal{C}_1 = \mathcal{C}_2$.

If $\mathcal{C}_1 = \mathcal{C}_2$ then from Remark 3 the associated developing map g is given by $g(z) = \psi(F(z))$, where $F(z)$ is a meromorphic function in D_ε^*. In addition, the meromorphic function $F(z)$ cannot have an essential singularity at the origin. Otherwise, since $F(\overline{z}) = \overline{F(z)}$, $g(z)$ would take every value of $\overline{\mathbb{C}}$, except at most two points, infinity many times in $\{z \in D_\varepsilon^* :\ \mathrm{Im}(z) \geq 0\}$ what would contradict the finite area condition.

If $\mathcal{C}_1 \cap \mathcal{C}_2 = \{p_1, p_2\}$, a similar argument to the one we just used for the case that $\mathcal{C}_1 \cap \mathcal{C}_2 = \{p_0\} \subset \overline{\mathbb{C}}$ lets us prove that there exists a unique $i_0 \in \{1, 2\}$ such that $\lim_{z\to 0} g(z) = p_{i_0}$. Thus, g can be continuously extended to the origin with $g(0) = p_{i_0}$ and $F(z)$ does not have an essential singularity at 0. Let us outline this argument.

Take again a sequence $\{z_n\} \to 0$ and $0 < \delta$ satisfying $\delta < d(p_1, p_2)/3$. Consider D_i the open disk centered at p_i and radius $\delta/2$, and let

$$\widehat{\delta} = \min\{d(\mathcal{C}_1 \backslash (D_1 \cup D_2), \mathcal{C}_2), d(\mathcal{C}_1, \mathcal{C}_2 \backslash (D_1 \cup D_2))\}.$$

Arguing as before, we can show that there exists a unique $i_0 \in \{1, 2\}$ such that $g(|z_n|), g(-|z_n|) \in D_{i_0}$ for n sufficiently large.

Hence, every point z_n with n sufficiently large satisfies $d(g(z_n), p_{i_0}) < \delta$, since otherwise $d(g(z_n), p_{i_0}) \geq \delta$, and so $L(|z_n|) \geq \delta$, which contradicts that $\lim_{r\to 0} L(r) = 0$. Therefore, $\lim_{z\to 0} g(z) = p_{i_0}$.

In order to finish the proof, let us show that the Schwarzian derivative $Q(z)$ of $g(z)$ has at most a pole of order two at the origin. Again by the invariance of the Schwarzian derivative under Möbius transformations, we only need to do the computation for $g(z) = z^\gamma F(z)$. And since $F(z)$ has no essential singularity at the origin,

$$\frac{g''(z)}{g'(z)} = \frac{z^2 F''(z) + 2\gamma z F'(z) + (\gamma - 1)\gamma F(z)}{z\,(zF'(z) + \gamma F(z))}$$

has at most a pole of order one there. Thus, from (12), $Q(z)$ has at most a pole of order two at the origin. This completes the proof of Theorem 3. □

Theorem 3 shows that for classifying the solutions to (L)-(4), it suffices to determine when the functions v given by (10) in terms of a developing map g as in the statement of Theorem 3 verify (4). We do this next.

First, assume that g is given by case (i) in Theorem 2, where F has at 0 either a pole or a finite value. If we let A, B, C, D denote the coefficients of the Möbius transformations defining g, with $AD - BC = 1$, a computation gives

$$e^v = \frac{4|\gamma z^{\gamma-1} F(z) + z^\gamma F'(z)|^2}{(|Cz^\gamma F(z) + D|^2 + K|Az^\gamma F(z) + B|^2)^2}.$$

If F has a finite value (resp. a pole) at 0, and if $|D|^2 + K|B|^2 \neq 0$ (resp. $|C|^2 + K|A|^2 \neq 0$), it is easy to check that near 0 we have $e^v = |z|^{2\alpha} a(z)$ where $\alpha > -1$ and $a(z)$ is continuous at 0, with $a(0) \neq 0$. Thus (4) holds in these cases.

We suppose now that F has a finite value at 0 and $|D|^2 + K|B|^2 = 0$. If $K = 0$, then $D = 0$ and $e^v = a(z)|z|^{-2(\gamma+k+1)}$ with $a(z)$ continuous at 0 and $a(0) \neq 0$, where k is the order of the zero that F has at the origin (if $F(0) \neq 0$ then $k = 0$). Thus (4) does not hold. If $K = -1$, the conditions $AD - BC = 1$ and $|D|^2 = |B|^2$ give that $\delta = -A\overline{B} + C\overline{D}$ is such that $|\delta| = 1$, and so we can estimate

$$\begin{aligned}
\int_{D_\varepsilon^+} e^v &= \int_{D_\varepsilon^+} \frac{4|\gamma z^{\gamma-1} F(z) + z^\gamma F'(z)|^2}{|z|^{2\gamma}|F|^2 \left((|C|^2 - |A|^2)|z|^\gamma |F(z)| + 2\mathrm{Re}(\delta \frac{F(z) z^\gamma}{|F(z)||z|^\gamma})\right)^2} \\
&\geq \int_{D_\varepsilon^+} \frac{4|\gamma z^{\gamma-1} F(z) + z^\gamma F'(z)|^2}{|z|^{2\gamma}|F|^2 \left(|\,|C|^2 - |A|^2\,|\,|z|^\gamma |F(z)| + 2\right)^2} \\
&\geq R \int_0^\varepsilon \frac{dr}{r} = \infty,
\end{aligned}$$

for a certain constant $R > 0$, where $r = |z|$. So, (4) does not hold in this case.

We consider now the situation when F has a pole of order $k \geq 1$ and $|C|^2 + K|A|^2 = 0$. If $K = 0$, we easily check that close to the origin, $e^v = a(z)|z|^{2(\gamma-k-1)}$ with $a(z)$ continuous at 0 and $a(0) \neq 0$. So, (4) does not hold since $k > \gamma$. If $K = -1$ and $\delta := -A\overline{B} + C\overline{D}$, we also have $|\delta| = 1$, and since $k > \gamma$ we can estimate as before

$$\begin{aligned} \int_{D_\varepsilon^+} e^v &= \int_{D_\varepsilon^+} \frac{4|\gamma z^{\gamma-1}F(z) + z^\gamma F'(z)|^2}{|F|^2|z|^{2\gamma}\left(\frac{|D|^2-|B|^2}{|F||z|^\gamma} + 2\mathrm{Re}(\delta\frac{F(z)z^\gamma}{|F||z|^\gamma})\right)^2} \\ &\geq R\int_0^\varepsilon \frac{dr}{r} = \infty, \end{aligned} \tag{28}$$

for a certain constant $R > 0$, where $r = |z|$, and so (4) does not hold.

Next, assume that g is given by case (ii) in Theorem 2, where again F has at 0 a pole or a finite value. Arguing as before, since

$$e^v = \frac{4|F'(z) + 1/z|^2}{(|C(F(z) + \log z) + D|^2 + K|A(F + \log z) + B|^2)^2}, \tag{29}$$

we see that if $|C|^2 + K|A|^2 \neq 0$ and F has a finite value (resp. a pole of order $k \geq 1$) at 0, then near 0 we have $e^v = |z|^{-2}(\ln|z|)^{-4}a(z)$ (resp. $e^v = |z|^{2(k-1)}a(z)$) where $a(z)$ is continuous at 0, with $a(0) \neq 0$. Again, (4) holds automatically.

Suppose now that $|C|^2 + K|A|^2 = 0$. If $K = 0$, it is immediate to see that (4) does not hold. So, we are left with the case $K = -1$ and $|A| = |C|$. Observe that this condition implies that $g(0) \in \mathbb{S}^1$. As g is given by case (ii) in Theorem 2, the images of $(0, \varepsilon)$ and $(-\varepsilon, 0)$ by g are two circle arcs meeting tangentially at one point of $\mathbb{S}^1$.

Assume first that F has a finite value at 0. If the constant $\delta = -A\overline{B} + C\overline{D}$ is such that $\mathrm{Re}\,\delta = 0$, then we easily check that close to the origin

$$e^v = \frac{4|F'(z) + 1/z|^2}{(|D|^2 - |B|^2 - 2\mathrm{Im}\delta(\mathrm{Im}F(z) + \arg z))^2} \geq R|F'(z) + 1/z|^2$$

for some $R > 0$. Thus (4) does not hold. On the other hand, if $\mathrm{Re}\delta \neq 0$ the asymptotic behavior of e^v is

$$e^v = \frac{|F' + 1/z|^2}{(\ln|z|)^2(\frac{|D|^2-|B|^2}{2\ln|z|} + \mathrm{Re}\delta(1 + \frac{\mathrm{Re}F(z)}{\ln|z|}) - \mathrm{Im}\delta\frac{(\mathrm{Im}F(z)+\arg z)}{\ln|z|})^2} = \frac{a(z)}{|z|^2(\ln|z|)^2},$$

where $a(z)$ is continuous at 0 and $a(0) \neq 0$. Then, we can easily check that (4) holds. Furthermore, a computation shows that $c_1 = -c_2 \in (-2, 2)$.

Now, assume that F has a pole at 0, $K=-1$ and $|A|=|C|$. If $\delta=-A\overline{B}+C\overline{D}$, we can proceed as we did before and estimate

$$\begin{aligned}\int_{D_\varepsilon^+} e^v &= \int_{D_\varepsilon^+} \frac{4|F'(z)+1/z|^2}{|F+\log z|^2\left(\frac{|D|^2-|B|^2}{|F+\log z|}+2\mathrm{Re}(\delta(\frac{F(z)+\log z}{|F+\log z|})\right)^2} \\ &\geq R\int_0^\varepsilon \frac{dr}{r}=\infty,\end{aligned}$$

for a certain constant $R>0$, where $r=|z|$. Again, (4) does not hold.

This completes the classification of all the solutions to (L)-(4). In particular, we have the following description of the asymptotic behavior of such solutions.

Corollary 1. *Let $v\in C^2(\overline{D_\varepsilon^+}\setminus\{0\})$ be a solution to (L)-(4). There are three possible asymptotic behaviors for v at 0:*

1. $\lim_{z\to 0}|z|^{-2\alpha}e^v\neq 0$, *for some $\alpha>-1$, i.e. $e^v|dz|^2$ has at 0 a boundary conical singularity.*
2. $\lim_{z\to 0}|z|^2(\ln|z|)^4e^v\neq 0$.
3. $\lim_{z\to 0}|z|^2(\ln|z|)^2e^v\neq 0$.

Here, the last case happens only when $K=-1$ and the boundary has infinite length around 0. In this last situation, it holds $c_1=-c_2\in(-2,2)$.

6 Global solutions: proof of Theorem 4.

In Section 2 we described in detail a large family of explicit solutions to (P) with finite area: the *canonical solutions*. We prove next that these are actually the only solutions to (P) with finite area.

Theorem 4. *Any solution to (P) satisfying the finite energy condition* (2) *is a canonical solution.*

Proof. Let v be a solution of (P) such that

$$\int_{\mathbb{R}_+^2} e^v\,dx\,dy<\infty,$$

and let $Q(z)$ be the Schwarzian map associated to v. From Theorem 3, $Q(z)$ has at most a pole of order two at the origin.

We observe that the new meromorphic function $h(w) = g(z)$ with $z = -1/w$ is also the developing map of a solution $\widehat{v}$ to (P). Thus, since

$$\int_{\mathbb{R}^2_+} e^{\widehat{v}}\, dx\, dy = \int_{\mathbb{R}^2_+} e^{v}\, dx\, dy < \infty,$$

we obtain again from Theorem 3 that the Schwarzian derivative $\widehat{Q}(w)$ has at most a pole of order two at the origin.

By using that $w^4\,\widehat{Q}(w) = Q(z)$, we conclude that $Q(z)$ is a holomorphic function in $\mathbb{C}^*$ with at most a pole of order two at the origin and at least a zero of order two at infinity. Therefore,

$$Q(z) = \frac{c}{z^2}, \qquad z \in \mathbb{C}^*$$

for a certain constant $c \in \mathbb{C}$.

The solutions $g(z)$ of the Schwarzian equation (12) for $Q(z) = c/z^2$ are well known. They are given by $g(z) = \psi(\log(z))$ if $2c = 1$ and by $g(z) = \psi(z^\gamma)$ if $2c = 1 - \gamma^2 \neq 1$, where ψ is an arbitrary Möbius transformation. In the latter case, if follows from Theorem 3 that in our situation γ must be a real constant. In fact, up to composition with the Möbius transformation $z \to 1/z$ if necessary, we can assume $\gamma > 0$.

Finally, in order to finish the proof, we compute the solutions v to our problem depending of the value of $g(z)$.

Let

$$g(z) = \psi(z^\gamma) = \frac{A\, z^\gamma + B}{C\, z^\gamma + D}$$

with $AD - BC = 1$.

Then, from (10),

$$e^v = \frac{4\gamma^2 |z|^{2(\gamma-1)}}{(K|B|^2 + |D|^2 + (KA\overline{B} + C\overline{D})z^\gamma + (K\overline{A}B + \overline{C}D)\overline{z}^\gamma + (K|A|^2 + |C|^2)|z|^{2\gamma})^2}\,.$$

If $K|A|^2 + |C|^2 = 0$, an argument as in (28) proves that $\int_{\mathbb{R}^2_+} e^v < \infty$. On the other hand, if $K|A|^2 + |C|^2 \neq 0$ we can take

$$\lambda = \frac{1}{|\,K|A|^2 + |C|^2\,|}, \qquad z_0 = -\frac{K\overline{A}B + \overline{C}D}{K|A|^2 + |C|^2} \tag{30}$$

and so we have

$$e^v = \frac{4\lambda^2\gamma^2 |z|^{2(\gamma-1)}}{(K\lambda^2 + |z^\gamma - z_0|^2)^2}\,,$$

that is, we obtain the canonical solution (5), as wished.

Now, let us consider the case

$$g(z) = \psi(\log z) = \frac{A\,\log z + B}{C\,\log z + D}. \tag{31}$$

with $AD - BC = 1$.

Then, from (10),

$$\begin{aligned} e^v &= 4/(|z|^2(K|B|^2 + |D|^2 + (KA\overline{B} + C\overline{D})\log z + (K\overline{A}B + \overline{C}D)\log \overline{z} + \\ &\quad +(K|A|^2 + |C|^2)|\log z|^2)^2). \end{aligned}$$

If $K|A|^2 + |C|^2 = 0$, the function v has infinite area in $\mathbb{R}^2_+$. This follows directly from our discussion after Theorem 3 except when $\mathrm{Re}\delta \neq 0$ and $K = -1$, where here $\delta := -A\overline{B} + C\overline{D}$. But in that case, using that $|A| = |C|$, the condition that the map g in (31) must satisfy $|g(z)| < 1$ for every $z \in \mathbb{C}^+$ leads to the inequality

$$2\mathrm{Re}\delta \ln |z| > |B|^2 - |D|^2 + 2\mathrm{Im}\delta\, \mathrm{arg} z,$$

which cannot hold since $\ln |z| : \mathbb{C}^+ \to \mathbb{R}$ is surjective. This proves the claim.

If $K|A|^2 + |C|^2 \neq 0$ the function v can be rewritten as

$$e^v = \frac{4\,\lambda^2}{|z|^2(K\lambda^2 + |\log z - z_0|^2)^2},$$

where λ and z_0 are chosen as in (30). This completes the proof of Theorem 4. $\square$

As a consequence, we get:

Corollary 2. *Given $K \in \{-1, 0, 1\}$, $c_1, c_2 \in \mathbb{R}$, there exists a solution to the problem (P) with finite area if and only if*

- $K = 1$, *or*
- $K = 0$ *and* $c_i < 0$ *for some* $i \in \{1, 2\}$, *or*
- $K = -1$ *and one of these conditions are satisfied:*

$$c_1 < -2, \qquad \textit{or} \qquad c_2 < -2, \qquad \textit{or} \qquad c_1 + c_2 < 0.$$

Proof. It suffices to prove the result for canonical solutions. Consider first of all a canonical solution given by (5) and write $z_0 = r_0 e^{i\theta_0}$. We already explained in Section 2 the restrictions between the parameters appearing in (5) for the solution to be well defined. Also, the relationship between the constants c_i and these parameters is given in Lemma 1. From this, we see directly that there are always canonical solutions with $K = 1$ for every $c_1, c_2 \in \mathbb{R}$, of the form (5).

In the cases $K = 0, -1$, the situation is more restrictive. Label $x = \theta_0$, $y = \theta_0 - \pi\gamma$ and $R_0 = 2r_0/\lambda$. From Lemma 1 we see that

$$c_1 = R_0 \sin(x) \qquad c_2 = -R_0 \sin(y). \tag{32}$$

By our analysis in Section 2, we have:

- if $K = -1$, then $0 < \alpha_0 < y < x < 2\pi - \alpha_0$ where $\alpha_0 \in (0, \pi/2)$ and $R_0 = 2/\sin(\alpha_0) > 2$. So, if $\alpha_0 < y \leq \pi/2$ (resp. $3\pi/2 \leq x < 2\pi - \alpha_0$) we get $c_2 < -2$ (resp. $c_1 < -2$), and if $\pi/2 < y < x < 3\pi/2$ we have $c_1 + c_2 < 0$. Conversely, assume that c_1 and c_2 satisfy the restrictions above, choose $R_0 > 2$ such that $c_1/R_0, c_2/R_0 \in [-1, 1]$, and call $\sin(\alpha_0) = 2/R_0$. Then, it is clear that some choices of $x = \arcsin(c_1/R_0)$ and $y = \arcsin(-c_2/R_0)$ satisfy $\alpha_0 < y < x < 2\pi - \alpha_0$. So we can find z_0 and λ as in Section 2, such that (7) holds.

- If $K = 0$, then $0 < y < x < 2\pi$. Thus, a simple analysis shows that c_1 and c_2 can not be positive simultaneously. The converse is analogous to the previous $K = -1$ case.

Finally, assume that the canonical solution is given by (6). Again, there are no restrictions if $K = 1$. On the other hand, from (8) in the cases $K = 0$ and $K = -1$, the condition $c_1 + c_2 > 0$ must be satisfied. Moreover, because of the restriction we had for this kind of solutions, at least one c_i must be strictly negative (strictly less than -2 in the case $K = -1$). The converse is trivial. $\square$

7 Uniqueness of polygonal circular metrics

Let us start by stating the following result, which follows from the proof of Theorem 3 and the subsequent discussion.

Corollary 3. *Let $v \in C^2(\overline{D_\varepsilon^+} \setminus \{0\})$ be solution to (L) that satisfies the finite energy condition (4), and let $g : \overline{D_\varepsilon^+} \setminus \{0\} \to \Sigma_K \subseteq \overline{\mathbb{C}}$ denote its developing map, which is a local diffeomorphism. Then:*

1. *The image $g(I_\varepsilon^+)$ lies on a circle $\mathcal{C}_1$, and the image $g(I_\varepsilon^-)$ lies on another circle $\mathcal{C}_2$, such that $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$ (possibly $\mathcal{C}_1 = \mathcal{C}_2$).*

2. *The geodesic curvature of $\mathcal{C}_i$, when parametrized as $g(s, 0)$, is constant of value $-c_i/2$, for the metric ds_K^2 in (9).*

3. g extends continuously to the origin, with $g(0) \in \mathcal{C}_1 \cap \mathcal{C}_2 \subset \overline{\mathbb{C}}$.

4. The Schwarzian map $Q = v_{zz} - v_z^2/2$ extends holomorphically to D_ε^* with $Q(\bar{z}) = \overline{Q(z)}$, and has at the origin at most a pole of order two.

5. If $K = 0$, then $g(0) \in \mathbb{C}$. If $K = -1$ and $g(0) \in \partial\mathbb{D} \equiv \mathbb{S}^1$, then $\mathcal{C}_1$ and $\mathcal{C}_2$ are tangent at $g(0)$, and are not *arcs of horocycles.*

From this result, it is not difficult to classify the conformal metrics of constant curvature K and finite area on $\mathbb{R}^2_+$ that have a finite number of boundary singularities on the real axis, and constant geodesic curvature along each boundary arc. From an analytical point of view, this corresponds to classifying the solutions $v \in C^2(\overline{\mathbb{R}^2_+} \setminus \{q_1, \dots, q_{n-1}\})$ with $\int_{\mathbb{R}^2_+} e^v < \infty$ to the Neumann problem

$$\begin{cases} \Delta v + 2Ke^v = 0 & \text{in } \mathbb{R}^2_+ = \{(s,t) \in \mathbb{R}^2 : \ t > 0\}, \\ \dfrac{\partial v}{\partial t} = c_j \, e^{v/2} & \text{on } I_j \subset \mathbb{R} \equiv \partial\mathbb{R}^2_+, \qquad c_j \in \mathbb{R}, \end{cases} \tag{33}$$

where $I_j := (q_j, q_{j+1})$, $j = 0, \dots, n-1$ and $q_0 = -\infty < q_1 < \cdots < q_{n-1} < q_n = \infty$.

There are obvious examples of this type of conformal metrics on $\mathbb{R}^2_+$. To see this, we denote

$$\widetilde{\Sigma_K} = \begin{cases} \Sigma_K & \text{if } K = 1, 0, \\ \overline{\Sigma_K} \equiv \overline{\mathbb{D}} & \text{if } K = -1, \end{cases}$$

and we consider a polygon in $\widetilde{\Sigma_K} \subseteq \overline{\mathbb{C}}$ whose edges are circular arcs, and a conformal mapping from $\mathbb{R}^2_+$ into the region bounded by it (there are two such regions if $\widetilde{\Sigma_K} = \overline{\mathbb{C}}$). In the case that $K = -1$ we will allow that these polygons have some vertices at $\mathbb{S}^1 \equiv \partial\widetilde{\Sigma_{-1}}$, as long as the edges common to any of such vertices are tangent at the vertex, and are not pieces of horocycles. Then, the induced metric on $\mathbb{R}^2_+$ from ds^2_K via this conformal mapping gives a metric in the above conditions. (That the area is finite when $K = -1$ and the polygon has ideal vertices is proved in the discussion after Theorem 3).

Also, in the case $K = 1$ (and so $\widetilde{\Sigma_K} = \overline{\mathbb{C}}$) we may compose with a suitable branched covering of $\overline{\mathbb{C}}$ to obtain other conformal metrics with the desired properties, as we explained in Subsection 2.2.

Still, there exist many other conformal metrics on $\mathbb{R}^2_+$ of finite area, constant geodesic curvature on the boundary, and a finite number of boundary singularities. In order to explain how to construct them, we give first some definitions.

Definition 3. *By a* piecewise regular closed curve *in $\widetilde{\Sigma_K}$ we mean a continuous map $\alpha : \mathbb{S}^1 \to \widetilde{\Sigma_K}$ such that α is smooth and regular everywhere except at a finite number of points $\theta_1, \dots, \theta_n \in \mathbb{S}^1$. By a* piecewise regular parametrization *of α we mean a composition $\beta = \alpha \circ \phi : \mathbb{S}^1 \to \widetilde{\Sigma_K}$, where ϕ is a diffeomorphism of $\mathbb{S}^1$.*

Let $A_j \subset \mathbb{S}^1$, $j \in \{1, \dots, n\}$, be the arc between θ_j and θ_{j+1} (we define $\theta_{n+1} := \theta_1$). Then α will be called an immersed circular polygon *in $\widetilde{\Sigma_K}$ if each regular open arc $\alpha|_{A_j}$ has constant geodesic curvature in Σ_K, and in the case that $K = -1$ and $\alpha(\theta_j) \in \mathbb{S}^1$ the arcs $\alpha|_{A_j}$ and $\alpha|_{A_{j-1}}$ are tangent at $\alpha(\theta_j)$ and are not pieces of horocycles.*

Observe that we allow the curve α to have self-intersections, even along each regular arc $A_j \subset \mathbb{S}^1$. We now introduce a concept from differential topology.

Definition 4. *A piecewise regular closed curve $\alpha : \mathbb{S}^1 \to \widetilde{\Sigma_K}$ is* Alexandrov embedded *(or simply* A-embedded*) if there exists a continuous map $G : \overline{\mathbb{D}} \to \widetilde{\Sigma_K}$ such that $G \in C^2(\overline{\mathbb{D}} \setminus \{p_1, \dots, p_n\})$ for some $p_1, \dots, p_n \in \mathbb{S}^1$, and:*

1. *For every $z \in \mathbb{D}$ it holds that $G(z) \in \Sigma_K$ and G is a local diffeomorphism around z.*
2. *$G|_{\mathbb{S}^1} : \mathbb{S}^1 \to \widetilde{\Sigma_K}$ is a piecewise regular parametrization of α.*

Example 1. *Any circular polygon without self-intersections in $\widetilde{\Sigma_K}$ is A-embedded. Also, given two points $p, q \in \mathbb{C}$, if γ_1 (resp. γ_2) is an oriented geodesic arc from p to q (resp. q to p), then $\gamma_1 \cup \gamma_2$ is a circular polygon, which is not A-embedded in $\mathbb{C}$, but it is A-embedded in $\overline{\mathbb{C}}$. Two further examples are given in Figure 1.*

We can now associate a conformal metric of constant curvature K in $\mathbb{R}^2_+$ to any immersed circular polygon that is A-embedded. Indeed, let $d\sigma^2$ denote the metric $d\sigma^2 = G^*(ds^2_K)$ induced on $\Gamma := \overline{\mathbb{D}} \setminus \{p_1, \dots, p_n\}$. It is then clear that Γ with the complex structure induced by $d\sigma^2$ is conformally equivalent to $\overline{\mathbb{D}} \setminus \{\theta_1, \dots, \theta_n\}$ for some $\theta_1, \dots, \theta_n \in \mathbb{S}^1$, or alternatively, to $\overline{\mathbb{R}^2_+} \setminus \{q_1, \dots, q_{n-1}\}$ for some $q_1 < \cdots < q_{n-1}$. Consequently, $d\sigma^2$ produces on $\mathbb{R}^2_+$ a conformal metric $ds^2 = e^v|dz|^2$ on $\overline{\mathbb{R}^2_+} \setminus \{q_1, \dots, q_{n-1}\}$ that has finite area, constant geodesic curvature on each boundary component, and $n-1$ boundary singularities along the real axis (we have n singularities if we also count the one placed at ∞).

Definition 5. *Any such metric ds^2 on $\mathbb{R}^2_+$ will be called a* circular polygonal metric *on $\mathbb{R}^2_+$.*

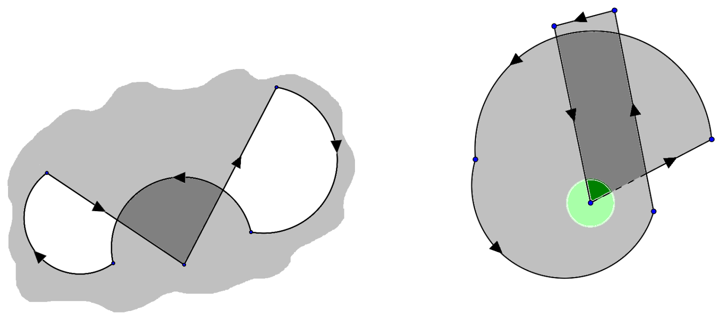

Figure 1: Two examples of circular polygonal contours that are not embedded. The first one is A-embedded in $\overline{\mathbb{C}}$ but not A-embedded in $\mathbb{C}$. The second one is A-embedded in $\mathbb{C}$ and $\overline{\mathbb{C}}$. The green circle indicates that the angle at that vertex is greater than 2π.

Remark 4. *The metrics on $\mathbb{R}^2_+$ that we considered after equation* (33)*, starting from a polygon in $\widetilde{\Sigma_K}$ whose edges are circle arcs, are examples of circular polygonal metrics. In that situation, the map G is given by an adequate conformal equivalence from $\mathbb{D}$ into the region bounded by this polygon. Note that if $K = 1$, the freedom of composing with suitable branched coverings of $\overline{\mathbb{C}}$ only gives different choices of G associated to the Alexandrov embedded polygonal boundary. Thus, even the metrics involving such branched coverings are trivially circular polygonal metrics as in Definition 5.*

Once here, we have the following consequence of Corollary 3:

Corollary 4. *Let $ds^2 = e^v|dz|^2$ be a conformal metric of constant curvature K and finite area in $\mathbb{R}^2_+$. Assume that ds^2 extends smoothly to $\overline{\mathbb{R}^2_+} \setminus \{q_1, \dots, q_{n-1}\}$ for some $q_1 < \cdots < q_{n-1} \in \mathbb{R}$, so that the geodesic curvature of each boundary arc in $\mathbb{R} \equiv \partial\mathbb{R}^2_+$ is constant. Then ds^2 is a circular polygonal metric in $\mathbb{R}^2_+$.*

Proof. Let $g : \mathbb{R}^2_+ \equiv \mathbb{C}_+ \to \Sigma_K \subseteq \overline{\mathbb{C}}$ be the developing map of v, let $\Psi : \mathbb{D} \to \mathbb{C}_+$ be a Möbius transformation giving a conformal equivalence, and define $G := g \circ \Psi : \mathbb{D} \to \Sigma_K$. Then, by Corollary 3 G extends continuously to $\overline{\mathbb{D}}$, and is a local diffeomorphism around each $z \in \mathbb{D}$. Again by Corollary 3, it is clear that $G|_{\mathbb{S}^1}$ is a piecewise regular parametrization of an immersed circular polygon α in $\widetilde{\Sigma_K}$, and that $G \in C^2(\overline{\mathbb{D}} \setminus \{p_1, \dots, p_n\})$, where the points $p_j \in \mathbb{S}^1$ are given by $\Psi(p_j) = q_j$ for $j = 1, \dots, n-1$, and $\Psi(p_n) = \infty$. Hence, α is Alexandrov-embedded.

Finally, since g is a local isometry (see Remark 1), we conclude that $ds^2 = e^v|dz|^2$ is indeed a circular polygonal metric on $\mathbb{R}^2_+$. $\square$

Remark 5. *Corollary 4 together with Theorem 4 explain the geometric interpretation of the canonical solutions that we pointed out without proof in Subsection 2.2.*

Corollary 4 provides a satisfactory geometric classification of all the finite area solutions to problem (33). Let us now describe such solutions from an analytic point of view, using for that Corollary 3 and some classical arguments of the conformal mapping problem from the upper half-plane to a circular polygonal domain in $\mathbb{C}$, see [Neh] for instance. We shall focus on the $K = 1$ case, although many of the next statements also hold when $K \leq 0$.

Corollary 5. *Let $v \in C^2(\overline{\mathbb{R}^2_+} \setminus \{q_1, \dots, q_{n-1}\})$ be a solution to (33) for $K = 1$ that satisfies $\int_{\mathbb{R}^2_+} e^v < \infty$. Then the Schwarzian map $Q = v_{zz} - v_z^2/2$ of v is given by*

$$Q = \sum_{i=1}^{n-1} \left(\frac{\alpha_i}{(z - q_i)^2} + \frac{\beta_i}{z - q_i} \right) \tag{34}$$

where $\alpha_i, \beta_i \in \mathbb{R}$ with $\alpha_i \leq 1/2$, $i = 1, \dots, n-1$, satisfy the following conditions:

$$\sum_{i=1}^{n-1} \beta_i = 0, \qquad \sum_{i=1}^{n-1} (\alpha_i + q_i \beta_i) \leq 1/2. \tag{35}$$

Conversely, if Q is as in (34)-(35), then there is a solution $v \in C^2(\overline{\mathbb{R}^2_+} \setminus \{q_1, \dots, q_{n-1}\})$ to problem (33) for $K = 1$ that satisfies $\int_{\mathbb{R}^2_+} e^v < \infty$. Moreover, the family of such solutions with the same Q is generically three-dimensional.

Proof. Let v be a solution to problem (33) satisfying $\int_{\mathbb{R}^2_+} e^v < \infty$. By Corollary 3, the Schwarzian map Q of v is holomorphic on $\mathbb{C} \setminus \{q_1, \dots, q_{n-1}\}$, and has at most a pole of order two at each q_i. So, clearly Q is of the form

$$Q = \sum_{i=1}^{n-1} \left(\frac{\alpha_i}{(z - q_i)^2} + \frac{\beta_i}{z - q_i} \right) + p(z)$$

for some $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, \dots, n-1$, and for some polynomial $p(z)$ with real coefficients.

Let now $g : \mathbb{C}_+ \to \overline{\mathbb{C}}$ denote the developing map of v, which satisfies $\{g, z\} = Q$. As v has finite area around each q_i, by Theorem 3 we know that g is a Möbius transformation of a function of the form $(z-q_i)^\lambda F(z)$ or $F(z)+\log(z-q_i)$ near each q_i, where F is holomorphic on a punctured neighborhood of q_i, and has at worst a pole at q_i. Noting that the Scwharzian derivative is invariant by Möbius transformations, a simple computation shows that the coefficient α_i in (34) satisfies $\alpha_i \leq 1/2$.

The rest of restrictions come from the fact that, by finite area, the holomorphic quadratic differential Qdz^2 has at ∞ at most a pole of order two. If we let $w=-1/z$, then by conformal invariance $Q(z)dz^2 = \widetilde{Q}(w)dw^2$ where

$$\widetilde{Q}(w) = \frac{1}{w^4} Q(-1/w).$$

So, again by Theorem 3 and the previous computation, the finite area condition at infinity is that there exists $\lim_{w\to 0} w^2 \widetilde{Q}(w) = \alpha_n$ for some $\alpha_n \in (-\infty, 1/2]$. By computing the first terms in the Taylor expansion of $Q(-1/w)$, we easily see that this happens if and only if $p=0$ and α_i, β_i satisfy the conditions (35). This completes the first part of the proof.

Conversely, let Q be as in (34)-(35), and let g be a solution to $\{g,z\}=Q$ in $\mathbb{C}_+$. By construction, g is a locally injective meromorphic function on $\mathbb{C}_+$, unique up to Möbius transformations, and which extends smoothly to $\overline{\mathbb{C}_+} \setminus \{q_1, \dots, q_{n-1}, \infty\}$. As Q is real on the real axis, we deduce from the equation $\{g,s\} = Q(s)$ on $\mathbb{R}$ that $g(s)$ lies on a circle in $\overline{\mathbb{C}}$ for each interval in $\mathbb{R} \setminus \{q_1, \dots, q_{n-1}\}$. All of this shows that the map $v \in C^2(\overline{\mathbb{R}^2_+} \setminus \{q_1, \dots, q_{n-1}\})$ given by

$$e^v = \frac{4|g'|^2}{(1+|g|^2)^2}$$

is a solution to (33) for $K=1$. We only have left to show that $\int_{\mathbb{R}^2_+} e^v < \infty$, for what we only need to prove this condition around each q_i and around ∞.

Let us fix q_i, $i \in \{1, \dots, n-1\}$, and consider the complex ODE $y'' + \frac{1}{2}Qy = 0$. As Q has at worst a pole of order two at q_i, it is a classical result that a fundamental system of solutions (y_1, y_2) of this equation around q_i is

$$y_1(z) = (z-q_i)^{\lambda_1} a_1(z), \qquad y_2(z) = (z-q_i)^{\lambda_2} a_2(z) + k y_1(z) \log(z-q_i),$$

where $k \in \mathbb{C}$, $a_1(z), a_2(z)$ are holomorphic on a neighborhood of q_i with $a_i(0) \neq 0$ for $i=1,2$, and λ_1, λ_2 are solutions of the *indicial equation*

$$\lambda^2 - \lambda + \frac{\alpha_i}{2} = 0.$$

Here α_i is the coefficient of Q in q_i given by (34). Note that from $\alpha_i \leq 1/2$ we deduce that $\lambda_1, \lambda_2 \in \mathbb{R}$, and we may assume that $\lambda_1 \leq \lambda_2$. Therefore, $k \neq 0$ if and only if $\lambda_2 - \lambda_1 \in \mathbb{N}$.

Also, it is a classical result that the quotient y_2/y_1 provides a solution to $\{g, z\} = Q$, that is, a developing map for the solution v. Thus, depending on whether $\lambda_2 - \lambda_1 \in \mathbb{N}$ or not, g is of the form

$$g(z) = F(z) + \log(z - q_i) \qquad \text{or} \qquad g(z) = (z - q_i)^{\lambda_2 - \lambda_1} F(z)$$

for some meromorphic function F around q_i such that $F(\bar{z}) = \overline{F(z)}$. By Theorem 3 and its subsequent discussion, we see then that $\int e^v < \infty$ on the half-disk $D^+(q_i, \varepsilon) \subset \mathbb{R}^2_+$ for $\varepsilon > 0$ small enough (note that we are assuming that $K = 1$).

The same argument can be done at ∞, this time using the additional conditions (35) and the conformal change $w = -1/z$, as we did before. This concludes the proof of existence.

Finally, observe that the solution g to $\{g, z\} = Q$ is unique up to Möbius transformations, so there is a real 6-parameter family of possible choices for g. As the developing map of a solution v to (1) is defined up to the change (11), we obtain generically a 3-parameter family of solutions to (33) for $K = 1$ with the same Q. This concludes the proof of the result.

$\square$

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