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Isolated singularities of graphs in warped products and Monge-Ampère equations

José A. Gálvez, Asun Jiménez, and Pablo Mira

Abstract. We study graphs of positive extrinsic curvature with a non-removable isolated singularity in 3-dimensional warped product spaces, and describe their behavior at the singularity in several natural situations. We use Monge-Ampère equations to give a classification of the surfaces in 3-dimensional space forms which are embedded around a non-removable isolated singularity and have a prescribed, real analytic, positive extrinsic curvature function at every point. Specifically, we prove that this space is in one-to-one correspondence with the space of regular, analytic, strictly convex Jordan curves in the 2-dimensional sphere $\mathbb{S}^2$.

1. Introduction

This paper investigates the geometry of graphs of positive extrinsic curvature (not necessarily constant) in three-dimensional warped product spaces $M^2 \times_f \mathbb{R}$ around a non-removable isolated singularity of the graph. The aim here is twofold.

Our first objective is to describe from a purely geometric point of view the behavior of a graph in the above conditions around an isolated singularity. For instance, we will show that these graphs, if bounded, extend continuously with bounded gradient to the singularity. We will also prove that, in Hadamard manifolds, if the graph is not bounded around the singularity, then the height function of the graph has limit $\pm\infty$ at the singular point.

Our second objective is to classify the embedded isolated singularities of prescribed, analytic, positive extrinsic curvature in 3-dimensional space forms $\mathbb{M}^3(c)$ of constant curvature c. For that purpose we will regard $\mathbb{M}^3(c)$ as a warped product manifold and show that the prescribed curvature equation in warped products is an elliptic equation of Monge-Ampère type. Then, we will generalize some aspects of our study in [**GJM2**] of isolated singularities of Monge-Ampère equations in order to classify the previous class of embedded isolated singularities in $\mathbb{M}^3(c)$ in terms of the class of regular, analytic, strictly convex Jordan curves in $\mathbb{S}^2$.

The first objective above will be carried out in Section 2, which can be read independently from the rest of the paper. Specifically, in Section 2 we will prove the following three results on the geometry of isolated singularities for graphs of positive extrinsic curvature in warped product three-manifolds.

Theorem 1. *Let Σ be a graph in $M^2 \times_f \mathbb{R}$ with $K_{\rm ext} > 0$. Assume that Σ has an isolated singularity at $p_0 \in M^2$ and is bounded around p_0. Then Σ extends across p_0 as a continuous graph and is uniformly non-vertical.*

Theorem 2. *Let Σ be a graph with $K_{\rm ext} > 0$ in the Riemannian product space $M^2 \times \mathbb{R}$ (i.e. $f = 1$). Assume that Σ has an isolated singularity at $p_0 \in M^2$. Then Σ extends across p_0 as a continuous graph and is uniformly non-vertical.*

Theorem 3. *Let Σ be a graph with $K_{\rm ext} > 0$ in a warped product space $M^2 \times_f \mathbb{R}$ which is a Hadamard manifold. Assume that Σ has an isolated singularity at $p_0 \in M^2$ and that Σ is* not *bounded around p_0. Then, if z denotes the height function of Σ,*

$$\lim_{p \to p_0} z(p) = \pm\infty, \qquad p \in M^2.$$

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In Sections 3 and 4, as a preparation for our main result in Section 5, we will study the behavior of solutions to the general elliptic equation of Monge-Ampère type

$$\det(D^2 z + \mathcal{A}(x,y,z,Dz)) = \varphi(x,y,z,Dz) > 0 \tag{1.1}$$

around a non-removable isolated singularity, where $\mathcal{A}(x,y,z,Dz) \in \mathcal{M}_2(\mathbb{R})$ is a symmetric matrix. Again, these two sections can be read independently from the rest of the paper. We will show that when $\mathcal{A}, \varphi$ are real analytic, the graph $z = z(x,y)$ of any solution to (1.1) can be analytically parametrized as an embedding defined on an annulus, and so that this parametrization extends analytically to the boundary circle of the annulus which the parametrization collapses to the singularity (Lemma 3). Also, we will show (Theorem 4) that the limit gradient at the singularity is a regular analytic Jordan curve in $\mathbb{R}^2$ whose curvature does not change sign. These results extend to equation (1.1) some facts proved in [**GJM2**] for the *pure* Monge-Ampère equation, i.e. for the case $\mathcal{A} = 0$. Other results on isolated singularities of elliptic Monge-Ampère equations can be found in [**ACG, Bey1, Bey2, CaLi, GMM, GJM2, JiXi, Jor, ScWa**].

In Section 5 we give the classification of the surfaces in a space form $\mathbb{M}^3(c)$ that are embedded around an isolated singularity, and whose extrinsic curvature at each point is predetermined by a given positive, analytic function $\mathcal{K}$ defined on a neighborhood of the singular point in $\mathbb{R}^3$ (see Theorem 5). Let us note that, by the Gauss equation

$$K_G = K_{\text{ext}} + c, \tag{1.2}$$

prescribing the extrinsic curvature K_{ext} on a surface in a space form $\mathbb{M}^3(c)$ of constant curvature c is equivalent to prescribing the (intrinsic) Gaussian curvature K_G of the surface.

Theorem 5 extends the analogous classification result proved by the authors in [**GJM2**, Theorem 4] for the case $\mathbb{M}^3(c) = \mathbb{R}^3$. However, the extension of the classification to $\mathbb{H}^3$ and $\mathbb{S}^3$ is non-trivial. Indeed, while in $\mathbb{R}^3$ the extrinsic curvature K_{ext} of a graph $z = z(x,y)$ is given by the *pure* Monge-Ampère equation

$$z_{xx}z_{yy} - z_{xy}^2 = \varphi(x,y,z,z_x,z_y), \tag{1.3}$$

where $\varphi(x,y,z,z_x,z_y) = K_{\text{ext}}(1+z_x^2+z_y^2)^2$, in $\mathbb{H}^3$ and $\mathbb{S}^3$ the prescribed extrinsic curvature equation has the general form (1.1), i.e. $\mathcal{A} \neq 0$. This makes our analytic study of isolated singularities of Monge-Ampère equations in [**GJM2**] insufficient for our geometric purposes here, and justifies the need of the analytic results of Sections 3 and 4. Theorem 5 also extends results proved in [**GHM, GaMi**] for surfaces of constant curvature.

2. Isolated singularities of graphs in warped products

We devote this section to the proof of Theorems 1, 2 and 3 stated in the introduction.

Given a Riemannian surface (M^2, g) and a smooth function $f : \mathbb{R} \to (0,\infty)$, we define the *three-dimensional warped product* $M^2 \times_f \mathbb{R}$ as the Riemannian manifold $(M^2 \times \mathbb{R}, \langle , \rangle)$, where

$$\langle , \rangle = f(t) g + dt^2.$$

A surface Σ in $M^2 \times_f \mathbb{R}$ is a *graph* if $\pi : \Sigma \to \pi(\Sigma)$ is a diffeomorphism, where π stands for the projection $M^2 \times_f \mathbb{R} \to M^2$. If we choose coordinates (x,y) on a domain of M^2 that contains $\pi(\Sigma)$, then the graph Σ in the coordinates (x,y,t) is given by $t = z(x,y)$, where $z(x,y)$ is a smooth function. We will call z the *height function* of the graph.

Definition 1. *Let Σ be a smooth graph in $M^2 \times_f \mathbb{R}$ over a punctured disk $\mathcal{D}^* \subset M^2$ around some $p_0 \in M^2$. If Σ does not extend as a C^1 graph to $\mathcal{D}^* \cup \{p_0\}$, we will call p_0 an* isolated singularity *of Σ.*

In what follows we will denote by K_{ext} the *extrinsic curvature function* of the oriented graph Σ, i.e. the determinant of the second fundamental form II of Σ with respect to its first fundamental form: $K_{\text{ext}} = \det(II)/\det(I)$. Then, the condition $K_{\text{ext}} > 0$ is equivalent to the property that II is (positive or negative) definite at every point.

Given a graph $\Sigma \subset M^2 \times_f \mathbb{R}$, we can orient it so that its unit normal N satisfies $\langle N, \partial_t \rangle \in (0,1]$, where t is the vertical coordinate in $M^2 \times_f \mathbb{R}$. We will call $\nu := \langle N, \partial_t \rangle$ the *angle function* associated to Σ, and we will say that Σ is *uniformly non-vertical* if $\nu \geq c > 0$ in Σ.

As a preparation for the proofs of Theorems 1, 2 and 3, we consider next a special type of local coordinates (u,v,t) on a warped product space $M^2 \times_f \mathbb{R}$.

Let (M^2,g) be an oriented Riemannian surface and $p_0 \in M^2$. For a fixed unit vector $\xi \in T_{p_0}M$, let $\gamma(v)$ be the unique geodesic in M^2 with initial conditions $\gamma(0)=p_0$, $\gamma'(0)=\xi$. Let exp and J denote, respectively, the exponential map and the complex structure of (M^2,g). Then, for $\varepsilon>0$ small enough, the map

$$(u,v) \longmapsto \exp_{\gamma(v)}(uJ\gamma'(v))$$

defines a diffeomorphism from $R_\varepsilon := (-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)$ into a neighborhood $U \subset M^2$ of p_0, such that the metric g is expressed with respect to (u,v) as

$$g = du^2 + G(u,v)dv^2$$

for a positive smooth function $G(u,v)$ in R_ε with $G(0,v)=1$ for all $v \in (-\varepsilon,\varepsilon)$.

Observe that in these coordinates, each curve $v=$ const. in R_ε corresponds to a geodesic of (M^2,g).

Let now Σ be a graph in $M^2\times_f \mathbb{R}$ with an isolated singularity at p_0. If we parametrize $U\times\mathbb{R}$ in terms of the (u,v,t) coordinates defined above, then Σ is written in a neighborhood of p_0 as

$$\Sigma = \{(u,v,z(u,v)) : (u,v) \in \Omega \subset R_\varepsilon \setminus \{(0,0)\}\}$$

for some smooth function z defined on a punctured disk Ω centered at the origin.

Consider $\bar\partial_u, \bar\partial_v$ the tangent coordinate frame in Σ with respect to (u,v), i.e.

$$\bar\partial_u = \partial_u + z_u\partial_t, \qquad \bar\partial_v = \partial_v + z_v\partial_t,$$

and let η be the (non-unit) upwards pointing normal vector field to Σ

$$\eta := -z_u\partial_u - \frac{z_v}{G}\partial_v + f\partial_t.$$

Then, bearing in mind that the Levi-Civita connection ∇ in $M^2\times_f\mathbb{R}$ in the coordinates (u,v,t) satisfies

$$\nabla_{\partial_u}\partial_u = -\frac{f'(t)}{2}\partial_t, \qquad \nabla_{\partial_u}\partial_t = \frac{f'(t)}{2f(t)}\partial_u, \qquad \nabla_{\partial_t}\partial_t = 0,$$

a simple computation shows that the second fundamental form II of Σ verifies

$$\begin{aligned} II(\bar\partial_u,\bar\partial_u) &= \langle \nabla_{\bar\partial_u}\bar\partial_u, \tfrac{\eta}{||\eta||}\rangle \\ &= \tfrac{1}{||\eta||}\left(-f'(z)z_u^2 + f(z)\left(z_{uu} - \tfrac{f'(z)}{2}\right)\right). \end{aligned}$$

If we assume now that $K_{\rm ext}>0$ for Σ, then

$$-f'(z)z_u^2 + f(z)\left(z_{uu} - \frac{f'(z)}{2}\right) \neq 0 \tag{2.1}$$

for every $(u,v)\in\Omega$. Alternatively, we can rewrite (2.1) as

$$\frac{\partial}{\partial u}\left(\frac{z_u}{f(z)}\right) > \frac{f'(z)}{2f(z)} \quad \text{or else} \quad \frac{\partial}{\partial u}\left(\frac{z_u}{f(z)}\right) < \frac{f'(z)}{2f(z)}. \tag{2.2}$$

Proof of Theorem 1: We assume, for instance, that the first inequality in (2.2) holds (the argument is similar with the second inequality). As z is bounded by hypothesis, there is some $c_0\in\mathbb{R}$ such that, for $(u,v)\in\Omega\subset R_\varepsilon\setminus\{(0,0)\}$,

$$\frac{\partial}{\partial u}\left(\frac{z_u}{f(z)}\right) > \frac{f'(z)}{2f(z)} \geq c_0,$$

and therefore

$$\frac{\partial}{\partial u}\left(\frac{z_u}{f(z)} - c_0 u\right) > 0.$$

This condition easily implies that

$$\frac{z_u}{f(z)} - c_0 u$$

is bounded in Ω, from where z_u is also bounded in Ω.

Recall now that, by construction, the coordinates (u,v) depend on the arbitrary unit vector ξ which determines the ∂_v direction at $(u,v)=(0,0)$. So, a different choice $\widehat{\xi}$ of the vector ξ will result in new coordinates $(\widehat{u},\widehat{v})$ for which $z_{\widehat{u}}$ will be bounded. By choosing $\widehat{\xi}$ so that $\{\partial_u,\partial_{\widehat{u}}\}$ are linearly independent at $(0,0)$, it is easy to deduce then that z_v is also bounded around $(0,0)$.

On the other hand, a computation shows that the angle function $\nu=\langle N,\partial t\rangle$ of Σ satisfies

$$\nu^2=\langle N,\partial_t\rangle^2=\frac{1}{||\eta||^2}\langle\eta,\partial_t\rangle^2=\frac{f(z)}{f(z)+z_u^2+\frac{z_v^2}{G}}. \tag{2.3}$$

As z,z_u and z_v are bounded around $(0,0)$, we conclude from (2.3) that $\nu^2\geq c>0$ around $(0,0)$, i.e. Σ is uniformly non-vertical around the isolated singularity. Finally, as z_u and z_v are bounded in a punctured neighborhood of the origin we deduce in a standard way that z is continuous at $(0,0)$. This completes the proof of Theorem 1. $\Box$

Proof of Theorem 2: By the condition $f=1$, we get from (2.1) that z_{uu} has a constant sign on the punctured disk $\Omega\subset R_\varepsilon\setminus\{(0,0)\}$. Thus z_u is bounded in Ω. The rest of the argument is identical to the one used in the proof of Theorem 1. $\Box$

Proof of Theorem 3:

Remark 1. *Theorem 3 is also true if, instead of assuming that $M^2\times_f\mathbb{R}$ is a Hadamard manifold, we ask for the following property* (**P**) *to hold for some strongly convex geodesic disk $D_r\subset M^2$ of radius r centered at p_0.*

> Property **(P)**: *For any two points in $D_r\times\mathbb{R}\subset M^2\times_f\mathbb{R}$ there is a unique minimizing geodesic arc joining both points, and moreover, this geodesic arc is totally contained in $D_r\times\mathbb{R}$.*

Observe that any Hadamard manifold $M^2\times_f\mathbb{R}$ verifies property (**P**). *Basic examples of this type of manifolds are $\mathbb{R}^3$, $\mathbb{H}^3$ and $\mathbb{H}^2\times\mathbb{R}$.*

Let $D_r\subset M^2$ be a strongly convex geodesic disk of radius r centered at p_0, that is, for every two points $p_1,p_2\in\overline{D_r}$ there exists a unique minimizing geodesic γ joining p_1 to p_2 such that $\text{int}(\gamma)\subset D_r$. Assume that property (**P**) holds in $D_r\times\mathbb{R}\subset M^2\times_f\mathbb{R}$. We will assume without loss of generality that the graph Σ is well defined on $D_r\setminus\{p_0\}$, and the second fundamental form of Σ is positive definite for the upwards pointing unit normal N of Σ (recall that $K_{\text{ext}}>0$ on Σ). This implies, using that the vertical planes $\gamma\times\mathbb{R}$ over a geodesic $\gamma\subset M^2$ are totally geodesic surfaces in $M^2\times_f\mathbb{R}$, that any geodesic $\widetilde{\gamma}$ in $M^2\times_f\mathbb{R}$ which is tangent to Σ at some point $p\in\Sigma$ lies below Σ around p (note that $\widetilde{\gamma}\subset\gamma\times\mathbb{R}$ for some geodesic γ of M^2).

We define the *epigraph* of z by

$$\text{epi}(z)=\{(p,t)\in D_r\times\mathbb{R}:t\geq z(p),p\neq p_0\}.$$

Claim. $\overline{\text{epi}(z)}$ *is a convex subset of* $\overline{D_r}\times\mathbb{R}$.

Observe that the convexity notion makes sense since we are assuming that property (**P**) holds in $D_r\times\mathbb{R}\subset M^2\times_f\mathbb{R}$.

To prove this claim, we take two points (p_1,t_1), (p_2,t_2) in $\overline{\text{epi}(z)}\subset\overline{D_r}\times\mathbb{R}$ and prove that the unique geodesic Γ in $\overline{D_r}\times\mathbb{R}$ joining them is contained in $\overline{\text{epi(z)}}$. We distinguish several cases.

Case 1: the geodesic γ joining p_1 and p_2 ($p_1\neq p_2$) in $\overline{D_r}$ does not pass through p_0.

First note that if $p_1=p_2(\neq p_0)$, the geodesic Γ corresponds to a vertical segment, so the property holds.

Consider the totally geodesic plane over γ, that is, $\gamma\times\mathbb{R}\subset D_r\times\mathbb{R}$. Then, the geodesic Γ is contained in $\gamma\times\mathbb{R}$. Let $\alpha=(\gamma\times\mathbb{R})\cap\Sigma$.

Let θ_0 be the angle that the geodesic $\Gamma\subset\gamma\times\mathbb{R}$ makes with the vertical direction ∂_t at the point (p_1,t_1), and consider the family $\{\Gamma_\theta\}_{\theta\in[0,\theta_0]}$ of geodesics in $\gamma\times\mathbb{R}$ starting at (p_1,t_1) and making an angle θ with ∂_t at this initial point. Note that Γ_0 is $\{p_1\}\times[t_1,\infty)$, that the interior of Γ_0 does not intersect α, that $\Gamma_{\theta_0}=\Gamma$, and that all such geodesics only intersect at the initial point (p_1,t_1).

Once here, observe that the existence of a point $q\in\Gamma$ not lying in $\overline{\text{epi}(z)}$ would mean that α is *above* Γ around q in $\gamma\times\mathbb{R}$. But that would mean that some geodesic Γ_θ lies above α in

$\gamma \times \mathbb{R}$ and touches α tangentially at some point. This is a contradiction with the fact that Σ has positive definite second fundamental form for the upwards pointing unit normal, and hence lies locally above all its tangent geodesics. This completes the proof of the convexity of $\overline{\mathrm{epi}(z)}$ in Case 1.

Case 2: the geodesic γ joining p_1 and p_2 in $\overline{D_r}$ passes through p_0.

Let $\{x_n\}$ be a sequence of points in D_r with $x_n \to p_2$, and such that the geodesic in D_r joining x_n and p_1 does not pass through p_0. Then, we can also take $(t_n)_n$ such that $(x_n, t_n) \in \overline{\mathrm{epi}(z)}$ and $(x_n, t_n) \to (p_2, t_2)$ as $n \to \infty$. By Case 1, the geodesic Γ_n joining (p_1, t_1) with (x_n, t_n) is contained in $\overline{\mathrm{epi}(z)}$. Taking limits, Γ_n converge to the geodesic Γ joining (p_1, t_1) with (p_2, t_2). In particular, Γ is contained in $\overline{\mathrm{epi}(z)}$ as we wanted to prove. We remark that this argument also holds in the case $p_0 = p_2$.

Case 3: $p_1 = p_2 = p_0$.

In this case we take two points (p_0, t_1) and (p_0, t_2) in $\overline{\mathrm{epi}(z)}$ with $t_1 < t_2$. Take a sequence $(x_n, t_n) \to (p_0, t_1)$ with $(x_n, t_n) \in \mathrm{epi}(z)$ for all n. Then, the vertical segments

$$\Gamma_n := \{x_n\} \times [t_n, t_n + t_2 - t_1] \subset \overline{\mathrm{epi}(z)}$$

are geodesics, so by taking limits $\Gamma_n \to \Gamma = \{p_0\} \times [t_1, t_2]$, which is also contained in $\overline{\mathrm{epi}(z)}$.

Thus, we have proved that $\overline{\mathrm{epi}(z)}$ is a convex subset of $\overline{D_r} \times \mathbb{R}$. That is, we have finished the proof of the Claim above.

Let us observe now that there is some $t_0 \in \mathbb{R}$ such that $(p_0, t_0) \in \mathrm{int}(\overline{\mathrm{epi}(z)})$. For this, let $p_1, p_2 \in D_r$ such that p_0 lies in the geodesic γ joining p_1 with p_2. If we take $t_1 > z(p_1)$, then $(p_1, t_1) \in \mathrm{int}(\overline{\mathrm{epi}(z)})$ and the geodesic in $D_r \times \mathbb{R}$ joining (p_1, t_1) with $(p_2, z(p_2))$ passes through some point of the form (p_0, t_0). By standard convexity arguments, all the points in such a geodesic arc lie in $\mathrm{int}(\overline{\mathrm{epi}(z)})$ except the endpoints. In particular, $(p_0, t_0) \in \mathrm{int}(\overline{\mathrm{epi}(z)})$.

Consider now $(p_0, t_0) \in \mathrm{int}(\overline{\mathrm{epi}(z)})$, and let $\varepsilon > 0$ so that $D_\varepsilon \times (t_0 - \varepsilon, t_0 + \varepsilon) \subset \overline{\mathrm{epi}(z)}$, where $D_\varepsilon \subset D_r$ is a geodesic disk of radius ε centered at p_0. Then, $D_\varepsilon \times (t_0 - \varepsilon, \infty) \subset \overline{\mathrm{epi}(z)}$ from where it follows that $(p_0, t_1) \in \mathrm{int}(\overline{\mathrm{epi}(z)})$ for all $t_1 > t_0$. By convexity, there are two possibilities:

(i) $\overline{\mathrm{epi}(z)} \cap (\{p_0\} \times \mathbb{R}) = \{p_0\} \times [h, \infty)$ with $\{p_0\} \times (h, \infty) \subset \mathrm{int}(\overline{\mathrm{epi}(z)})$, or
(ii) $\overline{\mathrm{epi}(z)} \cap (\{p_0\} \times \mathbb{R}) = \{p_0\} \times \mathbb{R}$, with $\{p_0\} \times \mathbb{R} \subset \mathrm{int}(\overline{\mathrm{epi}(z)})$.

We will prove first of all that in case (i) above we have $\lim_{p \to p_0} z(p) = h$. Thus, case (i) cannot happen since we assumed that z is not bounded around p_0.

Let $\{p_n\}$ be a sequence of points in $D_r \setminus \{p_0\}$ converging to p_0. As we are in case (i), given $\delta > 0$ there exists some $\varepsilon > 0$ such that

$$(p_0, h + \delta) \in D_\varepsilon \times (h + \delta - \varepsilon, h + \delta + \varepsilon) \subset \overline{\mathrm{epi}(z)}$$

and

$$(p_0, h - \delta) \in D_\varepsilon \times (h - \delta - \varepsilon, h - \delta + \varepsilon) \subset (D_r \times \mathbb{R}) \setminus \overline{\mathrm{epi}(z)}.$$

In particular, we have

$$D_\varepsilon \times [h + \delta, \infty) \subset \mathrm{int}(\overline{\mathrm{epi}(z)}) \quad \text{and} \quad D_\varepsilon \times (-\infty, h - \delta] \subset \mathrm{int}((D_r \times \mathbb{R}) \setminus \overline{\mathrm{epi}(z)}).$$

Now, observe that, since $(p_n, z(p_n))$ lies in the boundary of $\mathrm{epi}(z)$, it holds $(p_n, z(p_n)) \notin \mathrm{int}(\overline{\mathrm{epi}(z)}) \cup \mathrm{int}((D_r \times \mathbb{R}) \setminus \overline{\mathrm{epi}(z)})$. Thus, choosing n_0 so that if $n \geq n_0$ we have $p_n \in D_\varepsilon$, we can conclude from the previous condition that $|z(p_n) - h| < \delta$ for every $n \geq n_0$.

This proves that $\lim_{p \to p_0} z(p) = h$. Hence case (i) cannot happen, i.e., the condition in case (ii) must hold.

Now, the same argument used above to prove that $\lim_{p \to p_0} z(p) = h$ in case (i) also shows that $\lim_{p \to p_0} z(p) = -\infty$ in case (ii). Observe that we would have obtained $\lim_{p \to p_0} z(p) = +\infty$ if we had worked with the opposite (downwards pointing) unit normal of Σ. This completes the proof of Theorem 3. □

We finish this section with the following result characterizing isolated singularities of graphs of positive extrinsic curvature in Hadamard warped products.

COROLLARY 1. *Let $\Sigma \subset M^2 \times_f \mathbb{R}$ be a graph with $K_{\mathrm{ext}} > 0$ over a punctured disk $D^* \subset M^2$ centered at $p_0 \in M^2$, and assume that $M^2 \times_f \mathbb{R}$ is a Hadamard manifold. Then, exactly one of the following three situations happen:*

(1) Σ extends as a smooth C^1-graph across p_0.
(2) Σ extends continuously (but not C^1-smoothly) across p_0, and is uniformly non-vertical at p_0.
(3) The height function of Σ tends to $+\infty$ or to $-\infty$ at p_0. In particular, the metric of Σ is complete around the puncture.

Moreover, assume that the sectional curvature of $M^2 \times_f \mathbb{R}$ is bounded from below on a tubular neighborhood $D_\varepsilon \times \mathbb{R}$ of the geodesic $\{p_0\} \times \mathbb{R}$ by a number $i(p_0) \in \mathbb{R}$. Then the third situation above cannot happen provided $K_{\rm ext} > -i(p_0)$.

Proof. By Theorems 1 and 3 we only need to prove the final assertion. This is a consequence of the Gauss equation for Σ in $M^2 \times_f \mathbb{R}$, which implies that if $K_{\rm ext} > -i(p_0)$, the Gaussian curvature of Σ is bounded from below by a positive constant around p_0.

But now, observe that in $M^2 \times_f \mathbb{R}$, the distance between (p_1, t_1) and (p_2, t_2) is at least $|t_2 - t_1|$. Hence, the fact that the height function of Σ tends to $\pm\infty$ around p_0 indicates that there exist points $p, q \in \Sigma$ arbitrarily far away from each other, and also from $\partial\Sigma$. This contradicts that on a surface with curvature bounded from below by some $c > 0$, the length of any minimizing geodesic is at most $\pi/\sqrt{c}$. $\Box$

Example 1. *The half-space model of the hyperbolic 3-space $\mathbb{H}^3 = (\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_3 > 0\}, \frac{1}{x_3^2}(dx_1^2 + dx_2^3 + dx_3^2))$ can be written as the warped product*

$$(\mathbb{R}^2 \times \mathbb{R}, e^{-2z}(dx^2 + dy^2) + dz^2). \tag{2.4}$$

Hence, $\mathbb{R}^2 \times_{e^{-2z}} \mathbb{R}$ is a Hadamard manifold. In this model, the graphs $z = z(x, y)$ correspond geometrically to graphs in $\mathbb{H}^3$ over horospheres. For every $R > 0$, the graph $z = z(x, y)$ of the function

$$z(x, y) = \log\left(R - \sqrt{R^2 - x^2 - y^2}\right), \qquad 0 < x^2 + y^2 < R^2,$$

is a piece of a horosphere Σ_R in $\mathbb{H}^3$. In particular, Σ_R has $K_{\rm ext} = 1$. Note that $(0, 0)$ is an isolated singularity of the graph, that $z \to -\infty$ as $(x, y) \to (0, 0)$, and that Σ_R is complete around the puncture. This provides an example of Case (3) in Corollary 1. Σ_R also shows that the inequality $K_{\rm ext} > -i(p_0)$ in the last assertion of Corollary 1 is necessary, and cannot be weakened to $K_{\rm ext} \geq -i(p_0)$.

3. Isolated singularities of the Monge-Ampère equation: preliminaries

Let us consider the general elliptic equation of Monge-Ampère type in dimension two, which is the following fully nonlinear PDE:

$$\det(D^2 z + \mathcal{A}(x, y, z, Dz)) = \varphi(x, y, z, Dz) > 0. \tag{3.1}$$

Here, $Dz, D^2 z$ denote respectively the gradient and the Hessian of z, and $\mathcal{A}(x, y, z, Dz) \in \mathcal{M}_2(\mathbb{R})$ is symmetric. The Monge-Ampère equation (3.1) can be rewritten as

$$A z_{xx} + 2B z_{xy} + C z_{yy} + z_{xx} z_{yy} - z_{xy}^2 = E, \tag{3.2}$$

where $A = A(x, y, z, z_x, z_y), \dots, E = E(x, y, z, z_x, z_y)$ are defined on an open set $\mathcal{U} \subset \mathbb{R}^5$ and satisfy on $\mathcal{U}$ the ellipticity condition

$$\mathcal{D} := AC - B^2 + E > 0. \tag{3.3}$$

We will study solutions z to the elliptic equation (3.2) around a non-removable isolated singularity, and for simplicity we will assume that this singularity is placed at $(0, 0)$. All our results can be trivially adapted if the singularity is placed elsewhere.

Convention: From now on we will use the following notations:

- $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < \rho^2\}$, a punctured disc centered at the origin.
- $\mathcal{U} \subset \mathbb{R}^5$ is an open set.
- $A, \dots, E$ are functions in $C^2(\mathcal{U})$, which satisfy in $\mathcal{U}$ the ellipticity condition (3.3).
- For a function $z \in C^2(\Omega)$, we define

$$\mathfrak{H} := \{(x, y, z(x, y), z_x(x, y), z_y(x, y)) : (x, y) \in \Omega\}. \tag{3.4}$$

DEFINITION 2. *We say that a solution* $z \in C^2(\Omega)$ *to* (3.2) *for the coefficients* $A, \ldots, E$ *is a* singular solution *of* (3.2) *in* Ω *if*

(1) z *is not* C^1 *at the origin.*
(2) $\overline{\mathfrak{H}}$ *is a compact subset of* $\mathcal{U}$.

In the case that the coefficients $A, \ldots, E$ are real analytic on $\mathcal{U}$, the solution z is also real analytic on Ω. Also, it can be easily proved that in the conditions of the definition, $z(x, y)$ extends *continuously* to $(0, 0)$.

From now on we will assume for simplicity that any singular solution to (3.2) *has been continuously extended to the origin by* $z(0, 0) = 0$, *and that* $\mathcal{H} \neq \emptyset$, *where*

$$\mathcal{H} := \mathcal{U} \cap \{(x_1, \ldots, x_5) \in \mathbb{R}^5 : x_1 = x_2 = x_3 = 0\}.$$

Also, we will be using the standard classical notation $p = z_x$, $q = z_y$, $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$.

Let $z \in C^2(\Omega)$ be a singular solution to (3.2). It follows then from the ellipticity condition (3.3) that the expression

$$ds^2 = \varepsilon\left((z_{xx} + C)dx^2 + 2(z_{xy} - B)dxdy + (z_{yy} + A)dy^2\right) \tag{3.5}$$

is a Riemannian metric on Ω for $\varepsilon = 1$ or $\varepsilon = -1$. Then, it is a well known fact that (Ω, ds^2) admits global conformal parameters $w := u + iv$ such that

$$ds^2 = \frac{\sqrt{\mathcal{D}}}{u_x v_y - u_y v_x}|dw|^2. \tag{3.6}$$

That is, there exists a C^2 diffeomorphism

$$\Phi : \Omega \to \Lambda := \Phi(\Omega) \subset \mathbb{R}^2, \qquad (x, y) \mapsto \Phi(x, y) = (u(x, y), v(x, y)) \tag{3.7}$$

satisfying

$$x_u y_v - x_v y_u > 0, \tag{3.8}$$

and the Beltrami system

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \frac{1}{\sqrt{\mathcal{D}}} \begin{pmatrix} s - B & -(C + r) \\ A + t & -(s - B) \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix}. \tag{3.9}$$

Here, Λ is a domain in $\mathbb{R}^2 \equiv \mathbb{C}$ which is conformally equivalent to either the punctured disc $\mathbb{D}^*$ or an annulus $\mathbb{A}_R = \{z \in \mathbb{C} : 1 < |z| < R\}$.

In this situation, motivated by [**HeB**], we introduce the following definition.

DEFINITION 3. *A solution* z *to* (3.2) *in* Ω *satisfies the* Heinz-Beyerstedt condition, *in short* HeB-condition, *if* A_p, $A_q + 2B_p$, $C_p + 2B_q$ *and* C_q *are Liptschitz continuous in* $\overline{\Omega}$ *when they are considered as functions of* x *and* y.

It was proved by Heinz and Beyerstedt (cf. [**HeB**, Lemma 3.3]) that if $z \in C^2(\Omega)$ is a singular solution to (3.2) which satisfies the HeB-condition, then Λ is conformally equivalent to some annulus $\mathbb{A}_R$.

Let us point out that the HeB-condition holds automatically for singular solutions of a wide class of Monge-Ampère equations, as explained in the next lemma.

LEMMA 1. *Suppose that the coefficients* $A, \ldots, E : \mathcal{U} \subset \mathbb{R}^5 \longrightarrow \mathbb{R}$ *in* (3.2) *satisfy the condition:*

The functions A_p, $A_q + 2B_p$, $C_p + 2B_q$ *and* C_q *do not depend on* p *and* q *in* $\mathcal{U}$. $(\star)$

Then any singular solution $z(x, y)$ *to* (3.2) *satisfies the HeB-condition in Definition 3.*

PROOF. We denote by $F(x, y, z)$ any of the functions in the statement of the condition $(\star)$. Then, F can be seen as a function $\widetilde{F}$ depending on the variables (x, y) using the composition $\widetilde{F}(x, y) = (F \circ G)(x, y)$ where $G(x, y) = (x, y, z(x, y))$. As $A, \ldots, E \in C^2(\mathcal{U})$, we have $F \in C^1(\mathcal{U})$ and $\widetilde{F} \in C^1(\Omega) \cup C^0(\overline{\Omega})$. Note that

$$\widetilde{F}_x(x, y) = F_x(x, y, z(x, y)) + F_z(x, y, z(x, y))z_x(x, y),$$

which is bounded in Ω by condition (2) in Definition 2. Analogously, $\widetilde{F}_y$ is also bounded in Ω. Hence, $\widetilde{F} \in C^{0,1}(\overline{\Omega})$ (see [**GiTr**, p. 154]). This concludes the proof of Lemma 1. $\square$

For later use, let us also point out the following basic result.

LEMMA 2. *Let $z \in C^2(\Omega)$ be a singular solution to* (3.2)*, where Ω is an open domain of $\mathbb{R}^2$ (not necessarily a punctured disc). Then, for sufficiently large constants $a, c > 0$, the function*

$$z^*(x,y) = z(x,y) + \frac{\varepsilon a}{2}x^2 + \frac{\varepsilon c}{2}y^2 \tag{3.10}$$

satisfies that its graph $\{(x,y,z^*(x,y)) : (x,y) \in \Omega\}$ is a locally convex surface in $\mathbb{R}^3$, where $\varepsilon \in \{-1,1\}$ is as in (3.5).

PROOF. We will assume $\varepsilon = 1$; the case $\varepsilon = -1$ is analogous. As the coefficients $A, \dots, E$ are bounded on $\mathfrak{H}$, we can find constants $a, c > 0$ such that

$$c - C > 0, \qquad a - A > 0, \qquad (c - C)(a - A) - B^2 > 0,$$

i.e. the matrix

$$\mathcal{N} = \begin{pmatrix} c - C & B \\ B & a - A \end{pmatrix} \tag{3.11}$$

is positive definite. Now, we may use the fact that ds^2 in (3.5) is positive definite to conclude that the symmetric bilinear form

$$ds^2 + (dx,dy)\mathcal{N}(dx,dy)^T = (dx,dy)\begin{pmatrix} r + c & s \\ s & t + a \end{pmatrix}(dx,dy)^T$$

is also positive definite, that is, it is a Riemannian metric on Ω.

On the other hand, a straightforward computation shows that the matrix of the second fundamental form of the graph of $z^*(x,y)$ in (3.10) with respect to the upwards-pointing unit normal is given by

$$\mathrm{II}^* = \frac{1}{\sqrt{1 + (p + cx)^2 + (q + ay)^2}}\begin{pmatrix} r + c & s \\ s & t + a \end{pmatrix},$$

which we have just proved is positive definite on Ω. In particular, the graph of $z^*(x,y)$ has positive curvature at every point, which proves the assertion. $\square$

4. The limit gradient of singular solutions

Let $z(x,y)$ be a singular solution to (3.2) that satisfies the HeB-condition, and let (u,v) be the conformal parameters associated to $z(x,y)$ introduced in the previous section. As explained there, (u,v) are defined on a domain $\Lambda \subset \mathbb{C}$ conformally equivalent to an annulus. Thus, we may assume Λ to be a quotient strip $\Gamma_R := \{w \in \mathbb{C} : 0 < \operatorname{Im} w < R\}/(2\pi\mathbb{Z})$. From now on (u,v) will denote the canonical coordinates in this strip, and the quantities x, y, z, p, q associated to the solution $z(x,y)$ will be seen as functions depending on (u,v). Note that all of them are 2π-periodic in the variable u, by construction.

Let $G = \{(x,y,z(x,y)) : (x,y) \in \Omega\} \subset \mathbb{R}^3$ be the graph of $z(x,y)$. Then we can parameterize G as

$$\psi(u,v) = (x(u,v), y(u,v), z(u,v)) : \Gamma_R \to G \subset \mathbb{R}^3, \tag{4.1}$$

so that ψ extends continuously to $\mathbb{R}$ with $\psi(u,0) = (0,0,0)$. We next prove a boundary regularity result for the map

$$\mathbf{z}(u,v) = (x(u,v), y(u,v), z(u,v), p(u,v), q(u,v)) : \Gamma_R \to \mathbb{R}^5. \tag{4.2}$$

LEMMA 3. *Let $z \in C^2(\Omega)$ be a singular solution to* (3.2) *that satisfies the HeB-condition, and assume that $A,\ B,\ C,\ E \in C^k(\mathcal{U})$, $k \geq 2$ (resp. $A,\ B,\ C,\ E \in C^\omega(\mathcal{U})$). Let (u,v) be the conformal coordinates in Γ_R associated to z as explained previously. Then the map $\mathbf{z}(u,v)$ in* (4.2) *extends as a $C^{k,\alpha}$ map $\forall \alpha \in (0,1)$ (resp. as a real analytic map) to $\Gamma_R \cup \mathbb{R}$.*

PROOF OF LEMMA 3. The proof in an adaptation of Claim 1 in [**GJM2**] to a more general context.

To start with, we follow a bootstrapping method. Consider an arbitrary point of $\mathbb{R}$, which we will suppose without loss of generality to be the origin. Also, consider for $0 < \delta < R$ the domain $\mathbb{D}^+ = \{(u,v) : 0 < u^2 + v^2 < \delta^2, v > 0\}$.

From (3.9) it follows that

$$\begin{aligned} p_u &= \sqrt{\mathcal{D}}y_v + By_u - Cx_u, \\ p_v &= -\sqrt{\mathcal{D}}y_u + By_v - Cx_v, \\ q_u &= -\sqrt{\mathcal{D}}x_v + Bx_u - Ay_u, \\ q_v &= \sqrt{\mathcal{D}}x_u + Bx_v - Ay_v. \end{aligned} \tag{4.3}$$

And by derivation in (4.3) we obtain (cf. [**HeB**])

$$\begin{array}{rcl} \Delta x &=& h_1(x_u^2 + x_v^2) + h_2(x_u y_u + x_v y_v) + h_3(y_u^2 + y_v^2) + h_4(x_u y_v - x_v y_u), \\ \Delta y &=& \widetilde{h}_1(x_u^2 + x_v^2) + \widetilde{h}_2(x_u y_u + x_v y_v) + \widetilde{h}_3(y_u^2 + y_v^2) + \widetilde{h}_4(x_u y_v - x_v y_u), \end{array} \tag{4.4}$$

where the coefficients $h_1 = h_1(x,y,z,p,q), \dots, \widetilde{h}_4 = \widetilde{h}_4(x,y,z,p,q)$ are

$$\begin{array}{rcl} h_1 &=& B_q - \frac{1}{2\mathcal{D}}(\mathcal{D}_x + \mathcal{D}_z p - \mathcal{D}_p C + \mathcal{D}_q B), \\ h_2 &=& -A_q - B_p - \frac{1}{2\mathcal{D}}(\mathcal{D}_y + \mathcal{D}_z q + \mathcal{D}_p B - \mathcal{D}_q A), \\ h_3 &=& A_p, \\ h_4 &=& \frac{1}{\sqrt{\mathcal{D}}}(A_x + B_y + A_z p + B_z q - A_p C + (A_q + B_p)B - B_q A - \frac{1}{2}\mathcal{D}_p), \\ \widetilde{h}_1 &=& C_q, \\ \widetilde{h}_2 &=& -B_q - C_p - \frac{1}{2\mathcal{D}}(\mathcal{D}_x + \mathcal{D}_z p - \mathcal{D}_p C + \mathcal{D}_q B), \\ \widetilde{h}_3 &=& B_p - \frac{1}{2\mathcal{D}}(\mathcal{D}_y + \mathcal{D}_z q + \mathcal{D}_p B - \mathcal{D}_q A), \\ \widetilde{h}_4 &=& \frac{1}{\sqrt{\mathcal{D}}}(C_y + B_x + C_z q + B_z p - B_p C + (B_q + C_p)B - C_q A - \frac{1}{2}\mathcal{D}_q), \end{array} \tag{4.5}$$

all of them evaluated at $\mathbf{z}(u,v) = (x,y,z,p,q)(u,v)$. Besides, from the Cauchy-Scwharz inequality we get $|x_u y_v - x_v y_u| \le \frac{1}{2}(|Dx|^2 + |Dy|^2)$ and $|x_u y_u + x_v y_v| \le \frac{1}{2}(|Dx|^2 + |Dy|^2)$.

Hence, letting $Y = (x,y) : \mathbb{D}^+ \longrightarrow \Omega$, we get from (4.4) and the fact that $h_1, \dots, \widetilde{h}_4$ are bounded (since $\overline{\mathfrak{H}}$ is a compact subset of $\mathcal{U}$) that

$$|\Delta Y| \le c(|Dx|^2 + |Dy|^2) \tag{4.6}$$

for some constant $c > 0$.

Note that $Y \in C^2(\mathbb{D}^+) \cap C^0(\overline{\mathbb{D}^+})$ with $Y(u,0) = (0,0)$ for all u. Hence, we can deduce from Heinz's Theorem in [**He**] that $Y \in C^{1,\alpha}(\overline{\mathbb{D}^+_\varepsilon})$ for all $\alpha \in (0,1)$, where $\mathbb{D}^+_\varepsilon = \mathbb{D}^+ \cap B(0,\varepsilon)$ for some $0 < \varepsilon < \delta$.

As the right hand side terms in (4.3) are bounded in $\overline{\mathbb{D}^+_\varepsilon}$, then $p,q \in W^{1,\infty}(\overline{\mathbb{D}^+_\varepsilon})$. Therefore, $p,q \in C^{0,1}(\overline{\mathbb{D}^+_\varepsilon})$ (see [**GiTr**, p. 154]).

Once here, noting that

$$z_u = px_u + qy_u, \qquad z_v = px_v + qy_v, \tag{4.7}$$

we obtain $z \in C^{1,\alpha}(\overline{\mathbb{D}^+_\varepsilon})$ $\forall \alpha \in (0,1)$. Thus, the right hand side functions in (4.3) are Hölder continuous of any order in $\overline{\mathbb{D}^+_\varepsilon}$, and so $p,q \in C^{1,\alpha}(\overline{\mathbb{D}^+_\varepsilon})$ $\forall \alpha \in (0,1)$.

With this, we get from (4.4) that ΔY is Hölder continuous in $\overline{\mathbb{D}^+_\varepsilon}$. A standard potential analysis argument (see [**GiTr**, Lemma 4.10]) ensures that $x,y \in C^{2,\alpha}(\overline{\mathbb{D}^+_{\varepsilon/2}})$, and by formula (4.3) we have $p,q \in C^{2,\alpha}(\overline{\mathbb{D}^+_{\varepsilon/2}})$. So, from (4.7), $z \in C^{2,\alpha}(\overline{\mathbb{D}^+_{\varepsilon/2}})$.

By applying the same argument to Y_u and Y_v, we obtain that $x,y,z,p,q \in C^{3,\alpha}(\overline{\mathbb{D}^+_{\varepsilon/4}})$. A recursive process then shows that $\mathbf{z} = (x,y,z,p,q)$ is $C^{k,\alpha}$ $\forall \alpha \in (0,1)$ (resp. C^∞) at the origin. As we can do the same argument for all points of $\mathbb{R}$ and not just the origin, we conclude that $\mathbf{z}(u,v) \in C^{k,\alpha}(\Gamma_R \cup \mathbb{R})$ (resp. $\mathbf{z}(u,v) \in C^\infty(\Gamma_R \cup \mathbb{R})$).

Finally suppose that A, B, C, E are analytic. A computation in the same spirit of formula (4.4) shows that the Laplacians of z, p, q are given by:

$$\begin{array}{rcl} \Delta p &=& (\sqrt{\mathcal{D}\circ \mathbf{z}})_u y_v - (\sqrt{\mathcal{D}\circ \mathbf{z}})_v y_u + (B\circ \mathbf{z})_u y_u + (B\circ \mathbf{z})_v y_v \\ & & +(B\circ \mathbf{z})\Delta y - (C\circ \mathbf{z})\Delta x - (C\circ \mathbf{z})_u x_u - (C\circ \mathbf{z})_v x_v, \\ \\ \Delta q &=& -(\sqrt{\mathcal{D}\circ \mathbf{z}})_u x_v + (\sqrt{\mathcal{D}\circ \mathbf{z}})_v x_u + (B\circ \mathbf{z})_u x_u + (B\circ \mathbf{z})_v x_v \\ & & +(B\circ \mathbf{z})\Delta x - (A\circ \mathbf{z})\Delta y - (A\circ \mathbf{z})_u y_u - (A\circ \mathbf{z})_v y_v, \\ \\ \Delta z &=& p_u x_u + p_v x_v + q_u y_u + q_v y_v + p\Delta x + q\Delta y. \end{array} \tag{4.8}$$

Therefore, $\mathbf{z}(u,v)$ satisfies

$$\Delta \mathbf{z} = h(\mathbf{z}, \mathbf{z}_u, \mathbf{z}_v) \tag{4.9}$$

where $h : \mathcal{O} \subset \mathbb{R}^{15} \to \mathbb{R}^5$ is a real analytic function on an open set $\mathcal{O}$ of $\mathbb{R}^{15}$ containing the closure of the bounded set $\{(\mathbf{z}, \mathbf{z}_u, \mathbf{z}_v)(u,v) : (u,v) \in \Gamma_R\}$. Moreover, noting that

$$\mathbf{z}(u,v) = (\psi(u,v), \phi(u,v)) : \Gamma_R \to \mathbb{R}^3 \times \mathbb{R}^2 \equiv \mathbb{R}^5$$

where $\psi(u,v) = (x(u,v), y(u,v), z(u,v))$ and $\phi(u,v) = (p(u,v), q(u,v))$, we find that $\mathbf{z}(u,v)$ is a solution to (4.9) that satisfies the mixed initial conditions

$$\left\{ \begin{array}{l} \psi(u,0) = (0,0,0), \\ \phi_v(u,0)^T = \begin{pmatrix} -C(0,0,0,\phi(u,0)) & B(0,0,0,\phi(u,0)) & 0 \\ B(0,0,0,\phi(u,0)) & -A(0,0,0,\phi(u,0)) & 0 \end{pmatrix} \psi_v(u,0)^T. \end{array} \right.$$

As we have shown that $\mathbf{z} \in C^\infty(\Gamma_R \cup \mathbb{R})$, we are in the conditions to apply Theorem 3 in [**Mu**] to $\mathbf{z}$ around every point in $\mathbb{R}$, and we can conclude that $\mathbf{z}$ is real analytic in $\Gamma_R \cup \mathbb{R}$. $\square$

Definition 4. *Let $z(x,y)$ be a singular solution to* (3.2). *We define the* limit gradient *of z at the origin to be the set $\gamma \subset \mathbb{R}^2$ of points $\xi \in \mathbb{R}^2$ for which there is a sequence $q_n \to (0,0)$ in Ω such that $(z_x, z_y)(q_n) \to \xi$.*

The following theorem is the main result of this section, and describes the geometry of the limit gradient of a singular solution to (3.2) that satisfies the HeB-condition, when the coefficients of (3.2) are real analytic.

Theorem 4. *Let $z(x,y)$ be a singular solution to* (3.2) *that satisfies the HeB-condition, and assume that the coefficients A, B, C, E of* (3.2) *are real analytic. Let γ denote the limit gradient of z at the origin.*

Then, γ is a regular, convex, real analytic Jordan curve in $\mathcal{H} \subset \mathbb{R}^2$.

Proof. Let $\gamma \subset \mathbb{R}^2$ denote the limit gradient of $z(x,y)$. By Lemma 3 we can extend the map $(p(u,v), q(u,v))$ analytically to $\Gamma_R \cup \mathbb{R}$, and so we may consider the 2π-periodic map $(\alpha(u), \beta(u)) := (p(u,0), q(u,0))$. As it is clear that $\gamma = \{(\alpha(u), \beta(u)) : u \in \mathbb{R}\}$, we can conclude that γ is a closed curve in $\mathbb{R}^2$, possibly with singularities. In particular, we may parameterize γ in an analytic, 2π-periodic way, as $\gamma(u) = (\alpha(u), \beta(u))$.

Claim 1. *$\gamma'(u)$ vanishes, at most, at four points in $[0, 2\pi)$.*

Proof of Claim 1. By contradiction, assume that $\gamma'(u_i) = (0,0)$ for $u_i \in [0, 2\pi)$, $i = 1, \ldots, 5$. Observe that $p_u(u_i, 0) = q_u(u_i, 0) = 0$ for $i = 1, \ldots, 5$. Then, since $x(u,0) = y(u,0) = 0$ for every $u \in \mathbb{R}$, by (4.3) we see that $Dx(u_i, 0) = Dy(u_i, 0) = 0$, $i = 1, \ldots, 5$. Consider now the Taylor series expansions of $x(u,v)$ and $y(u,v)$ at one of the singular points $(u_i, 0)$, and denote by $P_x(u,v)$ and $P_y(u,v)$ their respective lower order terms. Note that both $P_x(u,v)$ and $P_y(u,v)$ are homogeneous polynomial of degree at least two.

Next, recall that $x(u,v)$ and $y(u,v)$ satisfy the elliptic system of PDEs (4.4). By comparing power series expansions in this equation, it is easily seen that the polynomial of least degree among $P_x(u,v)$ and $P_y(u,v)$ must be harmonic (if they have the same degree, both of them are harmonic). Assume momentarily that it is $P_x(u,v)$. Hence the first term in the power series expansion of $x(u,v)$ at $(u_i, 0)$ is a harmonic polynomial of degree ≥ 2. In these conditions, it is well-known that the

$v = 0$ axis (which is a nodal curve of $x(u,v)$) is crossed at $(u_i,0)$ by at least another nodal curve of $x(u,v)$. If the least degree polynomial were $P_y(u,v)$, the same could be said about $y(u,v)$.

As there are five points $(u_i,0)$ by hypothesis, the previous argument shows the following fact: for at least one of $f(u,v) = x(u,v)$ or $f(u,v) = y(u,v)$, the $v = 0$ axis is crossed transversely by some nodal curve of $f(u,v)$ at three or more points $(u_i,0)$.

Assume for the moment that $f(u,v) = x(u,v)$; if $f(u,v) = y(u,v)$ the argument is analogous.

By construction, the map (4.1) is a diffeomorphism from $\Gamma_R = \Sigma_R/(2\pi\mathbb{Z})$ into the graph $G = \{(x,y,z(x,y)) : (x,y) \in \Omega\}$. As G is a graph, $G \cap \{x = 0\} \subset \mathbb{R}^3$ is formed by exactly two regular curves with an endpoint at the singularity $(0,0,0)$. In other words, the function $x(u,v)$ only has two nodal curves in Γ_R, which is a contradiction. This shows that $\gamma'(u)$ vanishes at most at four points in $[0,2\pi)$. This finishes the proof of Claim 1. □

Claim 2. *$\gamma(\mathbb{R}) \subset \mathbb{R}^2$ bounds a compact, strictly convex set of $\mathbb{R}^2$.*

Proof of Claim 2. Consider $z^*(x,y)$ as in (3.10), where the constants $a, c > 0$ satisfy the conditions of Lemma 2. Hence, z^* has positive Hessian at every point and an isolated singularity at the origin whose limit gradient is γ. For simplicity, assume for the proof of this Claim that $\varepsilon = 1$; the case $\varepsilon = -1$ can be proved analogously.

A parametrization of the graph of $z^*(x,y)$ is given by $\psi^* : \Gamma_R \to \mathbb{R}^3$,

$$\psi^*(u,v) = \left(x(u,v), y(u,v), z(u,v) + \frac{c}{2}x(u,v)^2 + \frac{a}{2}y(u,v)^2\right).$$

The upwards-pointing unit normal of ψ^* in Γ_R is given by

$$N^*(u,v) = \frac{1}{\sqrt{1+(p+cx)^2+(q+ay)^2}}(-p-cx, -q-ay, 1),$$

where x, y, p, q are evaluated at (u,v). Note that ψ^* extends analytically to $\mathbb{R}$.

We now consider the analytic *Legendre transform* of $\psi^*(u,v)$ (see [**LSZ**, p. 89]):

$$\mathcal{L}(u,v) = \left(-\frac{N_1^*}{N_3^*}, -\frac{N_2^*}{N_3^*}, -x\frac{N_1^*}{N_3^*} - y\frac{N_2^*}{N_3^*} - z^*\right) : \Gamma_R \to \mathbb{R}^3, \tag{4.10}$$

where $N^* = (N_1^*, N_2^*, N_3^*)$. Since $\psi^*(\Gamma_R)$ is a locally strictly convex graph, it is well known that so is $\mathcal{L}(\Gamma_R)$.

The upwards-pointing unit normal of $\mathcal{L}$ is

$$\mathcal{N}_{\mathcal{L}} = \frac{(-x,-y,1)}{\sqrt{1+x^2+y^2}} : \Gamma_R \to \mathbb{S}^2_+, \tag{4.11}$$

where x, y are evaluated at (u,v). Note that $\mathcal{L}$ is locally strictly convex and extends analytically to $\mathbb{R}$ with $\mathcal{L}(u,0)$ lying on the horizontal plane $z = 0$, that $\mathcal{N}_{\mathcal{L}}(u,0) = (0,0,1)$ and that $z^*_{xx} > 0$. This shows that, considering a smaller $R > 0$ if necessary, $\mathcal{L}(\Gamma_R)$ lies on the upper half-space of $\mathbb{R}^3$.

In this way, the intersection of $\mathcal{L}(\Gamma_R)$ with any horizontal plane $z = \varepsilon$ for $\varepsilon > 0$ small enough is a regular, strictly convex planar Jordan curve. Also, $\mathcal{L}(u,0) = (\gamma(u),0)$ is the limit of those horizonal convex intersection curves. As $\gamma(u)$ is analytic and non-constant (by Claim 1), we deduce then that $\gamma(\mathbb{R})$ bounds a compact strictly convex set of $\mathbb{R}^2$. This proves Claim 2.

□

Claim 3. *γ is a regular curve, that is, $\gamma'(u) \neq (0,0)$ for every $u \in \mathbb{R}$.*

Proof of Claim 3. Consider for $\theta \in [0,2\pi)$ the vertical half-plane Π_θ of $\mathbb{R}^3$ with boundary the z-axis and which contains the vector $(\cos\theta, \sin\theta, 0)$. Then, the intersection of Π_θ with the graph $z = z(x,y)$, $(x,y) \in D^*$, is a regular, real analytic curve, which can be seen as $\psi(\delta_\theta)$ where δ_θ is the real analytic curve in Γ_R given by

$$\delta_\theta = \{(u,v) \in \Gamma_R : (x(u,v), y(u,v)) = \lambda(u,v)(\cos\theta, \sin\theta) \text{ for some } \lambda(u,v) > 0\}.$$

Consider now the map $\mathcal{L}(u,v) : \Gamma_R \to \mathbb{R}^3$ defined in (4.10), and let us denote $\beta_\theta := \mathcal{L}(\delta_\theta)$. Note that β_θ is a regular, real analytic curve in $\mathcal{L}(\Gamma_R)$; moreover, it follows from (4.11) that $\mathcal{L}(u,v) \in \beta_\theta$ if and only if $-\mathcal{N}_{\mathcal{L}}(u,v) \in \mathbb{S}^2 \cap \Pi_\theta$.

Let now γ_ε denote the intersection of $\mathcal{L}(\Gamma_R)$ with the horizontal plane $z=\varepsilon$ in $\mathbb{R}^3$. Then there is some $\varepsilon_0>0$ small enough such that, for every $\varepsilon\in(0,\varepsilon_0]$, γ_ε is a regular strictly convex Jordan curve, since $\mathcal{L}(\Gamma_R)$ is a graph of positive curvature that lies on the upper half-space of $\mathbb{R}^3$ (see the proof of Claim 2). In particular, there exist exactly two points in γ_ε where the tangent lines to γ_ε are parallel to $(-\sin\theta,\cos\theta,0)$. At these points $\mathcal{N}_{\mathcal{L}}$ is orthogonal to $(-\sin\theta,\cos\theta,0)$, which means that either $\mathcal{N}_{\mathcal{L}}\in\mathbb{S}^2\cap\Pi_\theta$ or $-\mathcal{N}_{\mathcal{L}}\in\mathbb{S}^2\cap\Pi_\theta$ at each of these points, and that there are no more points in γ_ε with this property.

In other words, γ_ε intersects $\beta_\theta\cup\beta_{\theta+\pi}$ at exactly two points, so by continuity we deduce that for every $\theta\in[0,2\pi)$ and every $\varepsilon\in(0,\varepsilon_0]$ there is some $p_{\varepsilon,\theta}\in\mathbb{R}^3$ such that

$$\gamma_\varepsilon\cap\beta_\theta=\{p_{\varepsilon,\theta}\}.$$

As explained in the proof of Claim 2, γ is the limit of the curves γ_ε as $\varepsilon\to0$, and since γ is strictly convex there are exactly two points in γ at which the tangent lines to γ are parallel to $(-\sin\theta,\cos\theta,0)$. As for every $\varepsilon\in(0,\varepsilon_0]$ the points of γ_ε with that property are exactly $\{p_{\varepsilon,\theta},p_{\varepsilon,\theta+\pi}\}$, which lie respectively in β_θ and $\beta_{\theta+\pi}$, we can deduce that the regular real analytic curve β_θ extends continuously to the boundary of $\mathcal{L}(\Gamma_R)$. Now, since $\mathcal{L}(u,v)$ is an immersion not only in Γ_R but also in $\Gamma_R\cup\mathbb{R}$, and since by definition $\beta_\theta=\mathcal{L}(\delta_\theta)$, we deduce that δ_θ extends continuously to $\Gamma_R\cup\mathbb{R}$ for every $\theta\in[0,2\pi)$.

That is, for every $\theta\in[0,2\pi)$ there is a unique $u_\theta\in[0,2\pi)$ such that $\delta_\theta\cup\{(u_\theta,0)\}$ is a continuous curve in $\Gamma_R\cup\mathbb{R}$.

We can now finish the proof that $\gamma(u)$ is regular. Assume that $\gamma'(\overline{u_0})=(0,0)$ for some $\overline{u_0}\in[0,2\pi)$. Choose $\theta\in[0,2\pi)$ such that $\overline{u_0}\notin\{u_\theta,u_{\theta+\pi},u_{\theta\pm\pi/2}\}$, and define

$$x_\theta(u,v):=\cos\theta x(u,v)+\sin\theta y(u,v),\qquad y_\theta(u,v):=-\sin\theta x(u,v)+\cos\theta y(u,v).$$

Clearly, $x_\theta(u,0)=y_\theta(u,0)=0$ for every $u\in\mathbb{R}$, and from (4.3) we get that $Dx_\theta(\overline{u_0},0)=Dy_\theta(\overline{u_0},0)=(0,0)$.

Besides, a computation using (4.4) shows that $x_\theta(u,v)$ and $y_\theta(u,v)$ satisfy the system of elliptic PDEs

$$\begin{aligned}\Delta x_\theta &= H_1((x_\theta)_u^2+(x_\theta)_v^2)+H_2((x_\theta)_u(y_\theta)_u+(x_\theta)_v(y_\theta)_v)\\ &\quad +H_3((y_\theta)_u^2+(y_\theta)_v^2)+H_4((x_\theta)_u(y_\theta)_v-(x_\theta)_v(y_\theta)_u)\\ \Delta y_\theta &= \widetilde{H}_1((x_\theta)_u^2+(x_\theta)_v^2)+\widetilde{H}_2((x_\theta)_u(y_\theta)_u+(x_\theta)_v(y_\theta)_v)\\ &\quad +\widetilde{H}_3((y_\theta)_u^2+(y_\theta)_v^2)+\widetilde{H}_4((x_\theta)_u(y_\theta)_v-(x_\theta)_v(y_\theta)_u).\end{aligned}\tag{4.12}$$

Here, the coefficients $H_i:=H_i(u,v),\widetilde{H}_i:=\widetilde{H}_i(u,v)\in C^\omega(\Gamma_R\cup\mathbb{R})$ are given in terms of the functions in (4.5) by

$$\begin{aligned}H_1 &= (h_1\circ\mathbf{z})(\cos\theta)^3+(\widetilde{h}_1\circ\mathbf{z}+h_2\circ\mathbf{z})(\cos\theta)^2\sin\theta\\ &\quad+(h_3\circ\mathbf{z}+\widetilde{h}_2\circ\mathbf{z})\cos\theta(\sin\theta)^2+(\widetilde{h}_3\circ\mathbf{z})(\sin\theta)^3,\\ H_2 &= (h_2\circ\mathbf{z})(\cos\theta)^3+(-2h_1\circ\mathbf{z}+\widetilde{h}_2\circ\mathbf{z}+2h_3\circ\mathbf{z})(\cos\theta)^2\sin\theta\\ &\quad-(2\widetilde{h}_1\circ\mathbf{z}+h_2\circ\mathbf{z}-2\widetilde{h}_3\circ\mathbf{z})\cos\theta(\sin\theta)^2-(\widetilde{h}_2\circ\mathbf{z})(\sin\theta)^3,\\ H_3 &= (h_3\circ\mathbf{z})(\cos\theta)^3+(\widetilde{h}_3\circ\mathbf{z}-h_2\circ\mathbf{z})(\cos\theta)^2\sin\theta\\ &\quad+(h_1\circ\mathbf{z}-\widetilde{h}_2\circ\mathbf{z})\cos\theta(\sin\theta)^2+(\widetilde{h}_1\circ\mathbf{z})(\sin\theta)^3,\\ H_4 &= (h_4\circ\mathbf{z})(\cos\theta)^3+(\widetilde{h}_4\circ\mathbf{z})(\cos\theta)^2\sin\theta+(h_4\circ\mathbf{z})\cos\theta(\sin\theta)^2+(\widetilde{h}_4\circ\mathbf{z})(\sin\theta)^3,\end{aligned}$$

and by

$$
\begin{aligned}
\widetilde{H}_1 &= (\widetilde{h}_1\circ \mathbf{z})(\cos\theta)^3-(h_1\circ\mathbf{z}-\widetilde{h}_2\circ\mathbf{z})(\cos\theta)^2\sin\theta\\
&\quad +(\widetilde{h}_3\circ\mathbf{z}-h_2\circ\mathbf{z})\cos\theta(\sin\theta)^2-(h_3\circ\mathbf{z})(\sin\theta)^3,\\
\widetilde{H}_2 &= (\widetilde{h}_2\circ \mathbf{z})(\cos\theta)^3-(2\widetilde{h}_1\circ\mathbf{z}+h_2\circ\mathbf{z}-2\widetilde{h}_3\circ\mathbf{z})(\cos\theta)^2\sin\theta\\
&\quad +(2h_1\circ\mathbf{z}-\widetilde{h}_2\circ\mathbf{z}-2h_3\circ\mathbf{z})\cos\theta(\sin\theta)^2+(h_2\circ\mathbf{z})(\sin\theta)^3,\\
\widetilde{H}_3 &= (\widetilde{h}_3\circ \mathbf{z})(\cos\theta)^3-(h_3\circ\mathbf{z}+\widetilde{h}_2\circ\mathbf{z})(\cos\theta)^2\sin\theta\\
&\quad +(\widetilde{h}_1\circ\mathbf{z}+h_2\circ\mathbf{z})\cos\theta(\sin\theta)^2-(h_1\circ\mathbf{z})(\sin\theta)^3,\\
\widetilde{H}_4 &= (\widetilde{h}_4\circ \mathbf{z})(\cos\theta)^3-(h_4\circ\mathbf{z})(\cos\theta)^2\sin\theta+(\widetilde{h}_4\circ\mathbf{z})\cos\theta(\sin\theta)^2-(h_4\circ\mathbf{z})(\sin\theta)^3.
\end{aligned}
$$

Once here, we can repeat the argument that we did at the beginning of the proof with $x(u,v)$ and $y(u,v)$ using (4.4), but this time applied to $x_\theta(u,v)$, $y_\theta(u,v)$, and using (4.12). In this way, we conclude that at $(\overline{u_0},0)$ the real axis is crossed transversely by another nodal curve of either $x_\theta(u,v)$ or $y_\theta(u,v)$. But now we may observe that, by definition of δ_θ, the nodal set of $x_\theta(u,v)$ in Γ_R is given by $\delta_\theta\cup\delta_{\theta+\pi}$. As we proved above, this set can be continuously extended to $\Gamma_R\cup\mathbb{R}$, and the intersection of $\mathbb{R}$ with this extended set is exactly $\{(u_\theta,0),(u_{\theta+\pi},0)\}$. In the same way, the nodal set of $y_\theta(u,v)$ in Γ_R is $\delta_{\theta+\pi/2}\cup\delta_{\theta-\pi/2}$, and its continuous extension intersects the $v=0$ axis at $\{(u_{\theta+\pi/2},0),(u_{\theta-\pi/2},0)\}$.

Since we had chosen $\theta\in[0,2\pi)$ such that $\overline{u_0}\notin\{u_\theta,u_{\theta+\pi},u_{\theta\pm\pi/2}\}$, we conclude that no nodal curve of x_θ or y_θ can cross the u-axis at $(\overline{u_0},0)$. This contradiction finishes the proof of Claim 3 and shows that $\gamma(u)$ is regular (and hence, real analytic). □

□

Remark 2. *The previous proof shows that the limit gradient γ at an isolated singularity is a regular Jordan curve that is convex. In particular, the curvature of γ does not change sign. Moreover, for every $p\in\gamma$ the intersection of γ with the tangent line of γ at p is exactly $\{p\}$.*

However, the curvature of γ can vanish at isolated points. An example of this phenomenon can be found in [**GJM2**, Remark 3].

5. Classification of isolated singularities of prescribed curvature in space forms

In this section we use the results from Sections 3 and 4 to classify the embedded isolated singularities of prescribed, analytic, positive extrinsic curvature in 3-dimensional space forms. In Subsection 5.1 we state the classification theorem. Subsections 5.2 and 5.3 provide some needed auxiliary results on graphs in warped products of constant curvature. In Subsection 5.4 we prove the classification theorem.

5.1. Definitions and statement of the classification theorem. Let $\mathbb{M}^3=\mathbb{M}^3(c)$ be a 3-dimensional Riemannian space form of constant curvature $c\in\{-1,0,1\}$, i.e. $\mathbb{M}^3=\mathbb{R}^3,\mathbb{S}^3$ or $\mathbb{H}^3$ depending on whether $c=0,1$ or -1, respectively. We will consider coordinates (x,y,z) on $\mathbb{M}^3(c)$ by making the identification $\mathbb{M}^3(c)\equiv(\Omega_c,\langle,\rangle)$ where $\Omega_c=\mathbb{R}^3$ if $c=0,1$, $\Omega_c=\{(x_1,x_2,x_3):x_1^2+x_2^2+x_3^2<4\}$ if $c=-1$, and

$$\langle,\rangle=\frac{1}{(1+\frac{c}{4}(x_1^2+x_2^2+x_3^2))^2}\,(dx_1^2+dx_2^2+dx_3^2).$$

For $c=-1$ this is the usual Poincaré ball model of $\mathbb{H}^3$. For $c=1$, the coordinates (x_1,x_2,x_3) are given after stereographic projection, and so are defined on $\mathbb{S}^3\setminus\{\text{north}\}$.

Definition 5. *Given a smooth positive function $\mathcal{K}:\mathbb{M}^3\to(0,\infty)$, by an* embedded isolated singularity of prescribed curvature $\mathcal{K}$ *in $\mathbb{M}^3$ we mean an immersion $\psi:D\setminus\{q_0\}\to\mathbb{M}^3$ from a punctured disk into $\mathbb{M}^3$ such that:*

(1) The extrinsic curvature of ψ at each point $a\in\psi(D\setminus\{q_0\})$ is given by $\mathcal{K}(a)$.
(2) ψ extends continuously but not C^1-smoothly to D.
(3) ψ is an embedding around q_0.

Let us define for such a surface $\psi:D\setminus\{q_0\}\to\mathbb{M}^3$ the following two notions:

(1) The *canonical orientation* of ψ, that is, the orientation associated to ψ for which the second fundamental form of ψ is positive definite at every point.

(2) The *limit unit normal* of ψ at the singularity $p_0 = \psi(q_0)$, i.e. the set $\sigma \subset \{v \in T_{p_0}\mathbb{M}^3 : |v| = 1\}$ given as follows: $v \in T_{p_0}\mathbb{M}^3$ lies in σ if and only if there exists a sequence $\{q_n\}_n$ in $D \setminus \{q_0\}$ converging to q_0 such that the unit normal vectors $N(q_n)$ of ψ at q_n converge to v in $T\mathbb{M}^3$.

Remark 3. *From now on we will assume without loss of generality that the singularity p_0 is placed at the origin $(0,0,0)$ in the model $(\Omega_c, \langle,\rangle)$ for $\mathbb{M}^3$ explained above. In this way, as $\langle,\rangle = dx_1^2 + dx_2^2 + dx_3^2$ at the origin, we see that $\{v \in T_{(0,0,0)}\mathbb{M}^3 : |v| = 1\}$ is canonically identified with the sphere $\mathbb{S}^2 = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$. In particular, the limit unit normal of ψ at the singularity will be regarded as a subset of $\mathbb{S}^2$.*

Once here, we can state the main result of this section.

Theorem 5. *Let $\mathcal{K} : \mathcal{O} \subset \mathbb{M}^3 \to (0, \infty)$ be a positive real analytic function defined on an open set $\mathcal{O} \subset \mathbb{M}^3$ containing a given point $p_0 \in \mathbb{M}^3$. Let $\mathcal{A}_1$ denote the class of all the canonically oriented surfaces Σ in $\mathbb{M}^3$ that have p_0 as an embedded isolated singularity, and whose extrinsic curvature at every point $a \in \Sigma \cap \mathcal{O}$ is given by $\mathcal{K}(a)$; here, we identify $\Sigma_1, \Sigma_2 \in \mathcal{A}_1$ if they overlap on an open set containing the singularity p_0.*

Then, the map that sends each surface in $\mathcal{A}_1$ to its limit unit normal at the singularity provides a one-to-one correspondence between $\mathcal{A}_1$ and the class $\mathcal{A}_2$ of regular, analytic, strictly convex Jordan curves in $\mathbb{S}^2$.

Remark 4. *By a* strictly convex *Jordan curve we mean a regular Jordan curve with the property that its geodesic curvature is non-zero at every point.*

Remark 5. *Theorem 5 generalizes* [**GHM**, Corollary 13] *(which covers the case $\mathcal{K} =$* const. *in $\mathbb{R}^3$),* [**GJM2**, Theorem 4] *(for arbitrarily prescribed analytic curvature in $\mathbb{R}^3$) and* [**GaMi**, Theorem 15] *(for flat surfaces in $\mathbb{H}^3$).*

5.2. The prescribed curvature equation in warped products. Let $\mathcal{W} \times_f \mathbb{R}$ be a three-dimensional warped product space, where $\mathcal{W} \subset \mathbb{R}^2$ is a neighborhood of a point $p_0 \in \mathcal{W}$ endowed with a conformal metric $g = \lambda(dx^2 + dy^2)$ for some function $\lambda > 0$. Then, a computation shows that the extrinsic curvature K_{ext} of an immersed graph $z = z(x, y)$ in $\mathcal{W} \times_f \mathbb{R}$ is given by the Monge-Ampère equation

$$Ar + 2Bs + Ct + rt - s^2 = E, \tag{5.1}$$

for the coefficients

$$\begin{aligned} A &= \frac{p\lambda_x}{2\lambda} - \frac{q\lambda_y}{2\lambda} - \frac{q^2 f'}{f} - \frac{f'}{2}\lambda, \quad & B &= \frac{p\lambda_y}{2\lambda} + \frac{q\lambda_x}{2\lambda} + \frac{pqf'}{f}, \\ C &= -\frac{p\lambda_x}{2\lambda} + \frac{q\lambda_y}{2\lambda} - \frac{p^2 f'}{f} - \frac{f'}{2}\lambda, \quad & E &= K_{\text{ext}}(f\lambda + p^2 + q^2)^2 - AC + B^2, \end{aligned} \tag{5.2}$$

where $p := z_x,\ q := z_y,\ r := z_{xx},\ s := z_{xy},\ t := z_{yy}$.

If we substitute the extrinsic curvature K_{ext} in (5.2) by a smooth function $\mathcal{K}(x, y, z, z_x, z_y) > 0$, then (5.1) becomes a general equation of Monge-Ampère type (3.2), which is elliptic since

$$\mathcal{D} := AC - B^2 + E = K_{\text{ext}}(f\lambda + p^2 + q^2)^2 > 0. \tag{5.3}$$

Therefore, the problem of prescribing the extrinsic curvature of graphs in warped products depends on solving a general elliptic equation of Monge-Ampère type.

We focus next on the three-dimensional space forms $\mathbb{M}^3 = \mathbb{M}^3(c)$, as explained in Subsection 5.1. It is well known that these spaces admit several expressions as warped product manifolds, see e.g. [**GJM**] for some of them. Here, we will use the following ones:

(i) The hyperbolic 3-space admits *cylindrical coordinates* given by the warped product model

$$\left(\mathbb{D}_2 \times \mathbb{R}, \frac{\cosh^2(z)}{(1 - \frac{1}{4}(x^2 + y^2))^2}(dx^2 + dy^2) + dz^2\right), \tag{5.4}$$

where $\mathbb{D}_2 = \{(x, y) : x^2 + y^2 < 4\}$. In this model, the slices $z =$ constant are totally geodesic *parallel* hyperbolic planes, while the vertical lines are geodesics orthogonal to these totally geodesic planes. Clearly, we can build this model with respect to any given totally geodesic hyperbolic plane of $\mathbb{H}^2$.

In this model, the graphs $z = z(x, y)$ correspond to geodesic graphs over totally geodesic hyperbolic planes in the usual sense.

(ii) Analogously to model (i), we can construct for the 3-sphere $\mathbb{S}^3$ a rotationally invariant warped product model, as
$$\mathbb{S}^3 \setminus \{\text{north}, \text{south}\} \equiv (\mathbb{S}^2 \times (-\pi/2, \pi/2), \cos^2(z) g_{\mathbb{S}^2} + dz^2),$$
where $g_{\mathbb{S}^2}$ is the standard metric of $\mathbb{S}^2$. Again, this model can be built with respect to any totally geodesic sphere in $\mathbb{S}^3$. Note that after stereographic projection from $\mathbb{S}^2$ into $\mathbb{R}^2$, this model can be written in coordinates as
$$\left(\mathbb{R}^2 \times (-\pi/2, \pi/2), \frac{\cos^2(z)}{(1 + \frac{1}{4}(x^2 + y^2))^2} (dx^2 + dy^2) + dz^2\right), \tag{5.5}$$
on the complement of one half of a great circle in $\mathbb{S}^3 \setminus \{\text{north}, \text{south}\}$. In these coordinates, as happened in (i), the graphs $z = z(x, y)$ correspond to geodesic graphs over totally geodesic spheres of $\mathbb{S}^3$ in the usual sense.

(iii) In Euclidean space $\mathbb{R}^3$, any choice of orthogonal coordinates (x, y, z) trivially provide warped product coordinates, as
$$\left(\mathbb{R}^2 \times \mathbb{R}, dx^2 + dy^2 + dz^2\right). \tag{5.6}$$

Therefore, graphs of positive extrinsic curvature in a space form endowed with one of these warped metrics will satisfy (5.1)-(5.2).

Let us observe that a surface Σ in $\mathbb{M}^3$ is a geodesic graph over some totally geodesic surface $M^2 \subset \mathbb{M}^3$ (i.e. M^2 is a plane in $\mathbb{R}^3$, a totally geodesic $\mathbb{H}^2$ in $\mathbb{H}^3$ or a totally geodesic $\mathbb{S}^2$ in $\mathbb{S}^3$) if and only if Σ can be written as a graph $z = z(x, y)$ for some coordinates (x, y, z) as in (5.4), (5.5) or (5.6), with the surface M^2 corresponding in these coordinates to the slice $z = 0$.

5.3. Embedded isolated singularities and graphs. Let Σ be an embedded isolated singularity of positive extrinsic curvature in $\mathbb{M}^3 = \mathbb{M}^3(c)$, and let $p_0 \in \mathbb{M}^3$ denote the singular point of Σ.

Lemma 4. *Σ is, around p_0, a geodesic graph over some totally geodesic surface $M^2 \subset \mathbb{M}^3$.*

Proof. If $c = 0$ (i.e. $\mathbb{M}^3 = \mathbb{R}^3$), the result was proved in [**GaMi**, Theorem 13]. When $c \neq 0$ (i.e. $\mathbb{M}^3 = \mathbb{S}^3$ or $\mathbb{H}^3$), the result can be reduced to the $c = 0$ case. Indeed, recall first of all the classical result that there exist totally geodesic embeddings from $\mathbb{H}^3$ and the hemisphere $\mathbb{S}^3_+$ into $\mathbb{R}^3$. These embeddings preserve geodesics and convexity. In particular, they preserve the properties of being an embedded isolated singularity of positive extrinsic curvature, and of being a geodesic graph over some totally geodesic surface. Therefore, the result in $\mathbb{H}^3$ and $\mathbb{S}^3$ follows from the result in $\mathbb{R}^3$. □

The previous lemma shows that Σ can be viewed around p_0 as a graph $z = z(x, y)$ with respect to a system of coordinates (x, y, z) of $\mathbb{M}^3$ as in (5.4), (5.5) or (5.6) (depending on whether $c = -1$, $c = 1$ or $c = 0$, respectively). We may also assume that the singularity p_0 corresponds to $(0, 0, 0)$ in these coordinates.

We next establish how to relate explicitly the limit unit normal $\sigma \subset \mathbb{S}^2$ of Σ at p_0 with the limit gradient $\gamma \subset \mathbb{R}^2$ of Σ at the origin, when Σ is viewed as a graph $z = z(x, y)$ as explained above. Specifically, with the terminology above, we prove:

Lemma 5. *$\sigma \subset \mathbb{S}^2$ is a regular analytic Jordan curve of non-vanishing geodesic curvature if and only if $\gamma \subset \mathbb{R}^2$ is so.*

Proof. Consider the model $(\Omega_c, \langle,\rangle) \equiv \mathbb{M}^3(c)$ for $\mathbb{M}^3$ explained in Subsection 5.1.

Let $\Psi : \Omega_c \subset \mathbb{R}^3 \to \mathbb{R}^3$ be the map defining the change of coordinates from (x_1, x_2, x_3) to coordinates (x, y, z) as in (5.4), (5.5) or (5.6), for which Σ is a graph $z = z(x, y)$. By writing $\Psi(x_1, x_2, x_3) = (x, y, z)$ we observe that $\Psi(0, 0, 0) = (0, 0, 0)$, and that there exists a positively oriented orthonormal basis $\{e_1, e_2, e_3\}$ of $T_{(0,0,0)}\Omega_c \equiv \mathbb{R}^3$ such that
$$d\Psi_{(0,0,0)}(e_1) = (1, 0, 0), \qquad d\Psi_{(0,0,0)}(e_2) = (0, 1, 0) \qquad d\Psi_{(0,0,0)}(e_3) = (0, 0, 1). \tag{5.7}$$

Note that in the (x,y,z) coordinates Σ is a graph $z=z(x,y)$ with an isolated singularity at the origin. Also, observe that the warped metric in (5.4) (resp. (5.5), (5.6)) can be written as

$$f(z)\lambda(x,y)(dx^2+dy^2)+dz^2 \tag{5.8}$$

for adequate positive functions $f(z)$, $\lambda(x,y)$ depending on the value of c. From here and with this notation, a computation shows that the unit normal N to Σ in these coordinates is given by

$$N(x,y)=\frac{(-z_x,-z_y,f\lambda)}{\sqrt{f^2\lambda^2+f\lambda(z_x^2+z_y^2)}}. \tag{5.9}$$

From this formula, and since $f(0)=\lambda(0,0)=1$, we clearly see that a vector $w=(w_1,w_2)\in\mathbb{R}^2$ is contained in the limit gradient $\gamma\subset\mathbb{R}^2$ of $z(x,y)$ at the singularity if and only if the vector given in coordinates (x,y,z) by

$$\frac{1}{\sqrt{1+w_1^2+w_2^2}}\,(-w_1,-w_2,1)$$

is contained in the limit unit normal of Σ at the singularity. Using now (5.7) we deduce that the limit unit normal $\sigma\subset\mathbb{S}^2$ in the original (x_1,x_2,x_3) coordinates for $\mathbb{M}^3$ is given by the set of points $a\in\mathbb{S}^2$ of the form

$$a=\frac{1}{\sqrt{1+w_1^2+w_2^2}}\left(-w_1e_1-w_2e_2+e_3\right), \tag{5.10}$$

where $w=(w_1,w_2)$ is a point of the limit gradient $\gamma\subset\mathbb{R}^2$.

Finally, it is well known (and also easy to prove by a direct computation) that a curve $\gamma(t)=(\alpha(t),\beta(t))$ in $\mathbb{R}^2$ is regular and has positive (resp. negative) geodesic curvature if and only if the curve $\sigma(t)\subset\mathbb{S}^2$ given for some positively oriented orthonormal basis $\{e_1,e_2,e_3\}$ of $\mathbb{R}^3$ by

$$\sigma(u)=\frac{1}{\sqrt{1+\alpha(u)^2+\beta(u)^2}}\,(-\alpha(u)e_1-\beta(u)e_2+e_3) \tag{5.11}$$

is regular and has positive (resp. negative) geodesic curvature in $\mathbb{S}^2$. This fact together with the relation (5.10) proves Lemma 5.

$\square$

5.4. Proof of the classification theorem. In this subsection we will prove Theorem 5, i.e. we will show that the map which sends each $\Sigma\in\mathcal{A}_1$ to its limit unit normal $\sigma\subset\mathbb{S}^2$ is a bijective correspondence between $\mathcal{A}_1$ and $\mathcal{A}_2$.

To start, let $\Sigma\in\mathcal{A}_1$, i.e. Σ is a canonically oriented embedded isolated singularity in $\mathbb{M}^3$ of prescribed analytic curvature $\mathcal{K}>0$. We assume that the singularity p_0 is placed at $(0,0,0)$ in the canonical coordinates (x_1,x_2,x_3) for $\mathbb{M}^3$ explained in Subsection 5.1, and we denote by σ the limit unit normal to Σ at the singularity.

The key point of the proof is the following claim:

Claim 4. *In the conditions above, σ is a regular, analytic, strictly convex Jordan curve in $\mathbb{S}^2$, i.e. $\sigma\in\mathcal{A}_2$.*

Proof of Claim 4. As explained in Subsection 5.3, the surface Σ can be seen around $(0,0,0)$ as a graph $z=z(x,y)$ with an isolated singularity at the origin with respect to the coordinates (x,y,z) in (5.8). Moreover, as Σ has prescribed extrinsic curvature $\mathcal{K}$, it follows that $z(x,y)$ satisfies the elliptic equation of Monge-Ampère type given by (5.1)-(5.2) with $K_{\rm ext}=\mathcal{K}(x,y,z(z,y))>0$.

Besides, observe that in equation (5.1)-(5.2), the functions A_p, A_q+2B_p, C_p+2B_q and C_q do not depend on p and q. By Lemma 1, this implies that $z(x,y)$ satisfies the HeB-condition in Definition 3. Hence, we are in the conditions to use our analytic study of Section 4. In particular, we can parametrize Σ around $(0,0,0)$ with respect to the coordinates (x,y,z) as an analytic map

$$\psi(u,v)=(x(u,v),y(u,v),z(u,v)) \tag{5.12}$$

defined on a quotient strip $\Gamma_R=\{w\in\mathbb{C}:0<{\rm Im}w<R\}/(2\pi\mathbb{Z})$, so that ψ extends analytically to $\mathbb{R}$ with $\psi(u,0)=(0,0,0)$. Here, the coordinates (u,v) are conformal for the Riemannian metric εds^2 given by (3.5) in terms of the coefficients A,B,C in (5.2).

From the expression (5.8) for the metric of $\mathbb{M}^3$ in the (x,y,z) coordinates we see that the unit normal of ψ is given in these coordinates by

$$N(u,v)=\frac{(-p(u,v),-q(u,v),f\lambda)}{\sqrt{f^2\lambda^2+f\lambda(p(u,v)^2+q(u,v)^2)}}, \tag{5.13}$$

where $f(u,v):=f(z(u,v))$ and $\lambda(u,v):=\lambda(x(u,v),y(u,v))$.

By Lemma 3, ψ,N are C^ω in $\Gamma_R\cup\mathbb{R}$. Moreover, as $f(u,0)=\lambda(u,0)=1$ we see that

$$N(u,0)=\frac{(-\alpha(u),-\beta(u),1)}{\sqrt{1+\alpha(u)^2+\beta(u)^2}},$$

where we are denoting $\alpha(u)=p(u,0)$, $\beta(u)=q(u,0)$. In particular, as the limit gradient at the singularity is given by $\gamma(u)=(\alpha(u),\beta(u))$, it follows from Theorem 3 that $N(u,0)$ is a 2π-periodic regular analytic curve in $\{(x,y,z):x^2+y^2+z^2=1\}$.

At this point, a straightforward computation using the expression (5.13) for the unit normal of $\psi(u,v)$ shows that the metric εds^2 is conformally equivalent to the second fundamental form II of $\psi(u,v)$. Note that II is positive definite since Σ is canonically oriented. Thus, if we write $w=u+iv$, we can express the first and second fundamental forms of $\psi(u,v)$ as

$$\left\{\begin{array}{rcl} I=\langle d\psi,d\psi\rangle & = & Q\,dw^2+2\mu|dw|^2+\overline{Q}d\overline{w}^2,\\ II=-\langle d\psi,dN\rangle & = & 2\rho|dw|^2,\end{array}\right. \tag{5.14}$$

where $Q:\Gamma_R\cup\mathbb{R}\to\mathbb{C}$ is given by $Q:=\langle\psi_w,\psi_w\rangle$ (here we are denoting $\partial_w:=(\partial_u-i\partial_v)/2$), and $\mu,\rho:\Gamma_R\cup\mathbb{R}\to(0,\infty)$ are a pair of positive real functions. Note that Q,μ,ρ are C^ω in $\Gamma_R\cup\mathbb{R}$ since ψ,N have that property.

By (5.3), the extrinsic curvature $K_{\rm ext}$ of $\psi(u,v)$ is

$$K_{\rm ext}(u,v)=\frac{\mathcal{D}\circ\mathbf{z}(u,v)}{(f\lambda+p(u,v)^2+q(u,v)^2)^2},$$

where $\mathbf{z}(u,v)$ is given by (4.2). In particular, $K_{\rm ext}(u,v)$ extends analytically as a positive function to $\Gamma_R\cup\mathbb{R}$.

Denote $K:=K_{\rm ext}(u,v)$, and let $\times$ be the exterior product associated to the warped metric $\langle,\rangle$ in (5.8). Then, a direct calculus using (5.13) shows that

$$N\times N_u=-\sqrt{K}\psi_v,\qquad N\times N_v=\sqrt{K}\psi_u, \tag{5.15}$$

hold in $\Gamma_R\cup\mathbb{R}$. Therefore,

$$\begin{array}{rcl} Q(u,0) & = & \frac{1}{4}\left(\langle\psi_u,\psi_u\rangle-\langle\psi_v,\psi_v\rangle-2i\langle\psi_u,\psi_v\rangle\right)(u,0)\\ & = & -\frac{1}{4}\langle\psi_v,\psi_v\rangle(u,0)=\frac{-1}{4K}\langle N\times N_u,N\times N_u\rangle(u,0)\\ & = & \frac{-1}{4K}\langle N_u,N_u\rangle(u,0).\end{array} \tag{5.16}$$

Recall that $N(u,0)$ is a regular curve. Then, by choosing a smaller $R>0$ if necessary, we may assume that Q never vanishes on $\Gamma_R\cup\mathbb{R}$. Hence, since $K=\det(II)/\det({\rm I})$ on Γ_R, we obtain from (5.14) that

$$\rho^2=K(\mu^2-|Q|^2)\qquad\text{in }\Gamma_R\cup\mathbb{R}. \tag{5.17}$$

This shows the existence of a real analytic function ω in $\Gamma_R\cup\mathbb{R}$ such that $\mu=|Q|\cosh\omega$ and $\rho=\sqrt{K}|Q|\sinh\omega$. We observe that $\omega>0$ on Γ_R and $\omega(u,0)=0$ for every $u\in\mathbb{R}$. In particular, (5.14) can be rewritten as

$$\left\{\begin{array}{rcl} I=\langle d\psi,d\psi\rangle & = & Q\,dw^2+2|Q|\cosh\omega|dw|^2+\overline{Q}d\overline{w}^2,\\ II=-\langle d\psi,dN\rangle & = & 2\sqrt{K}|Q|\sinh\omega|dw|^2.\end{array}\right. \tag{5.18}$$

If $c=0$, it was shown in [**Bob**], pp. 118-119 (see also [**GJM2**]) that the Gauss-Codazzi equations for ψ imply that the function ω satisfies

$$\omega_{w\overline{w}}+U_{\overline{w}}-V_w+K|Q|\sinh\omega=0, \tag{5.19}$$

where

$$U=\frac{-K_{\overline{w}}Q}{4K|Q|}\sinh\omega, \qquad V=\frac{K_w\overline{Q}}{4K|Q|}\sinh\omega. \tag{5.20}$$

When $c\neq 0$ the Codazzi equation associated to (5.18) does not vary, while the Gauss equation gives $K_{\rm ext}+c=K_G$ where K_G is the Gaussian curvature of I. From here and (5.19), we easily see that for any $c\in\{-1,0,1\}$, the function ω verifies

$$\omega_{w\overline{w}}+U_{\overline{w}}-V_w+(K+\varepsilon)|Q|\sinh\omega=0, \tag{5.21}$$

where U,V are given by (5.20). Once here, we see that (5.21) is an elliptic PDE of the type

$$\Delta\omega+a_1\,\omega_u\cosh\omega+a_2\,\omega_v\cosh\omega+a_3\sinh\omega=0, \tag{5.22}$$

where $a_i=a_i(u,v)\in C^\omega(\Gamma_R\cup\mathbb{R})$. Also, note that $\omega=0$ is a solution to (5.22).

Denote $\eta(u):=N(u,0)$, and observe that the exterior product $\times$ of $\mathbb{M}^3$ at the origin is just the usual vector product of $\mathbb{R}^3$ in (x,y,z) coordinates, since the metric (5.8) at the origin is written as $dx^2+dy^2+dz^2$. Then, by (5.15), (5.16),

$$\begin{aligned}
\langle\eta'',\eta\times\eta'\rangle &= \langle N_{uu},N\times N_u\rangle(u,0)=\sqrt{K}\langle N\times\psi_{uv},N\times N_u\rangle(u,0)\\
&= \sqrt{K}\langle\psi_{uv},N_u\rangle(u,0)=\sqrt{K}\left(\tfrac{\partial}{\partial v}(\langle\psi_u,N_u\rangle)-\langle\psi_u,N_{uv}\rangle\right)(u,0)\\
&= \sqrt{K}\tfrac{\partial}{\partial v}(\langle\psi_u,N_u\rangle)(u,0)=-2K|Q|\omega_v\cosh\omega(u,0)=-2K|Q|\omega_v(u,0)\\
&= -\tfrac{1}{2}\langle\eta',\eta'\rangle\omega_v(u,0).
\end{aligned}$$

Therefore,

$$\omega_v(u,0)=-\frac{2\langle\eta''(u),\eta(u)\times\eta'(u)\rangle}{\langle\eta'(u),\eta'(u)\rangle}. \tag{5.23}$$

Let us recall at this point that the real axis is a nodal curve of ω. Since ω is a solution to the elliptic PDE (5.22), by Theorem † in [**HaWi**] we deduce that, at the points $(u,0)$ where $\omega_v(u,0)=0$ there exists at least one nodal curve of ω that crosses the real axis at a definite angle. But this situation is impossible, since $\omega>0$ in Γ_R. Therefore we see that $\omega_v(u,0)>0$ for every u, and so by (5.23) we see that $\eta(u)$ is a regular, analytic Jordan curve in $\mathbb{S}^2$ of strictly negative geodesic curvature at every point.

But now, observe that $\eta(u)=N(u,0)$ is simply the expression with respect to the coordinates (x,y,z) of the limit unit normal of Σ at the singularity. Then, as explained in Subsection 5.3, the limit unit normal $\sigma\subset\mathbb{S}^2$ in the canonical initial coordinates (x_1,x_2,x_3) of $\mathbb{M}^3$ is given by

$$\sigma(u)=\eta_1(u)e_1+\eta_2(u)e_2+\eta_3(u)e_3$$

for some positively oriented orthonormal basis $\{e_1,e_2,e_3\}$ of $\mathbb{R}^3$. Thus, $\sigma(u)$ is also a regular analytic Jordan curve in $\mathbb{S}^2$ of strictly negative geodesic curvature at every point. This proves Claim 4. □

Claim 4 shows that the mapping sending each $\Sigma\in\mathcal{A}_1$ to its limit unit normal $\sigma\subset\mathbb{S}^2$ is a well defined mapping from $\mathcal{A}_1$ to $\mathcal{A}_2$. So, in order to prove Theorem 5 it remains to check that this map is bijective.

Surjectivity is a consequence of [**GJM2**, Corollary 1], as follows.

Consider $\sigma\in\mathcal{A}_2$, parametrized as a regular, analytic 2π-periodic curve $\sigma(u):\mathbb{R}/2\pi\mathbb{Z}\to\mathbb{S}^2$ of negative geodesic curvature. Let $\{e_1,e_2,e_3\}$ be a positively oriented orthonormal frame of $\mathbb{R}^3$ for which $\sigma(u)$ can be written as (5.11) for some $\alpha(u),\beta(u):\mathbb{R}/2\pi\mathbb{Z}\to\mathbb{R}$. It follows then from the last part of the proof of Lemma 5 that $\gamma(u):=(\alpha(u),\beta(u))$ is a regular, analytic 2π-periodic curve of negative geodesic curvature in $\mathbb{R}^2$.

Consider coordinates (x, y, z) on $\mathbb{M}^3$ as in (5.8) associated to the basis $\{e_1, e_2, e_3\}$ (i.e. so that (5.7) holds), and the elliptic equation of Monge-Ampère type (5.1)-(5.2) with $K_{\rm ext} = \mathcal{K}(x, y, z) > 0$. This equation has analytic coefficients. Therefore, item (4) in [**GJM2**, Theorem 3] ensures that there is a solution $z(x, y)$ to this equation with an isolated singularity at the origin, and whose limit gradient at the singularity is the curve γ. Moreover, $z_{xx} + C > 0$ holds, which means that the second fundamental form of the graph $z = z(x, y)$ is positive definite with respect to its usual orientation. In other words, the graph $z = z(x, y)$ in $\mathbb{M}^3$ is a canonically oriented embedded isolated singularity Σ of prescribed curvature $\mathcal{K}$ around $(0, 0, 0)$. Moreover, it follows then from the constructive procedure in the proof of Lemma 5 that the limit unit normal of Σ at the singularity is precisely the curve $\sigma \subset \mathbb{S}^2$ we started with. This proves that the considered map $\mathcal{A}_1 \to \mathcal{A}_2$ is surjective.

We only have left to prove injectivity. Let $\Sigma_1, \Sigma_2 \in \mathcal{A}_1$ with the same limit unit normal $\sigma \subset \mathbb{S}^2$ at the singularity $(0, 0, 0)$. By the results in Subsection 5.3 it follows that there exist coordinates (x, y, z) in $\mathbb{M}^3$ as in (5.8) so that Σ_1, Σ_2 are graphs $z = z_1(x, y), z = z_2(x, y)$ with an isolated singularity at the origin, and with the same limit gradient γ at the singularity. As both z_1, z_2 satisfy the elliptic equation of Monge-Ampère type (5.1)-(5.2) with $K_{\rm ext} = \mathcal{K}(x, y, z) > 0$, it follows that both of them also satisfy the HeB-condition (by Lemma 1). Therefore, both z_1, z_2 are in the conditions of our analysis in Section 4. In particular, we can consider for both z_1, z_2 the associated maps $\mathbf{z}_1(u, v)$, $\mathbf{z}_2(u, v)$ given by (4.2). Note that

$$\mathbf{z}_1(u, 0) = (0, 0, 0, \alpha_1(u), \beta_1(u)), \qquad \mathbf{z}_2(u, 0) = (0, 0, 0, \alpha_2(u), \beta_2(u)),$$

where $(\alpha_1(u), \beta_1(u))$ and $(\alpha_2(u), \beta_2(u))$ are regular parametrizations of γ. Also note that we showed in our proof above of the fact that the map $\mathcal{A}_1 \to \mathcal{A}_2$ is well defined that the parametrization of the limit gradient given by $(p(u, 0), q(u, 0))$ is oriented so that it has negative curvature. Thus, both $(\alpha_i(u), \beta_i(u))$, $i = 1, 2$, have this property. So, up to an orientation preserving reparametrization of one of them, which simply means a 2π-periodic conformal reparametrization of the parameters (u, v) in Γ_R (by definition, the parameters (u, v) were defined up to this type of conformal reparametrization), we get

$$\mathbf{z}_1(u, 0) = \mathbf{z}_2(u, 0) = (0, 0, 0, \alpha(u), \beta(u)),$$

where $(\alpha(u), \beta(u))$ is a regular, analytic parametrization of γ. It also follows from (4.3) and (4.7) that $(\mathbf{z}_1)_v(u, 0) = (\mathbf{z}_2)_v(u, 0)$. Hence, both $\mathbf{z}_1, \mathbf{z}_2$ are solutions to the analytic elliptic system (4.9) with the same analytic initial conditions. By uniqueness of the solution to the Cauchy problem for (4.9), we get $\mathbf{z}_1(u, v) = \mathbf{z}_2(u, v)$. In particular, by looking at the first three coordinates of this equality, we get that the graphs $z = z_1(x, y)$ and $z = z_2(x, y)$ (i.e. suitable subsets of Σ_1 and Σ_2) overlap on a neighborhood of the singularity. This proves injectivity and finishes the proof of Theorem 5.

Remark 6. *Given a regular, analytic, strictly convex Jordan curve $\sigma \subset \mathbb{S}^2$ and some analytic positive function $\mathcal{K}$ on $\mathbb{M}^3$, the proof of the surjectivity of the map $\mathcal{A}_1 \to \mathcal{A}_2$ in the proof of Theorem 5 actually provides a construction process of the unique canonically oriented embedded isolated singularity Σ around $(0, 0, 0) \in \mathbb{M}^3$ with prescribed extrinsic curvature $\mathcal{K}$ which has σ as its limit unit normal at the singularity.*

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José A. Gálvez, Departamento de Geometría y Topología, Universidad de Granada, 18071 Granada, Spain

E-mail address: jagalvez@ugr.es

Asun Jiménez, Departamento de Geometria, IME, Universidade Federal Fluminense, 24.020-140 Niterói, Brazil

E-mail address: asunjg@vm.uff.br

Pablo Mira, Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, 30203 Cartagena, Murcia, Spain.

E-mail address: pablo.mira@upct.es