The Liouville equation in an annulus

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Abstract

We give the solutions to the Liouville equation in an annulus $\mathcal{A}$ of $\mathbb{R}^2$ that satisfy a certain Neumann condition on each component of $\partial\mathcal{A}$. As a consequence, we classify all the metrics of constant curvature in $\mathcal{A}$ that have constant geodesic curvature on $\partial\mathcal{A}$.

1 Introduction

In this paper we study the following elliptic Neumann problem in $\mathbb{R}^2 \equiv \mathbb{C}$:

$$\begin{cases} \Delta u + 2Ke^u = 0, & \text{in } \mathcal{A} = \{z \in \mathbb{C} : \ e^{-r\pi} < |z| < 1\}, \\ \dfrac{\partial u}{\partial \nu_1} = c_1 e^{\frac{u}{2}} + 2, & \text{on } C_1 = \{z \in \mathbb{C} : \ |z| = 1\}, \\ \dfrac{\partial u}{\partial \nu_2} = c_2 e^{\frac{u}{2}} - 2e^{r\pi}, & \text{on } C_2 = \{z \in \mathbb{C} : \ |z| = e^{-r\pi}\}. \end{cases} \qquad (P)$$

Here ν_i denotes the interior unit normal to C_i, $i = 1, 2$ respectively, and $r > 0$ is a constant. Moreover, we suppose up to dilation that $K = \{-1, 0, 1\}$.

The solutions to (P) provide conformal metrics $e^u|dz|^2$ on $\overline{\mathcal{A}}$ such that $(\mathcal{A}, e^u|dz|^2)$ has constant curvature K on $\mathcal{A}$, and constant geodesic curvature $-c_i/2$ on $C_i \subset \partial\mathcal{A}$ for $i = 1, 2$. And conversely, if Σ is a compact surface diffeomorphic to a closed annulus, and $d\sigma^2$ is a Riemannian surface of constant curvature K on Σ and constant geodesic curvature on each boundary component of $\partial\Sigma$, then $(\Sigma, d\sigma^2)$ is isometric to $(\mathcal{A}, e^u|dz|^2)$ for some solution u to (P) with adequate constants c_1, c_2 and r.

The equation $\Delta u + 2Ke^u = 0$ is called the Liouville equation. An important property of the Liouville equation is that it is conformally invariant. Actually, since

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different values of r provide annuli that are not conformally equivalent, we are considering a family of problems that are also non-conformally equivalent.

The problem of finding what are the conformal Riemannian metrics on a domain Ω having constant curvature K, and constant geodesic curvature along each boundary component of $\partial\Omega$ has been widely studied when $\Omega = \mathbb{R}^2_+$. In the case that the metric extends smoothly to the whole $\mathbb{R}$, it was fully solved by Zhang [Zha] (in the finite energy case) and Gálvez-Mira [GaMi] (in general), as an extension of previous results in [LiZha, Ou] (see also [ChLi, ChWa, HaWa]). More recently this problem has been generalized to the case when the metric has a singularity at the origin and possibly different values of the geodesic curvature on $\mathbb{R}^-$ and $\mathbb{R}^+$. This is a work of Jost, Wang and Zhou in [JWZ] and Gálvez, Mira and the author in [GJM]. The case of the metrics of constant curvature over non simply connected domains is studied in the works of Chou and Wan [ChWa] (in the case of the punctured disk) and Brito, Hounie and Leite [BHL] (in the general case). Finally, in [GaMi] the authors use the results in $\mathbb{R}^2_+$ and a lifting to the universal cover to give a complete classification of the metrics in the punctured unit disk $\mathbb{D}^*$ that have constant curvature and constant geodesic curvature on the boundary.

Our goal in this paper is to classify all the solutions of (P) and deduce for what values of K, c_1 and c_2 such solutions do exist. This classification is given as our main result in Theorem 1 which is stated and proved in Section 2. The consequences concerning the possible values of the constants c_1 and c_2 and the existence result will be given in the Lemma 2 and Corollary 1 in Section 3. Finally, we recall that discs of constant curvature with constant geodesic curvature on the boundary were classified geometrically (see for example [HaWa]). They are isometric to spherical caps, planar or hyperbolic discs, respectively if $K = 1, 0, -1$. In Section 4 we give the analogous result in the case of an annulus $\mathcal{A}$ with constant geodesic curvature on the boundary. We show that all the solutions in Theorem 1 correspond to one of the canonical geometric situations described in Section 4.

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2 Analytic description of the solutions to (P)

Theorem 1. *Any solution to (P) is given by one of the following expressions, where $z = Re^{i \arg z}$.*

1.
$$e^u = \frac{4\gamma^2\lambda^2 R^{2(\gamma-1)}}{(K\lambda^2 + |R^\gamma e^{i\gamma \arg z} - z_0|^2)^2} \tag{1}$$
with $\gamma > 0$*,* $\lambda > 0$ *and* $z_0 \in \mathbb{C}$ *such that (i) if* $K = 0$ *and* $z_0 \neq 0$ *then* $|z_0| \notin [e^{-r\gamma\pi}, 1]$ *and* $\gamma \in \mathbb{N}$*; (ii) if* $K = -1$ *and* $z_0 \neq 0$*, then* $|z_0| \notin [e^{-r\pi\gamma} - \lambda, 1 + \lambda]$ *and* $\gamma \in \mathbb{N}$ *and (iii) if* $K = -1$ *and* $z_0 = 0$*, then* $\lambda \notin [e^{-r\gamma\pi}, 1]$.

2. If $K = 0$
$$e^u = 4\lambda^2 R^{2(\gamma-1)}, \tag{2}$$
for some $\lambda > 0$*,* $\gamma \geq 0$.

3. If $K = -1$
$$e^u = \frac{4}{R^2(\lambda + 2\log R)^2}, \tag{3}$$
where $\lambda \notin [0, 2\pi r]$*, or*
$$e^u = \frac{\gamma^2}{R^2(\cos(\theta - \gamma \log R))^2} \tag{4}$$
where $0 < \gamma < 1/r$ *and* $\theta \in \mathbb{R}$ *is such that* $\pi/2 + k\pi \notin [\theta, \theta + \gamma r\pi]$ $\forall k \in \mathbb{Z}$ *and* $\cos(\theta) > 0$*, or*
$$e^u = \frac{4\gamma^2 R^{2(\gamma-1)}}{(\lambda + 2R^\gamma \cos(\theta + \gamma \arg z))^2} \tag{5}$$
with $\gamma \in \mathbb{N}$ *and* $\lambda \notin [-2, 2]$.

In order to prove Theorem 1 we need to introduce some preliminaries. For that, we will identify from now on $\mathbb{R}^2$ and $\mathbb{C}$, and write $w = s + it \equiv (s,t)$ or $z = x + iy \equiv (x,y)$ for points in the domain of a solution to the Liouville equation. We will also denote as $\mathcal{Q}(K)$ the 2-dimensional space form of constant curvature $K \in \{-1, 0, 1\}$, which will be viewed as (Σ_K, ds_K^2) where
$$\Sigma_K = \begin{cases} \overline{\mathbb{C}} & \text{if } K = 1, \\ \mathbb{C} & \text{if } K = 0, \\ \mathbb{D} \subset \mathbb{C} & \text{if } K = -1, \end{cases}$$
and ds_K^2 is the Riemannian metric on Σ_K given by
$$ds_K^2 = \frac{4|d\zeta|^2}{(1 + K|\zeta|^2)^2}. \tag{6}$$

The following classical result, mainly due to Liouville [Li] (see also [Bry, ChWa]), shows the relationship between the Liouville equation and complex analysis.

Theorem 2. *Let* $u : \Omega \subset \mathbb{R}^2 \equiv \mathbb{C} \to \mathbb{R}$ *denote a solution to* $\Delta u + 2Ke^u = 0$ *in a simply connected domain* Ω. *Then there exists a locally univalent meromorphic function* g *(holomorphic with* $1 + K|g|^2 > 0$ *if* $K \leq 0$*) in* Ω *such that*

$$u = \log \frac{4|g'|^2}{(1 + K|g|^2)^2}. \tag{7}$$

Conversely, if g *is a locally univalent meromorphic function (holomorphic with* $1 + K|g|^2 > 0$ *if* $K \leq 0$*) in* Ω, *then* (7) *is a solution to* $\Delta u + 2Ke^u = 0$ *in* Ω.

Observe that the function g in the above theorem, which is called the *developing map* of the solution, is unique up to a Möbius transformation of the form

$$g \mapsto \frac{\alpha g - \bar{\beta}}{\varepsilon \beta g + \bar{\alpha}}, \qquad |\alpha|^2 - \varepsilon |\beta|^2 = 1. \tag{8}$$

These transformations are isometries of $\mathcal{Q}(\varepsilon)$.

Remark 1. *From a geometric point of view, if* $u \in C^2(\Omega)$ *is a solution to* $\Delta u + 2Ke^u = 0$, *then its developing map* $g : \Omega \subseteq \mathbb{C} \to \Sigma_K \subseteq \overline{\mathbb{C}}$ *provides a local isometry from* $(\Omega, e^u |dz|^2)$ *to* $\mathcal{Q}(K) \equiv (\Sigma_K, ds_K^2)$, *where* ds_K^2 *is given by* (6).

Although Theorem 2 is only valid for simply connected domains, we will be able to apply formula (7) by passing to the universal cover of $\mathcal{A}$, in order to obtain the solutions of (P). The way to do this will be shown in Lemma 1. The same method is used for solving the Liouville equation in other non simply connected domains (see [BHL] and [GaMi]).

There is another holomorphic function attached to any solution u of the Liouville equation that will be important in our study. We will denote it by Q, and it is given by the formulas below, where g is the developing map of u:

$$Q := u_{zz} - \frac{1}{2} u_z^2 = \{g, z\} := \left(\frac{g_{zz}}{g_z}\right)_z - \frac{1}{2}\left(\frac{g_{zz}}{g_z}\right)^2. \tag{9}$$

Here, by definition $u_z = (u_x - i u_y)/2$ (and $g_z = g'$), and $\{g, z\}$ is the *Schwarzian derivative* of the meromorphic function g with respect to z. Observe that Q is holomorphic, i.e. it does not have poles, and it does not depend on the choice of the developing map g. We will call it the *Schwarzian map* associated to the solution u.

Lemma 1. *Solving problem* (P) *is equivalent to obtaining the solutions of*

$$\begin{cases} \Delta v + 2Ke^v = 0 & \text{in } \Gamma = \{w = s + it \in \mathbb{C} : \; 0 < \operatorname{Im} w < \pi\}, \\ \dfrac{\partial v}{\partial t} = c_1 e^{v/2} & \text{on } \mathbb{R}, \\ \dfrac{\partial v}{\partial t} = -c_2 e^{v/2} & \text{on } \mathbb{R} + \pi i, \end{cases} \tag{$\widetilde{P}$}$$

that are $(2\pi/r)$-periodic. Specifically, the solutions of (P) are given by the formula (7) where $g = \widetilde{g} \circ \Phi^{-1}$, with $\Phi : \Gamma \to \mathcal{A}$ given by $\Phi(w) = e^{irw}$, and $\widetilde{g}$ the developing map associated to $(\widetilde{P})$.

Proof. It is clear that Φ defines a conformal diffeomorphism between $\mathcal{A}$ and the quotient $\Gamma/\sim$, where $w \sim w' \Leftrightarrow w' = w + \frac{2\pi}{r}\mathbb{Z}$.

On the other hand, it is well known (see [BHL]), that if $\Phi : \Omega_2 \to \Omega_1$ is a conformal map between two domains, the solutions of the Liouville equation in Ω_2 are given by

$$v = u \circ \Phi + 2 \log |\Phi'| \tag{10}$$

where u is a solution in Ω_1. Moreover, the developing map associated to u can be written as $g(z) = \widetilde{g}(\Phi^{-1}(z))$. In general, if Φ is a covering map, g is multivalued unless Ω_1 is simply connected.

Then, to prove the Lemma, we only must check that if u is a solution of (P) then

$$v(s,t) = u(\Phi(s,t)) + 2 \log r - 2rt \tag{11}$$

is a solution of $(\widetilde{P})$ which is $(2\pi/r)$-periodic. Conversely, if v is a $(2\pi/r)$-periodic solution of $(\widetilde{P})$, then

$$u(x,y) = v(\Phi^{-1}(x,y)) - 2 \log r - \log(x^2 + y^2) \tag{12}$$

is a solution of (P). But this a simple computation, taking into account the following facts.

(i) Formula (11) comes from (10), and (12) is just its inverse.

(ii) Φ is $(2\pi/r)$-periodic and Φ^{-1} is multivalued in the following way: $\Phi^{-1}(x,y) = \Phi^{-1}(x,y) + 2\pi/r$.

(iii) It holds

$$\frac{\partial u}{\partial \nu_1} = -xu_x - yu_y \quad \text{on } C_1 \quad \text{and} \quad \frac{\partial u}{\partial \nu_2} = e^{\pi r}(xu_x + yu_y) \quad \text{on } C_2.$$

□

Proof of Theorem 1. Let $v \in \mathcal{C}^2(\overline{\Gamma})$ be a $(2\pi/r)$-periodic solution of $(\widetilde{P})$. Then, its associated Schwarzian derivative $\widetilde{Q} = v_{ww} - \frac{1}{2} v_w^2$ will be also $(2\pi/r)$-periodic. Moreover, because of the boundary conditions in $(\widetilde{P})$ we have that

$$\begin{aligned} \operatorname{Im} \widetilde{Q}(s,0) &= -\tfrac{1}{2}\left(\tfrac{c_1}{2} v_s(s,0) e^{v(s,0)/2} - \tfrac{c_1}{2} v_s(s,0) e^{v(s,0)/2}\right) = 0, \\ \operatorname{Im} \widetilde{Q}(s,\pi) &= -\tfrac{1}{2}\left(-\tfrac{c_2}{2} v_s(s,\pi) e^{v(s,\pi)/2} + \tfrac{c_2}{2} v_s(s,\pi) e^{v(s,\pi)/2}\right) = 0. \end{aligned} \tag{13}$$

So, by the Schwarz reflection principle for harmonic functions, we can extend $\operatorname{Im}\widetilde{Q}$ from $\overline{\Gamma}$ to $\mathbb{C}$ as a $(2\pi i)$-periodic function. As $\operatorname{Im}\widetilde{Q}$ is also $(2\pi/r)$-periodic, we deduce from (13) that $\operatorname{Im}\widetilde{Q}=0$.

Consequently, $\widetilde{Q}=c=$ constant for a certain $c\in\mathbb{R}$. It is well known that the solutions of the Schwarzian equation $\{\widetilde{g},w\}=c$ are of the form $\widetilde{g}(w)=\psi(w)$, if $c=0$, or $\widetilde{g}(w)=\psi(e^{\sqrt{-2c}w})$ if $c\neq 0$, where ψ is a Möbius transformation.

Thus, by Lemma 1 the developing map $g:\mathcal{A}\longrightarrow\overline{\mathbb{C}}$ associated to the solutions of (P) can be written as

$$g(z)=\frac{Az^{-i\frac{\sqrt{-2c}}{r}}+B}{Cz^{-i\frac{\sqrt{-2c}}{r}}+D},\quad \text{if } c\neq 0, \tag{14}$$

or

$$g(z)=\frac{-Ai\log z/r+B}{-Ci\ \log z/r+D},\quad \text{if } c=0, \tag{15}$$

for some $A,B,C,D\in\mathbb{C}$ with $AD-BC=1$. Here $\psi(\xi)=\frac{A\xi+B}{C\xi+D}$.

We will denote $\gamma=\sqrt{-2c}/r$ if $c<0$ or $i\gamma=\sqrt{-2c}/r$ if $c>0$. So, from (14) and (15) we obtain the following expressions.

If $c>0$

$$e^u=\frac{4\gamma^2|z^{\gamma-1}|^2}{(K|B|^2+|D|^2+(KA\overline{B}+C\overline{D})z^\gamma+(K\overline{A}B+\overline{C}D)\bar{z}^\gamma+(K|A|^2+|C|^2)|z|^{2\gamma})^2}, \tag{16}$$

if $c<0$

$$e^u=\frac{4\gamma^2|z^{-i\gamma-1}|^2}{(K|B|^2+|D|^2+(KA\overline{B}+C\overline{D})z^{-i\gamma}+(K\overline{A}B+\overline{C}D)\bar{z}^{i\gamma}+(K|A|^2+|C|^2)|z^{-i\gamma}|^2)^2}, \tag{17}$$

and if $c=0$

$$\begin{aligned}e^u=&\ 4/(r^2|z|^2(K|B|^2+|D|^2-i(KA\overline{B}+C\overline{D})\log z/r+i(K\overline{A}B+\overline{C}D)\log\bar{z}/r\\&+(K|A|^2+|C|^2)|\log z|^2/r^2)^2).\end{aligned} \tag{18}$$

We determine now which of them are valid solutions in term of the constants A,B,C,D.

Assume first of all $K|A|^2+|C|^2\neq 0$. Then, we can take

$$\lambda=\frac{1}{|\,K|A|^2+|C|^2\,|},\qquad z_0=-\frac{K\overline{A}B+\overline{C}D}{K|A|^2+|C|^2}, \tag{19}$$

so that (16), (17) and (18) yield respectively as

$$e^u=\frac{4\gamma^2\lambda^2|z^{\gamma-1}|^2}{(K\lambda^2+|z^\gamma-z_0|^2)^2}, \tag{20}$$

$$e^u = \frac{4\gamma^2\lambda^2|z^{-i\gamma-1}|^2}{(K\lambda^2 + |z^{-i\gamma} - z_0|^2)^2}, \tag{21}$$

$$e^u = \frac{4\lambda^2}{r^2|z|^2(K\lambda^2 + |-\frac{i}{r}\log z - z_0|^2)^2}. \tag{22}$$

Observe that due to the behaviour of the function $\arg(z)$ in $\mathcal{A}$, $z^{-i\gamma} = z^{-i\gamma}e^{2\pi\gamma}$. Thus, in (21), the multivaluation of the numerator cannot be compensated with the multivaluation of the denominator and so this metric is excluded. In the same way, it is easy to see that (22) is never well defined in $\mathcal{A}$. Hence we also exclude it. On the other hand, (20) is well defined only when we are in one of the following cases.

- If $z_0 = 0$ and $K\lambda^2 + |z^\gamma|^2 \neq 0$. The last condition is always satisfied in $\mathcal{A}$ if $K = 1, 0$. In the case $K = -1$ it is equivalent to the condition $\lambda \notin [e^{-\pi r\gamma}, 1]$. Such solutions are always radially symmetric.
- If $z_0 \neq 0$, $\gamma \in \mathbb{N}$ and $K\lambda^2 + |z^\gamma - z_0|^2 \neq 0$. The last condition is always satisfied if $K = 1$. However, if $K = 0$ it is equivalent to the condition $|z_0| \notin [e^{-r\gamma\pi}, 1]$, and if $K = -1$, it reduces to $|z_0| \notin [e^{-r\gamma\pi} - \lambda, 1 + \lambda]$. This solutions are not radially symmetric.

Hence, we have obtained all the solutions of the first type as stated in Theorem 1.

Let us consider now the case $K|A|^2 + |C|^2 = 0$, and so it must hold $K = 0, -1$. Then, writing

$$KA\overline{B} + C\overline{D} = d, \tag{23}$$

(16) can be simplified as

$$e^u = \frac{4\gamma^2|z|^{2(\gamma-1)}}{(K|B|^2 + |D|^2 + 2|d||z|^\gamma \cos(\arg d + \gamma \arg z))^2}.$$

Because of the condition $K|A|^2 + |C|^2 = 0$ we have that $d = 0$ if $K = 0$ and $|d| = 1$ if $K = -1$. Thus, if $K = 0$, e^u is well defined if and only if $D \neq 0$. We obtain then the solutions (2) (for $\gamma > 0$). When $K = -1$ we have to impose that $\gamma \in \mathbb{N}$ and $|D|^2 - |B|^2 \notin [-2, 2]$ for e^u to be well defined. This solution correspond to the not radially symmetric solution (5).

Now, if d is as in (23) then (17) can be written as

$$e^u = \frac{4\gamma^2 e^{2(\gamma \arg z - \log|z|)}}{(K|B|^2 + |D|^2 + 2|d|e^{\gamma\arg(z)}\cos(\arg(d) - \gamma\log|z|))^2}.$$

Then it is easy to see that if $K = 0$, and so $d = 0$, the function e^u is not well defined. If $K = -1$ we need that $|B|^2 = |D|^2$ and $\pi/2 + k\pi \notin [\arg(d), \arg(d) + \gamma r\pi]$

$\forall k \in \mathbb{Z}$ (in particular $\gamma < 1/r$) in order that e^u is well defined in $\mathcal{A}$. As the condition $g(1) = \psi(1) \in \overline{\mathbb{D}}$ is necessary, we have that $\cos(\arg(d)) > 0$. We obtain then the radially symmetric solutions in (4).

Finally, from (23), the expression of (18) reduces to

$$e^u = \frac{4}{|z|^2(r(K|B|^2 + |D|^2) + 2|d|(\sin(\arg d)\log|z| + \arg z \cos(\arg d)))^2}.$$

If $K = 0$, as $d = 0$, this conformal factor is well defined provided that $D \neq 0$. Thus, calling $\lambda^2 = 1/(r|D|)^2$ we obtain the solutions in (2) (for $\gamma = 0$). If $K = -1$ we need to impose that $\arg d = \pi/2 + k\pi$ for some $k \in \mathbb{Z}$, that is, $d = (-1)^k i$, and that $(-1)^k(|D|^2 - |B|^2) \notin [0, 2\pi]$. Calling $r(|D|^2 - |B|^2)(-1)^k = \lambda$ we obtain the solutions in (3). This concludes the proof of Theorem 1.

3 Necessary and sufficient conditions for existence

The following Lemma follows from a simple computation that we omit.

Lemma 2. *Let $u \in \mathcal{C}^2(\overline{\mathcal{A}})$ be a solution to (P) given by one of the expressions (1)-(5) in Theorem 1. Then, its associated constants $c_1, c_2 \in \mathbb{R}$ are given as follows.*

- *For u as in (1),*

$$c_1 = S\frac{-K\lambda^2 - |z_0|^2 + 1}{\lambda}, \qquad c_2 = S\frac{e^{r\pi\gamma}(K\lambda^2 + |z_0|^2) - e^{-r\pi\gamma}}{\lambda},$$

  where

$$S = \begin{cases} \operatorname{sign}(1-\lambda) & \text{if } K = -1,\ z_0 = 0 \\ 1 & \text{otherwise} \end{cases}$$

- *For u as in (2), $c_1 = -\frac{\gamma}{\lambda}$ and $c_2 = \frac{e^{\pi r\gamma}\gamma}{\lambda}$.*

- *For u as in (3), $c_1 = 2\operatorname{sign}(\lambda)$, and $c_2 = -2\operatorname{sign}(\lambda)$.*

- *For u as in (4), $c_1 = 2\sin(\theta)$, and $c_2 = -2\sin(\theta + r\pi\gamma)$.*

- *For u as in (5), $c_1 = -|\lambda|$, and $c_2 = |\lambda|e^{\pi r\gamma}$.*

Now, we use Lemma 2 to deduce for which values of K, c_1 and c_2 a solution of (P) exists.

Corollary 1. *Given $c_1, c_2 \in \mathbb{R}$ there exists a solution to problem (P) if and only if*

- $K = 1$ *and* $c_1 + c_2 > 0$.
- $K = 0$ *and (i)* $c_1 + c_2 > 0$ *with* $c_i < 0$ *for some* $i = \{1, 2\}$ *, or (ii)* $c_1 = 0 = c_2$.
- $K = -1$ *and (i)* $c_1 + c_2 > 0$ *with* $c_1 < -2$ *and* $c_2 > 2$ *(or with* $c_1 > 2$ *and* $c_2 < -2$*), or (ii)* $c_1 = \pm 2$ *and* $c_2 = \mp 2$*, or (iii)* $c_1 + c_2 < 0$ *with* $0 \leq |c_i| < 2$ *for both* $i = \{1, 2\}$.

Proof. In the case $K = 1$ all the solutions are given by (1). Then, since

$$c_1 = \frac{-\lambda^2 - |z_0|^2 + 1}{\lambda}, \qquad c_2 = \frac{e^{r\pi\gamma}(\lambda^2 + |z_0|^2) - e^{-r\pi\gamma}}{\lambda}, \tag{24}$$

a simple computation shows that $c_1 + c_2 > 0$. Conversely, if we consider c_1 and c_2 such that $c_1 + c_2 > 0$, taking $z_0 = 0$ it is easy to obtain two constants $\lambda, \gamma > 0$ such that formula (24) is satisfied. Thus, for $K = 1$, the condition $c_1 + c_2 > 0$ is also sufficient for the existence of a solution. Moreover, these constants are completely determined by the conformal structure of $\mathcal{A}$ (given by r). Observe that, if the solution is given by (1), for all the values $K = 1, 0, -1$ we had the restriction $\gamma \in \mathbb{N}$ if $z_0 \neq 0$. Therefore, in such cases the choice of c_1 and c_2 will have another technical restriction in terms of the conformal structure, in order to obtain a solution.

In the case $K = 0$ we have more possibilities, since the solutions to (P) are given by either (1) or (2). If the solution is given by (1), then

$$c_1 = \frac{-|z_0|^2 + 1}{\lambda}, \qquad c_2 = \frac{e^{r\pi\gamma}|z_0|^2 - e^{-r\pi\gamma}}{\lambda}, \tag{25}$$

and because of the restrictions in Theorem 1 we deduce that $c_1 + c_2 > 0$ and that (i) if $|z_0| > 1$, then $c_1 < 0$ and $c_2 > 0$ and (ii) if $|z_0| < e^{-r\pi\gamma}$ then $c_1 > 0$ and $c_2 < 0$. Conversely, if we have c_1 and c_2 such that $c_1 + c_2 > 0$, $c_1 < 0$ and $c_2 > 0$, then we can chose $z_0 = 0$ and find $\lambda, \gamma > 0$ (unique for each fixed $r > 0$) such that (25) holds.

On the other hand, if the solution is given by (2) and $\gamma = 0$, as $c_1 = 0 = c_2$, the metric always exists, given any parameter $\lambda > 0$, for any conformal structure. If $\gamma \neq 0$ in (2), then

$$c_1 = \frac{-\gamma}{\lambda}, \qquad c_2 = \frac{e^{r\pi\gamma}\gamma}{\lambda}, \tag{26}$$

and so $c_1 + c_2 > 0$, $c_1 < 0$ and $c_2 > 0$. Hence, given c_1 and c_2 under these assumptions, we trivially find $\lambda, \gamma > 0$ such that (26) is satisfied. Looking at the solution (2) and the solution (1) when $z_0 = 0$ we see they differ by an inversion, that is, the role of c_1 and c_2 in (26) and (25) is interchanged.

If $K=-1$, the solution u to (P) can be given by formulas (1) or (3)-(5). If u is given by (1), then from Lemma 2 we know that

$$c_1 = S\frac{\lambda^2 - |z_0|^2 + 1}{\lambda}, \qquad c_2 = S\frac{e^{r\pi\gamma}(-\lambda^2 + |z_0|^2) - e^{-r\pi\gamma}}{\lambda}, \tag{27}$$

and a simple computation shows that $c_1 + c_2 > 0$. Moreover, from the restrictions in Theorem 1, we have that:

(i) if $z_0 \neq 0$, then $|z_0| > 1 + \lambda$, and so $c_1 < -2$ and $c_2 > 2$, or $|z_0| < e^{-r\pi\gamma} - \lambda$ and therefore $c_1 > 2$ and $c_2 < -2$,

(ii) if $z_0 = 0$, then either $\lambda > 1$ and so $c_1 < -2$ and $c_2 > 2$, or $\lambda < e^{-r\pi\gamma}$ and then $c_1 > 2$ and $c_2 < -2$.

Conversely, consider c_1 and c_2 such that $c_1 + c_2 > 0$ and $c_1 < -2$ (resp. $c_1 > 2$) and $c_2 > 2$ (resp. $c_2 < -2$). Then, at least for the case $z_0 = 0$, we can always find (just by solving a second order equation) parameters $\gamma > 0$ and $\lambda > 1$ (resp. $\lambda < e^{-r\pi\gamma}$) such that (27) holds.

By means of Lemma 2, if $c_1 = \pm 2$ and $c_2 = \mp 2$, we can always obtain a solution of type (3) for a convenient choice of $\lambda \notin [0, 2\pi r]$ provided that the equalities $c_1 = 2\mathrm{sign}(\lambda)$ and $c_2 = -2\mathrm{sign}(\lambda)$ are satisfied.

If the solution u is given by (4), we have

$$c_1 = 2\sin(\theta), \quad c_2 = -2\sin(\theta + r\pi\gamma), \tag{28}$$

for a certain $\theta \in \mathbb{R}$ and $\gamma > 0$ under the restrictions

$$\pi/2 + k\pi \notin [\theta, \theta + \gamma r\pi], \ \forall k \in \mathbb{Z}, \quad \cos(\theta) > 0. \tag{29}$$

Thus it is easy to deduce that $c_1 + c_2 < 0$ and that $0 \leq |c_i| < 2$ for both $i = \{1, 2\}$. Conversely, because of the behaviour of the sin and cos functions, given c_1 and c_2 under these assumptions we can always find $\theta \in \mathbb{R}$ and $\gamma > 0$ such that (29) and (28) are satisfied.

Finally, if the solution is given by (5), as

$$c_1 = -|\lambda|, \quad c_2 = |\lambda| e^{\pi r\gamma}, \tag{30}$$

we are led again to the relation $c_1 + c_2 > 0$; and since in this case $\lambda \notin [-2, 2]$, then $c_1 < -2$ and $c_2 > 2$. But it is easy to see from (30) that we have a restriction involving the conformal structure. Only when $c_1/c_2 = -e^{\pi r\gamma}$ for a certain $\gamma \in \mathbb{N}$ we will obtain solutions of type (5). Therefore, not all the conformal structures are admissible for the existence of such solutions. □

4 Classification of constant curvature annuli

As explained in the introduction, the solutions to (P) provide Riemannian annuli of constant curvature with constant geodesic curvature on each boundary component, and viceversa. Next, we use this connection to give the geometric counterpart to Theorem 1.

From a geometric point of view, there are several ways to produce constant curvature annuli with constant geodesic curvature on each boundary component, as we explain next.

(1) First of all, one has the induced metric of any annulus $\mathcal{A}'$ in $\mathcal{Q}(K)$ whose boundary consists of two disjoint circles. Observe that by composing with a finite-folded covering map of this annulus $\mathcal{A}'$ we also obtain conformal metrics in the same conditions. The conformal structure of such metrics depend on the covering number.

(2) Second, assume that $\mathcal{A}'$ is a radially symmetric annulus in $\mathcal{Q}(K)$. That is, its boundary consists of two circles C_1', C_2', and $\mathcal{A}'$ is foliated by geodesic arcs in $\mathcal{Q}(K)$ that meet both C_1' and C_2' orthogonally. Then, we may consider the sector of $\mathcal{A}'$ bounded by two of these radial geodesics, which make some angle γ, possibly greater than 2π. After identifying those geodesics, the quotient space is a topological annulus and the metric ds_K^2 restricted to $\mathcal{A}'$ projects to a well-defined metric of constant curvature K on this quotient. Hence, we obtain a conformal metric satisfying the desired conditions.

One can make similar constructions in the following cases:

(3) when $K = 0$, by considering a strip in $\mathbb{R}^2$ instead of a radially symmetric annulus, and identifying two different line segments orthogonal to the boundary of the strip.

(4) when $K = -1$ by considering the region of $\mathcal{Q}(-1) \equiv \mathbb{D}$ bounded by two horocycles with the same ideal point $p \in \mathbb{S}^1$, together with two geodesic arcs in $\mathcal{Q}(-1)$ starting at p, and identifying these arcs.

(5) when $K = -1$ by considering the region of $\mathcal{Q}(-1) \equiv \mathbb{D}$ bounded by two arcs of circle with common ideal endpoints $p_1,\ p_2 \in \mathbb{S}^1$, together with two geodesic arcs in $\mathcal{Q}(-1)$ which meet the previous two circles orthogonally and that we identify.

With this, let us prove as a consequence of Theorem 1 that these five types of Riemannian annuli provide all possible conformal metrics of constant curvature K on an annulus $\mathcal{A}$, with constant geodesic curvature on $\partial\mathcal{A}$.

Theorem 3. *Let $(\Sigma, d\sigma^2)$ be a Riemannian surface diffeomorphic to a closed annulus. Assume $d\sigma^2$ has constant curvature on Σ, and constant geodesic curvature along each boundary component of $\partial\Sigma$. Then, $(\Sigma, d\sigma^2)$ is isometric to one of the five examples of Riemannian annuli described above.*

Proof. Up to a conformal change of coordinates, we may view $(\Sigma, d\sigma^2)$ as $(\mathcal{A}, e^u|dz|^2)$ where $\mathcal{A} = \{z \in \mathbb{C} : \ e^{-r\pi} < |z| < 1\}$ for some $r > 0$, and u is a solution to (P) for some adequate constants K, c_1, c_2. We now analyze from a geometric point of view the possible expressions for u, as given by Theorem 1.

We consider first the solutions given by (1). In this case we know by the proof of Theorem 1 that the associated developing map is $g(z) = \psi(z^\gamma)$ with $\gamma > 0$ and $\psi(\xi) = \frac{A\xi+B}{C\xi+D}$ a certain Möbius transformation.

If $z_0 \neq 0$, we had the restriction $\gamma \in \mathbb{N}$. Thus, g is univalued on $\mathcal{A}$. Moreover, $g(\mathcal{A})$ is a topological annulus $\mathcal{A}'$ in $\mathcal{Q}(K)$ whose boundary consists in two circles, and the map g defines a γ-folded covering map from $\mathcal{A}$ into $\mathcal{A}'$. Thus, by Remark 1, we see that $(\mathcal{A}, e^u|dz|^2)$ is isometric to the annulus $\mathcal{A}'$ endowed with the metric ds_K^2, covered a number $\gamma \in \mathbb{N}$ of times. That is, $(\mathcal{A}, e^u|dz|^2)$ is isometric to the first type of canonical annuli of constant curvature defined before.

Now suppose we are in the case $z_0 = 0$, and so γ is not necessarily an integer. The multivalued function z^γ maps $\mathcal{A}$ into a piece of the annulus $\mathcal{B} = \{\xi \in \mathbb{C} : e^{-r\pi\gamma} < |\xi| < 1\}$ bounded by the segment $[e^{-r\pi\gamma}, 1]$ and $R_{2\pi\gamma}([e^{-r\pi\gamma}, 1])$, where R_t denotes from now on the rotation of angle t. These two segments correspond to the splitting by z^γ of the segment $[e^{-r\pi}, 1]$. Because of the condition $z_0 = 0$ we have from (19) that $K\overline{A}B + \overline{C}D = 0$. Then it is easy to prove that for each $\theta \in \mathbb{R}$, $\psi \circ R_\theta \circ \psi^{-1} = \phi$ where ϕ is an isometry of $\mathcal{Q}(K)$ described in (8). That is, for each $\theta \in \mathbb{R}$, $g(e^{i\theta}z) = \phi_\theta(g(z))$ for a certain isometry of $\mathcal{Q}(K)$, ϕ_θ. On the other hand, observe that this kind of metrics coincide with one of the canonical solutions that solve the Newmann problem in $\mathbb{R}^2_+$, given by formula (5) in [GJM]. Thus, we have from Lemma 1 in [GJM] that since $[e^{-r\pi}, 1] \subset \mathbb{R}_+$ then $g([e^{-r\pi}, 1])$ is a geodesic arc in $\mathcal{Q}(K)$. Hence, $g(\mathcal{A})$ is a piece of an annulus $\mathcal{A}'$ which is radially symmetric, that is, foliated by geodesic arcs meeting $\partial\mathcal{A}'$ orthogonally. These geodesic arcs are the image of the segments orthogonal to $\partial\mathcal{A}$ that foliate $\mathcal{A}$. Such a piece, $g(\mathcal{A})$, is bounded by two of those geodesic arcs that make an angle $2\pi\gamma > 0$, where γ can be greater than 1. As we explained before, they correspond to the splitting of the segment $[e^{-r\pi}, 1]$. We see then that $(\mathcal{A}, e^u|dz|^2)$ is isometric to the domain $g(\mathcal{A})$, where the the extremal geodesic arcs are identified, endowed with the projection of ds_K^2. Thus, these solutions correspond to the canonical annuli of type (2) mentioned before.

Consider now the case $K = 0$ when the solution is given by (2) with $\gamma \neq 0$. We have again, by the proof of Theorem 1, that $g(z) = \psi(z^\gamma)$ where now the Möbius transformation $\psi(\xi) = \frac{A\xi+B}{C\xi+D}$ satisfies that $C = 0$, that is, it is the composition of a dilation with an isometry of $\mathcal{Q}(0) \equiv \mathbb{R}^2$. Thus, $g(\mathcal{A})$ lies in an annulus $\mathcal{A}'$, which is radially symmetric in $\mathcal{Q}(0)$, and the image of the segments orthogonal to $\partial\mathcal{A}$ that foliate $\mathcal{A}$ are segments (and so geodesic arcs) in $\mathcal{A}'$ which are orthogonal to $\partial\mathcal{A}'$. As in the case of the solutions of type (1), $g(\mathcal{A})$ is a portion of such an annulus $\mathcal{A}'$ delimited by two of those segments which correspond to the split of the segment $[e^{-r\pi}, 1]$ and that make an angle $2\pi\gamma$, possibly greater than 2π. Hence, we are lead again in the case of annulus of type (2).

If $\gamma = 0$ in the formula (2) we know that $g(z) = \psi(-\frac{i}{r}\log z)$ where, as before, ψ is the composition of a dilation with an isometry of $\mathcal{Q}(0)$. The multivalued function $-\frac{i}{r}\log z$ maps $\mathcal{A}$ into the strip $\Gamma = \{w \in \mathbb{C} : \ 0 < \text{Im}w < \pi\}$ where the segment $[e^{-\pi r}, 1]$ splits into the vertical segments $S_1 = \{\xi \in \overline{\Gamma} : \ \text{Re}\xi = 0\}$ and $S_2 = \{\xi \in \overline{\Gamma} : \ \text{Re}\xi = 2\pi/r\}$. So, $g(\mathcal{A})$ is a piece of the strip $\psi(\Gamma)$ bounded by the segments $\psi(S_1)$ and $\psi(S_2)$. This solution makes $(\mathcal{A}, e^u|dz|^2)$ isometric to the domain $g(\mathcal{A})$, where $\psi(S_1)$ and $\psi(S_1)$ are identified, endowed with the projection of ds_0^2. Thus, it corresponds to the annulus of type (3) described before.

Assume next that $K = -1$ and the solution is given by formula (3). Then the developing map associated to it is $g(z) = \psi(-\frac{i}{r}\log z)$ where $\psi(\xi) = \frac{A\xi+B}{C\xi+D}$ is a Möbius transformation satisfying $|A| = |C|$, i.e. it maps the point of infinity into a point $p = \frac{A}{C} \in \partial\mathbb{D}$. Thus, we deduce that $g(\mathcal{A})$ lies in $\psi(\Gamma)$, which is the region limited by two horocycles $\mathcal{C}_1$ and $\mathcal{C}_2$ that are tangent at p. Observe also that the image by ψ of the vertical segments foliating Γ will be arcs of curves that start at p and which are orthogonal to both $\mathcal{C}_1$ and $\mathcal{C}_2$. Hence they are geodesic arcs that foliate the region between $\mathcal{C}_1$ and $\mathcal{C}_2$. Two of those geodesic arcs, corresponding to the splitting of the segment $[e^{-r\pi}, 1]$, are identified to obtain the quotient which, with the projected metric ds_{-1}^2, is isometric to $(\mathcal{A}, e^u|dz|^2)$. These solutions correspond with the annuli of type (4) mentioned before.

If $K = -1$ and the solution is given by the formula (4), then the associated developing map is $g(z) = \psi(z^{-i\gamma})$ where $\gamma < 1/r$ and the Möbius transformation $\psi(\xi) = \frac{A\xi+B}{C\xi+D}$ is such that $|A| = |C|$ and $|B| = |D|$. Note that the multivalued function $z^{-i\gamma}$ maps C_1 into the segment $S_1 = [1, e^{2\pi\gamma}]$ and C_2 into its rotated $S_2 = R_{\pi\gamma r}([1, e^{2\pi\gamma}])$. And the two arcs of circle centered at the origin with radii 1 and $e^{2\pi\gamma}$ respectively that join the endpoints of S_1 and S_2, correspond to the splitting by the function $z^{-i\gamma}$ of the segment $[e^{-r\pi}, 1]$. Thus $\mathcal{A}$ is mapped by $z^{-i\gamma}$ into the region delimited by S_1, S_2 and this two arcs of circle. On the other hand, it is easy to check that ψ maps the line passing trough the origin corresponding to the arguments $\pi/2 - \theta$ and $-\pi/2 - \theta$ (where θ is the parameter appearing in (4)) into $\partial\mathbb{D}$. As a consequence, all the circles centered at the origin (since they are orthogonal to such a line) will be mapped by ψ into geodesics of $\mathcal{Q}(-1) \equiv \mathbb{D}$ that will foliate $g(\mathcal{A})$. By all the reasoning before, we deduce that g maps $\mathcal{A}$ into the region bounded by (i) two arcs of circles (the image of C_1 and C_2) that meet at two points $p_1 = A/C, p_2 = B/D \in \partial\mathbb{D}$ with angle $\pi\gamma r$ and, (ii) two geodesic arcs orthogonal to them. These geodesic arcs that we identify correspond to the splitting by g of the segment $[e^{-r\pi}, 1]$. Hence, $(\mathcal{A}, e^u|dz|^2)$ is isometric to this quotient of $g(\mathcal{A})$ endowed with the projection of the metric ds^2_{-1}. It is then a Riemannian annulus of type (5).

Finally, if $K = -1$ and the solution is given by formula (5), we have again $g(z) = \psi(z^\gamma)$. Now $\gamma \in \mathbb{N}$ and $\psi(\xi) = \frac{A\xi+B}{C\xi+D}$ is such that $|A| = |C|$. Thus we can deduce as before that g is a γ-folded covering map from $\mathcal{A}$ into an annulus $\mathcal{A}' \subset \mathcal{Q}(-1)$. In this case, the boundary of $\mathcal{A}'$ is intersected orthogonally by two curves with common ideal point $p = \frac{A}{C} \in \partial\mathbb{D}$. These curves are the image by g of the real and the imaginary axis. Thus, we are lead again in the case of annulus of type (1).

$\square$

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