THE SEMICONTINUITY LEMMA

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Abstract. In this note we present a proof of the well-know semicontinuity lemma in the separable case.

1. Introduction

There exists two basic topological results, elsewhere know as The Semicontinuity Lemmas, that have been widely used especially in the theory of Dynamical Systems. These results loosely says that semicontinuous maps defined on a complete metric (or topological) space and taking values in $\mathbb{R}$ or in the Hausdorff topology of some metric space (see the definitions bellow) is continuous in most points of the domain, from the topological viewpoint.

As we said, this result has been used in a variety of works of the theory of Dynamical Systems, and nowdays there exists an intire branch of this theory, called generic dynamics, dedicated to study properties of "most" systems, from the topological viewpoint. For instance, see [1], [5], [7], [3] and [2], just to cite a few. For some nice account of generic dynamics see [6].

While the main topological property of $\mathbb{R}$ used the proof of the semicontinuity lemma in its first case is separability, in the literature, see for instance [4], pages 70-71, the statement for maps taking values in the Hausdorff space requires campactness.

In this note we shall remark that compactness in the second case is not necessary, only separability as in the real case. For the sake of completeness we shall also present the proof in the real case.

Let us give the precise definitions and statements.

For simplicity we shall only deal with domains that are metric spaces. The reader is invited to provide the modifications required when the domain is only topological.

Date: February 16, 2012.

1.1. **Definition.** *Given a metric space X, a function $\Gamma : X \to \mathbb{R}$ is said to be* lower semicontinuous *in a point $x \in X$ if given $\varepsilon > 0$ there exists $\delta > 0$ such that for every $y \in X$ with $d(x, y) < \delta$ then $\Gamma(y) > \Gamma(x) - \varepsilon$. Similarly, we say that Γ is* upper semicontinuous *in x if given ε there exists $\delta > 0$ such that $\Gamma(y) < \Gamma(x) + \varepsilon$, whenever $d(x, y) < \delta$.*

When a function $\Gamma : X \to \mathbb{R}$ is lower/upper semicontinuous in every point we just say that Γ is lower/upper semicontinuous.

It is obvious that a function that is both, lower and upper semi continuous, is continuous. Recall that a set in a metric space X is called residual if it contains a countable intersection of open and dense subsets.

1.2. **Theorem** (The Semicontinuity Lemma-Real Case)**.** *Let X be a metric space and consider $\Gamma : X \to \mathbb{R}$ a lower-semicontinuous function. Then, there is a residual set of points in X where Γ is continuous.*

If M is a metric space, we denote by C_M the space of compact subsets of M endowed with the Hausdorff metric. Given A, B compact subsets of M, the Hausdorff distance between them, wich we denote by $d_H(A, B)$ is the maximum of the numbers $d_A(B)$ and $d_B(A)$, where

$$d_A(B) = \sup \{d(b, A);\ b \in B\},$$

and

$$d_B(A) = \sup \{d(a, B);\ a \in A\}.$$

Is an easy matter to show that (C_M, d_H) is a metric space.

1.3. **Definition.** *We say that a map $\Gamma : X \to C_M$ is* lower semicontinuous *in a point $x \in X$ if for every open set U that intersects $\Gamma(x)$ there exists a neighborhood $\mathcal{U}$ of x in X such that if $y \in \mathcal{U}$ then $\Gamma(y) \cap U \neq \emptyset$. We say that Γ is* upper semicontinuous *in x if for every neighborhood U of $\Gamma(x)$ there exists a neighborhood $\mathcal{U}$ of x in X such that $y \in \mathcal{U}$ implies $\Gamma(y) \subset U$.*

Note that $\Gamma : X \to C_M$ is lower semicontinuous in a point x if and only if for every $\varepsilon > 0$ there exists $\delta > 0$, such that for every $y \in X$ with $d(y, x) < \delta$ we have $d_{\Gamma(y)}(\Gamma(x)) < \varepsilon$. In a similar way, Γ is upper semicontinuous in x, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(y, x) < \delta$ implies $d_{\Gamma(x)}(\Gamma(y)) < \varepsilon$. Thus, again Γ is continuous if, and only if, it is both, lower and upper semicontinuous.

1.4. **Theorem** (The Semicontinuity Lemma-The Hausdorff Case)**.** *Let X be a metric space and M a separable metric space and consider $\Gamma : X \to C_M$ a lower-semicontinuous map. Then, there is a residual set of points in X where Γ is continuous.*

If X is complete the Baire Category Theorem implies that the set of points in wich Γ is continuous is dense.

2. Proofs

We start with a short proof of the real case. It is enough to prove that the set of points in wich Γ is not continuous is a subset of a meager set, i.e. a countable union of closed sets with empty interior. Take a point $x \in X$ in wich Γ is not continuous. By the hypoteseis, we have that Γ is not upper semicontinuous in x, so there exists an $\varepsilon > 0$ and a sequence $x_n \to x$ such that $\Gamma(x_n) \geq \Gamma(x) + \varepsilon$. We can take a rational number q such that $\Gamma(x) < q < \Gamma(x) + \varepsilon$. Define the set

$$B_q = \{a \in X;\ \Gamma(a) \leq q\}.$$

Since Γ is lower semicontinuous the comlement of B_q is open, and so B_q is closed. Note that $x \in F_q := B_q - \mathrm{Int}(B_q)$. Since F_q is a closed set with empty interior, and the set of rational numbers is countable, we are done.

A carefull look for this proof shows that it has two main insights. The first one is to note that given a point of discontinuity for the function, the lower semicontinuity together with the absense of upper semicontinuity naturally allows us to construct closed sets with empty interior that contains this point, namelly the set F_q, wich can be constructed with *any* number $\Gamma(x) < q < \Gamma(x) + \varepsilon$. The second insight is that the separability of $\mathbb{R}$ allows us choose contably many such numbers q.

Now, trying to implement the same ideia, we turn to the more envolved proof of the Hausdorff case.

Proof of the Hausdorff case. Since M separable, there is a countable dense subset $A \subset X$.

Again, it suffices to prove that the set

$$\mathcal{D} = \{x \in X; \Gamma \text{ is not continuous in } x\}$$

is contained in a meager set.

Take a point $x \in \mathcal{D}$. Then, by the hypothesis on Γ, x is not an upper-semicontinuity point of Γ. Thus, there exists $\varepsilon > 0$ such that there is a sequence $x_n \to x$ with

$$\Gamma(x_n) \cap (M - U_\varepsilon(\Gamma(x))) \neq \emptyset, \quad \text{for every } n > 0,$$

where $U_\varepsilon(\Gamma(x))$ stands for the ε-neighborhood of $\Gamma(x)$.

Since $\Gamma(x)$ is compact and disjoint from the boundary $\partial U_\varepsilon(\Gamma(x))$ of $U_\varepsilon(\Gamma(x))$, wich is a closed set, there is a number $\beta > 0$ such that

$$d(\Gamma(x), \partial U_\varepsilon(\Gamma(x))) \geq \beta.$$

Take $\delta = \frac{\beta}{10}$ and, by compactness, consider a finite cover of $\Gamma(x)$ by balls $B(y_i, \delta)$, with $y_i \in \Gamma(x)$, $i = 1, ..., l = l(x)$. In each such ball there is a point $a_i \in A$, and we have that $\Gamma(x) \subset \cup_{i=1}^{l} B(a_i, 2\delta)$. We can choose a rational number $2\delta < r < 3\delta$ in such a way that

$$B(a_i, 2\delta) \subset B(a_i, r) \subset B(a_i, 3\delta),$$

for every $i = 1, ..., l$. Then, since $3\delta < \beta$, the set $C_{l,\{a_i\},r} := \bigcup_{i=1}^{l} B(a_i, r)$ satisfies

$$\Gamma(x) \subset C_{l,\{a_i\},r} \subset U_\varepsilon(\Gamma(x)).$$

Now, we consider the set

$$B_{l,\{a_i\},r} = \{y \in X; \Gamma(y) \cap (M - C_{l,\{a_i\},r}) = \emptyset\}.$$

Note that, by the lower semi-continuity of Γ, this set is closed. Indeed, the lower semi-continuity of Γ says precisely that the complement of $B_{l,\{a_i\},r}$ is an open set. Since $\Gamma(x) \subset C_{l,\{a_i\},r}$, $x \in B_{l,\{a_i\},r}$, but, since $x_n \to x$ and $\Gamma(x_n) \cap (M - U_\varepsilon(\Gamma(x))) \neq \emptyset$, for every n, we have that $x \notin \operatorname{int} B_{l,\{a_i\},r}$.

This shows that x belongs to the closed set with empty interior

$$F_{l,\{a_i\},r} = B_{l,\{a_i\},r} - \operatorname{int} B_{l,\{a_i\},r}.$$

Moreover, the colection of sets $B_{l,\{a_i\},r}$ is countable, and thus the colectin of sets $F_{l,\{a_i\},r}$ is also countable. Indeed, for each fixed pair $l \in \mathbb{N}$ and $r \in \mathbb{Q}^+$ one can find a surjection between the colection of sets $B_{l,\{a_i\},r}$ and the product $A \times ... \times A$, with l factors, wich is a countable set. So, the whole colection of sets $B_{l,\{a_i\},l}$ can be seen as a countable union of countable sets, and therefore is countable, as claimed.

Thus, we have proved that

$$\mathcal{D} \subset \bigcup_{l,\{a_i\},r} F_{l,\{a_i\},r},$$

and the later is a meager set. This completes the proof. □

References

[1] Abdenur, F. Bonatti, C Crovisier, S. *Nonuniform hyperbolicity for C^1 generic diffeomorphisms*. Israel Journal of Mathematics 183, pp. 1-60, 2011

[2] Araújo, A. *Existência de atratores hiperbólicos para difeomorfismos de superfícies*. Tese, IMPA, 1987.

[3] Bonatti, C. Díaz, L. Viana, M. *Dynamics beyond uniform hyperbolicity. A global geometric and probabilistic perspective*. Encyclopaedia of Mathematical Sciences, 102. Mathematical Physics, III. Springer-Verlag, Berlin, 2005

[4] Kuratowski, K. *Topology II*. Academic Pres-PWN-Polishi Sci. Publishers, Warszawa, 1968.

[5] Mañé, R. *An ergodic closing lemma*. Ann. of Math (2), (1982), no. 3, v.116, 503-540.

[6] Obata, D. *Resultados na Teoria de Dinâmica Genérica*. Monografia de Iniciação Científica, UFRJ, 2010.

[7] Pujals, E. and Sambarino, M. *Homoclinic tangencies and hyperboli- city for surface diffeomorphisms*. Ann. of Math. (2) v.151, (2000), no. 3, 961-1023.

[8] Robinson, C. *Dynamical systems. Stability, symbolic dynamics, and chaos*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[9] Santiago, B. *Hiperbolicidade Essencial em Superfícies*. Dissertação de Mestrado, UFRJ, 2011.

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