EXISTENCE OF ATTRACTORS, HOMOCLINIC TANGENCIES AND SINGULAR-HYPERBOLICITY FOR FLOWS

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Abstract. We prove that every C^1 generic three-dimensional flow has either infinitely many sinks, or, finitely many hyperbolic or singular-hyperbolic attractors whose basins form a full Lebesgue measure set. We also prove in the orientable case that the set of accumulation points of the sinks of a C^1 generic three-dimensional flow has no dominated splitting with respect to the linear Poincaré flow. As a corollary we obtain that every three-dimensional flow can be C^1 approximated by flows with homoclinic tangencies or by singular-Axiom A flows. These results extend [3], [6], [20] and solve a conjecture in [17].

1. Introduction

Araujo's Theorem [3] asserts that a C^1 generic surface diffeomorphism has either infinitely many sinks (i.e. attracting periodic orbits), or, finitely many hyperbolic attractors whose basins form a full Lebesgue measure set. In the recent paper [4] the authors were able to extend this result from surface diffeomorphisms to three-dimensional flows without singularities. More precisely, they proved that a C^1 generic three-dimensional flow without singularities either has infinitely many sinks, or, finitely many hyperbolic attractors whose basins form a full Lebesgue measure set. The present paper goes beyond and extend [4] to the singular case. Indeed, we prove that every C^1 generic three-dimensional flow has either infinitely many sinks, or, finitely many hyperbolic or singular-hyperbolic attractors whose basins form a full Lebesgue measure set. The arguments used in the proof will imply in the orientable case that the set of accumulation points of the sinks of a C^1 generic three-dimensional flow has no dominated splitting with respect to the linear Poincaré flow. From this we obtain that every three-dimensional flow can be C^1 approximated by flows with homoclinic tangencies or by singular-Axiom A flows. This last result extends [6], [20] and solves a conjecture in [17]. Let us state our results in a precise way.

By a *three-dimensional flow* we mean a C^1 vector fields on compact connected boundaryless manifolds M of dimension 3. The corresponding space equipped with the C^1 vector field topology will be denoted by $\mathfrak{X}^1(\mathrm{M})$. The flow of $X \in \mathfrak{X}^1(\mathrm{M})$ is denoted by X_t, $t \in \mathbb{R}$. A subset of $\mathfrak{X}^1(\mathrm{M})$ is *residual* if it is a countable intersection of open and dense subsets. We say that a C^1 *generic three-dimensional flow satisfies a certain property* P if there is a residual subset $\mathcal{R}$ of $\mathfrak{X}^1(\mathrm{M})$ such that P holds for every element of $\mathcal{R}$. The closure operation is denoted by $\mathrm{Cl}(\cdot)$.

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By a *critical point* of X we mean a point x which is either *periodic* (i.e. there is a minimal $t_{x,X} > 0$ satisfying $X_{t_{x,X}}(x) = x$) or *singular* (i.e. $X(x) = 0$). The *eigenvalues* of a critical point x are defined respectively as those of the linear automorphism $DX_{t_{x,X}}(x) : T_xM \to T_xM$ not corresponding to $X(x)$, or, those of $DX(x)$. A critical point is a *sink* if its eigenvalues are less than 1 in modulus (periodic case) or with negative real part (singular case). A *source* will be a sink for the time reversed flow $-X$. Denote by $\mathrm{Sink}(X)$ and $\mathrm{Source}(X)$ the set of sinks and sources of X respectively.

Given a point x we define the *omega-limit set*,

$$\omega(x) = \left\{ y \in M : y = \lim_{t_k \to \infty} X_{t_k}(x) \text{ for some integer sequence } t_k \to \infty \right\}.$$

(when necessary we shall write $\omega_X(x)$ to indicate the dependence on X.) We call a subset $\Lambda \subset M$ *invariant* if $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$; and *transitive* if there is $x \in \Lambda$ such that $\Lambda = \omega(x)$. The *basin* of any subset $\Lambda \subset M$ is defined by

$$W^s(\Lambda) = \{y \in M : \omega(y) \subset \Lambda\}.$$

(Sometimes we write $W^s_X(\Lambda)$ to indicate dependence on X). An *attractor* is a transitive set A exhibiting a neighborhood U such that

$$A = \bigcap_{t \geq 0} X_t(U).$$

A compact invariant set Λ is *hyperbolic* if there are a continuous DX_t-invariant tangent bundle decomposition $T_\Lambda M = E^s_\Lambda \oplus E^X_\Lambda \oplus E^u_\Lambda$ over Λ and positive numbers K, λ such that E^X_x is generated by $X(x)$,

$$\|DX_t(x)/E^s_x\| \leq Ke^{-\lambda t} \quad \text{and} \quad \|DX_{-t}(x)/E^u_{X_t(x)}\| \leq K^{-1}e^{\lambda t}, \quad \forall (x,t) \in \Lambda \times \mathbb{R}^+.$$

On the other hand, a *dominated splitting* $E \oplus F$ for X over an invariant set I is a continuous tangent bundle DX_t-invariant splitting $T_IM = E_I \oplus F_I$ for which there are positive constants K, λ satisfying

$$\|DX_t(x)/E_x\| \cdot \|DX_{-t}(X_t(x))/F_{X_t(x)}\| \leq Ke^{-\lambda t}, \qquad \forall (x,t) \in I \times \mathbb{R}^+.$$

In this case we say that the dominating subbundle E_I is *contracting* if

$$\|DX_t(x)/E_x\| \leq Ke^{-\lambda t}, \qquad \forall (x,t) \in I \times \mathbb{R}^+$$

The central subbundle F_I is said to be *volume expanding* if

$$|\det DX_t(x)/F_x|^{-1} \leq Ke^{-\lambda t}, \qquad \forall (x,t) \in I \times \mathbb{R}^+.$$

A compact invariat set is *partially hyperbolic* if it has a dominated splitting with contracting dominating direction. We say that a partially hyperbolic set is *singular-hyperbolic for X* if its singularities are all hyperbolic and its central subbundle is volume expanding. A *hyperbolic* (resp. *singular-hyperbolic*) *attractor* for X is an attractor which is simultaneously a hyperbolic (resp. singular-hyperbolic) set for X.

With these definitions we can state our first result.

Theorem A. *A C^1 generic three-dimensional flow has either infinitely many sinks, or, finitely many hyperbolic or singular-hyperbolic attractors whose basins form a full Lebesgue measure set.*

The method of the proof of the above result (based on [18]) will imply the following result for three-dimensional flows on orientable manifolds. Denote by $\mathrm{Sing}(X)$ the set of singularities of X. Given $\Lambda \subset M$ we denote $\Lambda^* = \Lambda \setminus \mathrm{Sing}(X)$.

We define the vector bundle N^X over M^* whose fiber at $x \in M^*$ is the the orthogonal complement of $X(x)$ in T_xM. Denoting the projection $\pi_x : T_xM \to N_x^X$ we define the *Linear Poincaré flow* (LPF) $P_t^X : N^X \to N^X$ by $P_t^X(x) = \pi_{X_t(x)} \circ DX_t(x)$, $t \in \mathbb{R}$. An invariant set Λ of X *has a LPF-dominated splitting* if $\Lambda^* \neq \emptyset$ and there exist a continuous tangent bundle decomposition $N_{\Lambda^*}^X = N_{\Lambda^*}^{s,X} \oplus N_{\Lambda^*}^{u,X}$ with $dimN_x^{s,X} = dimN_x^{u,X} = 1$ ($\forall x \in \Lambda^*$) and $T > 0$ such that

$$\left\|P_T^X(x)/N_x^{s,X}\right\| \left\|P_{-T}^X(X_T(x))/N_{X_T(x)}^{u,X}\right\| \leq \frac{1}{2}, \qquad \forall x \in \Lambda^*.$$

Theorem B. *If X is a C^1 generic three-dimensional flow of a orientable manifold, then neither* $\mathrm{Cl}(\mathrm{Sink}(X)) \setminus \mathrm{Sink}(X)$ *nor* $\mathrm{Cl}(\mathrm{Source}(X)) \setminus \mathrm{Source}(X)$ *have LPF-dominated splitting.*

As an application we obtain a solution for Conjecture 1.3 in [17]. A periodic point x of X is a *saddle* if it has eigenvalues of modulus less and bigger than 1 simultaneously. Denote by $\mathrm{PSaddle}(X)$ the set of periodic saddles of X. As is well known [13], through any $x \in \mathrm{PSaddle}(X)$ it passes a pair of invariant manifolds, the so-called strong stable and unstable manifolds $W^{ss}(x)$ and $W^{uu}(x)$, tangent at x to the eigenspaces corresponding to the eigenvalue of modulus less and bigger than 1 respectively. Saturating these manifolds with the flow we obtain the stable and unstable manifolds $W^s(x)$ and $W^u(x)$ respectively. A *homoclinic point* associated to x is a point q where these last manifolds meet. We say that q is a *transverse homoclinic point* if $T_qW^s(x) \cap T_qW^u(x)$ is the one-dimensional subspace generated by $X(q)$ and a *homoclinic tangency* otherwise.

We define the *nonwandering set* $\Omega(X)$ as the set of points p such that for every $T > 0$ and every neighborhood U of p there is $t > T$ satisfying $X_t(U) \cap U \neq \emptyset$.

Following [17], we say that X is *singular-Axiom A* if there is a finite disjoint union

$$\Omega(X) = \Lambda_1 \cup \cdots \cup \Lambda_r,$$

where each Λ_i for $1 \leq i \leq r$ is a transitive hyperbolic set (if $\Lambda_i \cap \mathrm{Sing}(X) = \emptyset$) or a singular-hyperbolic attractor for either X or $-X$ (otherwise).

With these definitions we can state the following corollary.

Corollary 1.1. *Every three-dimensional flow can be C^1 approximated by a flow exhibiting a homoclinic tangency or by a singular-Axiom A flow.*

Proof. Passing to a finite covering if necessary we can assume that M is orientable. Let $R(M)$ denote the set of three-dimensional flows which cannot be C^1 approximated by ones with homoclinic tangencies. As is well-known [6], $\mathrm{Cl}(\mathrm{PSaddle}(X))$ has a LPF-dominated splitting for every C^1 generic $X \in R(M)$. Furthermore,

$$(\mathrm{Cl}(\mathrm{Sink}(X)) \setminus \mathrm{Sink}(X)) \cup (\mathrm{Cl}(\mathrm{Source}(X)) \setminus \mathrm{Source}(X)) \subset \mathrm{Cl}(\mathrm{PSaddle}(X))$$

Combining this inclusion with Theorem B we obtain $\mathrm{Cl}(\mathrm{Sink}(X)) \setminus \mathrm{Sink}(X) = \mathrm{Cl}(\mathrm{Source}(X)) \setminus \mathrm{Source}(X) = \emptyset$, and so, $\mathrm{Sink}(X) \cup \mathrm{Source}(X)$ consists of finitely many orbits, for every C^1-generic $X \in R(M)$. Now we obtain that X is singular-Axiom A by Theorem A in [18]. □

2. Proof of theorems A and B

Let X be a three-dimensional flow. Denote by $\mathrm{Crit}(X)$ the set of critical points.

Recall that a periodic point saddle if it has eigenvalues of modulus less and bigger than 1 simultaneously. Analogously for singularities by just replace 1 by 0 and the eigenvalues by their corresponding real parts. Denote by $\mathrm{Sink}(X)$ and $\mathrm{Saddle}(X)$ the set of sinks and saddles of X respectively.

A critical point x is *dissipative* if the product of its eigenvalues (in the periodic case) or the divergence $\operatorname{div} X(x)$ (in the singular case) is less than 1 (resp. 0). Denote by $\mathrm{Crit}_d(X)$ the set of dissipative critical points. We define the *dissipative region* by $\mathrm{Dis}(X) = \mathrm{Cl}(\mathrm{Crit}_d(X))$.

For every subset $\Lambda \subset M$ we define the *weak basin* by

$$W^s_w(\Lambda) = \{x \in M : \omega(x) \cap \Lambda \neq \emptyset\}.$$

(This is often called *weak region of attraction* [7].) With these notations we obtain the following result. Its proof is similar to the corresponding one in [4]:

Theorem 2.1. *There is a residual subset $\mathcal{R}_6$ of three-dimensional flows X for which $W^s_w(\mathrm{Dis}(X))$ has full Lebesgue measure.*

The *homoclinic class* associated to $x \in \mathrm{PSaddle}(X)$ is the closure of the set of transverse homoclinic points q associated to x. A homoclinic class of X is the homoclinic class associated to some saddle of X.

Given a homoclinic class $H = H_X(p)$ of a three-dimensional flow X we denote by $H_Y = H_Y(p_Y)$ the continuation of H, where p_Y is the analytic continuation of p for Y close to X (c.f. [19]).

The following lemma was also proved in [4]. In its statement Leb denotes the normalized Lebesgue measure of M.

Lemma 2.2. *There is a residual subset $\mathcal{R}_{12}$ of three-dimensional flows X such that for every hyperbolic homoclinic class H there are an open neighborhood $\mathcal{O}_{X,H}$ of f and a residual subset $\mathcal{R}_{X,H}$ of $\mathcal{O}_{X,H}$ such that the following properties are equivalent:*

1. $\mathrm{Leb}(W^s_Y(H_Y)) = 0$ *for every* $Y \in \mathcal{R}_{X,H}$.
2. H *is not an attractor.*

We say that a compact invariant set Λ of a three-dimensionmal flow X *has a spectral decomposition* if there is a disjoint decomposition

$$\Lambda = \bigcup_{i=1}^{r} H_i$$

into finitely many disjoint homoclinic classes H_i, $1 \leq i \leq r$, each one being either hyperbolic (if $H_i \cap \mathrm{Sing}(X) = \emptyset$) or a singular-hyperbolic attractor for either X or $-X$ (otherwise).

Now we prove the following result which is similar to one in [4] (we include its proof for the sake of completeness). In its statement $\mathrm{PSaddle}_d(X)$ denotes the set of periodic dissipative saddles of a three-dimensional flow X.

Theorem 2.3. *There is a residual subset $\mathcal{R}_{11}$ of three-dimensional flows Y such that if $\mathrm{Cl}(\mathrm{PSaddle}_d(Y))$ has a spectral decomposition, then the following properties are equivalent for every homoclinic H associated to a dissipative periodic saddle:*

(a) $\mathrm{Leb}(W^s_Y(H)) > 0$.

(b) *H is either hyperbolic attractor or a singular-hyperbolic attractor for Y.*

Proof. Let $\mathcal{R}_{12}$ be as in Lemma 2.2. Define the map $S : \mathfrak{X}^1(\mathrm{M}) \to 2^M_c$ by $S(X) = \mathrm{Cl}(\mathrm{PSaddle}_d(X))$. This map is clearly lower-semicontinuous, and so, upper semicontinuous in a residual subset $\mathcal{N}$ (for the corresponding definitions see [14], [15]).

By Lemma 2.4 there is a residual subset $\mathcal{L}$ of three-dimensional flows X for which every singular-hyperbolic attractor with singularities of either X or $-X$ has zero Lebesgue measure.

By the flow-version of the main result in [1], there is a residual subset $\mathcal{R}_7$ of three-dimensional flows X such that for every singular-hyperbolic attractor C for X (resp. $-X$) there are neighborhoods $U_{X,C}$ of C, $\mathcal{U}_{X,C}$ of X and a residual subset $\mathcal{R}^0_{X,C}$ of $\mathcal{U}_{X,C}$ such that for all $Y \in \mathcal{R}^0_{X,C}$ if $Z = Y$ (resp. $Z = -Y$) then

$$C_Y = \bigcap_{t\geq 0} Z_t(U_{X,C}) \text{ is a singular-hyperbolic attractor for } Z. \tag{1}$$

Define $\mathcal{R} = \mathcal{R}_{12} \cap \mathcal{N} \cap \mathcal{L} \cap \mathcal{R}_7$. Clearly $\mathcal{R}$ is a residual subset of three-dimensional flows. Define

$$\mathcal{A} = \{f \in \mathcal{R} : \mathrm{Cl}(\mathrm{PSaddle}_d(X)) \text{ has no spectral decomposition}\}.$$

Fix $X \in \mathcal{R}\backslash\mathcal{A}$. Then, $X \in \mathcal{R}$ and $\mathrm{Cl}(\mathrm{PSaddle}_d(X))$ has a spectral decomposition

$$\mathrm{Cl}(\mathrm{PSaddle}_d(X)) = \left(\bigcup_{i=1}^{r_X} H^i\right) \cup \left(\bigcup_{j=1}^{a_X} A^j\right) \cup \left(\bigcup_{k=1}^{b_X} R^k\right)$$

into hyperbolic homoclinic classes H_i ($1 \leq i \leq r_X$), singular-hyperbolic attractors A^j for X ($1 \leq j \leq a_X$), and singular-hyperbolic attractors R^k for $-X$ ($1 \leq k \leq b_X$).

As $X \in \mathcal{R}_{12} \cap \mathcal{R}_7$, we can consider for each $1 \leq i \leq r_X$, $1 \leq j \leq a_X$ and $1 \leq k \leq b_X$ the neighborhoods $\mathcal{O}_{X,H^i}$, $\mathcal{U}_{X,A^j}$ and $\mathcal{U}_{X,R^k}$ of X as well as their residual subsets $\mathcal{R}_{X,H^i}$, $\mathcal{R}^0_{X,A^j}$ and $\mathcal{R}^0_{X,R^k}$ given by Lemma 2.2 and (1) respectively.

Define

$$\mathcal{O}_X = \left(\bigcap_{i=1}^{r_X} \mathcal{O}_{X,H^i}\right) \cap \left(\bigcap_{j=1}^{a_X} \mathcal{U}_{X,A^j}\right) \cap \left(\bigcap_{k=1}^{b_X} \mathcal{U}_{X,R^k}\right)$$

and

$$\mathcal{R}_X = \left(\bigcap_{i=1}^{r_X} \mathcal{R}_{X,H^i}\right) \cap \left(\bigcap_{j=1}^{a_X} \mathcal{R}^0_{X,A^j}\right) \cap \left(\bigcap_{k=1}^{b_X} \mathcal{R}^0_{X,R^k}\right).$$

Clearly $\mathcal{R}_X$ is residual in $\mathcal{O}_X$.

From the proof of Lemma 2.2 in [4] we obtain for each $1 \leq i \leq r_X$ a compact neighborhood $U_{X,i}$ of H^i such that

$$H^i_Y = \bigcap_{t\in\mathcal{R}} Y_t(U_{X,i}) \quad \text{is hyperbolic and equivalent to } H^i, \qquad \forall Y \in \mathcal{O}_{Y,H^i}. \tag{2}$$

As $X \in \mathcal{N}$, S is upper semicontinuous at X so we can further assume that

$$\mathrm{Cl}(\mathrm{PSaddle}_d(Y)) \subset \left(\bigcup_{i=1}^{r_X} U_{X,i}\right) \cup \left(\bigcup_{j=1}^{a_X} U_{X,A^j}\right) \cup \left(\bigcup_{k=1}^{b_X} U_{X,R^k}\right), \qquad \forall Y \in \mathcal{O}_X.$$

It follows that

$$\text{Cl}(\text{PSaddle}_d(Y)) = \left(\bigcup_{i=1}^{r_X} H_Y^i\right) \cup \left(\bigcup_{j=1}^{a_X} A_Y^j\right) \cup \left(\bigcup_{k=1}^{b_X} R_Y^k\right), \qquad \forall Y \in \mathcal{R}_X. \tag{3}$$

Next we take a sequence $X^i \in \mathcal{R} \setminus \mathcal{A}$ which is dense in $\mathcal{R} \setminus \mathcal{A}$.

Replacing $\mathcal{O}_{X^i}$ by $\mathcal{O}'_{X^i}$ where

$$\mathcal{O}'_{X^0} = \mathcal{O}_{X^0} \text{ and } \mathcal{O}'_{X^i} = \mathcal{O}_{X^i} \setminus \left(\bigcup_{j=0}^{i-1} \mathcal{O}_{X^j}\right), \text{ for } i \geq 1,$$

we can assume that the collection $\{\mathcal{O}_{X^i} : i \in \mathbb{N}\}$ is pairwise disjoint.

Define

$$\mathcal{O}_{12} = \bigcup_{i\in\mathbb{N}} \mathcal{O}_{X^i} \quad \text{and} \quad \mathcal{R}'_{12} = \bigcup_{i\in\mathbb{N}} \mathcal{R}_{X^i}.$$

We claim that $\mathcal{R}'_{12}$ is residual in $\mathcal{O}_{12}$.

Indeed, for all $i \in \mathbb{N}$ write $\mathcal{R}_{X^i} = \bigcap_{n\in\mathbb{N}} \mathcal{O}_i^n$, where $\mathcal{O}_i^n$ is open-dense in $\mathcal{O}_{X^i}$ for every $n \in \mathbb{N}$. Since $\{\mathcal{O}_{X^i} : i \in \mathbb{N}\}$ is pairwise disjoint, we obtain

$$\bigcap_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} \mathcal{O}_i^n \subset \bigcup_{i\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \mathcal{O}_i^n = \bigcup_{i\in\mathbb{N}} \mathcal{R}_{X^i} = \mathcal{R}'_{12}.$$

As $\bigcup_{i\in\mathbb{N}} \mathcal{O}_{X^i}^n$ is open-dense in $\mathcal{O}_{12}$, $\forall n \in \mathbb{N}$, we obtain the claim.

Finally we define

$$\mathcal{R}_{11} = \mathcal{A} \cup \mathcal{R}'_{12}.$$

Since $\mathcal{R}$ is a residual subset of three-dimensional flows, we conclude as in Proposition 2.6 of [16] that $\mathcal{R}_{11}$ is also a residual subset of three-dimensional flows.

Take $Y \in \mathcal{R}_{11}$ such that $\text{Cl}(\text{PSaddle}_d(Y))$ has a spectral decomposition and let H be a homoclinic class associated to a dissipative saddle of Y. Then, $H \subset \text{Cl}(\text{PSaddle}_d(Y))$ by Birkhoff-Smale's Theorem [12]. Since $\text{Cl}(\text{PSaddle}_d(Y))$ has spectral decomposition, we have $Y \notin \mathcal{A}$ so $Y \in \mathcal{R}'_{12}$ thus $Y \in \mathcal{R}_X$ for some $X \in \mathcal{R} \setminus \mathcal{A}$. As $Y \in \mathcal{R}_X$, (3) implies $H = H_Y^i$ for some $1 \leq i \leq r_X$ or $H = A_Y^j$ for some $1 \leq j \leq a_X$ or $H = R_Y^k$ for some $1 \leq k \leq b_X$.

Now, suppose that $\text{Leb}(W_Y^s(H)) > 0$. Since $Y \in \mathcal{R}_X$, we have $Y \in \mathcal{R}^0_{X,R^k}$ for all $1 \leq k \leq b_X$. As $X \in \mathcal{L}$, and $W_Y^s(R_Y^k) \subset R_Y^k$ for every $1 \leq k \leq b_X$, we conclude by Lemma 2.4 that $H \neq R_Y^k$ for every $1 \leq k \leq b_X$.

If $H = A_Y^j$ for some $1 \leq j \leq a_X$ then H is an attractor and we are done. Otherwise, $H = H_Y^i$ for some $1 \leq i \leq r_X$. As $Y \in \mathcal{R}_X$, we have $Y \in \mathcal{R}_{X,H^i}$ and, since $f \in \mathcal{R}_{12}$, we conclude from Lemma 2.2 that H^i is an attractor. But by (2) we have that H_Y^i and H^i are equivalent, so, H_Y^i is an attractor too and we are done. □

We shall need the following lemma which was essentially proved in [5].

Lemma 2.4. *There is a residual subset $\mathcal{L}$ of three-dimensional flows X for which every singular-hyperbolic attractor with singularities of either X or $-X$ has zero Lebesgue measure.*

Proof. As in [5], for any open set U and any three-dimensional vector field Y, let $\Lambda_Y(U) = \bigcap_{t\in\mathbb{R}} Y_t(U)$ be the maximal invariant set of Y in U. Define $\mathcal{U}(U)$ as the set of flows Y such that $\Lambda_Y(U)$ is a singular-hyperbolic set with singularities of Y. It follows that $\mathcal{U}(U)$ is open in $\mathfrak{X}^1(\mathrm{M})$.

Now define $\mathcal{U}(U)_n$ as the set of $Y \in \mathcal{U}(U)$ such that $\mathrm{Leb}(\Lambda_Y(U)) < 1/n$. It was proved in [5] that $\mathcal{U}(U)_n$ is open and dense in $\mathcal{U}(U)$.

Define $\mathcal{R}(U)_n = \mathcal{U}(U)_n \cup (\mathfrak{X}^1(M) \setminus \mathrm{Cl}(\mathcal{U}(U))$ which is open and dense set in $\mathfrak{X}^1(M)$. Let $\{U_m\}$ be a countable basis of the topology, and $\{O_m\}$ be the set of finite unions of such U_m's. Define

$$\mathcal{L} = \bigcap_m \bigcap_n \mathcal{R}(O_m)_n.$$

This is clearly a residual subset of three-dimensional flows. We can assume without loss of generality that $\mathcal{L}$ is symmetric, i.e., $X \in \mathcal{L}$ if and only if $-X \in \mathcal{L}$. Take $X \in \mathcal{L}$. Let Λ be a singular-hyperbolic attractor for X. Then, there exists m such that $\Lambda = \Lambda_X(O_m)$. Then $X \in \mathcal{U}(O_m)$ and so $X \in \mathcal{U}(O_m)_n$ for every n thus $\mathrm{Leb}(\Lambda) = 0$. Analogously, since $\mathcal{L}$ is symmetric, we obtain that $\mathrm{Leb}(\Lambda) = 0$ for every singular-hyperbolic attractor with singularities of $-X$. □

In the sequel we obtain the following key result representing the new ingredient with respect to [4]. Its proof will use the methods in [18]. In its statement card(Sink(X)) denotes the cardinality of the set of *different* orbits of a three-dimensional flow X contained in Sink(X).

Theorem 2.5. *There is a residual subset $\mathcal{Q}$ of three-dimensional flows X such that if* $\mathrm{card}(\mathrm{Sink}(X)) < \infty$*, then* $\mathrm{Cl}(\mathrm{PSaddle}_d(X))$ *has a spectral decomposition.*

Proof. First we state some useful notiations.

Given a three-dimensional flow Y and a point p we denote by $O_Y(p) = \{Y_t(p) : t \in \mathbb{R}\}$ the Y-orbit of p. If $p \in \mathrm{PSaddle}_d(Y)$ we denote by $E^{s,Y}_p$ and $E^{u,Y}_p$ the eigenspaces corresponding to the eigenvalues of modulus less and bigger than 1 respectively.

Denote by $\lambda(p, Y)$ and $\mu(p, Y)$ the eigenvalues of p satisfying

$$|\lambda(p, Y)| < 1 < |\mu(p, Y)|.$$

Define the *index* of a singularity σ as the number $Ind(\sigma)$ of eigenvalues with negative real part.

We say that a singularity σ of Y is *Lorenz-like for* Y if its eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are real and satisfy $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$ (up to some order). It follows in particular that σ is *hyperbolic* (i.e. without eigenvalues of zero real part) of index 2. Furthermore, the invariant manifold theory [13] implies the existence of *stable* and *unstable* manifolds $W^{s,Y}(\sigma)$, $W^{u,Y}(\sigma)$ tangent at σ to the eigenvalues $\{\lambda_2, \lambda_3\}$ and λ_1 respectively. There is an additional invariant manifold $W^{ss,Y}(\sigma)$, the *strong stable manifold*, contained in $W^{s,Y}(\sigma)$ and tangent at σ to the eigenspace corresponding to λ_1. We shall denote by $E^{ss,Y}_\sigma$ and $E^{cu,Y}_\sigma$ the eigenspaces associated to the set of eigenvalues λ_2 and $\{\lambda_3, \lambda_1\}$ respectively.

Let $S(M)$ be the set of three-dimensional flows X with $\mathrm{card}(\mathrm{Sink}(X)) < \infty$ such that

$$\mathrm{card}(\mathrm{Sink}(Y)) = \mathrm{card}(\mathrm{Sink}(X)), \qquad \text{for every } Y \text{ close to } X.$$

Every $X \in S(M)$ satisfies the following properties:

- There is a LPF-dominated splitting over $\mathrm{PSaddle}^*_d(X) \setminus \mathrm{Sing}(X)$, where $\mathrm{PSaddle}^*_d(X)$ denotes the set of points x for which there are sequences $Y_k \to X$ and $x_k \in \mathrm{PSaddle}_d(X_k)$ such that $x_k \to x$ (c.f. [21]).
- There are a neighborhood $\mathcal{U}_X$, $0 < \lambda < 1$ and $\alpha > 0$ such that if $(p, Y) \in \mathrm{PSaddle}_d(Y) \times \mathcal{U}_X$, then
  (a) 1. $|\lambda(p, Y)| < \lambda^{t_{p,Y}}$,
      2. $|\mu(p, Y)| > \lambda^{-t_{p,Y}}$.
  (b) $\mathrm{angle}(E^{s,Y}_p, E^{u,Y}_p) > \alpha$.

Indeed, the first property follows from the proof of Proposition 5.3 in [4] and the second from the proof of Theorem 3.6 in [18] (see also the proof of lemmas 7.2 and 7.3 in [4]).

In addition to this we also have the existence of a residual subset of three-dimensional flows $\mathcal{R}_7$ such that every $X \in S(M) \cap \mathcal{R}_7$ satisfies that:

- Every $\sigma \in \mathrm{Sing}(X) \cap \mathrm{Cl}(\mathrm{PSaddle}_d(X))$ with $Ind(\sigma) = 2$ is Lorenz-like for X and satisfies $\mathrm{Cl}(\mathrm{PSaddle}_d(X)) \cap W^{ss,X}(\sigma) = \{\sigma\}$.
- Every $\sigma \in \mathrm{Sing}(X) \cap \mathrm{Cl}(\mathrm{PSaddle}_d(X))$ with $Ind(\sigma) = 1$ is Lorenz-like for $-X$ and satisfies $\mathrm{Cl}(\mathrm{PSaddle}_d(X)) \cap W^{uu,X}(\sigma) = \{\sigma\}$, where $W^{uu,X}(\sigma) = W^{ss,-X}(\sigma)$.

Indeed, as in the remark after Lemma 2.13 in [8], there is a residual subset $\mathcal{R}_7$ of three-dimensional flows X such that every $\sigma \in \mathrm{Sing}(X)$ accumulated by periodic orbits is Lorenz-like for either X or $-X$ depending on whether σ has three real eigenvalues $\lambda_1, \lambda_2, \lambda_3$ satisfying either $\lambda_2 < \lambda_3 < 0 < \lambda_1$ or $\lambda_2 < 0 < \lambda_3 < \lambda_1$ (up to some order).

Now, take $X \in S(M) \cap \mathcal{R}_7$. Since $X \in S(M)$, we have that $\mathrm{PSaddle}^*_d(X) \setminus \mathrm{Sing}(X)$ has a LPF-dominated splitting and then $\mathrm{Cl}(\mathrm{PSaddle}_d(X)) \setminus \mathrm{Sing}(X)$ also does because $\mathrm{Cl}(\mathrm{PSaddle}_d(X)) \subset \mathrm{PSaddle}^*_d(X)$. Therefore, if $\sigma \in \mathrm{Sing}_2(X) \cap \mathrm{Cl}(\mathrm{PSaddle}_d(X))$, Proposition 2.4 in [10] implies that σ has three different real eigenvalues $\lambda_1, \lambda_2, \lambda_3$ satisfing $\lambda_2 < \lambda_3 < 0 < \lambda_1$ (up to some order). Since $X \in \mathcal{R}_7$, we conclude that σ is Lorenz-like for X. To prove $\mathrm{Cl}(\mathrm{PSaddle}_d(X)) \cap W^{ss,X}(\sigma) = \{\sigma\}$ we assume by contradiction that this is not the case. Then, there is $x \in (\mathrm{Cl}(\mathrm{PSaddle}_d(X)) \cap W^{ss,X}(\sigma)) \setminus \{\sigma\}$. Choose sequences $x_n \in \mathrm{Cl}(\mathrm{PSaddle}_d(X))$ and $t_n \to \infty$ such that $x_n \to x$ and $X_{t_n}(x_n) \to y$ for some $y \in W^{u,X}(\sigma) \setminus \{\sigma\}$. Let $N^{s,X} \oplus N^{u,X}$ denote the LPF-dominated splitting of $\mathrm{Cl}(\mathrm{PSaddle}_d(X)) \setminus \mathrm{Sing}(X)$. We have $N^{s,X}_x = N_x \cap W^{s,X}(\sigma)$ by Proposition 2.2 in [10] and so N_{x_n} tends to be tangent to $W^{s,X}(\sigma)$ as $n \to \infty$. On the other hand, Proposition 2.4 in [10] says that $N^{s,X}_y$ is almost parallel to $E^{ss,X}_\sigma$. Therefore, the directions $N^{s,X}_{X_{t_n}(x_n)}$ tends to have positive angle with $E^{ss,X}_\sigma$. But using that $\lambda_2 < \lambda_3$ we can see that $N^{s,X}_{x_n} = P_{-t_n}(X_{t_n}(x_n)) N^{s,X}_{X_{t_n}(x_n)}$ tends to be transversal to $W^{s,X}(\sigma)$ nearby x. As this is a contradiction, we obtain the result. The second property can be proved analogously.

On the other hand, there is another residual subset $\mathcal{Q}_1$ of three-dimensional flows for which every compact invariant set without singularities but with a LPF-dominated splitting is hyperbolic.

Indeed, by Lemma 3.1 in [8] we have that there is a residual subset $\mathcal{Q}_1$ of three-dimensional flows for which every transitive set without singularities but with a LPF-dominated splitting is hyperbolic. Fix $X \in \mathcal{Q}_1$ and a compact invariant set

Λ without singularities but with a LPF-dominated splitting $N^X_\Lambda = N^{s,X}_\Lambda \oplus N^{u,X}_\Lambda$. Suppose by contradiction that Λ is not hyperbolic. Then, by Zorn's Lemma, there is a minimally nonhyperbolic set $\Lambda_0 \subset \Lambda$ (c.f. p.983 in [20]). Assume for a while that Λ_0 is not transitive. Then, $\omega(x)$ and $\alpha(x) = \omega_{-X}(x)$ are proper subsets of Λ_0, $\forall x \in \Lambda_0$. Therefore, both sets are hyperbolic and then we have

$$\lim_{t\to\infty} \|P^X_t(x)/N^{s,X}_x\| = \lim_{t\to\infty} \|P^X_{-t}(x)/N^{u,X}_x\| = 0, \qquad \forall x \in \Lambda_0,$$

which easily implies that Λ_0 is hyperbolic. Since this is a contradiction, we conclude that Λ_0 is transitive. As $X \in \mathcal{Q}_1$ and Λ_0 has a LPF-dominated splitting (by restriction), we conclude that Λ_0 is hyperbolic, a contradiction once more proving the result.

Next we recall that a compact invariant set Λ of a flow X is *Lyapunov stable for* X if for every neighborhood U of Λ there is a neighborhood $V \subset U$ of Λ such that $X_t(V) \subset U$, for all $t \geq 0$.

It follows from [9], [17] that there is a residual subset $\mathcal{D}$ of three-dimensional flows X such that if $\sigma \in \mathrm{Sing}(X) \cap \mathrm{Cl}(\mathrm{PSaddle}_d(X))$ and $Ind(\sigma) = 2$, then $\mathrm{Cl}(W^u(\sigma))$ is a Lyapunov stable set for X with dense singular unstable branches contained in $\mathrm{Cl}(\mathrm{PSaddle}_d(X))$. Analogously, if $Ind(\sigma) = 1$, then $\mathrm{Cl}(W^s(\sigma))$ is a Lyapunov stable set for $-X$ with dense singular stable branches contained in $\mathrm{Cl}(\mathrm{PSaddle}_d(X))$.

From these properties we derive easily that every $X \in S(M) \cap \mathcal{R}_7 \cap \mathcal{D}$ and every $\sigma \in \mathrm{Sing}(X) \cap \mathrm{Cl}(\mathrm{PSaddle}_d(X))$ satisfies one of the following alternatives:

(c) If $Ind(\sigma) = 2$, then every $\sigma' \in \mathrm{Sing}(X) \cap \mathrm{Cl}(W^u(\sigma))$ is Lorenz-like for X.
(d) If $Ind(\sigma) = 1$, then every $\sigma' \in \mathrm{Sing}(X) \cap \mathrm{Cl}(W^s(\sigma))$ is Lorenz-like for $-X$.

Given a three-dimensional flow Y we define

$$E^{cu,Y}_p = E^{u,Y}_p \oplus E^Y_p, \qquad \forall p \in \mathrm{PSaddle}_d(Y).$$

We claim that there is a residual subset of three-dimensional flows $\mathcal{R}_{15}$ such that for every $X \in S(M) \cap \mathcal{R}_{15}$ and every $\sigma \in \mathrm{Sing}(X) \cap \mathrm{Cl}(\mathrm{PSaddle}_d(X))$ there are neighborhoods $\mathcal{V}_X$ of X, U_σ of σ and $\beta_\sigma > 0$ such that if $Y \in \mathcal{V}_X$ and $x \in \mathrm{PSaddle}_d(Y)$ satisfies $O_Y(x) \cap U_\sigma \neq \emptyset$, then

$$\mathrm{angle}(E^{s,Y}_x, E^{cu,Y}_x) > \beta_\sigma, \qquad \text{if } Ind(\sigma) = 2 \tag{4}$$

and

$$\mathrm{angle}(E^{s,-Y}_x, E^{cu,-Y}_x) > \beta_\sigma, \qquad \text{if } Ind(\sigma) = 1. \tag{5}$$

(This step corresponds to Theorem 3.7 in [18].)

Indeed, we just take $\mathcal{R}_{15} = \mathcal{Q}_1 \cap \mathcal{D} \cap \mathcal{R}_7 \cap \mathcal{I}$ where $\mathcal{I}$ is the set of upper semicontinuity points of the the map $\varphi : X \mapsto \mathrm{Cl}(\mathrm{PSaddle}_d(X))$.

To prove (4) it suffices to show the following assertions, correponding to propositions 4.1 and 4.2 of [18] respectively, for any $X \in S(M) \cap \mathcal{R}_{15}$ and $\sigma \in \mathrm{Sing}(X) \cap \mathrm{Cl}(\mathrm{PSaddle}_d(X))$ with $Ind(\sigma) = 2$ ($B_\delta(\cdot)$ denotes the δ-ball operation):

A1. Given $\epsilon > 0$ there are a neighborhood $\mathcal{V}_{X,\sigma}$ of X and $\delta > 0$ such that for all $Y \in \mathcal{V}_{X,\sigma}$ if $p \in \mathrm{PSaddle}_d(Y) \cap B_\delta(\sigma_Y)$ then
(a) $\mathrm{angle}(E^{s,Y}_p, E^{ss,Y}_{\sigma_Y}) < \epsilon$;
(b) $\mathrm{angle}(E^{cu,Y}_p, E^{cu,Y}_{\sigma_Y}) < \epsilon$.

A2. Given $\delta > 0$ there are a neighborhoof $\mathcal{O}$ of X and $C > 0$ such that if $Y \in \mathcal{O}$ and $p \in \mathrm{PSaddle}_d(Y)$ with $dist(p, \mathrm{Sing}(X) \cap \mathrm{Cl}(\mathrm{PSaddle}_d(X))) > \delta$, then

$$\mathrm{angle}(E^{s,Y}_p, E^{cu,Y}_p) > C.$$

To prove A1-(a) we proceed as in p. 417 of [9]. By contradiction suppose that it is not true. Then, there are $\gamma > 0$ and sequences $Y^n \to X$, $p_n \in \mathrm{PSaddle}_d(Y^n) \to \sigma$ such that

$$\mathrm{angle}(E^{s,Y^n}_{p_n}, E^{ss,Y^n}_{\sigma_{Y^n}}) > \gamma, \qquad \forall n \in \mathbb{N}.$$

As in [18] we take small cross sections $\Sigma^s_{\delta,\delta'}$ and Σ^u_δ located close to the singularities in $\mathrm{Cl}(W^u(\sigma))$ all of which are Lorenz-like (by (c) above). It turns out that since $p_n \to \sigma$, there are times $t_n \to \infty$ satisfying $q_n = Y^n_{t_n}(p_n) \in \Sigma^u_\delta$. Using the above inequality we obtain

$$\mathrm{angle}(E^{s,Y^n}_{q_n}, E^{Y^n}_{q_n}) \to 0.$$

Next consider the first $s_n > 0$ such that

$$\tilde{q}_n = Y^n_{s_n}(q_n) \in \Sigma^s_{\delta,\delta'}.$$

We obtain

$$\mathrm{angle}(E^{s,Y^n}_{\tilde{q}_n}, E^{Y^n}_{\tilde{q}_n}) \to 0. \tag{6}$$

To see why, we assume two cases: either s_n is bounded or not. If it does, then the above limit follows from the corresponding one for q_n. If not, we consider a limit point q of the sequence $Y^n_{\frac{s_n}{2}}(q_n)$ with $s_n \to \infty$. After observing that the X-orbit of q cannot accumulate any index 1 singularity we obtain easily that $q \in \Gamma$, where

$$\Gamma = \bigcap_{t\in\mathbb{R}} X_t \left(\mathrm{Cl}(\mathrm{PSaddle}_d(X)) \setminus B_{\delta^*}(\mathrm{Sing}(X) \cap \mathrm{Cl}(\mathrm{PSaddle}_d(X)))\right),$$

for some $\delta^* > 0$ small. Clearly Γ is a compact invariant subset of X contained in $\mathrm{Cl}(\mathrm{PSaddle}_d(X)) \setminus \mathrm{Sing}(X)$. Since $X \in S(M)$, we have that Γ has a LPF-dominated splitting, and so, it is hyperbolic because $X \in \mathcal{Q}_1$. This allows us to repeat the proof in p. 419 to obtain (6) which, together with (b) above, implies that $\mathrm{angle}(E^{u,Y^n}_{\tilde{q}_n}, E^{Y^n}_{\tilde{q}_n})$ is bounded away from zero. But now we consider the first positive time r_n satisfying $\tilde{\tilde{q}}_n = Y^n_{r_n}(\tilde{q}_n) \in \Sigma^u_\delta$. We get as in p. 419 in [18] that $\mathrm{angle}(E^{s,Y^n}_{\tilde{\tilde{q}}_n}, E^{Y^n}_{\tilde{\tilde{q}}_n}) \to 0$ and, since $\mathrm{angle}(E^{u,Y^n}_{\tilde{q}_n}, E^{Y^n}_{\tilde{q}_n})$ is bounded away from 0, we also obtain $\mathrm{angle}(E^{u,Y^n}_{\tilde{\tilde{q}}_n}, E^{Y^n}_{\tilde{\tilde{q}}_n}) \to 0$. All this together yield $\mathrm{angle}(E^{s,Y^n}_{\tilde{\tilde{q}}_n}, E^{u,Y^n}_{\tilde{\tilde{q}}_n}) \to 0$ which contradicts (b). This contradiction completes the proof of A1-(a). The bound in A1-(b) follows easily from the methods in [10]. This completes the proof of A1. A2 follows exactly as in p. 421 of [18]. Now A1 and A2 imply (4) as in [18]. To prove (5) we only need to repeat the above proof with $-Y$ instead of Y taking into account the symmetric relations below:

$$\lambda(p, -Y) = \mu^{-1}(p, Y),\ \mu(p, -Y) = \lambda^{-1}(p, Y),\ E^{s,-Y}_p = E^{u,Y}_p \text{ and } E^{u,-Y}_p = E^{s,Y}_p.$$

Once we prove (4) and (5) we use them together with (a) and (b), as in the proof of Theorem F in [9], to obtain that for every $X \in \mathcal{R}_{15} \cap S(M)$ there is a neighborhood $\mathcal{K}_X$, $0 < \rho < 1$, $c > 0$, $\delta > 0$ and $T_0 > 0$ satisfying the following properties for every $Y \in \mathcal{K}_X$ and every $x \in \mathrm{PSaddle}_d(Y)$ satisfying $t_{x,Y} > T_0$ and $O_Y(x) \cap B_\delta(\sigma) \neq \emptyset$:

- If $Ind(\sigma) = 2$, then
$$\|DY_T(p)/E^{s,Y}_p\| \cdot \|DY_{-T}(p)/E^{cu,Y}_{Y_{-T}(p)}\| \leq c\rho^T, \qquad \forall T > 0.$$
- If $Ind(\sigma) = 1$, then
$$\|D(-Y)_T(p)/E^{s,-Y}_p\| \cdot \|D(-Y)_{-T}(p)/E^{cu,-Y}_{(-Y)_{-T}(p)}\| \leq c\rho^T, \qquad \forall T > 0.$$

Since we can assume that X is Kupka-Smale (by the Kupka-Smale Theorem [12]), the set of periodic orbits with period $\leq T_0$ of X in $\text{PSaddle}_d(X)$ is finite. If one of these orbits (say O) do not belong to $\text{Cl}(\text{Cl}(\text{PSaddle}_d(X))\setminus\{x \in \text{PSaddle}_d(X) : t_x < T_0\})$ then it must happen that O is isolated in the sense that $\text{Cl}(\text{PSaddle}_d(X)) \setminus O$ is a closed subset. Therefore, up to a finite number of isolated periodic orbits, we can assume that the set $\text{PSaddle}^{T_0}_d(X) = \{p \in \text{PSaddle}_d(X) : t_{p,X} \geq T_0\}$ is dense in $\text{Cl}(\text{PSaddle}_d(X))$. Then, as in p.400 of [18] we obtain the following properties:

- If $Ind(\sigma) = 2$, then the splitting $E^{s,X} \oplus E^{cu,X}$ extends to a dominated splitting $E \oplus F$ for X over $\text{Cl}(W^u(\sigma))$ with $dim(E) = 1$ and $E^X \subset F$.
- If $Ind(\sigma) = 1$ the splitting $E^{s,-X} \oplus E^{cu,-X}$ extends to a dominated splitting $E \oplus F$ for $-X$ over $\text{Cl}(W^s(\sigma))$ with $dim(E) = 1$ and $E^{-X} \subset F$.

Therefore, we conclude from (c) and (d) above, lemmas 3.2 and 3.4 in [8] and Theorem D in [17] that if $X \in \mathcal{R}_{15} \cap S(M)$ and $\sigma \in \text{Sing}(X) \cap \text{Cl}(\text{PSaddle}_d(X))$, then:

- If $Ind(\sigma) = 2$, then $\text{Cl}(W^u(\sigma))$ is a singular-hyperbolic attractor for X.
- If $Ind(\sigma) = 1$, then $\text{Cl}(W^s(\sigma))$ is a singular-hyperbolic attractor for $-X$.

Next, we define $\phi : \mathfrak{X}^1(\text{M}) \to 2^M_c$ by $\phi(X) = \text{Cl}(\text{Sink}(X))$. This map is clearly lower semicontinuous, and so, upper semicontinuous in a residual subset $\mathcal{C}$ of $\mathfrak{X}^1(\text{M})$ ([15], [14]). If $X \in \mathcal{C}$ satisfies $\text{card}(\text{Sink}(X)) < \infty$, then the upper semicontinuity of ϕ at X do imply $X \in S(M)$.

Finally we define
$$\mathcal{Q} = \mathcal{R}_{15} \cap \mathcal{C}.$$
Clearly $\mathcal{Q}$ is a residual subset of three-dimensional flows.

Take $X \in \mathcal{Q}$ with $\text{card}(\text{Sink}(X)) < \infty$. Since $X \in \mathcal{C}$, we obtain $X \in S(M)$ thus $X \in \mathcal{R}_{15} \cap S(M)$. Then, if $\sigma \in \text{Sing}(X) \cap \text{Cl}(\text{PSaddle}_d(X))$, $\text{Cl}(W^u(\sigma))$ is singular-hyperbolic for X (if $Ind(\sigma) = 2$) and that $\text{Cl}(W^s(\sigma))$ is a singular-hyperbolic attractor for $-X$ (if $Ind(\sigma) = 1$).

Now we observe that if $p \in \text{PSaddle}_d(X)$ then $H(p) \subset \text{Cl}(\text{Saddle}_d(X))$ by the Birkhoff-Smale Theorem. From this we obtain

$$\text{Cl}(\text{PSaddle}_d(X)) = \text{Cl}\left(\bigcup\{H(p) : p \in \text{PSaddle}_d(X)\}\right). \tag{7}$$

We claim that the family $\{H(p) : p \in \text{PSaddle}_d(X)\}$ is finite. Otherwise, there is an infinite sequence $p_k \in \text{PSaddle}_d(X)$ yielding infinitely many distinct homoclinic classes $H(p_k)$. Consider the closure $\text{Cl}(\bigcup_k H(p_k))$, which is a compact invariant set contained in $\text{Cl}(\text{PSaddle}_d(X))$. If this closure does not contain any singularity, then it would be a hyperbolic set (this follows because $\mathcal{R}_{15} \subset \mathcal{Q}_1$). Since there are infinitely many distinct homoclinic classses in this closure, we obtain a contradiction proving that $\text{Cl}(\bigcup_k H(p_k))$ contains a singularity $\sigma \in \text{Cl}(\text{PSaddle}_d(X))$. If $Ind(\sigma) = 2$ then σ lies in $\text{Cl}(W^u(\sigma))$ which is an attractor, and so, we can assume that $H(p_k) \subset \text{Cl}(W^u(\sigma))$ for every k thus $H(p_k) = \text{Cl}(W^u(\sigma))$ for every k which is absurd. Analogously for $Ind(\sigma) = 1$ and the claim is proved. Combining with (7) we obtain the desired spectral decomposition. □

Proof of Theorem A. Define $\mathcal{R} = \mathcal{R}_6 \cap \mathcal{R}_{11} \cap \mathcal{Q}$, where $\mathcal{R}_6$, $\mathcal{R}_{11}$ and $\mathcal{Q}$ are the residual subsets given by theorems 2.1, 2.3 and 2.5 respectively. Suppose that $X \in \mathcal{R}$ has no infinitely many sinks. Then, $\mathrm{card}(\mathrm{Sink}(X)) < \infty$. Since $X \in \mathcal{Q}$, we conclude by Theorem 2.5 that $\mathrm{Cl}(\mathrm{PSaddle}_d(X))$ has a spectral decomposition. Since $X \in \mathcal{R}_{11}$, Theorem 2.3 implies that every homoclinic H associated to a dissipative periodic saddle of X with $\mathrm{Leb}(W^s_Y(H)) > 0$ is an attractor of X. Since $X \in \mathcal{R}_6$, we have that $\mathrm{Leb}(W^s_w(\mathrm{Dis}(X))) = 1$ by Theorem 2.1.

Now, we consider the following decomposition:

$$\mathrm{Dis}(X) = \mathrm{Cl}(\mathrm{Saddle}_d(X) \cap \mathrm{Sing}(X)) \cup \mathrm{Cl}(\mathrm{PSaddle}_d(X)) \cup \mathrm{Sink}(X),$$

valid in the Kupka-Smale case (which is generic). From this we obtain the union

$$W^s_w(\mathrm{Dis}(X)) = \left(\bigcup\{W^s(\sigma) : \sigma \in \mathrm{Saddle}_d(X) \cap \mathrm{Sing}(X) \text{ and } W^s_w(\sigma) = W^s(\sigma)\}\right) \cup$$
$$\left(\bigcup\{W^s_w(\sigma) : \sigma \in \mathrm{Saddle}_d(X) \cap \mathrm{Sing}(X) \text{ and } W^s_w(\sigma) \neq W^s(\sigma)\}\right) \cup$$
$$W^s_w(\mathrm{Cl}(\mathrm{PSaddle}_d(X))) \cup W^s(\mathrm{Sink}(X)).$$

But it is easy to check that the first element in the right-hand union above has zero Lebesgue measure and, by the Hayashi's connecting lemma [11], we can assume without loss of generality that every $\sigma \in \mathrm{Saddle}_d(X) \cap \mathrm{Sing}(X)$ satisfying $W^s_w(\sigma) \neq W^s(\sigma)$ lies in $\mathrm{Cl}(\mathrm{PSaddle}_d(X))$. Since $W^s_w(\mathrm{Dis}(X))$ has full Lebesgue measure, we conclude that

$$\mathrm{Leb}(W^s_w(\mathrm{Cl}(\mathrm{PSaddle}_d(X))) \cup W^s(\mathrm{Sink}(X))) = 1.$$

Now, we use the spectral decomposition

$$\mathrm{Cl}(\mathrm{PSaddle}_d(X)) = \bigcup_{i=1}^{r} H_i$$

into finitely many disjoint homoclinic classes H_i, $1 \leq i \leq r$, each one being either hyperbolic (if $H_i \cap \mathrm{Sing}(X) = \emptyset$) or a singular-hyperbolic attractor for either X or $-X$ (otherwise), yielding

$$\mathrm{Leb}\left(\left(\bigcup_{i=1}^{r} W^s_w(H_i)\right) \cup W^s(\mathrm{Sink}(X))\right) = 1.$$

But the results in Section 3 of [9] imply that each H_i can be written as $H_i = \Lambda^+ \cap \Lambda^-$, where $\Lambda^\pm$ is a Lyapunov stable set for $\pm X$. We conclude from Lemma 2.2 in [9] that $W^s_w(H_i) = W^s(H_i)$ thus

$$\mathrm{Leb}\left(\left(\bigcup_{i=1}^{r} W^s(H_i)\right) \cup W^s(\mathrm{Sink}(X))\right) = 1.$$

Let $1 \leq i_1 \leq \cdots \leq i_d \leq r$ be such that $\mathrm{Leb}(W^s(H_{i_k})) > 0$ for every $1 \leq k \leq d$. As the basin of the remainder homoclinic classes in the collection $H_1, \cdots, H_r$ are negligible, we can remove them from the above union yielding

$$\mathrm{Leb}\left(\left(\bigcup_{k=1}^{d} W^s(H_{i_k})\right) \cup \left(\bigcup_{j=1}^{l} W^s(s_j)\right)\right) = 1,$$

where the s_j's above correspond to the finitely many orbits of X in $\mathrm{Sink}(X)$. Since $f \in \mathcal{R}_{11}$, we have from Theorem 2.3 that H_{i_k} is an attractor which is either hyperbolic or singular-hyperbolic for X, $\forall 1 \leq k \leq d$. From this we obtain the result. $\square$

Proof of Theorem B. Suppose by contradiction that there is a C^1 generic three-dimensional flow of an orientable manifold such that $\mathrm{Cl}(\mathrm{Sink}(X))\setminus \mathrm{Sink}(X)$ has a LPF-dominated splitting. Then, $\mathrm{card}(\mathrm{Sink}(X))=\infty$ and X has finitely many periodic sinks with nonreal eigenvalues. Since X is C^1 generic, we obtain that the number of orbits of sinks with nonreal eigenvalues is locally constant at X. From this we can assume without loss of generality that every sink of a nearby flow is periodic with real eigenvalues. Furthermore, we obtain the following alternatives: If $Ind(\sigma)=2$, then σ is Lorenz-like for X and satisfies

$$(\mathrm{Cl}(\mathrm{Sink}(X))\setminus \mathrm{Sink}(X))\cap W^{ss,X}(\sigma)=\{\sigma\},$$

and, if $Ind(\sigma)=1$, then σ is Lorenz-like for $-X$ and satisfies

$$(\mathrm{Cl}(\mathrm{Sink}(X))\setminus \mathrm{Sink}(X))\cap W^{uu,X}(\sigma)=\{\sigma\}.$$

As before, these alternatives imply the following ones:

(1) If $Ind(\sigma)=2$, then every $\sigma'\in \mathrm{Sing}(X)\cap \mathrm{Cl}(W^u(\sigma))$ is Lorenz-like for X.
(2) If $Ind(\sigma)=1$, then every $\sigma'\in \mathrm{Sing}(X)\cap \mathrm{Cl}(W^s(\sigma))$ is Lorenz-like for $-X$.

For any $p\in \mathrm{Per}(X)$ we denote by $\lambda(p,X)$ and $\mu(p,X)$ the two eigenvalues of p so that

$$|\lambda(p,X)|\leq|\mu(p,X)|.$$

The corresponding eigenspaces will be denoted by $E^{-,X}_p$ and $E^{+,X}_p$. We have the symmetric relations

$$\lambda(p,-X)=\mu^{-1}(p,X), \mu(p,-X)=\lambda^{-1}(p,X), E^{-,-X}_p=E^{+,X}_p, E^{+,-X}_p=E^{-,X}_p.$$

We obtain from the fact that the number of sinks with nonreal eigenvalues is locally constant at X that there is a fixed number $0<\lambda<1$ and a neighborhood $\mathcal{U}_X$ of X satisfying:

(a) $\frac{|\lambda(p,Y)|}{|\mu(p,Y)|}\leq \lambda^{t_{p,Y}}$ and
(b) $\mathrm{angle}(E^{-,X}_p,E^{+,Y}_p)>\alpha$, for every $(p,Y)\in \mathrm{Sink}(Y)\times\mathcal{U}_X$.

Using these properties we obtain as in the proof of Theorem 2.5 that there are neighborhoods $\mathcal{V}_X$ of X, U_σ of σ and $\beta_\sigma>0$ such that if $Y\in\mathcal{V}_X$ and $x\in \mathrm{Sink}(Y)$ satisfies $O_Y(x)\cap U_\sigma\neq\emptyset$, then

$$\mathrm{angle}(E^{-,Y}_x,E^{cu,Y}_x)>\beta_\sigma, \qquad \text{if } Ind(\sigma)=2$$

and

$$\mathrm{angle}(E^{-,-Y}_x,E^{cu,-Y}_x)>\beta_\sigma, \qquad \text{if } Ind(\sigma)=1.$$

Consequently there are a neighborhood $\mathcal{K}_X$ of X, $0<\rho<1$, $c>0$, $\delta>0$ and $T_0>0$ satisfying the following properties for every $Y\in\mathcal{K}_X$ and every $x\in \mathrm{Cl}(\mathrm{Sink}(Y))\setminus \mathrm{Sink}(Y)$ satisfying $t_{x,Y}>T_0$ and $O_Y(x)\cap B_\delta(\sigma)\neq\emptyset$:

- If $Ind(\sigma)=2$, then

$$\|DY_T(p)/E^{-,Y}_p\|\cdot\|DY_{-T}(p)/E^{cu,Y}_{Y_{-T}(p)}\|\leq c\rho^T, \qquad \forall T>0.$$

- If $Ind(\sigma)=1$, then

$$\|D(-Y)_T(p)/E^{-,-Y}_p\|\cdot\|D(-Y)_{-T}(p)/E^{cu,-Y}_{(-Y)_{-T}(p)}\|\leq c\rho^T, \qquad \forall T>0.$$

Using these dominations as before we obtain the following:

- If $Ind(\sigma)=2$, then $\mathrm{Cl}(W^u(\sigma))$ is a singular-hyperbolic attractor for X.
- If $Ind(\sigma)=1$, then $\mathrm{Cl}(W^s(\sigma))$ is a singular-hyperbolic attractor for $-X$.

Since a singular-hyperbolic attractor for either X or $-X$ cannot be accumulated by sinks we conclude that

$$\mathrm{Sing}(X) \cap (\mathrm{Cl}(\mathrm{Sink}(X)) \setminus \mathrm{Sink}(X)) = \emptyset.$$

Since there is a LPF-dominated splitting, we conclude that $\mathrm{Cl}(\mathrm{Sink}(X)) \setminus \mathrm{Sink}(X)$ is a hyperbolic set. Since there are only a finite number of orbits of sinks in a neighborhood of a hyperbolic set, we conclude that $\mathrm{card}(\mathrm{Sink}(X)) < \infty$ which is absurd. This concludes the proof. □

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