On the density of singular hyperbolic three-dimensional vector fields: a conjecture of Palis

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Abstract. In this note we announce a result for vector fields on three-dimensional manifolds: those who are singular hyperbolic or exhibit a homoclinic tangency form a dense subset of the space of C^1-vector fields. This answers a conjecture by Palis. The argument uses an extension for local fibered flows of Mañé and Pujals-Sambarino's theorems about the uniform contraction of one-dimensional dominated bundles.

Sur la densité de l'hyperbolicité singulière pour les champs de vecteurs en dimension trois : une conjecture de Palis

Résumé. *Dans cette note, nous annonçons un résultat portant sur les champs de vecteurs des variétés de dimension 3 : ceux qui vérifient l'hyperbolicité singulière ou qui possèdent une tangence homocline forment un sous-ensemble dense de l'espace des champs de vecteurs C^1. Ceci répond à une conjecture de Palis. La démonstration utilise une généralisation pour les flots fibrés locaux des théorèmes de Mañé et Pujals-Sambarino traitant de la contraction uniforme de fibrés unidimensionnels dominés.*

1. Introduction

We are concerned with dynamics which are typical in the space of dynamical systems. Hyperbolic systems are natural candidates since they form an open set. However there are obstructions for hyperbolicity, such as homoclinic tangencies which produces rich behavior as Newhouse phenomena [13]. Palis [14, 15] conjectured that on surfaces, every diffeomorphism can be accumulated either by hyperbolic ones or by diffeomorphisms with a homoclinic tangency. Pujals and Sambarino [17] managed to prove the conjecture in the C^1 topology. For higher dimensions, other homoclinic bifurcations have to be included in Palis conjecture, such as heterodimensional cycles. For the case of vector fields, new kinds of bifurcations, associated to singularities, and called singular cycles, have to be introduced. A version of Palis conjecture for vector fields can be formulated as:

CONJECTURE 1. – (Palis [16]) – *Every vector field can be accumulated either by hyperbolic vector fields or by ones with a homoclinic bifurcation or with a singular cycle.*

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Vector fields admit robustly non-hyperbolic transitive attractors with singularities [1, 6]. Morales, Pacifico and Pujals [12] defined the *singular hyperbolicity* to characterize robust attractors with singularities in dimension 3. Palis gave a stronger version of his conjecture in dimension 3 (see also [2, 11]):

CONJECTURE 2. – (Palis [16]) – *Every vector field on a three dimensional manifold can be accumulated either by singular hyperbolic vector fields or by ones with a homoclinic tangency (associated to a non-singular periodic orbit).*

Arroyo and Rodriguez-Hertz [2] proved the first conjecture for three-dimensional manifolds in the C^1 topology. The goal of this work is to get a positive answer of the second one in the C^1 topology. Generalizations of singular hyperbolicity in higher dimension have been proposed [10, 18], but it is not clear for us what should be the generalization of Conjecture 2. Since Newhouse phenomenon requires C^2-smoothness, an even stronger result could be imagined, specific to the C^1-topology: singular hyperbolicy may be C^1-dense in the space of three dimensional vector fields.

2. Precise statements

Let M be a three-dimensional compact Riemannian manifold without boundary. A smooth vector field X on M generates a flow φ_t. A point x is *regular* if $X(x) \neq 0$; otherwise, x is *singular*. The set of singularities plays a particular role and is denoted by $\mathrm{Sing}(X)$. The derivative $\mathrm{D}\varphi_t$ of φ_t w.r.t. the space variable is called the *tangent flow*.

The dynamics of the flow is usually split in the following way. For any $\varepsilon > 0$, an *ε-pseudo orbit from x to y* is a sequence $\{x_i\}_{i=0}^n$ such that $x_0 = x$, $x_n = y$, $n \geqslant 1$ and such that $d(\varphi_{t_i}(x_i), x_{i+1}) < \varepsilon$ for any $i \in \{0, \dots, n-1\}$ and for some $\{t_i\}_{i=0}^{n-1}$ in $[1, 2]$. Any two points $x, y \in M$ are said to be *chain related* if for any $\varepsilon > 0$, there are ε-pseudo orbits from x to y and from y to x. If x is chain related to itself, then x is called a *chain recurrent point*. The (compact invariant) set of chain recurrent points is called the *chain recurrent set* of X. To be chain related is an equivalence relation on the chain recurrent set; the equivalence classes are called *chain recurrence classes* of X. They are compact, invariant (and pairewise disjoint).

Let Λ be an invariant compact set. An invariant continuous splitting $T_\Lambda M = E \oplus F$ (where E, F are non-trivial vector bundles) is *dominated* if there exist constants $C, \lambda > 0$ such that for any $t > 0$, any $x \in \Lambda$ and for any unit vectors $u \in E(x)$ and $v \in F(x)$, we have $\|\mathrm{D}\varphi_t(u)\| \leqslant C\mathrm{e}^{-\lambda t}\|\mathrm{D}\varphi_t(v)\|$.

The set Λ is *hyperbolic*, if there exist a continuous invariant splitting $T_\Lambda M = E^s \oplus \langle X \rangle \oplus E^u$ w.r.t. $\mathrm{D}\varphi_t$ and constants $C, \lambda > 0$ such that for any $x \in \Lambda$ and $t > 0$, one has $\|D\varphi_t|_{E^s(x)}\| \leqslant C\mathrm{e}^{-\lambda t}$ and $\|D\varphi_{-t}|_{E^u(x)}\| \leqslant C\mathrm{e}^{-\lambda t}$.

An attractor Λ is *singular hyperbolic* if there is a dominated splitting $T_\Lambda M = E^{ss} \oplus E^{cu}$ w.r.t. $\mathrm{D}\varphi_t$ and constants $C > 0$ and $\lambda > 0$ such that:

– Contraction: for any $t > 0$ and any $x \in \Lambda$, $\|\mathrm{D}\varphi_t|_{E^{ss}(x)}\| \leqslant C\mathrm{e}^{-\lambda t}$.

– Area-expansion: for any $t > 0$ and any $x \in \Lambda$, $|\mathrm{Det}\varphi_{-t}|_{E^{cu}(x)}| \leqslant C\mathrm{e}^{-\lambda t}$.

A transitive repeller is *singular hyperbolic* if it is a singular hyperbolic attractor for $-X$.

We say that X is *singular hyperbolic* if the chain recurrent set of X is the union of finitely pairewise disjoint invariant compact sets $\{\Lambda_i\}$ such that each Λ_i is a hyperbolic set, a singular hyperbolic attractor, or a singular hyperbolic repeller. X is *robustly* singular hyperbolic if any vector field C^1-close to X is singular hyperbolic.

We say that X has a *homoclinic tangency* if X has a (non-singular) hyperbolic periodic orbit γ such that the stable manifold of γ and the unstable manifold of γ have some non-transverse intersection.

We announce an answer to the above Palis conjecture 2 is:

MAIN THEOREM . – *In the C^1 topology, every three dimensional vector field can be accumulated by robustly singular hyperbolic vector fields, or by vector fields with homoclinic tangencies.*

For proving this theorem, we only need to prove the following (which was already obtained by Arroyo and Rodriguez-Hertz [2] for non-singular chain-recurrence classes).

MAIN THEOREM RESTATED . – *For C^1 generic vector field X on a three-dimensional manifold which is far from homoclinic tangencies, any chain-recurrence class is hyperbolic or is a singular hyperbolic attractor or repeller.*

3. Dominated splittings for tangent *vs* linear Poincaré flows

The dynamics induces another linear flow above the set of regular points. At any point $x \in M \setminus \mathrm{Sing}(X)$ we introduce the plane $\mathcal{N}_x = X(x)^\perp$ and define the normal bundle $\mathcal{N} = \coprod_{x \in M \setminus \mathrm{Sing}(X)} \mathcal{N}_x$. By orthogonal projection of the tangent flow $\mathrm{D}\varphi_t$, one gets the *linear Poincaré flow* ψ_t on $\mathcal{N}$.

Let C be any chain-recurrence class (which is not an isolated singularity) of a C^1-generic vector fields far from homoclinic tangencies. Using technics developed in the different works on robustly transitive sets [12, 7, 11] and Liao's estimation [8], Gan and Yang [5] have proved the following properties (up to replace X by $-X$):

1. Any non-isolated singularity $\sigma \in C$ is *Lorenz-like*: it has three real eigenvalues satisfying $\lambda_1 < \lambda_2 < 0 < -\lambda_2 < \lambda_3$.
   Moreover the (one-dimensional) strong stable manifold satisfies $W^{ss}(\sigma) \cap C = \{\sigma\}$.
2. The linear Poincaré flow on $C \setminus \mathrm{Sing(X)}$ has a dominated splitting.
3. [5, Theorem C]. If the tangent flow $\mathrm{D}\varphi_t$ on Λ has a dominated splitting and if C contains a singularity, then C is a singular hyperbolic attractor.

Thus our main theorem essentially reduces to compare the dominated splitting of the linear Poincaré flow and of the tangent flow. (Note that the following result goes beyond C^1-generic vector fields.)

THEOREM 1. – Dominated splitting for the tangent flow. – *Consider any C^3 three-dimensional vector field X and any compact invariant set Λ with the following properties:*
- *Every periodic orbit in Λ is a hyperbolic saddle.*
- *Every singularity $\sigma \in \Lambda$ is Lorenz-like and $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$.*
- *Λ does not contain a minimal repeller whose dynamics is the suspension of an irrational rotation of the circle.*

Then the tangent flow $\mathrm{D}\varphi_t$ on Λ has a dominated splitting if and only if the linear Poincaré flow on $\Lambda \setminus \mathrm{Sing(X)}$ has a dominated splitting.

4. Uniform contraction for dominated fibered dynamics

The existence of singularities introduces several difficulties: the regular orbits may separate when passing the singularities, which causes a lack of compactness and of uniformity. These problems have been overcome in previous works in two ways:
- by blowing up the singular set and extending the flow (introduced by Li-Gan-Wen [7]),
- by rescaling the flow near the singular set (in Liao's work [8] and in [5]).

These ideas may be applied to several flows associated to the initial flow φ_t: the flow itself, the sectional Poincaré flow, their tangent flows,...

In this work we get a compactification of the rescaled sectional Poincaré flow that we define now. For any regular points x and time $t_0 \in \mathbb{R}$, the flow φ_t induces a local holonomy map from a neighborhood of x in $\exp_x(\mathcal{N}_x)$ to a neighborhood of $\varphi_t(x)$ in $\exp_{\varphi_t(x)}(\mathcal{N}_{\varphi_t(x)})$. This induces a local map P_{t_0} from $\mathcal{N}_x$ to $\mathcal{N}_{\varphi_t(x)}$ which preserves 0 and gives a local flow P_t on the bundle $\mathcal{N}$ in a neighborhood of the 0-section. It is called the *sectional Poincaré flow*. Its linearization at the 0-section is the linear Poincaré flow ψ_t.

For these flows, we define the rescaled sectional Poincaré flow and the rescaled linear Poincaré flow as:

$$P_t^*(v_x) = \|X(\varphi_t(x))\|^{-1} \cdot P_t\big(\|X(x)\| \cdot v_x\big) \text{ and } \psi_t^*(v_x) = \frac{\|X(x)\|}{\|X(\varphi_t(x))\|} \cdot \psi_t(v_x).$$

Still ψ_t^* can be seen as the linearization of P_t^*.

THEOREM 2. – Compactification of the sectional Poincaré flow. – *Assume that the flow φ_t is C^r, $r \geqslant 2$, and let K be an invariant compact set whose singularities $\mathrm{Sing}(X) \cap K$ are hyperbolic. Then, there exists a flow $\widehat{\varphi}_t$ on a compact metric space $\widehat{K}$ and a local flow $\widehat{P}_t^*$ on a linear bundle $\widehat{\mathcal{N}}$ over the dynamics of $\widehat{\varphi}_t$ on $\widehat{K}$, which preserves the 0-section, such that:*

- *There exists an injective continuous fiber-preserving map $p\colon \mathcal{N} \to \widehat{\mathcal{N}}$ which conjugates the local flows P_t^* and $\widehat{P}_t^*$.*
- *$\widehat{P}_t^*$ is C^{r-1}-along fibers: it induces local C^{r-1}-diffeomorphisms $(\widehat{\mathcal{N}}_x, 0) \to (\widehat{\mathcal{N}}_{\widehat{\varphi}_t(x)}, 0)$ depending continuously on x for the C^{r-1}-topology.*

Theorem 1 can be translated for the flow $\widehat{P}_t^*$ with the following remarks:

- If the linear Poincaré flow ψ_t has a dominated splitting, then the 0-section in $\widehat{\mathcal{N}}$ has a dominated splitting $T\widehat{\mathcal{N}} = \widehat{E} \oplus \widehat{F}$ for the compactified rescaled flow $\widehat{P}_t^*$.
- If moreover $\widehat{E}$ is uniformly contracted by the linearization of $\widehat{P}_t^*$, then K has a dominated splitting $T_K M = E \oplus F$ for the tangent flow $\mathrm{D}\varphi_t$, with $\dim(E) = 1$ and $\mathbb{R}.X \subset F$.

Theorem 1 is thus a consequence of the theorem below for dominated local fibered flow $\widehat{P}_t^*$ on a bundle $\widehat{\mathcal{N}}$ with 2-dimensional fibers. It requires an important *compatibility assumption* which reflects the fact that $\widehat{P}_t^*$ has been obtained from the sectional Poincaré flow on a manifold:

1. There exists an open set $U \subset \widehat{K}$ and for close points $x, y \in U$ an identification map $\pi_{x,y}\colon \widehat{\mathcal{N}}_x \to \widehat{\mathcal{N}}_y$ which is "compatible" with the local flow $\widehat{P}_t^*$.
2. Along pieces of orbit in $\widehat{K} \setminus U$, the bundle $\widehat{E}$ is uniformly contracted.

THEOREM 3. – Uniform contraction of one-dimensional bundle for dominated local fibered flow. – *Let $\widehat{P}_t^*$ be a local fibered flow which is C^2-along fibers, on a bundle $\widehat{\mathcal{N}}$ with 2-dimensional fibers satisfying:*

- *the 0-section is invariant and has a dominated splitting $T\widehat{\mathcal{N}} = \widehat{E} \oplus \widehat{F}$;*
- *the compatibility assumption holds;*
- *E is contracted along each periodic orbit of the base space $\widehat{K}$ of $\widehat{\mathcal{N}}$;*
- *the base space $\widehat{K}$ of $\widehat{\mathcal{N}}$ does not contain any invariant compact set which is a repeller and whose dynamics is conjugated to the suspension of an irrational circle rotation.*

Then $\widehat{E}$ is uniformly contracted by the linearization of the flow $\widehat{P}_t^*$.

This theorem is a continuation of a sequence of works initiated by Mañé [9] and Pujals-Sambarino [17]: the later proves the hyperbolicity of dominated invariant compact set for surface diffeomorphisms. Compared with the usual Pujals-Sambarino arguments, we meet the following difficulties:

- We work with continuous time dynamics. It is not enough to consider the time-one map since the flow shears along the orbits. Bounded time evolution produces small shear, but we need to consider long-term behaviors. This difficulty already appears in Arroyo and Rodriguez-Hertz' result [2].
- There is no extension of Pujals-Sambarino result to general dominated fibered dynamics (as one can see considering a product of a minimal base dynamics with the identity along fibers). Our proof uses strongly the identification maps $\pi_{x,y}$.
- Since the identifications $\pi_{x,y}$ are only defined on U, we have to handle with the induced dynamics in U. This makes us to consider "induced hyperbolic returns" (as in [3]).

– The flow $\widehat{P}_t^*$ is locally defined and the global arguments of [17] have to be replaced by local ones.
– We need to construct some Markovian boxes in a non-symmetric setting: the bundles $\widehat{E}, \widehat{F}$ play different roles. We borrow some ideas from a recent work of Crovisier, Pujals and Sambarino [4].

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