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EXISTENCE OF ATTRACTORS FOR THREE-DIMENSIONAL FLOWS

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1. Introduction

Araujo proved in the eighties that every C^1 generic surface diffeomorphism has either infinitely many attracting periodic orbits (often called *sinks*) or else finitely many hyperbolic attractors whose basins form a full Lebesgue measure in the ambient manifold [2]. Therefore, hyperbolic attractors do exist for all C^1 generic surface diffeomorphisms (solving a question by René Thom [26]). It is natural to ask if these results hold for three-dimensional flows instead of surface diffeomorphisms. Although this is true for nonsingular flows (as proved recently in [5]), the answer for this question is negative. In fact, [22] used the geometric Lorenz attractor [15] in order to construct an open set of three-dimensional flows in the sphere S^3 for which there are no hyperbolic attractors. But this last example exhibits a singular-hyperbolic attractor (e.g. the geometric Lorenz one) whose basin has full Lebesgue measure. It is then reasonable to ask if Araujo's holds for three-dimensional flows but replacing the hyperbolic attractor alternative by the hyperbolic or singular-hyperbolic attractor one.

In this paper we will give positive answer for this last question. More precisely, we will prove that every C^1 generic three-dimensional flow has either infinitely many sinks or finitely many hyperbolic or singular-hyperbolic attractors whose basins form a full Lebesgue measure set. In particular, every C^1 generic three-dimensional flow carries hyperbolic or singular-hyperbolic attractors. The proof will use a recent result by Crovisier and Yang [12]. Let us state our results in a precise way.

Hereafter, the term *three-dimensional flow* will be referred to a C^1 vector field on a Riemannian compact connected boundaryless three-dimensional manifold M. The corresponding space equipped with the C^1 vector field topology will be denoted by $\mathfrak{X}^1(\mathrm{M})$. The flow of $X \in \mathfrak{X}^1(\mathrm{M})$ is denoted by X_t, $t \in \mathbb{R}$.

By a *critical point* of X we mean a point x satisfying $X_t(x) = x$ for some $t \geq 0$. If this is satisfied for every $t \geq 0$ we say that x is a *singularity*, otherwise it is a *periodic point*. For every periodic point we have a minimal $t > 0$ satisfying $X_t(x) = x$. The minimal of such t's is the period of x denoted by t_x (or $t_{x,X}$ to indicate X). We denote by $\mathrm{Crit}(X)$ the set of critical points, by $\mathrm{Sing}(X)$ the set of singularities and by $\mathrm{Per}(X)$ the set of periodic points thus $\mathrm{Crit}(X) = \mathrm{Sing}(X) \cup \mathrm{Per}(X)$. The *eigenvalues* of a critical point x are either those of the linear automorphism $DX_{t_x}(x) : T_xM \to T_xM$ not corresponding to the eigenvector $X(x)$ (periodic case) or those of $DX(x) : T_xM \to T_xM$ (singular case). We say that x is a *sink* if its

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eigenvalues either are less than 1 in modulus (periodic case) or else with negative real part (singular case). A *source* is a sink for the time-reversed flow $-X$. The set of sinks and sources of X will be denoted by $\mathrm{Sink}(X)$ and $\mathrm{Source}(X)$ respectively. A critical point is *hyperbolic* if it has no eigenvalues of modulus 1 (periodic case) or with zero real part (singular case).

For every point x we define its *omega-limit set*,

$$\omega(x) = \left\{ y \in M : y = \lim_{t_k \to \infty} X_{t_k}(x) \text{ for some integer sequence } t_k \to \infty \right\}.$$

(If necessary we shall write $\omega_X(x)$ to indicate the dependence on X.)

We say that $\Lambda \subset M$ is *invariant* if $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$; and *transitive* if there is $x \in \Lambda$ such that $\Lambda = \omega(x)$. The *basin* of any subset $\Lambda \subset M$ is defined by

$$W^s(\Lambda) = \{y \in M : \omega(y) \subset \Lambda\}.$$

(Sometimes we write $W^s_X(\Lambda)$ to indicate dependence on X). An *attractor* is a transitive set A exhibiting a neighborhood U such that

$$A = \bigcap_{t \geq 0} X_t(U).$$

A compact invariant set Λ is *hyperbolic* if there are a continuous DX_t-invariant tangent bundle decomposition $T_\Lambda M = E^s_\Lambda \oplus E^X_\Lambda \oplus E^u_\Lambda$ over Λ and positive numbers K, λ such that E^X_x is generated by $X(x)$, and for every $(x,t) \in \Lambda \times \mathbb{R}^+$ we have

$$\|DX_t(x)/E^s_x\| \leq Ke^{-\lambda t} \text{ and } \|DX_{-t}(x)/E^u_{X_t(x)}\| \leq Ke^{-\lambda t},$$

A *dominated splitting* for X over an invariant set I is a continuous tangent bundle DX_t-invariant splitting $T_I M = E_I \oplus F_I$ for which there are positive constants K, λ satisfying

$$\|DX_t(x)/E_x\| \cdot \|DX_{-t}(X_t(x))/F_{X_t(x)}\| \leq Ke^{-\lambda t}, \qquad \text{for all } (x,t) \in I \times \mathbb{R}^+.$$

We say that the dominating subbundle E above is *contracting* if

$$\|DX_t(x)/E_x\| \leq Ke^{-\lambda t}, \qquad \text{for all } (x,t) \in I \times \mathbb{R}^+$$

and that the central subbundle F is *volume expanding* if

$$|\det DX_t(x)/F_x| \geq K^{-1}e^{\lambda t}, \qquad \text{for all } (x,t) \in I \times \mathbb{R}^+.$$

A compact invariant set is *partially hyperbolic* if it has a dominated splitting with contracting dominating direction.

A partially hyperbolic set Λ is *singular-hyperbolic for X* if the singularities of X in Λ are all hyperbolic and the central subbundle F is volume expanding. A *hyperbolic* (resp. *singular-hyperbolic*) *attractor (for X)* is an attractor which is simultaneously a hyperbolic (resp. singular-hyperbolic) set for X.

We call $\mathcal{R} \subset \mathfrak{X}^1(\mathrm{M})$ *residual* if it is a countable intersection of open and dense subsets. We say that a C^1 *generic three-dimensional flow satisfies a certain property* P if there is a residual subset $\mathcal{R}$ of $\mathfrak{X}^1(\mathrm{M})$ such that P holds for every element of $\mathcal{R}$.

With these definitions we can state our main result.

Theorem A. *A C^1 generic three-dimensional flow has either infinitely many sinks or finitely many hyperbolic or singular-hyperbolic attractors whose basins form a full Lebesgue measure set of M.*

In particular, we obtain the existence of hyperbolic or singular-hyperbolic attractors for C^1 generic three-dimensional flows:

Corollary 1.1. *For every C^1 generic three-dimensional flow, there exists a hyperbolic or singular-hyperbolic attractor.*

To prove Theorem A we will need the existence of a spectral decomposition of a certain invariant set. To introduce it we will need some preliminars.

A critical point is a *saddle* if it has eigenvalues of modulus less and bigger than 1 simultaneously (periodic case) or with positive and negative real part simultaneously (singular case). The set of periodic saddles of X is denoted by $\mathrm{PSaddle}(X)$.

As is well known [18], through any $x \in \mathrm{PSaddle}(X)$ it passes a pair of invariant manifolds, the so-called strong stable and unstable manifolds $W^{ss}(x)$ and $W^{uu}(x)$, tangent at x to the eigenspaces corresponding to the eigenvalue of modulus less and bigger than 1 respectively. Saturating these manifolds with the flow we obtain the stable and unstable manifolds $W^s(x)$ and $W^u(x)$ respectively. A *homoclinic point* associated to x is a point q where $W^s(x)$ and $W^u(x)$ meet. We say that q is a *transverse homoclinic point* if $T_qW^s(x) \cap T_qW^u(x)$ is one-dimensional, otherwise we call it *homoclinic tangency*. The *homoclinic class* associated to x, denoted by $H(x)$, is the closure of the set of transverse homoclinic points associated to x. We write $H_X(x)$ to indicate dependence on X. By a homoclinic class we mean the homoclinic class associated to some saddle of X. We denoted by $\mathrm{Cl}(\cdot)$ the closure operator.

Definition 1.2. *A non-empty subset $\mathcal{P} \subset \mathrm{PSaddle}(X)$ is* homoclinically closed *if $H(p) \subset \mathrm{Cl}(\mathcal{P})$ for every $p \in \mathcal{P}$.*

Basic examples of homoclinically closed subsets are $\mathrm{PSaddle}(X)$ itself and also the set $\mathrm{PSaddle}_d(X)$ of *dissipative saddles*, i.e., those saddles for which the product of the eigenvalues is less than 1 in modulus. This follows from the Birkhoff-Smale Theorem [16].

Definition 1.3. *We say that a compact invariant set of X* has a spectral decomposition *if it is a finite disjoint union of transitive sets, each one being either hyperbolic or a singular-hyperbolic attractor for either X or $-X$.*

The following result will give a sufficient condition for existence of spectral decomposition for the closure of homoclinically closed subsets of saddles. Given $\Lambda \subset M$ we define $\Lambda^* = \Lambda \setminus \mathrm{Sing}(X)$. We define the vector bundle N^X over M^* whose fiber at $x \in M^*$ is the the orthogonal complement of $X(x)$ in T_xM.

Denoting the projection $\pi_x : T_xM \to N_x^X$ we define the *Linear Poincaré flow* (LPF), $P_t^X : N^X \to N^X$, by $P_t^X(x) = \pi_{X_t(x)} \circ DX_t(x)$ whenever $t \in \mathbb{R}$. We say that Λ of X *has a LPF-dominated splitting* if $\Lambda^* \neq \emptyset$ and there exist a continuous tangent bundle decomposition $N^X = N^{s,X} \oplus N^{u,X}$ over Λ^* with $dim N_x^{s,X} = dim N_x^{u,X} = 1$ (for every $x \in \Lambda^*$) and $T > 0$ such that $\left\| P_T^X(x)/N_x^{s,X} \right\| \left\| P_{-T}^X(Y_T(x))/N_{X_T(x)}^{u,X} \right\| \leq \frac{1}{2}$, $\forall x \in \Lambda^*$.

With these definitions we obtain the following result.

Theorem B. *Let X be a C^1 generic three-dimensional flow and $\mathcal{P} \subset \mathrm{PSaddle}(X)$ be homoclinically closed. If $\mathrm{Cl}(\mathcal{P})$ has a LPF-dominated splitting, then $\mathrm{Cl}(\mathcal{P})$ has a spectral decomposition.*

This result will be proved in the next section using the recent work [12] by Crovisier and Yang ([1]).

Theorem B is strong enough to solve a conjecture in [22]. Indeed, define the nonwandering set $\Omega(X)$ of a flow X as the set of those points $x \in M$ such that for every neighborhood U of x and every $T > 0$ there is $t > T$ such that $X_t(U) \cap U \neq \emptyset$. Clearly $\Omega(X)$ is a nonempty compact invariant set of X. We say that X is *singular-Axiom A* if $\Omega(X)$ has a spectral decomposition [22]. In that a case we say that X has no cycles if there are not finitely many nonsingular orbits joining the pieces of the spectral decomposition in a cyclic way. A *robustly singular-Axiom A flow* a flow for which every nearby flow is singular-Axiom A. Now we state the aforementioned conjecture in [22] (see Conjecture 1.3 in p. 1577 of [22]):

Conjecture 1.4. *Every three-dimensional flow can be C^1 approximated by a flow exhibiting a homoclinic tangency or by a singular-Axiom A flow without cycles.*

Let us prove this conjecture using Theorem B.

Define $R(M)$ as the (open) set of three-dimensional flows which cannot be C^1 approximated by flows with a homoclinic tangency. *The following sequence of assertions should be understood for C^1 generic $X \in R(M)$.*

As is well-known [7], Cl(PSaddle(X)) has a LPF-dominated splitting. Since PSaddle(p) is homoclinically closed by the Birkhoff-Smale Theorem [16], we conclude from Theorem B that Cl(PSaddle(X)) has a spectral decomposition. But from the arguments in [29] we see that

$$(\mathrm{Cl}(\mathrm{Sink}(X)) \setminus \mathrm{Sink}(X)) \cup (\mathrm{Cl}(\mathrm{Source}(X)) \setminus \mathrm{Source}(X)) \subset \mathrm{Cl}(\mathrm{PSaddle}(X)).$$

As Cl(PSaddle(X)) has spectral decomposition, there are only finitely many orbits of sinks or sources close to it. This together with the previous inclusion implies $\mathrm{Cl}(\mathrm{Sink}(X)) \setminus \mathrm{Sink}(X) = \mathrm{Cl}(\mathrm{Source}(X)) \setminus \mathrm{Source}(X) = \emptyset$, or, equivalently, that $\mathrm{Sink}(X) \cup \mathrm{Source}(X)$ consists of finitely many orbits. On the other hand, we have that

$$\Omega(X) = (\mathrm{Sink}(X) \cup \mathrm{Source}(X)) \cup \mathrm{Cl}(\mathrm{PSaddle}(X))$$

by Pugh's General Density Theorem [28]. Since Cl(PSaddle(X)) has a spectral decomposition and $\mathrm{Sink}(X) \cup \mathrm{Source}(X)$ consists of finitely many orbits, we conclude that X is singular-Axiom A. The nonexistence of cycles was proved earlier [10]. This ends the proof. □

Conjecture 1.4 is closely related to the recent result announced by Crovisier and Yang in [12]: Every three dimensional flow can be C^1 approximated by robustly singular hyperbolic flows, or by flows with a homoclinic tangency (they claim to have solved a conjecture by Jacob Palis [24]). Indeed, we do not know if the approximation by singular-Axiom A flows in the conjecture can be performed by robustly singular-Axiom A flows. We stress that, unlike Axiom A flows, the singular-Axiom A flows without cycles need not be robustly singular-Axiom A in general [23].

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[1]Arbieto and Santiago [4] will prove Theorem A and Corollary 1.1 without appealing to [12]

conversations and, specially, the last one by finding inaccuracies in an earlier version of this work.

2. Proof of Theorem B

In this section we shall prove Theorem B. For this we need some preliminary results. We start with the following consequence of Lemma 3.1 in [9].

Lemma 2.1. *Every compact invariant set without singularities but with a LPF-dominated splitting of a C^1 generic three-dimensional flow is hyperbolic.*

Proof. By Lemma 3.1 in [9] we have that there is a residual subset $\mathcal{Q}_1$ of three-dimensional flows for which every transitive set without singularities but with a LPF-dominated splitting is hyperbolic. Fix $X \in \mathcal{Q}_1$ and a compact invariant set Λ without singularities but with a LPF-dominated splitting $N^X_\Lambda = N^{s,X}_\Lambda \oplus N^{u,X}_\Lambda$. Suppose by contradiction that Λ is not hyperbolic. Then, by Zorn's Lemma, there is a minimally nonhyperbolic set $\Lambda_0 \subset \Lambda$ (c.f. p.983 in [29]). Assume for a while that Λ_0 is not transitive. Then, $\omega(x)$ and $\alpha(x) = \omega_{-X}(x)$ are proper subsets of Λ_0, for every $x \in \Lambda_0$. Therefore, both sets are hyperbolic and then we have

$$\lim_{t\to\infty} \|P^X_t(x)/N^{s,X}_x\| = \lim_{t\to\infty} \|P^X_{-t}(x)/N^{u,X}_x\| = 0, \quad \text{for all } x \in \Lambda_0,$$

which easily implies that Λ_0 is hyperbolic (see [13]). Since this is a contradiction, we conclude that Λ_0 is transitive. As $X \in \mathcal{Q}_1$ and Λ_0 has a LPF-dominated splitting (by restriction), we conclude that Λ_0 is hyperbolic, a contradiction once more proving the result. □

Let Y be a three-dimensional flow. We say that $\sigma \in \mathrm{Sing}(Y)$ is *Lorenz-like for Y* if its eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are real and satisfy $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$ (up to some order). The invariant manifold theory [18] asserts the existence of *stable* and *unstable* manifolds denoted by $W^s(\sigma)$, $W^u(\sigma)$ (or $W^{s,Y}(\sigma)$, $W^{u,Y}(\sigma)$ to emphasize Y) tangent at σ to the eigenvalues $\{\lambda_2, \lambda_3\}$ and λ_1 respectively. There is an additional invariant manifold $W^{ss,Y}(\sigma)$, the *strong stable manifold*, contained in $W^{s,Y}(\sigma)$ and tangent at σ to the eigenspace corresponding to λ_1.

As in the remark after Lemma 2.13 in [9] we obtain the following.

Lemma 2.2. *If X is a C^1 generic three-dimensional flow, then every $\sigma \in \mathrm{Sing}(X)$ accumulated by periodic orbits is Lorenz-like, for either X or $-X$, depending on whether σ has three real eigenvalues $\lambda_1, \lambda_2, \lambda_3$ satisfying either $\lambda_2 < \lambda_3 < 0 < \lambda_1$ or $\lambda_2 < 0 < \lambda_3 < \lambda_1$ (up to some order).*

We shall use the following standard definitions.

Definition 2.3. *The index $Ind(\sigma)$, of a singularity σ, is the number of eigenvalues with negative real part counted with multiplicity.*

Definition 2.4. *The Hausdorff distance of two compact sets A and B is given by*

$$d_h(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}.$$

It is well known that the space of compact subsets of M is compact with this distance. Clearly the Hausdorff limit of a sequence of periodic orbits is a nonempty compact invariant set. Moreover, for C^1 generic flows the Hausdorff limit of sequences of periodic orbits are characterized as the so-called chain-transitive sets [11], [14].

The next result detects when a singularity is Lorenz-like through its index.

Lemma 2.5. *Let H be the Hausdorff limit of a sequence of periodic orbits of a C^1 generic three-dimensional flow X. If H has a LPF-dominated splitting, then every singularity $\sigma \in H \cap \mathrm{Sing}(X)$ satisfies one of the following:*

(1) *If $Ind(\sigma) = 2$, then σ is Lorenz-like for X and $H \cap W^{ss,X}(\sigma) = \{\sigma\}$.*
(2) *If $Ind(\sigma) = 1$, then σ is Lorenz-like for $-X$ and $H \cap W^{ss,-X}(\sigma) = \{\sigma\}$.*

Proof. We only prove (1) because (2) is similar. Since H has a LPF-dominated splitting, we obtain from Proposition 2.4 in [13] that σ has three different real eigenvalues $\lambda_1, \lambda_2, \lambda_3$ satisfying $\lambda_2 < \lambda_3 < 0 < \lambda_1$ (up to some order). Since $\sigma \in H$, we obtain that σ is accumulated by periodic orbits. Then, σ is Lorenz-like for X by Lemma 2.2.

To prove $H \cap W^{ss,X}(\sigma) = \{\sigma\}$ we assume by contradiction that this is not the case. Then, there is $x \in H \cap W^{ss,X}(\sigma) \setminus \{\sigma\}$. Set $H = \lim_{n\to\infty} O_n$ where each O_n is a periodic orbit of X.

Choose sequences $x_n \in O_n$ and $t_n \to \infty$ such that $x_n \to x$ and $X_{t_n}(x_n) \to y$ for some $y \in W^{u,X}(\sigma) \setminus \{\sigma\}$. Let $N^{s,X} \oplus N^{u,X}$ denote the LPF-dominated splitting of H. Since H is clearly connected, we can assume without loss of generality that this splitting is defined in the union $\bigcup_n O_n$ (see Lemma 2.29 p.41 in [3]).

On the one hand, $N^{s,X}_x = N_x \cap W^{s,X}(\sigma)$ by Proposition 2.2 in [13] and so $N^{s,X}_{x_n}$ *tends to be tangent to $W^{s,X}(\sigma)$ at x for n large.*

On the other hand, Proposition 2.4 in [13] says that $N^{s,X}_y$ is almost parallel to $E^{ss,X}_\sigma$, and so, the directions $N^{s,X}_{X_{t_n}(x_n)}$ tends to be parallel to $E^{ss,X}_\sigma$.

Since $\lambda_2 < \lambda_3$ and $N^{s,X}_{x_n} = P_{-t_n}(X_{t_n}(x_n))N^{s,X}_{X_{t_n}(x_n)}$, we conclude that $N^{s,X}_{x_n}$ *tends to be transversal to $W^{s,X}(\sigma)$ at x for n large.*

Since these two behaviors are contradictory, we obtain the result. □

Recall that a compact invariant set Λ of a flow X is *Lyapunov stable for X* if for every neighborhood U of Λ there is a neighborhood $V \subset U$ of Λ such that $X_t(V) \subset U$, for all $t \geq 0$.

Let Λ be a compact invariant set with singularities (all hyperbolic) of X. We say that Λ *has dense singular unstable* (resp. *stable) branches* if for every $\sigma \in \Lambda \cap \mathrm{Sing}(X)$ one has $\Lambda = \omega(q)$ (resp. $\Lambda = \omega_{-X}(q)$) for all $q \in W^u(\sigma) \setminus \sigma$ (resp. $q \in W^s(\sigma) \setminus \{\sigma\}$).

The results in [10], [22] imply the following lemma.

Lemma 2.6. *If H is the Hausdorff limit of a sequence of periodic orbits of a C^1 generic three-dimensional flow X, then the following alternatives hold for every $\sigma \in H \cap \mathrm{Sing}(X)$:*

(1) *If $Ind(\sigma) = 2$, then $\mathrm{Cl}(W^u(\sigma))$ is a Lyapunov stable set for X with dense singular unstable branches and $\mathrm{Cl}(W^u(\sigma)) = H$.*
(2) *If $Ind(\sigma) = 1$, then $\mathrm{Cl}(W^s(\sigma))$ is a Lyapunov stable set for $-X$ with dense singular stable branches and $\mathrm{Cl}(W^s(\sigma)) = H$.*

Combining lemmas 2.5 and 2.6 we obtain the following result.

Corollary 2.7. *Let H be the Hausdorff limit of a sequence of periodic orbits of a C^1 generic three-dimensional flow X. If H has a LPF-dominated splitting, then one of the following alternatives hold:*

(1) *Every* $\sigma \in H \cap \mathrm{Sing}(X)$ *is Lorenz-like for* X *and* $H \cap W^{ss,X}(\sigma) = \{\sigma\}$.
(2) *Every* $\sigma \in H \cap \mathrm{Sing}(X)$ *is Lorenz-like for* $-X$ *and* $H \cap W^{ss,-X}(\sigma) = \{\sigma\}$.

This permits us to separate the Hausdorff limits (of sequences of periodic orbits) with both singularities and LPF-dominated splitting in two cases depending on whether there is a singularity of index 1 or 2.

Next we formulate the key result below by Crovisier and Yang.

Theorem 2.8 (Theorem 1 in [12])**.** *Let* Γ *be a compact invariant set with a LPF-dominated splitting of a* C^3 *three-dimensional flow* Y*. If every periodic point in* Γ *is hyperbolic saddle, every* $\sigma \in \Lambda \cap \mathrm{Sing}(Y)$ *is Lorenz-like satisfying* $W^{ss}(\sigma) \cap \Gamma = \{\sigma\}$ *and* Γ *does not contain a minimal repeller whose dynamics is the suspension of an irrational rotation of the circle, then* Γ *is* dominated *(i.e. has a dominated splitting) for* Y*.*

We shall use it to prove the following lemma.

Lemma 2.9. *Let* Λ *be a transitive set with a LPF-dominated splitting of a* C^1 *generic three-dimensional flow* X*. If every singularity* $\sigma \in \Lambda$ *is Lorenz-like for* X *satisfying* $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$*, then* Λ *is dominated for* X*.*

Proof. Indeed, the result is obtained as in the proof of Lemma 3.1 in [9] with Theorem 2.8 playing the role of Theorem B in [7]. We include details for the sake of completeness. For this we need some basic definitions.

A compact invariant set Λ is called *chain transitive* for a flow X if for any $\epsilon > 0$ and any $x, y \in \Lambda$ there are points $x_0, \cdots, x_n \in \Lambda$ and numbers $t_0, \cdots, t_{n-1} \in [1, \infty[$ such that $x_0 = x$, $x_n = y$ and $d(X_{t_i}(x_i), x_{i+1}) < \epsilon$ for all $0 \leq i \leq n-1$. For any $K \subset M$ we define $CR(X, K)$ as the set of those points x for which there is a chain-transitive set Λ satisfying $x \in \Lambda \subset K$. This is a compact invariant set of X contained in K. We also define the *maximal invariant set* of X in K:

$$\max(X, K) = \bigcap_{t \in \mathbb{R}} X_t(K).$$

Take a countable basis $\{U_n\}$ of M and let $\mathcal{O} = \{O_n\}$ be such that each $\mathcal{O}_n$ is a finite union of elements of $\{U_n\}$. For each n we define

$$\mathcal{D}_n = \{X \in \mathfrak{X}^1(\mathrm{M}) : CR(X, \mathrm{Cl}(O_n)) \text{ is } \emptyset \text{ or dominated for } X\},$$

and

$$\mathcal{N}_n = \{X \in \mathfrak{X}^1(\mathrm{M}) : CR(X, O_n) \text{ is neither dominated nor } \emptyset\}.$$

By lemmas 2.9 (which is true for dominated sets instead of hyperbolic sets) and 2.10 in [9] we have that $\mathcal{D}_n \cup \mathcal{N}_n$ is open and dense in $\mathfrak{X}^1(\mathrm{M})$. It follows that

$$\mathcal{G} = \bigcap_n (\mathcal{D}_n \cup \mathcal{N}_n)$$

is residual in $\mathfrak{X}^1(\mathrm{M})$. Let us prove that every $X \in \mathcal{G}$ satisfies the conclusion of the lemma. Indeed, take Λ as in the hypothesis of the lemma and suppose by contradiction that Λ is not dominated for X.

Take also n such that $\Lambda \subset O_n$ and a neighborhood $\mathcal{U}$ of X such that, for every $Y \in \mathcal{U}$, $\max(Y, \mathrm{Cl}(O_n))$ has a LPF-dominated splitting for Y and every $\sigma \in \max(Y, \mathrm{Cl}(O_n))$ is Lorenz-like satisfying

$$W^{ss,Y}(\sigma) \cap \max(Y, \mathrm{Cl}(O_n)) = \{\sigma\}.$$

Since Λ is not dominated for X (and $\emptyset \neq \Lambda \subset CR(X, \mathrm{Cl}(O_n))$) we see that $X \notin \mathcal{D}_n$. As $X \in \mathcal{G}$, we conclude that $X \in \mathcal{N}_n$. Now, take a C^3 Kupka-Smale flow $Y \in \mathcal{N}_n \cap \mathcal{U}$ having no minimal repellers whose dynamics is the suspension of a irrational rotation of the circle (the nonexistence of such dynamics is dense in any topology).

Since $Y \in \mathcal{N}_n$, one has that $CR(Y, \mathrm{Cl}(O_n))$ is not dominated. It follows from the definitions that $CR(Y, \mathrm{Cl}(O_n)) \cap (\mathrm{Sink}(Y) \cup \mathrm{Source}(Y))$ consists of isolated orbits. Therefore,

$$\Gamma = CR(Y, \mathrm{Cl}(O_n)) \setminus (\mathrm{Sink}(Y) \cup \mathrm{Source}(Y))$$

is a compact (and obviously invariant) set of Y. Since $CR(Y, \mathrm{Cl}(O_n))$ is not dominated, we have that Γ *is not dominated for* Y.

On the other hand, $Y \in \mathcal{U}$ thus $\max(Y, \mathrm{Cl}(O_n))$ has a LPF-dominated splitting and, also, every $\sigma \in Sing(Y) \cap \max(Y, \mathrm{Cl}(O_n))$ is Lorenz-like satisfying $W^{ss,Y}(\sigma) \cap \max(Y, \mathrm{Cl}(O_n)) = \{\sigma\}$. As $\Gamma \subset CR(Y, \mathrm{Cl}(O_n)) \subset \max(Y, \mathrm{Cl}(O_n))$ we conclude the same for Γ instead of $\max(Y, \mathrm{Cl}(O_n))$. Since Γ has neither sinks nor sources, we conclude from Theorem 2.8 that Γ *is dominated for* Y. This is a contradiction so Λ is dominated for X. The proof follows. $\square$

The above lemma implies the following result.

Proposition 2.10. *Let H be the Hausdorff limit of a sequence of periodic orbits of a C^1 generic three-dimensional flow X. If H has a LPF-dominated splitting, then H is a hyperbolic set (if $H \cap \mathrm{Sing}(X) = \emptyset$), a singular-hyperbolic attractor for X (if H contains a singularity of index 2) or a singular-hyperbolic attractor for $-X$ (otherwise).*

Proof. If $H \cap \mathrm{Sing}(X) = \emptyset$, then H is hyperbolic by Lemma 2.1. Now, suppose that H contains a singularity σ of index 2. Clearly, H is nontrivial (i.e. not equal to a single orbit) and by Corollary 2.7 we also have that it is the chain-recurrent class of σ. Since H contains a singularity of index 2, we have from the first alternative of Corollary 2.7 that every $\sigma \in H \cap \mathrm{Sing}(X)$ is Lorenz-like and satisfies $H \cap W^{ss,X}(\sigma) = \{\sigma\}$. Then, we can apply Lemma 2.9 to conclude that H has a dominated splitting for X. Since H is the chain recurrent class of σ we conclude from Theorem C in [14] that H is a singular-hyperbolic attractor for X. If H contains a singularity of index 1, then the same argument with $-X$ instead of X implies that H is a singular-hyperbolic attractor for $-X$. This concludes the proof. $\square$

Proof of Theorem B. Let X a C^1 generic three-dimensional flow and $\mathcal{P} \subset \mathrm{PSaddle}(X)$ be homoclinically closed. Suppose that $\mathrm{Cl}(\mathcal{P})$ has a LPF-dominated splitting.

By taking the Hausdorff limit of sequences of periodic orbits in $\mathcal{P}$ accumulating on the singularities of X in $\mathrm{Cl}(\mathcal{P})$ we obtain from Proposition 2.10 that every $\sigma \in \mathrm{Cl}(\mathcal{P}) \cap \mathrm{Sing}(X)$ belongs to a singular-hyperbolic attractor for either X or $-X$.

Using that $\mathcal{P}$ is homoclinically closed we obtain

$$\mathrm{Cl}(\mathcal{P}) = \mathrm{Cl}\left(\bigcup \{H(p) : p \in \mathcal{P}\}\right). \tag{1}$$

We claim that the family $\{H(p) : p \in \mathcal{P}\}$ is finite. Otherwise, there is an infinite sequence $p_k \in \mathcal{P}$ yielding infinitely many distinct homoclinic classes $H(p_k)$. Consider the closure $\mathrm{Cl}(\bigcup_k H(p_k))$, which is a compact invariant set contained in $\mathrm{Cl}(\mathcal{P})$. If this closure does not contain any singularity, then it would be a hyperbolic set by Lemma 2.1. Since the number of homoclinic classes contained in any hyperbolic set is finite, we obtain a contradiction proving that $\mathrm{Cl}(\bigcup_k H(p_k))$ contains a singularity $\sigma \in \mathrm{Cl}(\mathcal{P})$. But, as we have seem, any of these singularities belong to a singular-hyperbolic attractor for either X or $-X$. Since there are finitely many singularities, it must exist distinct k, k' satisfying $H(p_k) = H(p_{k'})$. But this is an absurd, so the claim follows. Combining the claim with (1) and the well-known fact that the homoclinic classes are pairwise disjoint [10] we obtain the desired spectral decomposition. □

3. Proof of Theorem A

In this section we shall prove our main result. We start with some useful definitions.

Let X be a three-dimensional flow. Recall that a periodic point saddle if it has eigenvalues of modulus less and bigger than 1 simultaneously. Analogously for singularities by just replace 1 by 0 and the eigenvalues by their corresponding real parts. Denote by $\mathrm{Saddle}(X)$ the set of saddles of X.

A critical point x is *dissipative* if the product of its eigenvalues (in the periodic case) or the divergence $\operatorname{div} X(x)$ (in the singular case) is less than 1 (resp. 0). Denote by $\mathrm{Crit}_d(X)$ the set of dissipative critical points. Define the *dissipative region* by $\mathrm{Dis}(X) = \mathrm{Cl}(\mathrm{Crit}_d(X))$.

For every subset $\Lambda \subset M$ we define

$$W^s_w(\Lambda) = \{x \in M : \omega(x) \cap \Lambda \neq \emptyset\}.$$

(This is often called *weak region of attraction* [8].)

The following result was proved in [5] in the nonsingular case. The proof in the general case is similar.

Theorem 3.1. *Let X be a C^1 generic three-dimensional flow. Then, $W^s_w(\mathrm{Dis}(X))$ has full Lebesgue measure.*

Given a homoclinic class $H = H_X(p)$ of a three-dimensional flow X we denote by $H_Y = H_Y(p_Y)$ the continuation of H, where p_Y is the analytic continuation of p for Y close to X (c.f. [25]).

The following lemma was also proved in [5]. In its statement Leb denotes the normalized Lebesgue measure of M.

Lemma 3.2. *If X is a C^1 generic three-dimensional flow, then for every hyperbolic homoclinic class H there are an open neighborhood $\mathcal{O}_{X,H}$ of f and a residual subset $\mathcal{R}_{X,H}$ of $\mathcal{O}_{X,H}$ such that the following properties are equivalent:*

1. $\mathrm{Leb}(W^s_Y(H_Y)) = 0$ *for every* $Y \in \mathcal{R}_{X,H}$.
2. *H is not an attractor.*

We also need the following lemma essentially proved in [6].

Lemma 3.3. *If X is a C^1 generic three-dimensional flow, then every singular-hyperbolic attractor with singularities of either X or $-X$ has zero Lebesgue measure.*

Proof. Given $U \subset M$ we define $\mathcal{U}(U)$ as the set of flows Y such that $\max(Y, U)$ is a singular-hyperbolic set with singularities of Y. We shall assume that U is open. It follows that $\mathcal{U}(U)$ is open in $\mathfrak{X}^1(\mathrm{M})$.

Now define $\mathcal{U}(U)_n$ as the set of $Y \in \mathcal{U}(U)$ such that $\mathrm{Leb}(\max(Y, U)) < 1/n$. It was proved in [6] that $\mathcal{U}(U)_n$ is open and dense in $\mathcal{U}(U)$.

Define $\mathcal{R}(U)_n = \mathcal{U}(U)_n \cup (\mathfrak{X}^1(M) \setminus \mathrm{Cl}(\mathcal{U}(U))$ which is open and dense set in $\mathfrak{X}^1(M)$. Let $\{U_m\}$ be a countable basis of the topology, and $\{O_m\}$ be the set of finite unions of such U_m's. Define

$$\mathcal{L} = \bigcap_m \bigcap_n \mathcal{R}(O_m)_n.$$

This is clearly a residual subset of three-dimensional flows. We can assume without loss of generality that $\mathcal{L}$ is symmetric, i.e., $X \in \mathcal{L}$ if and only if $-X \in \mathcal{L}$. Take $X \in \mathcal{L}$. Let Λ be a singular-hyperbolic attractor for X. Then, there exists m such that $\Lambda = \Lambda_X(O_m)$. Then $X \in \mathcal{U}(O_m)$ and so $X \in \mathcal{U}(O_m)_n$ for every n thus $\mathrm{Leb}(\Lambda) = 0$. Analogously, since $\mathcal{L}$ is symmetric, we obtain that $\mathrm{Leb}(\Lambda) = 0$ for every singular-hyperbolic attractor with singularities of $-X$. $\square$

Now we prove the following result which is similar to one in [5] (we include its proof for the sake of completeness). In its statement $\mathrm{PSaddle}_d(X)$ denotes the set of periodic dissipative saddles of a three-dimensional flow X.

Theorem 3.4. *Let Y be a C^1 generic three-dimensional flow. If $\mathrm{Cl}(\mathrm{PSaddle}_d(Y))$ has a spectral decomposition, then every homoclinic class H associated to a dissipative periodic saddle satisfying $\mathrm{Leb}(W^s_Y(H)) > 0$ is either a hyperbolic attractor or a singular-hyperbolic attractor for Y.*

Proof. Define the map $S : \mathfrak{X}^1(\mathrm{M}) \to 2^M_c$ by $S(X) = \mathrm{Cl}(\mathrm{PSaddle}_d(X))$. This map is clearly lower-semicontinuous, and so, upper semicontinuous in a residual subset $\mathcal{N}$ (for the corresponding definitions see [19], [20]).

By the flow-version of the main result in [1], there is a residual subset $\mathcal{R}_7$ of three-dimensional flows X such that for every singular-hyperbolic attractor C for X (resp. $-X$) there are neighborhoods $U_{X,C}$ of C, $\mathcal{U}_{X,C}$ of X and a residual subset $\mathcal{R}^0_{X,C}$ of $\mathcal{U}_{X,C}$ such that for all $Y \in \mathcal{R}^0_{X,C}$ if $Z = Y$ (resp. $Z = -Y$) then

$$C_Y = \bigcap_{t \geq 0} Z_t(U_{X,C}) \text{ is a singular-hyperbolic attractor for } Z. \tag{2}$$

Define $\mathcal{R} = \mathcal{N} \cap \mathcal{R}_7$. Clearly $\mathcal{R}$ is a residual subset of three-dimensional flows. Define

$$\mathcal{A} = \{f \in \mathcal{R} : \mathrm{Cl}(\mathrm{PSaddle}_d(X)) \text{ has no spectral decomposition}\}.$$

Fix $X \in \mathcal{R} \setminus \mathcal{A}$. Then, $X \in \mathcal{R}$ and $\mathrm{Cl}(\mathrm{PSaddle}_d(X))$ has a spectral decomposition

$$\mathrm{Cl}(\mathrm{PSaddle}_d(X)) = \left(\bigcup_{i=1}^{r_X} H^i \right) \cup \left(\bigcup_{j=1}^{a_X} A^j \right) \cup \left(\bigcup_{k=1}^{b_X} R^k \right)$$

into hyperbolic homoclinic classes H_i $(1 \leq i \leq r_X)$, singular-hyperbolic attractors A^j for X $(1 \leq j \leq a_X)$, and singular-hyperbolic attractors R^k for $-X$ $(1 \leq k \leq b_X)$.

As $X \in \mathcal{R}_7$, we can consider for each $1 \leq i \leq r_X$, $1 \leq j \leq a_X$ and $1 \leq k \leq b_X$ the neighborhoods $\mathcal{O}_{X,H^i}$, $\mathcal{U}_{X,A^j}$ and $\mathcal{U}_{X,R^k}$ of X as well as their residual subsets $\mathcal{R}_{X,H^i}$, $\mathcal{R}^0_{X,A^j}$ and $\mathcal{R}^0_{X,R^k}$ given by Lemma 3.2 and (2) respectively.

Define

$$\mathcal{O}_X = \left(\bigcap_{i=1}^{r_X} \mathcal{O}_{X,H^i} \right) \cap \left(\bigcap_{j=1}^{a_X} \mathcal{U}_{X,A^j} \right) \cap \left(\bigcap_{k=1}^{b_X} \mathcal{U}_{X,R^k} \right)$$

and

$$\mathcal{R}_X = \left(\bigcap_{i=1}^{r_X} \mathcal{R}_{X,H^i}\right) \cap \left(\bigcap_{j=1}^{a_X} \mathcal{R}^0_{X,A^j}\right) \cap \left(\bigcap_{k=1}^{b_X} \mathcal{R}^0_{X,R^k}\right).$$

Clearly $\mathcal{R}_X$ is residual in $\mathcal{O}_X$.

From the proof of Lemma 3.2 in [5] we obtain for each $1 \leq i \leq r_X$ a compact neighborhood $U_{X,i}$ of H^i such that

$$(3) \quad H^i_Y = \bigcap_{t\in\mathcal{R}} Y_t(U_{X,i}) \text{ is hyperbolic and equivalent to } H^i, \text{ for all } Y \in \mathcal{O}_{Y,H^i}.$$

As $X \in \mathcal{N}$, S is upper semicontinuous at X so we can further assume that

$$\mathrm{Cl}(\mathrm{PSaddle}_d(Y)) \subset \left(\bigcup_{i=1}^{r_X} U_{X,i}\right) \cup \left(\bigcup_{j=1}^{a_X} U_{X,A^j}\right) \cup \left(\bigcup_{k=1}^{b_X} U_{X,R^k}\right), \text{ for all } Y \in \mathcal{O}_X.$$

It follows that

$$(4) \quad \mathrm{Cl}(\mathrm{PSaddle}_d(Y)) = \left(\bigcup_{i=1}^{r_X} H^i_Y\right) \cup \left(\bigcup_{j=1}^{a_X} A^j_Y\right) \cup \left(\bigcup_{k=1}^{b_X} R^k_Y\right), \text{ for all } Y \in \mathcal{R}_X.$$

Next we take a sequence $X^i \in \mathcal{R} \setminus \mathcal{A}$ which is dense in $\mathcal{R} \setminus \mathcal{A}$.

Replacing $\mathcal{O}_{X^i}$ by $\mathcal{O}'_{X^i}$ where

$$\mathcal{O}'_{X^0} = \mathcal{O}_{X^0} \text{ and } \mathcal{O}'_{X^i} = \mathcal{O}_{X^i} \setminus \left(\bigcup_{j=0}^{i-1} \mathcal{O}_{X^j}\right), \text{ for } i \geq 1,$$

we can assume that the collection $\{\mathcal{O}_{X^i} : i \in \mathbb{N}\}$ is pairwise disjoint.

Define

$$\mathcal{O}_{12} = \bigcup_{i\in\mathbb{N}} \mathcal{O}_{X^i} \quad \text{ and } \quad \mathcal{R}'_{12} = \bigcup_{i\in\mathbb{N}} \mathcal{R}_{X^i}.$$

We claim that $\mathcal{R}'_{12}$ is residual in $\mathcal{O}_{12}$.

Indeed, for all $i \in \mathbb{N}$ write $\mathcal{R}_{X^i} = \bigcap_{n\in\mathbb{N}} \mathcal{O}^n_i$, where $\mathcal{O}^n_i$ is open-dense in $\mathcal{O}_{X^i}$ for every $n \in \mathbb{N}$. Since $\{\mathcal{O}_{X^i} : i \in \mathbb{N}\}$ is pairwise disjoint, we obtain

$$\bigcap_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} \mathcal{O}^n_i \subset \bigcup_{i\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \mathcal{O}^n_i = \bigcup_{i\in\mathbb{N}} \mathcal{R}_{X^i} = \mathcal{R}'_{12}.$$

As $\bigcup_{i\in\mathbb{N}} \mathcal{O}^n_{X^i}$ is open-dense in $\mathcal{O}_{12}$, $\forall n \in \mathbb{N}$, we obtain the claim.

Finally we define

$$\mathcal{R}_{11} = \mathcal{A} \cup \mathcal{R}'_{12}.$$

Since $\mathcal{R}$ is a residual subset of three-dimensional flows, we conclude as in Proposition 2.6 of [21] that $\mathcal{R}_{11}$ is also a residual subset of three-dimensional flows.

Take $Y \in \mathcal{R}_{11}$ such that $\mathrm{Cl}(\mathrm{PSaddle}_d(Y))$ has a spectral decomposition and let H be a homoclinic class associated to a dissipative saddle of Y. Then, $H \subset \mathrm{Cl}(\mathrm{PSaddle}_d(Y))$ by Birkhoff-Smale's Theorem [16].

Since $\mathrm{Cl}(\mathrm{PSaddle}_d(Y))$ has a spectral decomposition, we have $Y \notin \mathcal{A}$ so $Y \in \mathcal{R}'_{12}$ thus $Y \in \mathcal{R}_X$ for some $X \in \mathcal{R} \setminus \mathcal{A}$. As $Y \in \mathcal{R}_X$, (4) implies $H = H^i_Y$ for some $1 \leq i \leq r_X$ or $H = A^j_Y$ for some $1 \leq j \leq a_X$ or $H = R^k_Y$ for some $1 \leq k \leq b_X$.

Now, suppose that $\mathrm{Leb}(W^s_Y(H)) > 0$. Since $Y \in \mathcal{R}_X$, we have $Y \in \mathcal{R}^0_{X,R^k}$ for all $1 \le k \le b_X$. As $W^s_Y(R^k_Y) \subset R^k_Y$ for every $1 \le k \le b_X$, we conclude by Lemma 3.3 that $H \ne R^k_Y$ for every $1 \le k \le b_X$.

If $H = A^j_Y$ for some $1 \le j \le a_X$ then H is an attractor and we are done. Otherwise, $H = H^i_Y$ for some $1 \le i \le r_X$. As $Y \in \mathcal{R}_X$, we have $Y \in \mathcal{R}_{X,H^i}$ and, since $f \in \mathcal{R}_{12}$, we conclude from Lemma 3.2 that H^i is an attractor. But by (3) we have that H^i_Y and H^i are equivalent, so, H^i_Y is an attractor too and we are done. □

Proof of Theorem A. Let X be a C^1 generic three-dimensional flow with only a finite number of sinks. Then, we can prove as in [5] or [29] that $\mathrm{Cl}(\mathrm{PSaddle}_d(X))$ has a LPF-dominated splitting. Since $\mathrm{PSaddle}_d(X)$ is homoclinically closed by the Birkhoff-Smale Theorem [16], we conclude from Theorem B that $\mathrm{Cl}(\mathrm{PSaddle}_d(X))$ has a spectral decomposition.

Since X is C^1 generic we can assume that X is Kupka-Smale too. Then, we have the following decomposition:

$$\mathrm{Dis}(X) = \mathrm{Cl}(\mathrm{Saddle}_d(X) \cap \mathrm{Sing}(X)) \cup \mathrm{Cl}(\mathrm{PSaddle}_d(X)) \cup \mathrm{Sink}(X)$$

yielding

$$W^s_w(\mathrm{Dis}(X)) = \left(\bigcup\{W^s(\sigma) : \sigma \in \mathrm{Saddle}_d(X) \cap \mathrm{Sing}(X) \text{ and } W^s_w(\sigma) = W^s(\sigma)\}\right)$$
$$\cup \left(\bigcup\{W^s_w(\sigma) : \sigma \in \mathrm{Saddle}_d(X) \cap \mathrm{Sing}(X) \text{ and } W^s_w(\sigma) \ne W^s(\sigma)\}\right) \cup$$
$$W^s_w(\mathrm{Cl}(\mathrm{PSaddle}_d(X))) \cup W^s(\mathrm{Sink}(X)).$$

One can esasily check that the first element in the above union has zero measure.

Moreover, by the Hayashi's Connecting Lemma [17], we can assume without loss of generality that every $\sigma \in \mathrm{Saddle}_d(X) \cap \mathrm{Sing}(X)$ satisfying $W^s_w(\sigma) \ne W^s(\sigma)$ belongs to $\mathrm{Cl}(\mathrm{PSaddle}_d(X))$.

Since $W^s_w(\mathrm{Dis}(X))$ has full Lebesgue measure by Theorem 3.1, we conclude that

$$\mathrm{Leb}(W^s_w(\mathrm{Cl}(\mathrm{PSaddle}_d(X))) \cup W^s(\mathrm{Sink}(X))) = 1.$$

But, $\mathrm{Cl}(\mathrm{PSaddle}_d(X))$ has a spectral decomposition

$$\mathrm{Cl}(\mathrm{PSaddle}_d(X)) = \bigcup_{i=1}^{r} H_i$$

into finitely many disjoint homoclinic classes H_i, $1 \le i \le r$, each one being either hyperbolic (if $H_i \cap \mathrm{Sing}(X) = \emptyset$) or a singular-hyperbolic attractor for either X or $-X$ (otherwise). Replacing above we obtain

$$\mathrm{Leb}\left(\left(\bigcup_{i=1}^{r} W^s_w(H_i)\right) \cup W^s(\mathrm{Sink}(X))\right) = 1.$$

Now the results in Section 3 of [10] imply that each H_i can be written as $H_i = \Lambda^+ \cap \Lambda^-$, where $\Lambda^\pm$ is a Lyapunov stable set for $\pm X$. We conclude from Lemma 2.2 in [10] that $W^s_w(H_i) = W^s(H_i)$ thus

$$\mathrm{Leb}\left(\left(\bigcup_{i=1}^{r} W^s(H_i)\right) \cup W^s(\mathrm{Sink}(X))\right) = 1.$$

Let $1 \leq i_1 \leq \cdots \leq i_d \leq r$ be such that $\mathrm{Leb}(W^s(H_{i_k})) > 0$ for every $1 \leq k \leq d$. By Theorem 3.4 we have that each H_{i_k} for $1 \leq k \leq d$ is either a hyperbolic attractor or a singular-hyperbolic attractor for X.

As the basins of the remainder homoclinic classes in the collection $H_1, \cdots, H_r$ are negligible, we can remove them from the above union yielding

$$\mathrm{Leb}\left(\left(\bigcup_{k=1}^{d} W^s(H_{i_k})\right) \cup W^s(\mathrm{Sink}(X))\right) = 1.$$

Since each H_{i_k} is a hyperbolic or singular-hyperbolic attractor for X and $\mathrm{Sink}(X)$ consists of finitely many orbits, we are done. □

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