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On Araujo's Theorem for Flows

A. Arbieto · C. A. Morales · B. Santiago

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Abstract We prove that every C^1 generic three-dimensional flow without singularities has either infinitely many sinks or finitely many hyperbolic attractors whose basins form a full Lebesgue measure set.

Keywords Hyperbolic Attractor · Sink · Three-dimensional flow

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1 Introduction

Araujo announced in his thesis [4] that a C^1 generic surface diffeomorphism has either infinitely many sinks (i.e., attracting periodic orbits) or finitely many hyperbolic attractors whose basins form a full Lebesgue measure set. However, a gap was pointed out and the result was never published. Afterward, the fundamental work by Pujals and Sambarino [29] appeared, proving the relation between domination and hyperbolicity in their nowadays famous Theorem B. Simultaneously, they observed that their Theorem B implies a result closely related to Araujo's but in the C^2 class [30]. More recently, R. Potrie explained that the gap in Araujo's work can be solved with the aid of Pujals–Sambarino's Theorem B (see [27] or p.16 in [28]). He then used this issue to obtain a slightly weaker result in which the full Lebesgue measure condition is replaced by open-denseness. Finally, we can mention

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A. Arbieto (✉) · C. A. Morales · B. Santiago
Instituto de Matemática, Universidade Federal do Rio de Janeiro P. O. Box 68530, 21945-970
Rio de Janeiro, Brazil
e-mail: alexander.arbieto@gmail.com

C. A. Morales
e-mail: morales@impa.br

B. Santiago
e-mail: bruno_santiago@im.ufrj.br

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the proof of Araujo's Theorem based on Pujals–Sambarino's Theorem B and Mañé's C^2 connecting lemma [19] obtained in third author's dissertation [31].

In this paper, we shall extend Araujo's Theorem from surface diffeomorphisms to three-dimensional flows without singularities. More precisely, we prove that a C^1 generic three-dimensional flow without singularities has either infinitely many sinks or finitely many hyperbolic attractors whose basins form a full Lebesgue measure set. The proof given here still use Pujals–Sambarino's Theorem B (or its flow version [6]) but not Mañé's C^2 connecting lemma [19]. Our result also implies the same result for diffeomorphisms by the standard suspension procedure. It is worth to note that this result is false in higher dimensions, in light of the recent work [10]. Let us state it in a precise way.

Hereafter, the term *three-dimensional flow* will be referred to a C^1 *vector field on compact connected boundaryless manifolds M of dimension 3*. The corresponding space equipped with the C^1 vector field topology will be denoted by $\mathfrak{X}^1(\mathrm{M})$.

The flow of $X \in \mathfrak{X}^1(\mathrm{M})$ is denoted by X_t, $t \in \mathbb{R}$. By *singularity*, we mean a point x where X vanishes, i.e., $X(x) = 0$. A subset of $\mathfrak{X}^1(\mathrm{M})$ is *residual* if it is a countable intersection of open and dense subsets. We say that a C^1 *generic three-dimensional flow satisfies a certain property P* if there is a residual subset $\mathcal{R}$ of $\mathfrak{X}^1(\mathrm{M})$ such that P holds for every element of $\mathcal{R}$. The closure operation is denoted by $\mathrm{Cl}(\cdot)$.

Given $X \in \mathfrak{X}^1(\mathrm{M})$, we denote by $O_X(x) = \{X_t(x) : t \in \mathcal{R}\}$ the orbit of a point x. By an orbit of X, we mean a set $O = O_X(x)$ for some point x. A point x (and its corresponding orbit) is *periodic* if there is a minimal $t_x > 0$ satisfying $X_{t_x}(x) = x$ (notation $t_{x,X}$ indicates dependence on X). Clearly if x is periodic, then $DX_{t_x}(x) : T_x M \to T_x M$ is a linear automorphism having 1 as eigenvalue with eigenvector $X(x)$. The remainder eigenvalues (i.e., the ones not corresponding to $X(x)$) will be referred to as the *eigenvalues of x*. We say that a periodic point x is a *sink* if its eigenvalues are less than one (in modulus).

Given a point x, we define its *omega-limit set*,

$$\omega(x) = \left\{ y \in M : y = \lim_{t_k \to \infty} X_{t_k}(x) \text{ for some integer sequence } t_k \to \infty \right\}.$$

(when necessary we shall write $\omega_X(x)$ to indicate the dependence on X.) We call $\Lambda \subset M$ *invariant* if $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$; and *transitive* if there is $x \in \Lambda$ such that $\Lambda = \omega(x)$. The *basin* of any subset $\Lambda \subset M$ is defined by

$$W^s(\Lambda) = \{y \in M : \omega(y) \subset \Lambda\}.$$

(Sometimes we write $W^s_X(\Lambda)$ to indicate dependence on X). An *attractor* is a transitive set A exhibiting a neighborhood U such that

$$A = \bigcap_{t \geq 0} X_t(U).$$

A compact invariant set Λ of X is *hyperbolic* if there are a continuous invariant tangent bundle decomposition $T_\Lambda M = E^s_\Lambda \oplus E^X_\Lambda \oplus E^u_\Lambda$ over Λ and positive numbers K, λ such that E^X_x is generated by $X(x)$,

$$\|DX_t(x)/E^s_x\| \leq Ke^{-\lambda t} \quad \text{and} \quad \|DX_{-t}(x)/E^u_{X_t(x)}\| \leq K^{-1}e^{\lambda t}, \quad \forall (x,t) \in \Lambda \times \mathbb{R}^+.$$

With these definitions, we can state our result.

Theorem 1 *A C^1 generic three-dimensional flow without singularities has either infinitely many sinks or finitely many hyperbolic attractors whose basins form a full Lebesgue measure set.*

The proof we shall give here has some advantages if compared with the aforementioned works [27], [31]. Indeed, [27] is based on recent C^1 generic dynamical tools as [25], [8] (given genericity of flows with residual subsets of points with quasi-attracting omega-limit set), Pujals-Sambarino's Theorem B (or its variant in [2]), Proposition 1.4 in [13], and the existence of suitable ergodic measures supported on quasi-attractors (closely related to Lemma 2 below). It does not use Mañé's C^2 connecting lemma [19], but produce a weaker output, namely, open-denseness instead of full Lebesgue measure. On the other hand, [31] follows the same arguments of the original one [4], but making use of Pujals-Sambarino's Theorem B to rule out certain intricate arguments and still using Mañé's C^2 connecting lemma to obtain the full Lebesgue measure condition.

Our proof instead fit nice in the flow context (this is interesting because some of the aforementioned tools may be difficult to extend for flows even in the nonsingular case) is suitable to extend to the singular case (to be carried out in the forthcoming paper [5]) and avoid the use of Mañé's C^2 connecting lemma.

The idea is as follows. A key object here is the dissipative periodic points, i.e., periodic points where the product of the eigenvalues are less than one in modulus. We shall prove two interesting properties of the closure of these points. The first one in Theorem 2 is that it intersects the omega-limit set of almost every point. This required a non-direct adaptation from diffeomorphisms to flows of a part of Araujo's Thesis [4] (see Lemma 2). The second one in Theorem 4 is that it is a hyperbolic set (assuming finiteness of sinks) and so decomposes into a finite union of sinks and hyperbolic homoclinic classes associated with dissipative saddles. These properties combined with Lemma 1 will imply that the Lebesgue measure of the union of the basins of the elements in the above decomposition is full. To rule out Mañé's C^2 connecting lemma, we prove in Theorem 3 that the attractors in the above decomposition are precisely those for which the basin has positive Lebesgue measure. This permits to delete the homoclinic classes with negligible basin in the decomposition to obtain the result.

It is worth noting that the result about omega-limit sets [25], [8] together with theorems 2 and 3, suggests the following question:

Question 1 Does every C^1 generic flow exhibits a full Lebesgue measure set of points for which the omega-limit set is Lyapunov stable?

Indeed, the result obtained from a positive answer could be used to adapt Potrie's argument [27] to obtain an alternative proof of Theorem 1.

2 Proof of Theorem 1

Hereafter, we will consider three-dimensional flows without singularities only.

First, we introduce some key objects. We say that a periodic point p is *dissipative* if $|\det DX_{t_p}(p)| < 1$. Denote by $\mathrm{Per}_d(X)$ the set of dissipative periodic points. A *saddle* of X is a periodic point having eigenvalues of modulus less and bigger than 1. We denote by $\mathrm{Saddle}(X)$ the set of saddles of X.

Definition 1 The set of *dissipative saddles* is $\mathrm{Saddle}_d(X) = \mathrm{Per}_d(X) \cap \mathrm{Saddle}(X)$.

We also recall that a *Kupka–Smale flow* is a flow for which every periodic orbit is *hyperbolic* (i.e., without eigenvalues of modulus 1) and the stable and unstable manifolds are in general position [15].

Clearly, $\mathrm{Cl}(\mathrm{Saddle}_d(X)) \cup \mathrm{Cl}(\mathrm{Sink}(f)) \subset \mathrm{Cl}(\mathrm{Per}_d(X))$. Moreover, if every periodic point is hyperbolic, then

$$\mathrm{Cl}(\mathrm{Per}_d(X)) = \mathrm{Cl}(\mathrm{Saddle}_d(X)) \cup \mathrm{Cl}(\mathrm{Sink}(X)). \tag{1}$$

In particular, this equality is true for *Kupka–Smale flows*,

In the next sections, we will show the existence of residual subsets $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, and $\mathcal{R}_4$ which will be given by Lemma 1, Theorem 2, Theorem 3, and Theorem 4, respectively. In the proof of the main Theorem, we will quote the properties given by these subsets which will be used. The residual subset $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3 \cap \mathcal{R}_4$. will be the residual subset required in the statement of the Theorem. So, we fix $X \in \mathcal{R}$ with finitely many sinks.

Theorem 4 will say that $\mathrm{Cl}(\mathrm{Saddle}_d(X))$ is hyperbolic.

Now, we recall the concepts of homoclinic classes and weak basins. Through any saddle x, it passes a pair of invariant manifolds, the so-called strong stable and unstable manifolds $W^{ss}(x)$ and $W^{uu}(x)$, tangent at x to the eigenspaces corresponding to the eigenvalue of modulus less and bigger than 1, respectively [16]. Saturating these manifolds with the flow, we obtain the stable and unstable manifolds $W^s(x)$ and $W^u(x)$, respectively. A *homoclinic point* associated to x is a point where these last manifolds meet whereas a homoclinic point q is *transverse* if $T_qM = T_qW^s(x) + T_qW^u(x)$ and $T_qW^s(x) \cap T_qW^u(x)$ is the one-dimensional space generated by $X(q)$.

Definition 2 The *homoclinic class* associated to x is the closure of the set of transverse homoclinic points q associated to x. A homoclinic class of X is the homoclinic class associated to some saddle of X.

Definition 3 For every subset $\Lambda \subset M$, the *weak basin of attraction* [7] is defined as follows:

$$W^s_w(\Lambda) = \{x \in M : \omega(x) \cap \Lambda \neq \emptyset\}.$$

Lemma 1 will say that if $X \in \mathcal{R}_1$ is Kupka–Smale with finitely many sinks then, applying (1), we obtain a finite disjoint collection of homoclinic classes $H_1, \cdots, H_r$ and sinks $s_1, \cdots, s_l$ satisfying

$$W^s_w(\mathrm{Cl}(\mathrm{Per}_d(X))) = \left(\bigcup_{i=1}^{r} W^s(H_i)\right) \cup \left(\bigcup_{j=1}^{l} W^s(s_j)\right).$$

Let $m(.)$ be the (normalized) Lebesgue measure induced by the Riemannian metric of M. Theorem 2 will say that if $X \in \mathcal{R}_2$ then $m(W^s_w(\mathrm{Cl}(\mathrm{Per}_d(X)))) = 1$. In particular,

$$m\left(\left(\bigcup_{i=1}^{r} W^s(H_i)\right) \cup \left(\bigcup_{j=1}^{l} W^s(s_j)\right)\right) = 1.$$

Let $1 \le i_1 \le \cdots \le i_d \le r$ be the set of integers such that $m(W^s(H_{i_k})) > 0$. Since the other ones has zero measure, we obtain:

$$m\left(\left(\bigcup_{k=1}^{d} W^s(H_{i_k})\right) \cup \left(\bigcup_{j=1}^{l} W^s(s_j)\right)\right) = 1.$$

Finally, Theorem 3 will say that if $X \in \mathcal{R}_3$, $\mathrm{Cl}\,\mathrm{Saddle}(X)$ is a hyperbolic set, and $m(W^s_X(H_{i_k})) > 0$, then H_{i_k} is a hyperbolic attractor for every $1 \le k \le d$. This completes the proof.

3 Measure and Decompositions of the Weak Basin of Dissipative Periodic Points

The first purpose of this section is, using results from [12], to obtain a decomposition of the weak basin of the closure of the set of dissipative periodic points in the presence of hyperbolicity.

Lemma 1 *There is a residual subset $\mathcal{R}_1$ of three-dimensional flows X such that if* $\mathrm{Cl}(\mathrm{Saddle}_d(X))$ *is hyperbolic and* $\mathrm{Sink}(X)$ *consists of finitely may orbits $s_1, \cdots, s_l$, then there is a finite disjoint collection of homoclinic class associated to a dissipative saddles $H_1, \cdots, H_r$ such that*

$$W^s_w(\mathrm{Cl}(\mathrm{Per}_d(X))) = \left(\bigcup_{i=1}^{r} W^s(H_i)\right) \cup \left(\bigcup_{j=1}^{l} W^s(s_j)\right).$$

Let us give some remarks. It follows easily from Birkhoff–Smale's Theorem [15] that every homoclinic class associated to a dissipative saddle is contained in $\mathrm{Cl}(\mathrm{Per}_d(X))$. Furthermore, if $\mathrm{Cl}(\mathrm{Saddle}_d(X))$ is hyperbolic and $\mathrm{Sink}(X)$ consists of finitely many orbits $s_1, \cdots, s_l$, then there is a finite disjoint union

$$\mathrm{Cl}(\mathrm{Per}_d(X)) = \left(\bigcup_{i=1}^{r} H_i\right) \cup \left(\bigcup_{j=1}^{l} s_j\right), \tag{2}$$

where each H_i is a homoclinic class associated to a dissipative saddle.

The following notions are from [12] and will be used in the next proof. Let Λ be a compact invariant set of X. We say that Λ is *Lyapunov stable for X* if for every neighborhood U of Λ there is a neighborhood $V \subset U$ of Λ such that $X_t(V) \subset U$, for all $t \geq 0$. We say that Λ is *neutral* if $\Lambda = \Lambda^+ \cap \Lambda^-$ where Λ^+ (resp. Λ^-) is a Lyapunov stable set for X (resp. $-X$).

Proof (Proof of Lemma 1) It follows from the results in Section 3 of [12] that there is a residual subset $\mathcal{R}_1$ of three-dimensional flows X whose homoclinic classes are all neutral. Now suppose that $X \in \mathcal{R}_1$ and that $\mathrm{Cl}(\mathrm{Saddle}_d(X))$ is hyperbolic. Then, we obtain a finite disjoint collection of homoclinic class associated to a dissipative saddles $H_1, \cdots, H_r$ satisfying (2). Since every H_i is neutral, we obtain the result from Lemma 2.2 of [12]. □

The second purpose of this section is to show that generically the Lebesgue measure of the weak basin of the closure of the dissipative periodic points is total. We recall that $m(\cdot)$ denotes the Lebesgue measure of M.

Theorem 2 *There is a residual subset $\mathcal{R}_2$ of three-dimensional flows X for which* $m(W^s_w(\mathrm{Cl}(\mathrm{Per}_d(X)))) = 1$

Let δ_p be the Dirac measure supported on a point p. For each flow X and $t > 0$, we define the Borel probability measure

$$\mu_{p,t} = \frac{1}{t}\int_0^t \delta_{X_s(p)} ds.$$

(The notation $\mu^X_{p,t}$ indicates dependence on X.)

Definition 4 $\mathcal{M}(p, X)$ is the set of Borel probability measures $\mu = \lim_{k\to\infty} \mu_{p,t_k}$ for some sequence $t_k \to \infty$.

Notice that each $\mu \in \mathcal{M}(p, X)$ is *invariant*, i.e., $\mu \circ X_{-t} = \mu$ for every $t \geq 0$. With these notations, we have the following lemma.

Lemma 2 *For every three-dimensional flow X, there is a full Lebesgue measure set L_X of points x satisfying*

$$\int \operatorname{div} X d\mu \leq 0, \qquad \forall \mu \in \mathcal{M}(x, X).$$

Proof For every $\delta > 0$, we define

$$\Lambda_\delta(X) = \{x : \exists N_x \in \mathbb{N} \text{ such that } |\det DX_t(x)| < (1+\delta)^t, \forall t \geq N_x\}.$$

We assert that $m(\Lambda_\delta(X)) = 1$ for every $\delta > 0$. This assertion is similar to one for surface diffeomorphisms given by Araujo [4].

To prove the assertion, we define

$$\Lambda_\rho(s) = \{x : \exists N_x \in \mathbb{N} \text{ such that } |\det DX_{ns}(x)| < (1+\rho)^{ns}, \forall n \geq N_x\}, \forall s, \rho > 0.$$

We claim that

$$m(\Lambda_\rho(s)) = 1, \qquad \text{for every } s, \rho > 0. \tag{3}$$

Indeed, take $\varepsilon > 0$ and for each integer n we define

$$\Omega(n) = \{x : |\det DX_{ns}(x)| \geq (1+\rho)^{ns}\}.$$

On the one hand, we get easily that

$$\Lambda_\rho(s) = \bigcup_{N\in\mathbb{N}} \left(\bigcup_{n\geq N} \Omega(n) \right)^c,$$

where $(\cdot)^c$ above denotes the complement operation. On the other hand,

$$1 = \int |\det DX_{ns}(x)| dm \geq \int_{\Omega(n)} |\det DX_{ns}(x)| dm \geq (1+\rho)^{ns} m(\Omega(n)).$$

Thus, $m(\Omega(n)) \leq \frac{1}{(1+\rho)^{ns}}$, for all n.

Take N large so that

$$\sum_{n=N}^{\infty} \frac{1}{(1+\rho)^{ns}} < \varepsilon.$$

Therefore,

$$m(\Lambda_\rho(s)) \geq 1 - m\left(\bigcup_{n\geq N} \Omega(n) \right) \geq 1 - \sum_{n=N}^{\infty} \frac{1}{(1+\rho)^{ns}} > 1 - \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we get (3). This proves the claim.

Now, we continue with the proof of the assertion.

Fix $0 < \rho < \delta$ and $\eta > 0$ such that

$$(1+\eta)(1+\rho)^t < (1+\delta)^t, \qquad \text{for every } t \geq 1.$$

Choose $0 < s < 1$ satisfying

$$|\det DX_r(y) - 1| \leq \eta, \qquad \text{for all } |r| \leq s \text{ and } y \in M.$$

Take $x \in \Lambda_\rho(s)$. Then, there is an integer $N_x > 1$ such that

$$|\det DX_{ns}(x)| < (1+\rho)^{ns}, \qquad \text{for every } n \geq N_x.$$

If $t \geq N_x$ let $n \geq N_x$ and $0 \leq r < s$ such that $ns \leq t < ns + r$. Thus,

$$|\det DX_t(x)| = |\det DX_{t-ns}(X_{ns}(x))| \cdot |\det DX_{ns}(x)| < (1+\eta)(1+\rho)^{ns}.$$

Then, the choice of η, ρ above yields $|\det DX_t(x)| < (1+\delta)^t$ for all $t \geq N_x$ proving

$$\Lambda_\rho(s) \subset \Lambda_\delta(X).$$

But (3) implies $m(\Lambda_\rho(s)) = 1$ so $m(\Lambda_\delta(X)) = 1$ proving the assertion.

To continue with the proof of the lemma, we notice that $\Lambda_{\delta'}(X) \subset \Lambda_\delta(X)$ whenever $\delta' \leq \delta$. It then follows from the assertion that L_X has full Lebesgue measure, where

$$L_X = \bigcap_{k \in \mathbb{N}^+} \Lambda_{\frac{1}{k}}(X).$$

Now, take $x \in L_X$, $\mu \in \mathcal{M}(x, X)$ and $\varepsilon > 0$. Fix $k > 0$ with $\log\left(1 + \frac{1}{k}\right) < \varepsilon$.

By definition, we have $x \in \Lambda_{\frac{1}{k}}(X)$ and so there is $N_x \in \mathbb{N}^+$ such that

$$|\det DX_t(x)|^{\frac{1}{t}} < 1 + \frac{1}{k}, \qquad \forall t \geq N_x.$$

Take a sequence $\mu_{x,t_i} \to \mu$ with $t_i \to \infty$. We can assume $t_i \geq N_x$ for all i. Applying Liouville's formula [20], we obtain

$$\int \operatorname{div} X d\mu = \lim_{i\to\infty} \int \operatorname{div} X d\mu_{x,t_i} = \lim_{i\to\infty} \frac{1}{t_i} \int_0^{t_i} \operatorname{div} X(X_s(x)) ds =$$

$$= \lim_{i\to\infty} \frac{1}{t_i} \log |\det DX_{t_i}(x)| \leq \log\left(1 + \frac{1}{k}\right) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the result. □

Given $x \in M$, we define N_x as the orthogonal complement of $X(x)$ in $T_x M$ (when necessary we will write N_x^X to indicate dependence on X). The union $N = \bigcup_{x \in M} N_x$ turns out to be a vector bundle with fiber N_x (called the normal bundle). Denote by $\pi_x^X : T_x M \to N_x$ the corresponding orthogonal projection.

Definition 5 The *Linear Poincaré flow* $P_t^X(x) : N_x \to N_{X_t(x)}$ of X is defined by

$$P_t^X(x) = \pi^X_{X_t(x)} \circ DX_t(x), \qquad \text{for all } (x,t) \in M \times \mathbb{R}.$$

We shall use the following version of the classical Franks' Lemma [14] (c.f. Appendix A in [9])

Lemma 3 (Franks' Lemma for flows) *For any flow X and every neighborhood $W(X)$ of X there is a neighborhood $W_0(X) \subset W(X)$ of X such that for any $T > 0$ there exists $\varepsilon > 0$ such that for any $Z \in W_0(X)$ and $p \in$ Per(Z), any tubular neighborhood U of $O_Z(p)$, any partition $0 = t_0 < t_1 < \ldots < t_n = t_{p,Z}$, with $t_{i+1} - t_i < T$ and any family of linear maps $L_i : N_{Z_{t_i}(p)} \to N_{Z_{t_{i+1}}(p)}$ satisfying*

$$\left\| L_i - P^Z_{t_{i+1}-t_i}(Z_{t_i}(p)) \right\| < \varepsilon, \qquad \textit{for any } 0 \leq i \leq n-1,$$

there exists $Y \in W(X)$ with $Y = Z$ along $O_Z(p)$ and outside U such that

$$P^Y_{t_{i+1}-t_i}(Y_{t_i}(p)) = L_i, \qquad \text{for any } 0 \leq i \leq n-1.$$

Proof (Proof of Theorem 2) Denote by 2^M_c the set of compact subsets of M. Let $S : \mathfrak{X}^1(\mathrm{M}) \to 2^M_c$ be defined by

$$S(X) = \mathrm{Cl}(\mathrm{Saddle}_d(X)) \cup \mathrm{Cl}(\mathrm{Sink}(X))$$

It follows easily from the continuous dependence of the eigenvalues of a hyperbolic periodic point with respect to X that this map is *lower-semicontinuous*, i.e., for every $X \in \mathfrak{X}^1(\mathrm{M})$ and every open set W with $S(X) \cap W \neq \emptyset$ there is a neighborhood $\mathcal{P}$ of X such that $S(Y) \cap W$ for all $Y \in \mathcal{P}$. From this and well-known properties of lower-semicontinuous maps [17], [18], we obtain a residual subset $\mathcal{A} \subset \mathfrak{X}^1(\mathrm{M})$ where S is *upper-semicontinuous*, i.e., for every $X \in \mathcal{A}$ and every compact subset K satisfying $S(X) \cap K = \emptyset$ there is a neighborhood $\mathcal{D}$ of X such that $S(Y) \cap K = \emptyset$ for all $Y \in \mathcal{D}$.

By the Ergodic Closing Lemma for flows (c.f. Theorem 3.9 in [35]), there is another residual subset $\mathcal{B}$ of three-dimensional flows X such that for every ergodic measure μ of X there are sequences $Y^k \to X$ and p_k (of periodic points of Y^k) such that $\mu^{Y^k}_{p_k, t_{p_k, Y^k}} \to \mu$.

By the Kupka–Smale Theorem [15], there is a residual subset of Kupka–Smale three-dimensional flows $\mathcal{KS}$.

Let $\mathcal{R}_2 = \mathcal{A} \cap \mathcal{B} \cap \mathcal{KS}$. which is also a residual subset of three-dimensional flows.

To prove the result, we only need to prove

$$L_X \subset W^s_w(\mathrm{Cl}(\mathrm{Per}_d(X))), \qquad \text{for all } X \in \mathcal{R}_2,$$

where L_X is the full Lebesgue measure set in Lemma 2.

If not, then there exist $X \in \mathcal{R}_2$ and $x \in L_X$ satisfying

$$\omega(x) \cap \mathrm{Cl}(\mathrm{Per}_d(X)) = \emptyset.$$

Since $X \in \mathcal{KS}$, we have $S(X) = \mathrm{Cl}(\mathrm{Per}_d(X))$. Then, since S is upper-semicontinuous on $X \in \mathcal{A}$, there exist neighborhoods U of $\omega(x)$ and $W(X)$ of X such that

$$U \cap (\mathrm{Saddle}_d(Z) \cup \mathrm{Sink}(Z)) = \emptyset, \qquad \text{for all } Z \in W(X). \tag{4}$$

Put $W(X)$ and $T = 1$ in Franks' Lemma for flows to obtain $\varepsilon > 0$ and a neighborhood $W_0(X) \subset W(X)$ of X. Set

$$C = \sup\{\|P^Z_t(x)\| : (Z, x, t) \in W(X) \times M \times [0, 1]\}$$

and fix $\delta > 0$ such that

$$|1 - e^{-\frac{\delta}{2}}| < \frac{\varepsilon}{C}.$$

Since M is compact, we have $\mathcal{M}(x, X) \neq \emptyset$ and so we can fix $\mu \in \mathcal{M}(x, X)$. Since $x \in L_X$, we have $\int \mathrm{div}\, X d\mu \leq 0$ by Lemma 2. By the Ergodic Decomposition Theorem [20], we can assume that μ is ergodic. Since $X \in \mathcal{B}$, there are sequences $Y^k \to X$ and p_k (of periodic points of Y^k) such that $\mu^{Y^k}_{p_k, t_{p_k, Y^k}} \to \mu$. It then follows from Liouville's Formula [20] that

$$0 \geq \int \mathrm{div}\, X d\mu = \lim_{k\to\infty} \int \mathrm{div}\, X d\mu^{Y^k}_{p_k, t_{p_k, Y^k}} =$$

$$\lim_{k\to\infty} \frac{1}{t_{p_k, Y^k}} \int_0^{t_{p_k, Y^k}} \mathrm{div}\, X(X_s(x)) ds = \lim_{k\to\infty} \frac{1}{t_{p_k, Y^k}} |\det P^{Y^k}_{t_{p_k, Y^k}}(p_k)|.$$

Thus,

$$\lim_{k\to\infty}\frac{1}{t_{p_k,Y^k}}|\det P^{Y^k}_{t_{p_k,Y^k}}(p_k)|\le 0.$$

Therefore, since $Y^k\to X$ and μ is supported on $\omega(x)\subset U$, we can fix k such that

$$p_k\in U,\quad Y^k\in W_0(X)\quad\text{and}\quad |\det P^{Y^k}_{t_{p_k,Y^k}}(p_k)|<e^{t_{p_k,Y^k}\delta}.$$

Once we fix this k, write $t_{p_k,Y^k}=n+r$ where $n\in\mathbb{N}^+$ is the integer part of t_{p_k,Y^k} and $0\le r<1$. This induces the partition $0=t_0<t_1<\ldots<t_{n+1}=t_{p_k,Y^k}$ given by $t_i=i$ for $1\le i\le n$. Clearly, $t_{i+1}-t_i\le 1$ for $0\le i\le n$.

Define the linear maps $L_i:N^{Y^k}_{Y^k_{t_i}(p)}\to N^{Y^k}_{Y^k_{t_{i+1}}(p)}$ by

$$L_i=e^{-\frac{\delta}{2}}P^{Y^k}_{t_{i+1}-t_i}(Y^k_{t_i}(p_k)),\qquad\text{for every }0\le i\le n.$$

A direct computation shows

$$\left\|L_i-P^{Y^k}_{t_{i+1}-t_i}(Y^k_{t_i}(p_k))\right\|\le|1-e^{-\frac{\delta}{2}}|C<\varepsilon,\qquad\text{for every }0\le i\le n.$$

Then, by Franks' Lemma for flows, there exists $Z\in W(X)$ with $Z=Y^k$ along $O_{Y^k}(p_k)$ such that

$$P^Z_{t_{i+1}-t_i}(Z_{t_i}(p_k))=L_i,\qquad\text{for every }0\le i\le n.$$

Consequently, $t_{p_k,Z}=t_{p_k,Y^k}$ and also $P^Z_{t_{p_k,Z}}(p_k)=e^{-t_{p_k,Y^k}\frac{\delta}{2}}P^{Y^k}_{t_{p_k,Y^k}}(p_k)$ thus

$$|\det P^Z_{t_{p_k,Z}}(p_k)|=e^{-t_{p_k,Y^k}\delta}|\det P^{Y^k}_{t_{p_k,Y^k}}(p_k)|<1.$$

Up to a small perturbation, if necessary, we can assume that p_k has no eigenvalues of modulus 1. Then, $p_k\in\mathrm{Saddle}_d(Z)\cup\mathrm{Sink}(Z)$ by the previous inequality which implies $p_k\in U\cap(\mathrm{Saddle}_d(Z)\cup\mathrm{Sink}(Z))$. But $Z\in W(X)$ so we obtain a contradiction by (4) and the result follows. $\square$

Remark 1 Although we state the results in this section for 3-dimensional flows (which is the context of the main Theorem), we remark that all of the results in this section are true also on any higher dimensional manifold, with the same proofs.

4 Lebesgue measure of the basin of hyperbolic homoclinic classes

This section is devoted to the proof of the following result.

Theorem 3 *There is a residual subset $\mathcal{R}_3$ of three-dimensional flows Y such that if* $\mathrm{Cl}(\mathrm{Saddle}_d(Y))$ *is hyperbolic, then the following properties are equivalent for every homoclinic H associated to a dissipative saddle of Y:*

(a) $m(W^s_Y(H))>0$.
(b) *H is an attractor of Y.*

For this, we need the lemma below. Given a homoclinic class $H=H_X(p)$ of a three-dimensional flow X, we denote by $H_Y=H_Y(p_Y)$ the continuation of H, where p_Y is the analytic continuation of p for Y close to X (c.f. [26]).

Lemma 4 *There is a residual subset $\mathcal{R}_3^*$ of three-dimensional flows X such that for every hyperbolic homoclinic class H there are an open neighborhood $\mathcal{O}_{X,H}$ of f and a residual subset $\mathcal{R}_{X,H}$ of $\mathcal{O}_{X,H}$ such that the following properties are equivalent:*

1. $m(W_Y^s(H_Y)) = 0$ *for every* $Y \in \mathcal{R}_{X,H}$.
2. H *is not an attractor.*

Proof As in Theorem 4 of [1], there is a residual subset $\mathcal{R}_3^*$ of three-dimensional flows X such that, for every homoclinic class H of X, the map $Y \mapsto H_Y$ varies continuously at X.

Now, let H be a hyperbolic homoclinic class of some $X \in \mathcal{R}_3^*$. Since H is hyperbolic, we have that H has the local product structure. From this and the flow version of Proposition 8.22 in [32], we have that H is *uniformly locally maximal*, i.e., there are a compact neighborhood U of H and a neighborhood $\mathcal{O}_{X,H}$ of X such that

(a) $H = \bigcap_{t\in\mathbb{R}} X_t(U)$.

(b) H is topologically equivalent to $\bigcap_{t\in\mathbb{R}} Y_t(U)$, $\forall Y \in \mathcal{O}_{X,H}$.

Since $X \in \mathcal{R}_3^*$, the map $Y \mapsto H_Y$ varies continuously at X. From this, we can assume up to shrinking $\mathcal{O}_{X,H}$ if necessary that $H_Y \subset U$, and so, $H_Y \subset \bigcap_{t\in\mathbb{R}} Y_t(U)$, for every $Y \in \mathcal{O}_{X,H}$. Now, we use the equivalence in (b) above and the transitivity of homoclinic classes to conclude that $\bigcap_{t\in\mathbb{R}} Y_t(U)$ is a transitive set of Y. Hence, $H_Y = \bigcap_{t\in\mathbb{R}} Y_t(U)$. We conclude that

(c) $H_Y = \bigcap_{t\in\mathbb{R}} Y_t(U)$ is hyperbolic and topologically equivalent to H, for every $Y \in \mathcal{O}_{X,H}$.

We claim that if H is not an attractor, then there is a residual subset $\mathcal{L}_{X,H}$ of $\mathcal{O}_{X,H}$ such that

$$m(W_Y^s(H_Y)) = 0, \qquad \text{for every } Y \in \mathcal{L}_{X,H}. \tag{5}$$

Indeed, define

$$\Lambda_Y^N = \bigcap_{0\le t\le N} Y_{-t}(U), \qquad \text{for all } (Y, N) \in \mathcal{O}_{X,H} \times (\mathbb{N} \cup \{\infty\}),$$

and

$$\mathcal{U}^\varepsilon = \{Y \in \mathcal{O}_{X,H} : m(\Lambda_Y^\infty) < \varepsilon\}, \qquad \forall \varepsilon > 0.$$

We assert that $\mathcal{U}^\epsilon$ is open and dense in $\mathcal{O}_{X,H}$, $\forall \epsilon > 0$. To prove it, we use an argument from [3].

For the openness, take $\varepsilon > 0$ and $Y \in \mathcal{U}^\varepsilon$. It follows from the definitions that there is N large such that $m(\Lambda_Y^N) < \varepsilon$.

Set $\varepsilon_1 = \varepsilon - m(\Lambda_Y^N)$ thus $\varepsilon_1 > 0$. Choose $\delta > 0$ such that

$$m(B_\delta(\Lambda_Y^N) \setminus \Lambda_Y^N) < \frac{\varepsilon_1}{2}.$$

(where $B_\delta(\cdot)$ denotes the δ-ball operation). Since N is fixed, we can select a neighborhood $\mathcal{U}_Y \subset \mathcal{O}_{X,H}$ of Y such that

$$\Lambda_Z^N \subset B_\delta(\Lambda_Y^N), \qquad \text{for every } Z \in \mathcal{U}_Y.$$

Therefore, for every $Z \in \mathcal{U}_Y$,

$$m(\Lambda_Z^\infty) \le m(\Lambda_Z^N) \le m(B_\delta(\Lambda_Y^N)) \le m(\Lambda_Y^N) + \frac{\epsilon_1}{2} \le \frac{m(\Lambda_Y^N) + \epsilon}{2} < \varepsilon.$$

This implies the openness of $\mathcal{U}^{\epsilon}$.

For the denseness, take $\mathcal{D}$ as the set of C^2 flows in $\mathcal{O}_{X,H}$. Clearly, $\mathcal{D}$ is dense in $\mathcal{O}_{X,H}$. Since H is not an attractor and conjugated to H_Y, we have that H_Y is not an attractor too, for all $Y \in \mathcal{O}_{X,H}$. In particular, no $Y \in \mathcal{O}_{X,H}$ has an attractor in U. Applying Corollary 5.7 in [11], we conclude that for every $Y \in \mathcal{D}$ we have $m(\Lambda_Y^{\infty}) = 0$.

From this, we have $\mathcal{D} \subset \mathcal{U}^{\varepsilon}$, for any $\varepsilon > 0$. As $\mathcal{D}$ is dense in $\mathcal{O}_{Y,H}$, we are done.

It follows from the assertion that the intersection

$$\mathcal{L}_{X,H} = \bigcap_{k\in\mathbb{N}^+} \mathcal{U}^{\frac{1}{k}}$$

is residual in $\mathcal{O}_{X,H}$. Moreover, for any $Y \in \mathcal{L}_{X,h}$, we have $m(\Lambda_Y^{\infty}) = 0$.

Since every $Y \in \mathcal{L}_{X,H}$ is C^1, we also obtain

$$m\left(\bigcup_{n=0}^{\infty} Y_{-n}(\Lambda_Y^{\infty})\right) = 0, \qquad \text{for every } Y \in \mathcal{L}_{X,h}.$$

But clearly $W_Y^s(H_Y) = \bigcup_{n=0}^{\infty} Y_{-n}(\Lambda_Y^{\infty})$ so (5) holds and the claim follows.

Now, we define

$$\mathcal{R}_{X,H} = \begin{cases} \mathcal{L}_{X,H}, & \text{if } H \text{ is not an attractor} \\ \mathcal{O}_{X,H}, & \text{otherwise} \end{cases}$$

Suppose that $m(W_Y^s(H_Y)) = 0$ for every $Y \in \mathcal{R}_{X,H}$. If H were an attractor, then H_Y also is by equivalence thus $m(W_Y^s(H_Y)) > 0$, for any $Y \in \mathcal{O}_{X,H}$, yielding a contradiction. Therefore, H cannot be an attractor.

If, conversely, H is not an attractor, then $\mathcal{R}_{X,H} = \mathcal{L}_{X,H}$ and so $m(W_Y^s(H_Y)) = 0$ for every $Y \in \mathcal{R}_{X,H}$ by (5). This completes the proof. □

Proof (Proof of Theorem 3) Let $\mathcal{R}_3^*$ be as in Lemma 4. Define the map $S : \mathfrak{X}^1(\mathrm{M}) \to 2_c^M$ by $S(X) = \mathrm{Cl}(\mathrm{Saddle}_d(X))$. As before, this map is clearly lower-semicontinuous, and so, upper semicontinuous in a residual subset $\mathcal{A}$. We take the residual subset $\mathcal{R} := \mathcal{R}_3^* \cap \mathcal{A}$. Define

$$A = \{X \in \mathcal{R} : \mathrm{Cl}(\mathrm{Saddle}_d(X)) \text{ is not hyperbolic}\}.$$

Fix $X \in \mathcal{R} \setminus A$. Then, $\mathrm{Cl}(\mathrm{Saddle}_d(X))$ is hyperbolic and so there are finitely many disjoint homoclinic class associated to a dissipative saddles $H^1, \cdots, H^{r_X}$ (all hyperbolic) satisfying

$$\mathrm{Cl}(\mathrm{Saddle}_d(X)) = \bigcup_{i=1}^{r_X} H^i.$$

As $X \in \mathcal{R}_3^*$, we can consider for each $1 \leq i \leq r_X$ the neighborhood $\mathcal{O}_{X,H^i}$ of X as well as its residual subset $\mathcal{R}_{X,H^i}$ given by Lemma 4.

Define,

$$\mathcal{O}_X = \bigcap_{i=1}^{r_X} \mathcal{O}_{X,H^i} \quad \text{and} \quad \mathcal{R}_X = \bigcap_{=1}^{r_X} \mathcal{R}_{X,H^i}.$$

Recalling (c) in the proof of Lemma 4, we obtain for each $1 \leq i \leq r_X$ a compact neighborhood U_{X,H^i} of H^i such that

$$H_Y^i = \bigcap_{t\in\mathcal{R}} Y_t(U_{X,H^i}) \quad \text{is hyperbolic and equivalent to } H^i, \quad \text{for any } Y \in \mathcal{O}_{Y,H^i}.$$

As $X \in \mathcal{A}$, S is upper semicontinuous at X. So, we can further assume that

$$\mathrm{Cl}(\mathrm{Saddle}_d(Y)) \subset \bigcup_{i=1}^{r_X} U_{X,H^i}, \qquad \text{for any } Y \in \mathcal{O}_X.$$

This easily implies

$$\mathrm{Cl}(\mathrm{Saddle}_d(Y)) = \bigcup_{i=1}^{r_X} H_Y^i, \qquad \text{for every } Y \in \mathcal{O}_X. \tag{6}$$

Define

$$\mathcal{O} = \bigcup_{X \in \mathcal{R} \setminus A} \mathcal{O}_X \quad \text{and} \quad \mathcal{R}_3^{**} = \bigcup_{X \in \mathcal{R} \setminus A} \mathcal{R}_X.$$

We have that $\mathcal{O}$ is open and $\mathcal{R}_3^{**}$ is residual in $\mathcal{O}$.

Finally, we define

$$\mathcal{R}_3 = A \cup \mathcal{R}_3^{**}.$$

Since $\mathcal{R}$ is a residual subset of three-dimensional flows, we conclude from Proposition 2.6 in [23] that $\mathcal{R}_3$ also is.

Now, take a $Y \in \mathcal{R}_3$ such that $\mathrm{Cl}(\mathrm{Saddle}_d(Y))$ is hyperbolic and let H be a homoclinic class associated to a dissipative saddle of Y. Then, $H \subset \mathrm{Cl}(\mathrm{Saddle}_d(Y))$ from Birkhoff–Smale's Theorem [15].

Since $\mathrm{Cl}(\mathrm{Saddle}_d(Y))$ is hyperbolic, we have $Y \notin A$ so $Y \in \mathcal{R}_3^{**}$ thus $Y \in \mathcal{R}_X$ for some $X \in \mathcal{R} \setminus A$. As $\mathcal{R}_X \subset \mathcal{O}_X$, we obtain $Y \in \mathcal{O}_X$ thus (6) implies $H = H_Y^i$ for some $1 \le i \le r_X$.

If $m(W_Y^s(H)) > 0$, then $m(W_Y^s(H_Y^i)) > 0$. But, since $Y \in \mathcal{R}_X$ then $Y \in \mathcal{R}_{X,H^i}$. As $X \in \mathcal{R}_3^*$, we conclude from Lemma 4 that H^i is an attractor. But H^i and $H = H_Y^i$ are equivalent (6), so, H_Y^i is an attractor too and we are done. $\square$

Remark 2 Again, the results of this section are true for any higher dimensional manifolds, with the same proofs.

5 Hyperbolicity of the dissipative saddles

In this section, we shall prove the following result in whose statement $\mathrm{card}(\mathrm{Sink}(X))$ denotes the cardinality of the set of *different* orbits of a three-dimensional flow X on $\mathrm{Sink}(X)$.

Theorem 4 *There is a residual subset $\mathcal{R}_4$ of three-dimensional flows X such that if* $\mathrm{card}(\mathrm{Sink}(X)) < \infty$, *then* $\mathrm{Cl}(\mathrm{Saddle}_d(X))$ *is hyperbolic.*

In its proof, we shall use the following definitions and facts. We say that a point x is a *dissipative presaddle* of a three-dimensional flow X if there are sequences $Y_k \to X$ and $x_k \in \mathrm{Saddle}_d(X_k)$ such that $x_k \to x$. Compare with [34]. Denote by $\mathrm{Saddle}_d^*(X)$ the set of dissipative presaddles of X.

We shall need the following elementary property of the set of dissipative presaddles whose proof is a direct consequence of the definition.

Lemma 5 *For every three-dimensional flow Y and every neighborhood U of* $\mathrm{Saddle}_d^*(Y)$ *there is a neighborhood $\mathcal{V}_Y$ of Y such that* $\mathrm{Saddle}_d^*(Z) \subset U$, *for every $Z \in \mathcal{V}_Y$.*

We also need the auxiliary definition below.

Definition 6 We denote by $S(M)$ the set of three-dimensional flows X such that $\mathrm{card}(\mathrm{Sink}(X)) < \infty$ and $\mathrm{card}(\mathrm{Sink}(Y)) = \mathrm{card}(\mathrm{Sink}(X))$ for every flow Y that is C^1 close to X.

Recall that a compact invariant set Λ *has a dominated splitting* if there exist a continuous tangent bundle decomposition $N_\Lambda = E_\Lambda \oplus F_\Lambda$ and $T > 0$ such that, for every $p \in \Lambda$ we have

$$\left\| P_T^X(p)/E_p \right\| \left\| P_{-T}^X(X_T(p))/F_{X_T(p)} \right\| \leq \frac{1}{2}.$$

The following result can be proved with the techniques in [29].

Proposition 1 *If $X \in S(M)$, then* $\mathrm{Saddle}_d^*(X)$ *has a dominated splitting.*

Proof (Proof of Theorem 4) Define $\phi : \mathfrak{X}^1(\mathrm{M}) \to 2_c^M$ by $\phi(X) = \mathrm{Cl}(\mathrm{Sink}(X))$. This map is clearly lower semicontinuous, and so, upper semicontinuous in a residual subset $\mathcal{C}$ of $\mathfrak{X}^1(\mathrm{M})$. Define,

$$\mathcal{A} = \{X \in \mathcal{C} : X \text{ has infinitely many sinks}\}.$$

Fix $X \in \mathcal{C} \setminus \mathcal{A}$. Then, $X \in \mathcal{C}$ and $\mathrm{card}(\mathrm{Sink}(X)) < \infty$. Since ϕ is upper semicontinuous at X, we conclude that $\mathrm{card}(\mathrm{Sink}(Y)) = \mathrm{card}(\mathrm{Sink}(X))$ for every Y in a neighborhood $\mathcal{O}_X$ of X. We conclude that $\mathcal{O}_X \subset S(M)$.

By the Kupka–Smale theorem [15], we can find a dense subset $\mathcal{D}_X \subset \mathcal{O}_X$ formed by C^2 Kupka–Smale three-dimensional flows. Furthermore, we can assume that every $Y \in \mathcal{D}_X$ has neither normally contracting nor normally expanding irrational tori (see [6] for the corresponding definition).

Let us prove that $\mathrm{Saddle}_d^*(Y)$ hyperbolic for every $Y \in \mathcal{D}_X$. Take any $Y \in \mathcal{D}_X$. Then $Y \in S(M)$, and so, $\mathrm{Saddle}_d^*(Y)$ has a dominated splitting by Proposition 1. On the other hand, it is clear from the definition that every periodic point of Y in $\mathrm{Saddle}_d^*(Y)$ is a saddle. Then, Theorem B in [6] implies that $\mathrm{Saddle}_d^*(Y)$ is the union of a hyperbolic set and normally contracting irrational tori. Since no $Y \in \mathcal{D}_X$ has such tori, we are done.

We claim that every $Y \in \mathcal{D}_X$ exhibits an open neighborhood $\mathcal{V}_Y \subset \mathcal{O}_X$ such that $\mathrm{Saddle}_d^*(Z)$ is hyperbolic, for any $Z \in \mathcal{V}_Y$. Indeed, fix $Y \in \mathcal{D}_X$. Since $\mathrm{Saddle}_d^*(Y)$ is hyperbolic, we can choose a neighborhood U_Y of $\mathrm{Saddle}_d^*(Y)$ and a neighborhood $\mathcal{V}_Y$ of Y such that any compact invariant set of any $Z \in \mathcal{V}_Y$ is hyperbolic [15]. Applying Lemma 5, we can assume that $\mathrm{Saddle}_d^*(Z) \subset U_Y$, for every $Z \in \mathcal{V}_Y$, proving the claim.

Define

$$\mathcal{O}'_X = \bigcup_{Y \in \mathcal{D}_X} \mathcal{V}_Y.$$

Then, $\mathcal{O}_X$ is open and dense in $\mathcal{O}_X$. Define

$$\mathcal{O} = \bigcup_{X \in \mathcal{C} \setminus \mathcal{A}} \mathcal{O}_X \qquad \text{and} \qquad \mathcal{O}' = \bigcup_{X \in \mathcal{C} \setminus \mathcal{A}} \mathcal{O}'_X.$$

It turns out that $\mathcal{O}$ is open and that $\mathcal{A} \cup \mathcal{O}$ is a residual subset of three-dimensional flows. Since $\mathcal{O}'$ is open and dense in $\mathcal{O}$, we conclude that $\mathcal{R}_4 = \mathcal{A} \cup \mathcal{O}'$ is also a residual subset of three-dimensional flows (see Proposition 2.6 in [23]).

Now, take $Y \in \mathcal{R}_4$ with $\mathrm{card}(\mathrm{Sink}(Y)) < \infty$. Then, $Y \notin \mathcal{A}$ and so $Y \in \mathcal{O}'_X$ for some $X \in \mathcal{C} \setminus \mathcal{A}$. From this, we conclude that $\mathrm{Saddle}_d^*(Y)$ is hyperbolic. Since $\mathrm{Cl}(\mathrm{Saddle}_d(Y)) \subset \mathrm{Saddle}_d^*(Y)$, we are done. $\square$

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