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Lyapunov stability and sectional-hyperbolicity for higher-dimensional flows

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Abstract We study C^1-generic vector fields on closed manifolds without points accumulated by periodic orbits of different indices. We prove that these flows exhibit finitely many sinks and sectional-hyperbolic transitive Lyapunov stable sets whose basins form a residual subset of the ambient manifold. This represents a partial positive answer to conjectures in Arbieto and Morales (Proc Am Math Soc 141:2817–2827, 2013), the Palis conjecture Palis (Nonlinearity 21:T37–T43, 2008) and gives a flow version of Crovisier and Pujals (Essential hyperbolicity and homoclinic bifurcations: a dichotomy phenomenon/mechanism for diffeomorphisms, 2010).

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1 Introduction

Dynamical systems (i.e. vector fields or diffeomorphisms) on closed manifolds and, specifically, the C^1 generic ones, have been studied during these last fifty years or so. In fact, Pugh proved in the early sixties [29] that such systems display dense closed orbits in their nonwandering set. Moreover, Mañé [19] proved that a C^1 generic

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surface diffeomorphism is Axiom A or exhibits infinitely many attracting periodic orbits up to time reversing. Besides Araujo, in his thesis [4], proved that these diffeomorphisms have either infinitely many attracting periodic orbits or finitely many hyperbolic attractors whose basins form a full Lebesgue measure set. On the other hand, Hayashi and Mañé proved the celebrated Palis-Smale's C^1 *stability conjecture* [27] that all C^1 structural stable systems are Axiom A [13,18] and Mañé [19] initiated the study of what today's we call *star systems*, i.e., dynamical systems which cannot be C^1-approximated by ones exhibiting nonhyperbolic closed orbits. By noting that star flows may not be Axiom A (e.g. the *geometric Lorenz attractor* [1,11,12]), he asked if, on the contrary, all star diffeomorphisms on closed manifolds are Axiom A. Such a problem was solved in positive by Aoki and Hayashi [2,14]. This inspired Gan and Wen [8] to identify the presence of singularities in the preperiodic set (c.f. [31]) as the sole obstruction for a star flow to be Axiom A. In particular, they proved that all nonsingular star flows on closed manifolds are Axiom A. This solved in positive a conjecture by Liao and Mañé.

Meanwhile [25] introduced the notion of *singular-Axiom A flow* inspiried on both the Axiom A flows and the geometric Lorenz attractor. Based on techniques introduced by Hayashi and Mañé for the solution of the stability conjecture, it was proved in [23] that a C^1 generic vector field on a closed three-manifold either is singular-Axiom A or exhibits infinitely many attracting periodic orbits up to flow reversing. This result motivated the question whether analogous result holds for C^1 generic vector fields in higher dimensional manifolds, but negative results were then obtained. These results in turn motivated the notion of *sectional-Axiom A flow* [20] as a natural substitute of the singular-Axiom A flows in higher-dimensions. Unfortunately, results like [23] with the term sectional-Axiom A in place of singular-Axiom A are not longer true. Instead, the first author conjectured in [3] that a C^1 generic star flow on a closed manifold is sectional-Axiom A. If this conjecture were true, then it would be also true that all C^1-generic vector fields without points accumulated by hyperbolic periodic orbits of different Morse indices are sectional-Axiom A. It was this last assertion what was proved in [3] but when the singularities accumulated by periodic orbits have Morse index 1 or $n-1$.

In this paper we prove that all C^1 generic vector fields without points accumulated by periodic orbits of different indices on a closed manifold are essentially sectional-Axiom A. By this we mean that they come equipped with finitely many sinks and sectional-hyperbolic transitive Lyapunov stable sets for which the union of the basins is a residual subset. This result (which applies to the star flows with spectral decomposition) represents a partial positive answer to the aforementioned conjectures [3] and the Palis conjecture [26]. In addition, it represents a flow version of the result in [7] that every C^1 diffeomorphism of a closed manifold is approximated by another diffeomorphism with a homoclinic or heteroclinic bifurcation or by one which is essentially hyperbolic (i.e. exhibiting finitely many hyperbolic attractors for which the union of the basins is open and dense). Let us state our result in a precise way[1].

[1] Some related results have been appearing during the submission of this paper [30].

In what follows M will denote a *closed n-manifold*, i.e., a compact connected boundaryless Riemannian manifold of dimension $n \geq 3$. The space of C^1 vector fields in M will be denoted by $\mathcal{X}^1$. If $X \in \mathcal{X}^1$ we denote by X_t the flow generated by X in M. A *periodic orbit* (resp. *singularity*) of X is the orbit $\{X_t(p) : t \in \mathbb{R}\}$ of a point $p \in M$ satisfying $X_T(p) = p$ for some minimal $T > 0$ (resp. a zero of X). By a *closed orbit* we mean a periodic orbit or a singularity. Denote by $Sing_X(\Lambda)$ the set of singularities of X in a subset $\Lambda \subset M$.

Given $p \in M$ we define the *ω-limit set*

$$\omega(p) = \left\{ x \in M : x = \lim_{n\to\infty} X_{t_n}(p) \text{ for some sequence } t_n \to \infty \right\}.$$

A subset $\Lambda \subset M$ is *invariant* if $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$; *nontrivial* if it does not reduces to a single closed orbit; a *limit cycle* if $\Lambda = \omega(x)$ for some $x \in M$; *transitive* if $\Lambda = \omega(p)$ for some $p \in \Lambda$ and *Lyapunov stable* if for every neighborhood U of it there is a neighborhood $\Lambda \subset W \subset U$ such that $X_t(W) \subset U$ for all $t \geq 0$. Moreover, we say that Λ has *dense closed* (resp. *periodic) orbits* if the closed (resp. periodic) orbits of X in Λ are dense in Λ. We also define the *basin of attraction* of Λ by

$$W^s(\Lambda) = \{x \in M : \omega(x) \subset \Lambda\}.$$

A transitive set Λ will be called *attractor* if it exhibits a neighborhood U such that $\Lambda = \bigcap_{t\geq 0} X_t(U)$. On the other hand, a compact invariant set Λ is C^1 *robustly transitive* if there is a compact neighborhood U of Λ with $\Lambda = \bigcap_{t\in\mathbb{R}} X_t(U)$ such that $\Lambda(Y) = \bigcap_{t\in\mathbb{R}} Y_t(U)$ is a nontrivial transitive set of Y for every vector field Y that is C^1 close to X ($\Lambda(Y)$ is often referred to as the natural continuation of Λ).

Denote by $\|\cdot\|$ and $m(\cdot)$ the norm and the minimal norm induced by the Riemannian metric and by $Det(\cdot)$ the jacobian operation. We say that Λ is *hyperbolic* if there are a continuous invariant tangent bundle decomposition

$$T_\Lambda M = \hat{E}^s_\Lambda \oplus \hat{E}^X_\Lambda \oplus \hat{E}^u_\Lambda$$

and positive constants K, λ such that $\hat{E}^X_\Lambda$ is the subbundle generated by X,

$$\|DX_t(x)/\hat{E}^s_x\| \leq Ke^{-\lambda t} \quad \text{and} \quad m(DX_t(x)/\hat{E}^u_x) \geq K^{-1}e^{\lambda t},$$

for all $x \in \Lambda$ and $t \geq 0$. A closed orbit is hyperbolic if it does as a compact invariant set. We define the *Morse index* $I(O)$ of a hyperbolic closed orbit O by $I(O) = dim(E^s_x)$ for some (and hence for all) $x \in O$. In case O is a singularity σ we write $I(\sigma)$ instead of $I(\{\sigma\})$. A *sink* will be a hyperbolic closed orbit of maximal Morse index and a *source* is a sink for the time reverser vector field.

Given an invariant splitting $T_\Lambda M = E_\Lambda \oplus F_\Lambda$ over an invariant set Λ of a vector field X we say that the subbundle E_Λ *dominates* F_Λ if there are positive constants K, λ such that

$$\frac{\|DX_t(x)/E_x\|}{m(DX_t(x)/F_x)} \leq Ke^{-\lambda t}, \qquad \forall x \in \Lambda \text{ and } t \geq 0.$$

(In such a case we say that $T_\Lambda M = E_\Lambda \oplus F_\Lambda$ is a *dominated splitting*).

We say that Λ is *partially hyperbolic* if it has a dominated splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ whose dominating subbundle E^s_Λ is *contracting*, namely,

$$\|DX_t(x)/E^s_x\| \leq Ke^{-\lambda t}, \qquad \forall x \in \Lambda \text{ and } t \geq 0.$$

Moreover, we call the central subbundle E^c_Λ *sectionally expanding* if

$$dim(E^c_x) \geq 2 \quad \text{and} \quad |Det(DX_t(x)/L_x)| \geq K^{-1}e^{\lambda t}, \qquad \forall x \in \Lambda \text{ and } t \geq 0$$

and all two-dimensional subspace L_x of E^c_x.

We call *sectional-hyperbolic* any partially hyperbolic set whose singularities (if any) are hyperbolic and whose central subbundle is sectionally expanding [20].

Notice that $\mathcal{X}^1$ is a Baire space if equipped with the standard C^1 topology. We shall use consistently the expression *residual subset* which indicates a certain subset in a metric space which is a countable intersection of open and dense subsets. A fundamental property of the set of residual subsets is that it is closed under countable intersection. This property will be used implicitely along the proof of our theorem. We also use the customary expression C^1*-generic vector field* meaning *for every vector field in a residual subset of* $\mathcal{X}^1$.

With these definitions we can state our main result.

Theorem *Let $X \in \mathcal{X}^1$ be a C^1-generic vector field without points accumulated by hyperbolic periodic orbits of different Morse indices. Then, X has finitely many sinks and sectional-hyperbolic transitive Lyapunov stable sets for which the union of the basins is residual in M.*

The proof will use some recent results like [9,10,22,24]. It would be nice to obtain attractors instead of transitive Lyapunov stable sets in this theorem. Unfortunately, as asked in [5], it is unkown whether a sectional-hyperbolic transitive Lyapunov stable set is an attractor (even generically). Let us present a short application of our result.

We say that $X \in \mathcal{X}^1$ is a *star flow* if there is a neighborhood $\mathcal{U}$ of X such that every closed orbit of every $Y \in \mathcal{U}$ is hyperbolic. Recall that the *nonwandering set* of X is the set of points $p \in M$ such that for every neighborhood U of p and every $T > 0$ there is $t > T$ such that $X_t(U) \cap U \neq \emptyset$. We say that X has *spectral decomposition* if $\Omega(X)$ splits into finitely many disjoint transitive sets. Moreover, we say that X is a *sectional-Axiom A flow* if there is a finite disjoint union $\Omega(X) = \Omega_1 \cup \cdots \cup \Omega_k$ formed by transitive sets with dense closed orbits $\Omega_1, \ldots, \Omega_k$ such that, for all $1 \leq i \leq k$, Ω_i is either a hyperbolic set for X or a sectional-hyperbolic set for X or a sectional-hyperbolic set for $-X$. Clearly a sectional-Axiom A flow has a spectral decomposition but the converse is not necessarily true.

As already mentioned, the first author conjectured in [3] that all C^1-generic star flows on closed manifolds are sectional-Axiom A. A support for this conjecture is given below. Its proof follows from the Theorem and Lemma 16 in [3].

Corollary *A C^1-generic star flow with spectral decomposition has finitely many sinks and sectional-hyperbolic transitive Lyapunov stable sets with residual basin of attraction.*

2 Proof

Previously we state some basic results. The first one is the main result in [24].

Lemma 1 *For every C^1-generic vector field $X \in \mathcal{X}^1$ there is a residual subset R_X of M such that $\omega(x)$ is a Lyapunov stable set, $\forall x \in R_X$.*

With the same methods as in [6] and [24] it is possible to prove the following variation of this lemma. We shall use the standard stable and unstable manifold operations $W^s(\cdot)$, $W^u(\cdot)$ (c.f. [15]).

Lemma 2 *For every C^1-generic vector field $X \in \mathcal{X}^1$ and every hyperbolic closed orbit O of X the set $\{x \in W^u(O) \setminus O : \omega(x)$ is Lyapunov stable$\}$ is nonempty (it is indeed residual in $W^u(O)$).*

An extension to higher dimensions of the three-dimensional arguments in [22] (e.g. [17]) imply the following lemma.

Lemma 3 *A sectional-hyperbolic set Λ of $X \in \mathcal{X}^1$ contains only finitely many attractors, i.e., the collection $\{A \subset \Lambda : A$ is an attractor of $X\}$ is finite.*

The following concept comes from [9].

Definition 4 We say that a compact invariant set Λ of $X \in \mathcal{X}^1$ has a definite index $0 \leq Ind(\Lambda) \leq n-1$ if there are a neighborhood $\mathcal{U}$ of X in $\mathcal{X}^1$ and a neighborhood U of Λ in M such that $I(O) = Ind(\Lambda)$ for every hyperbolic periodic orbit $O \subset U$ of every vector field $Y \in \mathcal{U}$. In such a case we say that Λ is *strongly homogeneous (of index $Ind(\Lambda)$)*.

The importance of the strongly homogeneous property is given by the following result proved in [9]: If a strongly homogeneous sets Λ with singularities (all hyperbolic) of $X \in \mathcal{X}^1$ is C^1 robustly transitive, then it is partially hyperbolic for either X or $-X$ depending on whether

$$I(\sigma) > Ind(\Lambda), \qquad \forall \sigma \in Sing_X(\Lambda) \tag{1}$$

or

$$I(\sigma) \leq Ind(\Lambda), \qquad \forall \sigma \in Sing_X(\Lambda) \tag{2}$$

holds. This result was completed in [20] by proving that all such sets are in fact sectional-hyperbolic for either X or $-X$ depending on whether (1) or (2) holds. Another proof of this completion can be found in [10].

On the other hand, [3] observed that the completion in [20] (or [10]) is also valid for transitive sets with singularities (all hyperbolic of Morse index 1 or $n-1$) as soon as $n \geq 4$ and $1 \leq Ind(\Lambda) \leq n-2$. The proof is the same as [9] and [20] but with the preperiodic set playing the role of the natural continuation of a C^1 robustly transitive set.

Now we observe that such a completion is still valid for limit cycles or when the periodic orbits are dense. In other words, we have the following result.

Lemma 5 *If a strongly homogeneous set* Λ *with singularities (all hyperbolic) of* $X \in \mathcal{X}^1$ *satisfying* $1 \leq Ind(\Lambda) \leq n-2$ *is a limit cycle or has dense periodic orbits, then it is sectional-hyperbolic for either* X *or* $-X$ *depending on whether (1) or (2) holds.*

This lemma motivates the problem whether a strongly homogeneous set with hyperbolic singularities which is a limit cycle or has dense periodic orbits satisfies either (1) or (2). For instance, Lemma 3.3 of [10] proved this is the case for all C^1 robustly transitive strongly homogeneous sets. Similarly for strongly homogeneous limit cycles with singularities (all hyperbolic of Morse index 1 or $n-1$) satisfying $n \geq 4$ and $1 \leq Ind(\Lambda) \leq n-2$ (e.g. Proposition 7 in [3]). Consequently, all such sets are sectional-hyperbolic for either X or $-X$. See Theorem A in [10] and Corollary 8 in [3] respectively.

Unfortunately, (1) or (2) need not be valid for general strongly homogeneous sets with dense periodic orbits even if $1 \leq Ind(\Lambda) \leq n-2$. A counterexample is the nonwandering set of the vector field in S^3 obtained by gluing a Lorenz attractor and a Lorenz repeller as in p. 1576 of [23]. Despite, it is still possible to analyze the singularities of a strongly homogeneous set with dense periodic orbits even if (1) or (2) does not hold. For instance, adapting the proof of Lemma 2.2 in [10] (or the sequence of lemmas 4.1, 4.2 and 4.3 in [9]) we obtain the following result.

Lemma 6 *If* Λ *is a strongly homogeneous set with singularities (all hyperbolic) and dense periodic orbits of* $X \in \mathcal{X}^1$*, then every* $\sigma \in Sing_X(\Lambda)$ *satisfying* $I(\sigma) \leq Ind(\Lambda)$ *exhibits a dominated splitting* $\hat{E}^u_\sigma = E^{uu}_\sigma \oplus E^c_\sigma$ *with* $dim(E^{uu}_\sigma) = n - Ind(\Lambda) - 1$ *over* σ *such that the strong unstable manifold* $W^{uu}(\sigma)$ *tangent to* E^{uu}_σ *at* σ *(c.f. [15]) satisfies* $\Lambda \cap W^{uu}(\sigma) = \{\sigma\}$.

Now we can prove our result.

Proof of the Theorem Let $X \in \mathcal{X}^1$ be a C^1-generic vector field without points accumulated by hyperbolic periodic orbits of different Morse indices. By [3], since X is C^1 generic, it follows that if $\mathrm{Per}_i(X)$ denotes the union of the periodic orbits with Morse index i, then the closure $\mathrm{Cl}(\mathrm{Per}_i(X))$ is strongly homogeneous of index $Ind(\mathrm{Cl}(\mathrm{Per}_i(X))) = i$, $\forall 0 \leq i \leq n-1$. Moreover, X is a star flow and so it has finitely many singularities and also finitely many sinks and sources (c.f. [16,28]).

Let us prove that $\omega(x)$ is sectional-hyperbolic for all $x \in R_X$ where $R_X \subset M$ is the residual subset in Lemma 1. We can assume that $\omega(x)$ is nontrivial and has singularities for, otherwise, $\omega(x)$ is hyperbolic by Theorem B in [8] and the Pugh's closing lemma [29].

Since X is C^1 generic we can further assume that $\omega(x) \subset \mathrm{Cl}(\mathrm{Per}_i(X))$ for some $0 \leq i \leq n-1$ by the closing lemma once more. Since X has finitely many singularities sinks and sources we have $1 \leq i \leq n-2$ (otherwise $\omega(x)$ will be reduced to a singleton which is absurd). Since $\mathrm{Cl}(\mathrm{Per}_i(X))$ is strongly homogeneous of index i we have that $\omega(x)$ also does so $1 \leq Ind(\omega(x)) \leq n-2$. Then, since $\omega(x)$ is a limit cycle, we only need to prove by Lemma 5 that (1) holds for $\Lambda = \omega(x)$. To prove it we proceed as in Corollary B in [9], namely, suppose by contradiction that (1) does not hold. Then, there is $\sigma \in Sing_X(\Lambda)$ such that $I(\sigma) \leq Ind(\omega(x))$. Since $\omega(x) \subset \mathrm{Cl}(\mathrm{Per}_i(X))$ and $\mathrm{Cl}(\mathrm{Per}_i(X))$ is a strongly homogeneous set with singularities, all hyperbolic, in $\Omega(X)$ we have by Lemma 6 that there is a dominated splitting $\hat{E}^u_\sigma = E^{uu}_\sigma \oplus E^c_\sigma$ for which the associated strong unstable manifold $W^{uu}(\sigma)$ satisfies $\mathrm{Cl}(\mathrm{Per}_i(X)) \cap W^{uu}(\sigma) = \{\sigma\}$. However $W^{uu}(\sigma) \subset \omega(x)$ since $\sigma \in \omega(x)$ and $\omega(x)$ is Lyapunov stable. As $\omega(x) \subset \mathrm{Cl}(\mathrm{Per}_i(X))$ we conclude that $W^{uu}(\sigma) = \{\sigma\}$ so $\dim(E^{uu}_\sigma) = 0$. But $\dim(E^{uu}_\sigma) = n-i-1$ by Lemma 6 so $\dim(E^{uu}_\sigma) \geq n-n+2-1 = 1$ a contradiction. We conclude that (1) holds so $\omega(x)$ is sectional-hyperbolic for all $x \in R_X$.

Next we prove that $\omega(x)$ is transitive for $x \in R_X$. If $\omega(x)$ has no singularities, then it is hyperbolic and so a hyperbolic attractor of X. Otherwise, there is $\sigma \in Sing_X(\Lambda)$. By Lemma 2 we can select $y \in W^u(\sigma) \setminus \{\sigma\}$ with Lyapunov stable ω-limit set. On the other hand, $\omega(x)$ is Lyapunov stable and so $W^u(\sigma) \subset \omega(x)$. Then, we obtain $y \in \omega(x)$ satisfying $\omega(x) = \omega(y)$ thus $\omega(x)$ is transitive.

It remains to prove that X has only finitely many sectional-hyperbolic transitive Lyapunov stable sets. Suppose by absurd that there is an infinite sequence A_k of sectional-hyperbolic transitive Lyapunov stable sets. Clearly the members in this sequence must be disjoint, so, since there are finitely many singularities, we can assume that none of them have singularities. It follows that all these sets are hyperbolic and then they are all nontrivial hyperbolic attractors of X. In particular, every A_k has dense periodic orbits by the Anosov closing lemma. We can assume that there is $1 \leq i \leq n-2$ such that each Λ_k belong to $Cl(Per_i(X))$. Define

$$\Lambda = Cl\left(\bigcup_{k \in \mathbb{N}} A_k\right).$$

Notice that Λ contains infinitely many attractors (the A_k's say). Moreover, Λ is a strongly homogeneous set of index $Ind(\Lambda) = i$ with dense periodic orbits (since each A_k does).

Let us prove that Λ satisfies (1). Indeed, suppose by contradiction that it does not, i.e., there is $\sigma \in Sing_X(\Lambda)$ such that $I(\sigma) \leq Ind(\Lambda)$. By Lemma 6 there is a dominated splitting $\hat{E}^u_\sigma = E^{uu}_\sigma \oplus E^c_\sigma$ for which the associated strong unstable manifold $W^{uu}(\sigma)$ satisfies $\Lambda \cap W^{uu}(\sigma) = \{\sigma\}$.

Take a sequence $x_k \in A_k$ converging to some point $x \in W^s(\sigma) \setminus \{\sigma\}$. By Corollary 1 p. 949 in [10] there is a dominated splitting $D = \Delta^s \oplus \Delta^u$ for the linear Poincaré flow ψ_t which, in virtue of Lemma 2.2 in [10], satisfies $\lim_{t\to\infty} \psi_t(\Delta^u_x) = E^{uu}_\sigma$. Using exponential maps we can take a codimension one submanifold Σ orthogonal to X of the form $\Sigma = \Delta^s_x(\delta) \times \Delta^u_x(\delta)$ where $\Delta^*_x(\delta)$ indicates the closed δ-ball around x in Δ^*_x $(* = s, u)$. Since $\psi_t(\Delta^u_x) \to E^{uu}_\sigma$ as $t \to \infty$ we can assume by replacing x by $X_t(x)$

with $t > 0$ large if necessary that $\Delta^u_x(\delta)$ is almost parallel to E^{uu}_σ. In particular, since $\Lambda \cap W^{uu}(\sigma) = \{\sigma\}$, one has $(\partial\Delta^s_x(\delta) \times \Delta^u_x(\delta)) \cap \Lambda = \emptyset$ where $\partial(\cdot)$ indicates the boundary operation. Since both $\partial\Delta^s_x(\delta) \times \Delta^u_x(\delta)$ and Λ are closed we can arrange a neighborhood U of $\partial\Delta^s_x(\delta) \times \Delta^u_x(\delta)$ in Σ such that $U \cap \Lambda = \emptyset$.

Now we consider k large in a way that x_k is close to x. Replacing x_k by $X_t(x_k)$ with suitable t we can assume that $x_k \in \Sigma$. Since $x_k \in A_k$ and A_k is a hyperbolic attractor we can consider the intersection $S = W^u(x_k) \cap \Sigma$ of the unstable manifold of x_k and Σ. It turns out that S is the graph of a C^1 map $S : \Delta^u_x(\rho) \to \Delta^s_x(\delta)$ for some $0 < \rho \le \delta$ whose tangent space T_yS is almost parallel to Δ^u_x. We assert that $\rho = \delta$. Otherwise, it would exist some boundary point $z \in \partial S$ in the interior of Σ. Since A_k is a hyperbolic set and $z \in A_k$ we could consider as in [21] the unstable manifold $W^u(z)$ which will overlap $W^u(x)$. Since $z \in Int(W^u(z))$ and $W^u(z) \subset A_k$ (for Λ_k is an attractor) we would obtain that z is not a boundary point of S, a contradiction which proves the assertion. It follows from the assertion that A_k (and so Λ) would intersect U which is absurd since $U \cap \Lambda = \emptyset$. Thus (1) holds.

Then, Lemma 5 implies that Λ is sectional-hyperbolic for X and so Λ has finitely many attractors by Lemma 3. But, as we already observed, Λ contains infinitely many attractors so we obtain a contradiction. This contradiction proves the finiteness of sectional-hyperbolic transitive Lyapunov stable sets for X thus ending the proof of the theorem. $\square$

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