Commuting vector fields and generic dynamics

Bruno Rodrigues Santiago

Tese de Doutorado apresentada ao Programa de Pós-graduação do Instituto de Matemática, da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Doutor em Matemática.

Orientador: Alexander Eduardo Arbieto Mendoza

Co-orientador: Christian Bonatti

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[1] I should stress that these things go mainly in the direction Davi $- >$ Bruno.

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Commuting vector fields and generic dynamics

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This thesis deals with dynamical properties of vector fields and diffeomorphisms, in two different poins of view. The first is the point of view of group actions. We study actions of $\mathbb{R}^2$ on 3-manifolds, given by a pair of commuting vector fields. We are interested in the singularities of the foliation by orbits. We prove an existence result for 0-dimensional singularities (which correspond to a common zero for the pair of vector fields), under two assumptions: (1) there exists a compact region U such that X does not vanishes at the boundary of U and has a non zero Poincaré-Hopf index $\mathrm{Ind}(X, U)$; (2) all the singularities of the foliation (the 1-dimensional and 0-dimensional orbits) in U are contained in some embeded closed suface. This is a strong indication that the results in [Bo1] should hold for C^1 vector fields.

The second point of view is that of Baire genecity in the C^1 topology, for closed manifolds. For vector fields, we establish two generic results. We prove that every C^1 generic, non-singular, three dimensional vector field either has infinitely many periodic attractors or is *essentialy hyperbolic* i. e. has a finite number of hyperbolic attractors, whose basins cover a full Lebesgue measure subset of the ambient manifold. We also derive similar conclusions for a special class of star vector fields. Indeed, we prove that every C^1 generic vector field

X on a d-dimensional closed manifold M without points accumulated by periodic orbits of different indices has a finite number of transitive sectional-hyperbolic Lyapunov stable sets, whose basins form a residual subset of M.

Finally, we study mixing properties of C^1 generic diffeomorphisms on closed manifolds, giving necessary and suficient conditions in some cases. We first study ergodic implications of topological mixing for isolated homoclinic classes. We establish that for any generic diffeomorphism f, if the dynamics restricted to an isolated homoclinic class is topologically mixing then the Bernoulli measures are dense in space of invariant measures supported on the class. In particular, the set of weakly mixing measures contain a residual subset. Then, we turn to the problem of deciding whether a robustly transitive diffeomorphism must be topologically mixing. In this direction, we first prove that here exists an open and dense subset among robustly transitive diffeomorphisms far from homoclinic tangencies formed by diffeomorphisms such that the whole manifold is a homoclinic class. Then, we show that there is an open and dense subset among robustly transitive diffeomorphisms far from homoclinic tangencies formed by robustly topologically mixing diffeomorphisms.

Campos de vetores que comutam e dinâmica genérica

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Esta tese lida com propriedades dinâmicas de campos de vetores e difeomorfismos, sob dois pontos de vista diferentes. O primeiro é o ponto de vista das ações de gupos. Estudamos ações de $\mathbb{R}^2$ em 3-variedades, dadas por um par de campos de vetores X e Y que comutam. Estamos interessados nas singularidades da folheação por órbitas. Provamos um resutado de existência para singularidades de dimensão 0 (as quais correspondem a zeros comuns dos campos X e Y), sob duas hipóteses: (1) existe uma região compacta U, sobre o bordo da qual X não se anula, e tal que o índice de Poincaé-Hopf $\mathrm{Ind}(X, U)$ de X em U é não nulo; (2) todas as singularidades da folheação (aquelas de dimensão 1 e aquelas de dimensão 0) estão contidas numa superfície fechada mergulhada na variedade. Este resultado é uma indicação forte de que os teoremas em [Bo1] para campos analíticos na verdade devem valer para campos C^1.

O segundo ponto de vista é o dos conjuntos genéricos (segunda categoria) de Baire na topologia C^1. Para campos de vetores, nós estabelecemos dois resultados dentro dessa linha. Provamos que todo campo de vetores C^1 genérico, sem singularidades, em dimensão três, ou possui initos poços (atratores periódicos) ou é essencialmente hiperbólico, isto é, possui um número finito de atratores hiperbólicos cuja união das bacias cobre um conjunto com medida

Lebesgue total da variedade ambiente. Também derivamos conclusões similares para uma classe especial de campos estrela. De fato, nosso resultado é que todo campo de vetores C^1 genérico X numa variedade d-dimensional fechada M que não possui pontos acumulados por órbitas periódicas com índices diferentes possui um número finito de conjuntos Lyapunov estáveis transitivos seccionalmente hiperbólicos, cuja união das bacias forma um conjunto residual de M.

Finalmente, estudamos propriedade misturadoras de difeomorfismos C^1 genéricos em variedades fechadas, fornecendo condições necessárias e suficientes em alguns casos. Nosso primeiro objetivo é estudar implicações ergódicas da mistura topológica em classes homoclínicas isoladas. Estabelecemos que para todo difeomorfismo C^1 genérico f se a dinânica restrita a uma classe homoclínica isolada é topologicamente misturadora então as medidas de Bernoulli formam um subconjunto denso do espaç das medidas invariantes suportadas na classe. Em particular, as medidas fracamente misturadoras formam um residual do mesmo espaço. Depois disso, nos voltamos para o problema de decidir se um difeomorfismo robustamente transitivo deve ser topologicamente misturador. Nesta direção provamos que dentro dos difeomorfismos robustamente transitivos longe de tangências existe um conjunto aberto e denso formado por difeomorfismos para os quais a variedade toda é uma classe homoclínica. Então, mostramos que aberta e densamente difeomorfismos robustamente transitivos longe de tangências são topologicamente misturadores.

Contents

Chapter 1

Introduction

This thesis deals with dynamical properties of vector fields and diffeomorphisms, in two different points of view. The first is the point of view of group actions. The second is the point of view of Baire genericity in the C^1 topology.

In the first part we consider two commuting vector fields X and Y on a 3-manifold. This means that X is invariant under the flow of Y, and vice-versa. We explore the dynamical consequences of this simmetry and use them to derive conclusions about the topological behaviour of the vector fields.

In the second part we use perturbation tools, for C^1 vector fields and diffeomorphisms, and the wide variety of its consequences to study ergodic (Bernoulli measures, weakly mixing measures) and assymptotical topological properties (attractors, Lyapunov stability, robust transitivity, topological mixing).

Even though the problems atacked in these two parts are quite different in its goals, there is a common flavour in the approachs since the objects under consideration are the same. Nevertheless, for the sake of clarity, we shall introduce each part separately.

1.1 Commuting vector fields

Suppose we are given an action of $\mathbb{R}^n$ on a compact manifold M. This is equivalent to give n pairwise commuting vector fields $X^1, ..., X^n$. An interesting problem is to determine topological conditions on the manifold M which may ensure the existence of orbits with low dimensions.

For instance, in the case of actions of $\mathbb{R}$, this dates back to the classical Poincaré-Hopf theory which ensures existence of zeros for vector fields on closed manifolds with non zero Euler characteristic. There exists also a parallel theory of Lefschets index for $\mathbb{Z}$ actions.

The work of E. Lima, in the sixties [Li2],[Li1], solves the case of actions of $\mathbb{R}^n$ on closed surfaces, with non zero Euler characteristic, proving the existence of common zeros for the vector fields generating the action. Since then, much work has been done on the existence of fixed points for (even more general) actions on surfaces.

In the late eighties, [Bo2] proved that commuting diffeomorphisms of the sphere $\mathbb{S}^2$ which are C^1-close to the identity have a common fixed point. Later [Bo3] extended this result to any surface with non-zero Euler characteristic (see other generalizations in [DFF][Fi]). Then, Handel [Ha] provided a topological invariant in $\mathbb{Z}/2\mathbb{Z}$ for a pair of commuting diffeomorphisms of the sphere $\mathbb{S}^2$ whose vanishing guarantees a common fixed point. This was further generalized by Franks, Handel and Parwani [FHP] for any number of commuting diffeomorphisms on the sphere (see [Hi] and [FHP2] for generalizations on other surfaces).

For actions of continuous groups, we mention the work of Plante [Pl], who showed that if G is a connected finite-dimensional nilpotent Lie group and M is a compact surface with non-vanishing Euler characteristic, then every continuous action of G into M has a fixed point. It is worth to mention that the result do not hold for solvable groups, in general. Examples were constructed by Lima [Li2], Plante [Pl] and Hirsch and Weinstein [HW] even gave analytic actions of solvable groups without fixed points.

An amusing fact, which is worth mention, is that two commuting *continuous interval maps* may fail to have a common fixed point: an example is constructed in [Boy] of two continuous commuting, non-injective, maps of the interval which do not have a common fixed point.[1]

For actions on higher dimensional manifolds, much less is known about the existence of orbits of low dimensions. Molino and Turiel [MT1] proved that if M is a compact, connected manifold without boundary, of dimension $2d$, with Euler characteristic not zero, then any action of R^n on M has orbits of dimension les or equal than $d-1$. This statement gives Lima's theorem as a corollary. Later, they extended this by establishing an estimate for the minimum possible dimension for the orbits of an action of $\mathbb{R}^n$, in terms of the rank and the dimension of the manifold. Recall that the rank of M is maximal number of commuting vector fields $X^1, ..., X^k \in \mathfrak{X}^1(M)$ such that $X^1(x), ..., X^k(x)$ are linearly independent at any point $x \in M$. The result of [MT2] is that for a closed d-dimensional manifold M, with rank k, given a C^∞ action $\Phi : \mathbb{R}^n \times M \to M$, the minimum possible dimension of its orbits is less than $(d+k)/2$, with equality only if $M = \mathbb{T}^d$.

Concerning the existence of zero dimensional orbits for actions of $\mathbb{R}^n$ on manifolds of dimensions greater than two, the scenario is completely open. There are some special cases where one can prove existence of common zeros, as in the case of a pair of commuting vector fields X and Y, where Y is the suspension of a diffeomorphism of a closed surface (see Chapter 6 and Theorem 6.4.1 for details). Another (too) simple configuration would be the case where X and Y are everywhere collinear. This case has been treated in Lemme 1.c.1 of [Bo1] and the same proof holds at least in the C^d setting where d is the dimension of the ambient manifold. The only known general results comes from the work of Bonatti [Bo1].

Before stating the main result of [Bo1], we briefly recall the notion of the Poincaré-Hopf index $\mathrm{Ind}(X, U)$ of a vector field X on a compact region U whose boundary ∂U is disjoint from the set $\mathrm{Zero}(X)$. If U is a small compact neighborhood of an isolated zero p of the vector field X, then $\mathrm{Ind}(X, U)$ is just the classical Poincaré-Hopf index $\mathrm{Ind}(X, p)$ of X at p. For a general compact region U with $\partial U \cap \mathrm{Zero}(X) = \emptyset$, one considers a small perturbation Y of X with only finitely many isolated zeros in U. Then, we define the index $\mathrm{Ind}(X, U)$ as the sum of the Poincaré-Hopf indices $\mathrm{Ind}(Y, p)$, $p \in \mathrm{Zero}(Y) \cap U$. We refer the reader to Chapter 2 for details (in particular for the fact that $\mathrm{Ind}(X, U)$ does not depend on the

[1] Even though every pair of commuting diffeomorphisms of the interval must have a common fixed point, as it is easy to see.

perturbation Y of X).

The result of [Bo1] on 3-manifolds can be restated as follows:

Theorem. *Let M be a real analytic manifold of dimension $d \leq 4$. Let X and Y be two analytic commuting vector fields over M. Let U be a compact subset of M such that* $\mathrm{Zero}(X) \cap \partial U = \emptyset$*. If* $\mathrm{Ind}(X,U) \neq 0$ *then* $\mathrm{Zero}(X) \cap \mathrm{Zero}(Y) \cap U \neq \emptyset$

As a corollary of this result, one obtains the following[2]

Corollary. *Let M be a real analytic closed 4-manifold, with non vanishing Euler characteristic. Then, any analytic action of $\mathbb{R}^2$ over M has a fixed point.*

The result of Bonatti when M has dimension 2 is true in much more generality. For instance we shall see a proof of it when the vector fields are just C^1 (see Chapter 6 and Corollary 6.3.2). This motivates the following.

Conjecture. *Let X and Y be two C^1 commuting vector fields on a 3-manifold M. Let U be a compact subset of M such that* $\mathrm{Zero}(X) \cap \partial U = \emptyset$*. If* $\mathrm{Ind}(X,U) \neq 0$ *then* $\mathrm{Zero}(X) \cap \mathrm{Zero}(Y) \cap U \neq \emptyset$

This conjecture was stated as a problem in [Bo1].

In our paper [BS] jointly with Christian Bonatti, we tackled, in the C^1-setting, what was the main difficulty to solve this conjecture in the analytic case in [Bo1]. We explain now what was this difficulty in [Bo1]. A crucial role is played by the *collinearity locus*: set of points of M in which X and Y are collinear:

$$\mathrm{Col}(X,Y) := \{p \in M; \dim\left(\langle X(p), Y(p)\rangle\right) \leq 1\}.$$

Let us denote, for simplicity, $\mathrm{Col}(X,Y,U) = \mathrm{Col}(X,Y) \cap U$. In [Bo1] the assumption that the commuting vector fields are analytic is used to say that $\mathrm{Col}(X,Y,U)$ is either equals to U or is an analytic set of dimension 2. The case where $\mathrm{Col}(X,Y,U) = U$ admits a direct proof. In the other case, a simple argument allows to assume that $\mathrm{Col}(X,Y,U)$ is a surface.

[2] We shall see a proof of this implication in Chapter 6.

The main difficulty in [Bo1] consists in proving that, if $\text{Col}(X, Y, U)$ is a smooth surface and X and Y are analytic, then $\text{Zero}(X) \cap \text{Zero}(Y) \cap U \neq \emptyset$, provided that the index of X does not vanish on U.

Our result is the following

Theorem (Bonatti,S.)**.** *Let M be a 3-manifold and X and Y be two C^1 commuting vector fields over M. Let U be a compact subset of M such that* $\text{Zero}(Y) \cap U = \text{Zero}(X) \cap \partial U = \emptyset$. *Assume that the collinearity locus* $\text{Col}(X, Y, U)$ *is contained in a C^1-surface which is a closed submanifold of M. If* $\text{Ind}(X, U) \neq 0$ *then* $\text{Zero}(X) \cap \text{Zero}(Y) \cap U \neq \emptyset$.

The hypothesis " $\text{Col}(X, Y, U)$ *is contained in a C^1-surface*" consists in considering the simplest geometric configuration of $\text{Col}(X, Y, U)$ for which the conjecture is not trivial: if (X, Y, U) satisfies all the hypothesis of the conjecture, but not its conclusion then $\text{Col}(X, Y)$ cannot be "smaller" than a surface. More precisely, if $\text{Ind}(X, U) \neq 0$, the sets $\text{Zero}(X - tY)$ for small t are not empty compact subsets of $\text{Col}(X, Y, U)$, invariant by the flow of Y and therefore consist in orbits of Y. If X and Y are assumed without common zeros, every set $\text{Zero}(X - tY)$ consists on regular orbits of Y, thus is a 1-dimensional lamination. Furthermore, these laminations are pairwise disjoint and vary semi-continously with t.

Let us say a few words about our proof. It is a proof by contradiction, so we begin with a vector field X which has non zero Poincaré-Hopf index $\text{Ind}(X, U)$, in some compact region U, and commutes with a vector field Y, which has no zeros in U. The intuitive idea which guides the argument is that, at one hand, the vector field X needs to turn in all directions in a non-trivial way in order to have a non-zero index. On the other hand, X commutes with Y and therefore is invariant under the tangent flow of a non-zero vector field. We shall use this invariance of X under the flow of Y, together with the knowledge that X turns in all directions to achieve a contradiction.

Indeed, the main novelty of our work [BS] is the use of a *normal component* of X in the direction of Y and its relation with the first return time function and the holonomies of Y (see Lemma 6.1.11). We show that the Poincaré-Hopf index of X is determined by the rotation of the normal component. A non trivial rotation of the normal component implies that either the first return time is constant when restricted to the colinearity locus or that the Poincaré map of Y is close to the identity. In the former case one easily reachs a contradiction with the assumption that $\mathrm{Ind}(X, U) \neq 0$. In the latter case we analyse the dynamics of the first return map and conclude that the normal component cannot rotate, reaching a contradiction once more.

We believe that our approach will be useful to prove the conjecture, at least for C^2 vector fields. This is being carried out in a ongoing project with Christian Bonatti and Sébastien Alvarez.

Finally, let us put a general comment on this line of research. The accumulation of results proving the existence of common fixed points for commuting dynamical systems seems to indicate the possibility of a general phenomenon. However, our approach in Poincaré-Bendixson spirit has a difficulty which increases drastically with the ambient dimension. We hope that this results will motivate other attempts to study this phenomenon.

1.2 Generic dynamics

In the second part of the thesis we turn to properties of the spaces $\mathfrak{X}^1(M)$ of C^1 vector fields and $\mathrm{Diff}^1(M)$ of C^1 diffeomorphisms of some closed manifold M, which holds in some Baire generic set i.e. a countable intersection of dense open sets.

This study dates back to the work of Pugh [Pu1], where he used the C^1-closing lemma [Pu2] to show that for a Baire generic subset of $\mathrm{Diff}^1(M)$ every non-wandering point is accumulated by periodic orbits. Since Hayashi's monumental improvement of the closing lemma, the C^1-connecting lemma [H1], much work has been done and a panorama of the space of $\mathrm{Diff}^1(M)$ is emerging. A program of conjectures by Palis [Pa], latter extended by Bonatti [Bo4], tries to unify these results by dividing $\mathrm{Diff}^1(M)$ into open regions, whose union is dense, and such that the dynamics inside each region may (hopefully) be understood. Moreover, the advent of the connecting lemma made possible to settle many problems which were untouchable before.

The general goal in these problems is to describe the assimptotical behaviour of a generic $f \in \mathrm{Diff}^1(M)$, in the topological sense (structure of the non-wandering set, existence of attractors etc) and in the ergodic sense (existence of physical measures, structure of the space of invariant measures etc). We worked on some problems inserted within this general goal in our papers [AMS1],[AMS2] and [ACS]. Let us describe them and state our results below.

Existence of attractors

An attractor for a vector field or a diffeomorphism is an invariant set Λ which admits a neighborhood, called its *basin of attraction*, such that for any point in this neighborhood the ω-limit is contained in Λ. In particular, attractors carry the knowledge of the future dynamics of open sets and for this reason they are important objects in dynamics. Thus, a natural question is whether they are frequent or not among dynamical systems. It was conjectured by Thom that a generic $f \in \mathrm{Diff}^1(M)$ possess an attractor (see Problem 26 in [PaPu]).

The first progress on this conjecture of Thom was made by Araújo [A] in the late eighties. He proved that a C^1 generic surface diffeomorphism f, either has an infinity number of sinks (attractors which are periodic orbits of f) or is *essentially hyperbolic:* has a finite number of hyperbolic attractors whose basin of attraction cover a full Lebesgue measure subset of the surface. Thus, apart from the wild case of f having infinitely many sinks, the future of almost every point of the ambient surface is well understood.

This remarkable result was inspired by a previuous one, due to Mañé [Ma2]. He proved that a C^1 generic $f \in \mathrm{Diff}^1(M)$ either has an infinity number of sinks or sources, or it is *hyperbolic:* it satisfies Axiom A and the no cycles condition. Mañé's proof is based on the fact that if a C^1 generic $f \in \mathrm{Diff}^1(M)$ has finitely many sinks and sources then it satisfies the *star property:* every sufficiently small perturbation has only hyperbolic periodic orbits. Then, combining a very technical linear algebraic study, which showed that the periodic orbits of a star diffeomorphism have the domination property and uniform rates on the stable/unstable eigenspaces, together with his powerful ergodic closing lemma, Mañé concludes that every star diffeomorphism on surfaces is Axiom A with no cycles.

The proof of Araújo has two main ideas. First, the techniques employed by Mañé can be adapted to study the dissipative periodic orbits. If f is far from sinks then f looks like a star diffeomorphism close to the disspative periodic orbits. Indeed, any perturbation of f along a dissipative periodic orbit which creates another dissipative periodic orbit for some g close to f cannot create sinks nor sources. In this way Araújo adpted the arguments of Mañé to conclude domination and uniform rates over the dissipative periodic orbits.

However this was not enough to conclude hyperbolicity of the attractors. He had to develop a new result, which was the second main idea in his thesis. In this result, Araújo tries to obtain hyperbolicity directly from domination, for C^2 diffeomorphisms using distortion techniques.

This second part of Araújo's proof was pushed much further ten years later by Pujals and Sambarino (see Theorem B in [PS]). In fact, Pujals and Sambarino remark in their paper that there was a gap in Araújo's proof, which they managed to fix. Many years later, using Pujals and Sambarino's work, Potrie [Po1] in his PhD thesis presented a proof of a slightly weaker version of Araújo's result (with the full Lebesgue measure condition replaced by open and denseness) using new material from C^1 generic dynamics. In my master dissertation [San] I presented the original proof of Araújo, correcting the aforementioned gap with Theorem B of [PS].

Since surface diffeomorphisms have much similarity with flows of 3-dimensional vector fields, the works of Mañé and Araújo brought the natural question of the validity of analogous statements for vector fields on three manifolds, in particular if Thom's conjecture also holds within this context. However, the geometric Lorenz attractor [GuW] furnishes an example where the natural extensions of both results (Mañé's and Araújo's) fails at the same time: it is a system robustly non hyperbolic, and robustly with no sinks nor sources. Despite the similarities, the key point, which makes everything fail, is that the vector field has a zero accumulated by recurrent orbits. As is always the case when one finds some counter example, a question arises: is the presence of singularities the sole obstruction?

In 2006, Gan and Wen answered this affirmatively for vector fields with the star property. They proved that if $X \in \mathfrak{X}^1(M)$ has the star property and $\mathrm{Zero}(X) = \emptyset$ then X

stisfies Axiom A and the no cycles condition. Previously, Hayashi [H2] (before having the connecting lemma) extended to any dimensions Mañé's work on surfaces proving that every $f \in \mathrm{Diff}^1(M)$ with the star property is Axiom A with no cycles.

Very recently, Bonatti, Li and Yang [BLY] and Bonatti and Shinohara [BSh] provided counter examples to Thom's conjecture in higher dimensions. In [BLY] it is given a local residual subset of $\mathrm{Diff}^1(M)$ where every diffeomorphism in it has no attractors. If M has dimension bigger than four then [BLY] gives no attractor nor repeller, but in dimension three they only constructed generic diffeomorphisms without attractors but with (infinitely many) repellers. In [BSh] the authors settled the problem in dimension three [3], by giving locally residual subsets of $\mathrm{Diff}^1(M)$ without attractors or repellers. As a corollary of these works, one obtains that the property of having an attractor is not generic in $\mathfrak{X}^1(M)$, if M has dimension greater than or equal to 4.

Therefore, it remained open the question if a generic vector field on a three manifold has an attractor. In [AMS1] we answered affirmatively part of this question and extended Araújo's result to non singular vector fields. Our result is the following.

Theorem (Arbieto,Morales,S.). *Let M be a 3-manifold. Then, for a generic $X \in \mathfrak{X}^1(M)$ such that $\mathrm{Zero}(X) = \emptyset$, either X has infinitely many sinks or X has a finite number of hyperbolic attractors whose basins of attraction cover a full Lebesgue measure subset of M.*

This result implies Araújo's theorem by the the well-known method of suspensions. The proof we shall give here has some advantages if compared with the aforementioned works [Po1] (which is based in [Po2]) and [San]. Indeed, [Po2] is based on recent C^1 generic dynamical tools as [MP1] and [BC] (given genericity of diffeomorphisms with residual subsets of points with quasi-attracting ω-limit set), Pujals and Sambarino's Theorem B (or its variant in [AH]) Proposition 1.4 in [C] and the existence of suitable ergodic measures supported on quasi-attractors (closely related to Lemma 8.1.7). It does not use the C^2-connecting lemma of Mañé [Ma4] but produce a weaker output, namely, open-denseness instead of full Lebesgue

[3] It is worth to mention, however, that the result in [BLY], for dimensions bigger than 4 give non existence of attractors nor repellers C^r generically, for every $r \geq 1$. The result of [BSh] we mentioned gives only a C^1 residual, in dimension three. In particular, it is still an open question whether on a 3-manifold M there exist C^r locally generic diffeomorphisms on M without attractors or repellers, if $r \geq 2$. See Question 2 in [BSh].

measure. On the other hand, [San] follows the same arguments of the original one but making use of Pujals-Sambarino's Theorem B to rule out certain intricate arguments and still using Mañé's C^2-connecting lemma to obtain the full Lebesgue measure condition. Our proof instead fits nice in the flow context (this is interesting because some of the aforementioned tools may be difficult to extend for flows even in the nonsingular case) is suitable to extend to the singular case and avoid the use of Mañé's C^2-connecting lemma.

Further develoments

After submission of our paper [AMS1], Crovisier and Yang [CY] announced a result which is a version of Theorem of B of [PS] to vector fields with singularities. Soon after, Morales [M1] announced that with the result in [CY] it is possible to completely settle Thom's conjecture for vector fields on 3-manifolds. The statement in [M1] is that a C^1 generic $X \in \mathfrak{X}^1(M)$ either has an infinity number of sinks, or a finite number of attractors with a weaker form of hyperbolicity, adapted to the singularities in this case, namely sectional-hyperbolicity (also called singular-hyperbolicity in this three dimensional context). The proof of [M1] follows the same structure of the proof we shall present in Chapter 8, but basically invoking [CY] instead of [AH].

Star flows

The Lorenz attractor is a paradigmatic example of a robustly non-hyperbolic flow, when there are singularities approached by recurrent orbits. It is a a *star flow* i.e. every sufficiently small C^1 perturbation of it has only hyperbolic zeros and periodic orbits, and thus Hayashi's result [H2] does not hold for vector fields with singularities approached by periodic orbits. For a long time, it was completely unknown what should be the type of hyperbolicity of the Lorenz attractor, that would be responsible for its robust properties. This problem was settled by Morales, Pacífico and Pujals [MPP2], [MPP1]. They not only indentified the type of hyperbolicity but also showed that every robustly transitive set for a vector field in 3-dimensions should have the same type hyperbolicity of the Lorenz attractor.

This was the starting point in the study of the *sectional-hyperbolic* vector fields. The definition is that the tangent flow has a dominated splitting $E \oplus F$ such that E contracts and F expands area of every two dimensional subspaces, with uniform and constant rates. Later, Morales and Pacífico [MP2] showed that, even though a star flow may fail of being sectional-hyperbolic, a C^1 generic star flow on three dimensions is sectionally-hyperbolic. Moreover, it admits a spectral decomposition into finitely many hyperbolic non-singular homoclinic classes and sectional hyperbolic attractors and repellers with singularities.

This result brought the hope that one could find the type of hyperbolicity that a C^1 generic star flow should have. The combined efforts of the chinese school with Gan, Wen, Li and Zhu [GaLiW],[GaWZ] and the latin american school with Metzger and Morales [MeM] extended [MPP2], showing that any robustly transitive, strongly homogeneous set with only hyperbolic zeros is sectionally hyperbolic thus improving our understanding of this beautiful phenomenon of hyperbolic singularities accumulated by regular orbits. Based on this new tools, Arbieto and Morales [AM] were able to prove that a C^1 generic vector field without points accumulated by periodic orbits with different indeces[4] and whose singularities have codimension one is sectional Axiom A i.e. its non-wandering set admits a spectral decomposition into hyperbolic non-singular pieces and singular pieces which are sectionally-hyperbolic.

In our paper jointly with Arbieto and Morales [AMS2], we studied further C^1 generic vector field without points accumulated by periodic orbits with different indeces, considering the case of arbitrary codimension but obtaining a weaker conclusion.

Theorem (Arbieto,Morales,S.). *Let $X \in \mathfrak{X}^1(M)$ be a C^1-generic vector field without points accumulated by hyperbolic periodic orbits of different indices. Then, X has finitely many sinks and sectional-hyperbolic transitive Lyapunov stable sets for which the union of the basins is residual in M.*

The proof uses some recent results like [GaLiW], [GaWZ], [M2], [MP1]. It would be nice to obtain attractors instead of transitive Lyapunov stable sets in this theorem. Unfortunately, as asked in [CM], it is unkown whether a sectional-hyperbolic transitive Lyapunov stable set is an attractor (even generically).

[4]This condition implies the star property, and for this reason this class of vector fields is, sometimes, called by the unofficial term *superstar flow*.

Further developments

After the submission of our paper [AMS2], some recent striking new results about star flows apeared. First, Gan, Shi and Wen [GaSW] indentified the presence of singularities with different stable indices in a same chain recurrent class as the sole obstruction for the sectional hyperbolicity of a C^1-generic star flow. Using this, they obtained strong generalizations of parts of our result. They proved, for instance, that for a C^1 generic star flow *every* Lyapunov stable chain recurrent class is sectional hyperbolic, thus droping the stronger assumption that the vector field has no points accumulated by hyperbolic periodic orbits of different Morse indices.

More recently Adriana da Luz and Bonatti announced an unexpected example of a star flow which has, robustly, the precise obstruction to sectional hyperbolicity found in [GaSW]! Namely, they found an example of star flow which has two singularities p and q, with different stable indices, which are robustly in the same chain recurrent class. This surprising result re-opens the question: what is the type of hyperbolicity of a C^1 generic star flow?

Mixing-like properties

We now turn our attention to the following problem within the aforementioned framework of problems in generic dynamics: *given a C^1 generic $f \in \mathrm{Diff}^1(M)$, describe its space of invariant measures.* This problem is important because invariant measures help to describe the dynamics. To put it in a more concrete setting, we shall restrict f to some chain recurrence class.

Let us quote some key developments in the study of chain recurrence classes. This began once that Conley's Fundamental Theorem of Dynamical Systems appeared. It says that up to quotient these classes on points any dynamical system looks like a gradient dynamics.

However, some of these classes, called homoclinic classes, gained interest with the advent of Smale's Spectral Decomposition Theorem. Indeed, this theorem says that for Axiom A (hyperbolic) dynamics the non-wandering set splits into finitely many homoclinic classes. Moreover, each of these classes is isolated: it is the maximal invariant set of a neighbourhood of itself. Thus, these homoclinic classes are the sole chain recurrence classes of such dynamics.

Hence, the study of homoclinic classes, in non-hyperbolic situations, attracted the attention of many mathematicians, see [BDV] for a survey on the subject. In [S2], Sigmund made progress in the hyperbolic case. More precisely he proved that for any homoclinic class of an Axiom A diffeomorphism, the set of periodic measures, i.e. Dirac measures evenly distributed on a periodic orbit, is dense in the set of invariant measures. On the other hand, there is a refinement of the Spectral Decomposition Theorem, due to Bowen, which says that any such class of an Axiom A system splits into finitely many compact sets which are cyclically permuted by the dynamics and the dynamics of each piece, at the return, is topologically mixing, i.e. given two open sets U and V then the n-th iterate of U meets V *for every* n large enough. Using this, Sigmund in [S1] was able to prove that the set of Bernoulli measures is dense among the invariant measures. He also proved that weakly mixing measures contains a residual subset of invariant measures. Indeed, the set of weakly mixing measures is a countable intersection of open sets. We recall that a measure is Bernoulli if the system endowed with it is measure theoretically isomorphic to a Bernoulli shift.

In the non-hyperbolic case, [ABC] proved that for a generic diffeomorphism any isolated homoclinic class has periodic measures dense in the set of invariant measures, thus extending the first result of Sigmund mentioned above to the generic setting. In our paper with Arbieto and Catalan [ACS] we extended the second result of Sigmund mentioned above.

Theorem (Arbieto,Catalan,S.)**.** *For any generic diffeomorphism f, if the dynamics restricted to an isolated homoclinic class is topologically mixing then the Bernoulli measures are dense in space of invariant measures supported on the class. In particular, the set of weakly mixing measures contain a residual subset.*

The main tools employed here to prove this are the results from [ABC], mentioned above, the main theorem in [AC], and the *large periods property* which is a very elementary concept that we devised in order to detect mixing behavior. For instance, a dynamical system has *large periods property* if there are periodic points with any large enough period which are arbitrarily dense. The presence of this property implies that the system is topologically mixing. In the differentiable setting, we also define the *homoclinic large periods property* which only considers the homoclinically related periodic points. We prove that this property is robust, see Proposition 3.3.1.

In [AC], the authors use their main result to prove that any homoclinic class of a generic diffeomorphism has a spectral decomposition in the sense of Bowen, like discussed before. One of the motivations is that all known examples of robustly transitive diffeomorphisms are robustly topologically mixing.

So, in the same article the authors ask the following questions:

1. Is *every* robustly transitive diffeomorphism topologically mixing?
2. Failing that, is topological mixing at least a C^1 open and dense condition within the space of all robustly transitive diffeomorphisms?

Now, we point out that the results of section 2 of [AC] gives immediately the following result[5] (see also Remark **??**).

Theorem. *Let f be a generic diffeomorphism. If an isolated homoclinic class of f is topologically mixing then it is robustly topologically mixing.*

Actually, since the large periods property implies topological mixing, the robustness of this property could lead to another proof of the previous result, see Section 4.

We also attacked problem (2) above in [ACS]. It is natural to look for the global dynamics of the previous theorem instead of the semi-local dynamics. This leads us to a question posed in [BDV] (Problem 7.25, page 144): "For an open and dense subset of robustly transitive partially hyperbolic diffeomorphism: Is the whole manifold robustly a homoclinic class?". Recall by a result of [BC], for generic transitive diffeomorphisms, the whole manifold is a homoclinic class.

The next result gives a positive answer to Problem 7.25 of [BDV] (quoted above) *far from homoclinic tangencies*. A homoclinic tangency is a non-transversal intersection between the invariant manifolds of a hyperbolic periodic point. The result is the following:

[5] We would like to thank Prof. Sylvain Crovisier for pointing out this result to us.

Theorem (Arbieto,Catalan,S.). *There exists an open and dense subset among robustly transitive diffeomorphisms far from homoclinic tangencies formed by diffeomorphisms such that the whole manifold is a homoclinic class.*

This result together with the above result quoted from [AC] give us a partial answer to question (2) above, posed in [AC].

Theorem (Arbieto,Catalan,S.). *There is an open and dense subset among robustly transitive diffeomorphisms far from homoclinic tangencies formed by robustly topologically mixing diffeomorphisms.*

These two results were previously obtained by [BDU] for strongly partially hyperbolic diffeomorphisms with one dimensional center bundle, see also [HHU]. By strong partial hyperbolicity we mean partial hyperbolicity with both non-trivial extremal bundles such that the center bundle splits in one-dimensional subbundles in a dominated way. Actually, they obtain this proving that one of the strong foliations given by the partial hyperbolicity is minimal, which is a stronger property than topological mixing. In order to obtain this minimality they used arguments involving the accessibility property. We notice however that our results hold for diffeomorphisms with higher dimensional center directions. In Chpter 2, we present a way to produce such examples.

1.3 Orgnization of the thesis

The formal goal of the thesis is to report the research conducted during my PhD, so the proofs contained in the aforementioned papers [BS], [AMS1], [AMS2] and [ACS] are reproduced here, with some minor modifications which, hopefully, will improve their presentation. Nevertheless, I would like the text to readable by other mathematicians than just the jury of the thesis, with this goal in mind I prepared many sections intended as a preparation for the main arguments. In this way, the reader may choose to go directly to the proofs of the main results, checking some of the preliminary lemmas if he (or she) believes it is necessary, or even to read just the preliminary parts.

In Chapter 2 we collect all the necessary definitions and as usual fix the notations used in the sequel. In Chapter 3 we introduce homoclinic classes an we discuss mixin-like properties. In particular, we introduce a new definition, the *large periods property* and give some applications of it. We discuss the (elementary) part of the paper [AC] of Abdenur and Crovisier which is very important for our result [ACS] about denseness of Bernoulli for isolated generic homoclinic classes. In Chapter 4 continue the presentation of our paper [ACS] and prove mixing for an open and dense set robustly transitive systems far from tangencies.

We devote Chapter 5 to a detailed discussion of the index of a vector in a compact region. In the final section we prove an elementary result vey useful for degree calculations.

In Chapter 6 we make a pedestrian introduction to commuting vector fields, giving equivalent definitions and proving basic properties, which will be important in the sequel. We also present proofs of Lima's Theorem [Li1],[Li2]. A key part of this chapter is Lemma 6.3.3 (which is taken from [BS]), that we use to give another proof Lima's Theorem and will be usefull in our result with Bonatti too. The main source of difficulties in our proof of Lemma 6.3.3 is that Sard's theorem for real valued functions on a surface requires at least C^2. Even though it is still elementary, I believe that this proof of Lima's Theorem is knew.

Chapter 7 is devoted to the proof of existence of common zeros for commuting vector fields, under the assumption of non-vanishing index and that the colinearity locus is enclosed in a compact surface. In Chapter 8 we prove existece of attractors for non singular three dimensional vector fields. The presentation in this chapter is very different from the original article [AMS1], but the structure and the main arguments are the same.

In Chapter 9 we present the proof that a C^1-generic vector field without points accumulated by hyperbolic periodic orbits of different Morse indices has finitely many sinks and sectional-hyperbolic transitive Lyapunov stable sets for which the union of the basins is residual in M.

Chapter 2

The stage and the actors: manifolds, vector fields and diffeomorphisms

In this chapter we fix the notations used in this thesis and collect some preliminary materail that we shall use.

In all this work M^d denotes a d-dimensional Riemanninan manifold. We fix, once and for all, a Riemannnian metric in M, and denote by m the normalized Lebesgue measure induced by the volume form. We denote the space of diffeomorphisms of M by $\mathrm{Diff}^1(M)$, endowed with the C^1 topology. The space of vector fields over M, endowed with the C^1 topology, is denoted by $\mathfrak{X}^1(M)$.

2.1 Basic dynamical definitions

Whenever X is a vector field over M, we shall denote its flow by X_t. Given $x \in M$, the *orbit* of x, denoted by $O_X(p)$, is the set $\{X_t(x); t \in Dom(x)\}$, where $Dom(x) \subset \mathbb{R}$ is the maximal interval of definition of the integral curves of X, which start at x. A compact set $\Lambda \subset M$ is *invariant under the flow of* X if $X_t(\Lambda) = \Lambda$ for every $t \in \mathbb{R}$. Denote $\mathrm{Zero}(X) = \{x \in M; X(x) = 0\}$ and $\mathrm{Zero}(X, U) = \mathrm{Zero}(X) \cap U$, for any subset $U \subset M$.

A point $p \in M$ is said to be *periodic* for X if there exists $T > 0$ such that $X_T(p) = p$.

The set of all periodic points for X is denoted by $\mathrm{Per}(X)$. The minimum such T is called the *period* of p, and is denoted by $\tau(p)$.

Let $f : M \to M$ be a homeomorphism. Given $x \in M$, we define the *orbit* of x as the set $O(x) := \{f^n(x); n \in \mathbb{Z}\}$. The forward orbit of x is the set $O^+(x) := \{f^n(x); n \in \mathbb{N}\}$. In a similar way we define the backward orbit $O^-(x)$. If necessary, to emphasize the dependence of f, we may write $O_f(x)$. We say that $p \in M$ is a *periodic point* if $f^n(p) = p$ for some $n \geq 1$. The minimum of such n is called the *period* of p and it is denoted by $\tau(p)$.

Given $\Lambda \subset M$ we say that it is an *invariant* set if $f(\Lambda) = \Lambda$.

We recall the notions of transitivity and mixing. We say that f is *transitive* if there exists a point in M whose forward orbit is dense. This is equivalent to the existence of a dense backward orbit and is also equivalent to the following condition: for every pair U, V of open sets, there exists $n > 0$ such that $f^n(U) \cap V \neq \emptyset$. We say that f is *topologically mixing* if for every par U, V of open sets there exists $N_0 > 0$ such that $n \geq N_0$ implies $f^n(U) \cap V \neq \emptyset$.

2.1.1 Hyperbolic Periodic Points

Diffeomorphisms

Let $f \in \mathrm{Diff}^1(M)$ be fixed.

A periodic point of f is *hyperbolic* if the eigenvalues of $Df^{\tau(p)}(p)$ do not belong to S^1. As usual, $E^s(p)$ (resp. $E^u(p)$) denotes the eigenspace of the eigenvalues with norm smaller (resp. bigger) than one. This gives a $Df^{\tau(p)}$ invariant splitting of the tangent bundle over the orbit $O(p)$ of p. The *index* of a hyperbolic periodic point p is the dimension of the stable direction.

If p is a hyperbolic periodic point for f then every diffeomorphism g, C^1-close to f have also a hyperbolic periodic point close to p with same period and index, which is called the continuation of p for g, and it is denoted by $p(g)$.

The local stable and unstable manifolds of a hyperbolic periodic point p are defined as follows: given $\varepsilon > 0$ small enough, we set

$$W^s_{loc}(p) = \{x \in M;\ \ d(f^n(x), f^n(p)) \leq \varepsilon, \text{ for every } n \geq 0\} \text{ and}$$

$$W^u_{loc}(p) = \{x \in M;\ (f^{-n}(x), f^{-n}(p)) \leq \varepsilon, \text{ for every } n \geq 0\}.$$

They are embedded differentiable (as smooth as f) manifolds tangent at p to $E^s(p)$ and $E^u(p)$. This is the content of the so-called*stable manifold theorem*, see [dMP].

The stable and unstable manifolds are given by the saturations of the local manifolds. indeed,

$$W^s(p) = \bigcup_{n\geq 0} f^{-n\tau(p)}(W^s_{loc}(p)) \text{ and } W^u(p) = \bigcup_{n\geq 0} f^{n\tau(p)}(W^u_{loc}(p)).$$

The stable and unstable set of a hyperbolic periodic orbit, $O(p)$ are given by:

$$W^s(O(p)) = \bigcup_{j=0}^{\tau(p)-1} W^s(f^j(p)) \text{ and } W^u(O(p)) = \bigcup_{j=0}^{\tau(p)-1} W^u(f^j(p)).$$

Flows

We shall recall briefly how to define hyperbolicity of critical elements for flows. For further details, we invite the reader to consult [dMP]. Take $\sigma \in \mathrm{Zero}(X)$. We say that σ is *hyperbolic* if the map $DX(\sigma) : T_\sigma M \to T_\sigma M$ does not have 0 as an eigenvalue. In this case, the tangent space $T_\sigma M$ splits as direct sum $E^s \oplus E^u$, where E^s is spanned by the eigenspaces associated with negative eigenvalues while E^u is spanned by the eigenspaces associated with positive eigenvalues.

The local stable/unstable manifolds

$$W^s_{loc}(\sigma) = \{x \in M;\ \ d(X_t(x), \sigma) \leq \varepsilon, \text{ for every } t \geq 0\} \text{ and}$$

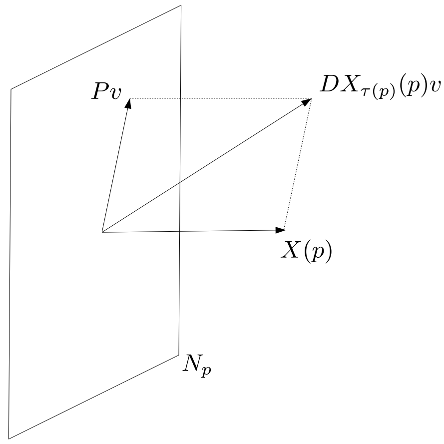
$$W^u_{loc}(p) = \{x \in M;\ (X_{-t}(x), \sigma) \leq \varepsilon, \text{ for every } t \geq 0\}$$

are embedded embedded differentiable (as smooth as X) manifolds tangent at σ to E^s and E^u. The stable and unstable manifolds are given by the saturations of the local stable unstable manifolds.

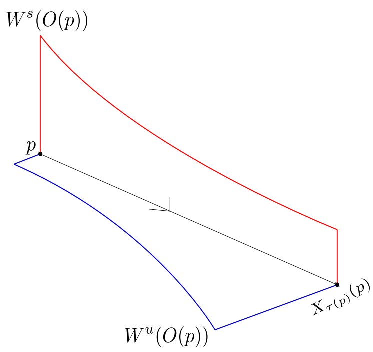
Consider now $p \in \mathrm{Per}(X)$. Since $X(X_t(p)) \neq 0$, for every $t \in \mathbb{R}$, there exists consider a subbundle N, transverse to X at every point in a neighborhood of $O_X(p)$. We define the *liner Poincaré map* $P : N_p \to N_p$ in the following way:

$$Pv = \pi \circ DX_{\tau(p)}(p)v,$$

for every $v \in N_p$, where $\pi : T_pM \to N_p$ is the projection parallel to X. The linear Poincaré flow will be an important tool in Chapter 8



We say that p is *hyperbolic* if P has no eigenvalues of modulus 1. The stable manifold theorem also applies: the sets $W^s(O(p)) = \{x \in M; d(X_t(x), X_t(p)) \to 0, \text{ as } t \to +\infty\}$ and $W^u(O(p)) = \{x \in M; d(X_t(x)X_t(p)) \to 0, \text{ as } t \to -\infty\}$ are immersed submanifolds of M which intersect transversely along $O(p)$.



2.1.2 Invariant Measures

A probability measure μ is f-invariant if $\mu(f^{-1}(B)) = \mu(B)$ for every measurable set B. An invariant measure is ergodic if the measure of any invariant set is zero or one. Let $\mathcal{M}(f)$

be the space of f-invariant *probability measures* on M, and let $\mathcal{M}_e(f)$ denote the ergodic elements of $\mathcal{M}(f)$.

For a periodic point p of f with period $\tau(p)$, we let μ_p denote the periodic measure associated to p, given by

$$\mu_p = \frac{1}{\tau(p)} \sum_{x \in O(p)} \delta_x$$

where δ_x is the Dirac measure at x.

Given an invariant measure μ, Oseledets' Theorem says that for almost every $x \in M$ one has a measurably varying splitting of the tangent bundle $TM = E_1 \oplus ... \oplus E_k$ such that if $v \in E_j$ then

$$\lambda(x, v) := \lim_{n \to \pm\infty} \frac{1}{n} \log \|Df^n(x)v\|$$

is well defined and does not depend on $v \in E_j$. In particular, one has measurable invariant functions $\lambda_j : M \to \mathbb{R}$, $j = 1, ..., k$ such that if $v \in E_j$ then $\lambda(x, v) = \lambda_j(x)$.

The number $\lambda_j(x)$ is called the *Lyapunov exponent* of f at x.

Now, let us define the notion of Bernoulli measure. We first recall the so-called Bernoulli shift. It is the homeomorphism $\sigma : \{1, ..., n\}^{\mathbb{Z}} \to \{1, ..., n\}^{\mathbb{Z}}$ defined by $\sigma(\{x_n\}) = \{x_{n+1}\}$. In $\{1, ..., n\}^{\mathbb{Z}}$ consider m_B the product measure with respect to the uniform probability in $\{1, ..., n\}$. It is easy to see that m_B is invariant under σ.

We say that $\mu \in \mathcal{M}(f)$ is a *Bernoulli measure* if (f, μ) is measure theoretically isomorphic to (σ, m_B).

2.1.3 Domination, hyperbolicity and beyond

Let $\Lambda \subset M$ to be invariant under a diffeomorphism f. Let E, F to be subbundles of $T_\Lambda M$ of the tangent bundle over Λ, invariant under Df and with trivial intersection at every $x \in \Lambda$. We say that E *dominates* F if there exists $N \in \mathbb{N}$ such that

$$\|Df^N(x)|_E\| \|Df^{-N}(f^N(x))|_F\| \leq \frac{1}{2},$$

for every $x \in \Lambda$. We say that Λ admits a *dominated splitting* if there exists a decomposition of the tangent bundle $T_\Lambda M = \bigoplus_{l=1}^k E_l$ such that E_l dominates E_{l+1}.

We say that a f-invariant subset Λ is *partially hyperbolic* if it admits a dominated splitting $T_\Lambda M = E^s \oplus E^c_1 \oplus \ldots \oplus E^c_k \oplus E^u$, with at least one of the extremal bundles being non-trivial, such that the extremal bundles have uniform contraction and expansion: there exist a constants $m \in \mathbb{N}$ such that for every $x \in M$:

- $\|Df^m(x)v\| \leq 1/2$ for each unitary $v \in E^s$,
- $\|Df^{-m}(x)v\| \leq 1/2$ for each unitary $v \in E^u$

and the other bundles, which are called center bundles, do not contracts neither expands.

If all center bundles are trivial, then Λ is called a *hyperbolic set*. Now, we say Λ is *strongly partially hyperbolic* if both extremal bundles and center bundle are non-trivial and moreover such that all of its center bundles are one-dimensional. In particular a strongly partially hyperbolic set is not hyperbolic.

We say that a diffeomorphism $f : M \to M$ is *partially hyperbolic* (resp. *strong partially hyperbolic*) if M is a partially hyperbolic (resp. strongly partially hyperbolic) set of f. When M is a hyperbolic set we say that f is *Anosov*.

We remark now that strongly partially hyperbolic diffeomorphisms are by definition far from homoclinic tangencies, since all central sub bundles have dimension one.

Examples of partially hyperbolic diffeomorphisms with higher dimensional central directions can be given by deforming some linear Anosov diffeomorphisms as in Mañé's example. For instance, let A be a linear Anosov diffeomorphism with eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < 1 < \lambda_4$ but, such that λ_2 and λ_3 are close to 1. Then we can create a pitchfork bifurcation, producing two fixed points p and q with eigenvalues $\mu_1(p) < 1 < \mu_2(p) < \mu_3(p) < \mu_4(p)$ and $\mu_1(q) < \mu_2(q) < 1 < \mu_3(q) < \mu_4(q)$, such that $\mu_3(q)$ is still close to 1. Moreover, as in Mañé's argument [Ma3] we can guarantee that this diffeomorphism is transitive. Now we can perform another pitchfork bifurcation on q producing two other fixed points q_1 and q_2 with eigenvalues $\mu_1(q_1) < \mu_2(q_1) < 1 < \mu_3(q_1) < \mu_4(q_1)$ and $\mu_1(q_2) < \mu_2(q_2) < \mu_3(q_2) < 1 < \mu_4(q_2)$. Once again, this diffeomorphism is transitive. Now, since the bifurcations preservers the center unstable leaves, we can guarantee that there exists a dominated splitting $E^s \oplus E^c_1 \oplus E^c_2 \oplus E^u$, where E^c_1 is related to μ_2 and E^c_2 is related to μ_3. As in Mañé's example, the unstable foliation will be minimal. In particular, it will be topologically mixing also.

Remark 2.1.1. If f is partially hyperbolic, by Theorem 6.1 of [HPS] there exist strong stable and strong unstable foliations that integrate E^s and E^u. More, precisely, for any point $x \in M$ there is a unique invariant local strong stable manifold $W^{ss}_{loc}(x)$ which is a smooth graph of a function $\phi_x : E^s \to E^c \oplus E^u$ (in local coordinates), and varies continuously with x. In particular, $W^{ss}_{loc}(x)$ has uniform size for every $x \in M$. The same holds for $W^{uu}_{loc}(x)$, integrating E^u.

Saturating these local manifolds, we obtain two foliations, that we denote by $\mathcal{F}^s$ and $\mathcal{F}^u$ respectively. Indeed, $\mathcal{F}^s(x) = \bigcup_{n\geq 0} f^{-n}(W^{ss}_{loc}(f^n(x))$. Analogous definition holds for $\mathcal{F}^u$.

Sectional hyperbolicity

Consider now a vector field $X \in \mathfrak{X}^1(M)$

Denote by $\|\cdot\|$ and $m(\cdot)$ the norm and the minimal norm induced by the Riemannian metric and by $\det(\cdot)$ the jacobian operation. We say that Λ is *hyperbolic* if there are a continuous invariant tangent bundle decomposition

$$T_\Lambda M = \hat{E}^s_\Lambda \oplus \hat{E}^X_\Lambda \oplus \hat{E}^u_\Lambda$$

and positive constants K, λ such that $\hat{E}^X_\Lambda$ is the subbundle generated by X,

$$\|DX_t(x)/\hat{E}^s_x\| \leq Ke^{-\lambda t} \quad \text{and} \quad m(DX_t(x)/\hat{E}^u_x) \geq K^{-1}e^{\lambda t},$$

for all $x \in \Lambda$ and $t \geq 0$. A closed orbit is hyperbolic if it does as a compact invariant set. We define the *Morse index* $I(O)$ of a hyperbolic closed orbit O by $I(O) = dim(E^s_x)$ for some (and hence for all) $x \in O$. In case O is a singularity σ we write $I(\sigma)$ instead of $I(\{\sigma\})$. A *sink* will be a hyperbolic closed orbit of maximal Morse index and a *source* is a sink for the time reverser vector field.

Given an invariant splitting $T_\Lambda M = E_\Lambda \oplus F_\Lambda$ over an invariant set Λ of a vector field X we say that the subbundle E_Λ *dominates* F_Λ if there are positive constants K, λ such that

$$\frac{\|DX_t(x)/E_x\|}{m(DX_t(x)/F_x)} \leq Ke^{-\lambda t}, \qquad \forall x \in \Lambda \text{ and } t \geq 0.$$

(In such a case we say that $T_\Lambda M = E_\Lambda \oplus F_\Lambda$ is a *dominated splitting*).

We say that Λ is *partially hyperbolic* if it has a dominated splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ whose dominating subbundle E^s_Λ is *contracting*, namely,

$$\|DX_t(x)/E^s_x\| \leq Ke^{-\lambda t}, \qquad \forall x \in \Lambda \text{ and } t \geq 0.$$

Moreover, we call the central subbundle E^c_Λ *sectionally expanding* if

$$dim(E^c_x) \geq 2 \quad \text{ and } \quad |\det(DX_t(x)/L_x)| \geq K^{-1}e^{\lambda t}, \qquad \forall x \in \Lambda \text{ and } t \geq 0$$

and all two-dimensional subspace L_x of E^c_x.

We call *sectional-hyperbolic* any partially hyperbolic set whose singularities (if any) are hyperbolic and whose central subbundle is sectionally expanding [MeM].

2.1.4 Robustness and Genericity

As mentioned before, we deal with the space $\text{Diff}^1(M)$ of C^1 diffeomorphisms over M endowed with the C^1-topology. This is a Baire space. Thus any residual subset, i.e. a countable intersection of open and dense sets, is dense. When a property P holds for any diffeomorphism in a fixed residual subset, we will say that P holds generically. Or even, that a generic diffeomorphisms exhibits the property P.

On the other hand, we say that a property holds robustly for a diffeomorphism f if there exists a neighborhood $\mathcal{U}$ of f such that the property holds for any diffeomorphism in $\mathcal{U}$.

In this way, we say that a diffeomorphism $f \in \text{Diff}^1(M)$ is *robustly transitive* if it admits a neighborhood entirely formed by transitive diffeomorphisms.

In this thesis we let $\mathcal{T}(M)$ denote the open set of $\text{Diff}^1(M)$ formed by robustly transitive diffeomorphisms which are far from tangencies. Notice that being far from tangencies is, by definition, an open condition. Also we define by $\mathcal{T}_{NH}(M)$ as the interior of robustly transitive strongly partially hyperbolic diffeomorphisms, which is a subset of $\mathcal{T}(M)$.

When dealing with properties which involves objects defined by the diffeomorphism itself we need to deal with the continuations of these objects.

For instance, when we say that a homoclinic class of f is robustly topologically mixing, we are fixing a hyperbolic periodic point p of f and a neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ the continuation $p(g)$ of p is defined and the homoclinic class $H(p(g), g)$ is topologically mixing, i.e. for any U and V open sets of $H(p(g), g)$ there exists $N > 0$ such that for any $n \geq N$ we have $g^n(U) \cap V \neq \emptyset$.

Another example of a robust property is given by the following well known result which says that partial hyperbolicity is a robust property.

Proposition 2.1.2 (p. 289 of [BDV]). *Let Λ be a (strongly) partially hyperbolic set for f. Then, there exists a neighborhood U of Λ and a C^1 neighborhood $\mathcal{U}$ of f such that every g-invariant set $\Gamma \subset U$, is (strongly) partially hyperbolic, for every $g \in \mathcal{U}$.*

Since the space $\mathfrak{X}^1(M)$ is also a Baire space, we can speak of generecity and robustness in $\mathfrak{X}^1(M)$ in the same way as above. In particular, we shall denote by $\mathfrak{X}^1_{NS}(M)$ the open subset of $\mathfrak{X}^1(M)$ formed by vector fields X such that $\mathrm{Zero}(X) = \emptyset$. In Chapter 8 we shall see a generic property of $\mathfrak{X}^1_{NS}(M)$ (i.e a property which holds in every element of the intersection between $\mathfrak{X}^1_{NS}(M)$ and some residual subset of $\mathfrak{X}^1(M)$).

2.2 Lyapunov stable sets

In this section we fix M, a d-dimension manifold and $X \in \mathfrak{X}^1(M)$. We shall define the invariant sets which describe the asymptotic topological

Given $p \in M$ we define the *ω-limit set*

$$\omega(p) = \left\{ x \in M : x = \lim_{n\to\infty} X_{t_n}(p) \text{ for some sequence } t_n \to \infty \right\}.$$

An invariant set $\Lambda \subset M$ is *nontrivial* if it does not reduces to a single closed orbit. We also define the *basin of attraction* of Λ by

$$W^s(\Lambda) = \{x \in M : \omega(x) \subset \Lambda\}.$$

A transitive set Λ will be called *attractor* if it exhibits a neighborhood U with the following properties: there exists $T > 0$ such that $X_T(\overline{U}) \subset U$ and $\Lambda = \bigcap_{t\geq 0} X_t(U)$. In particular, $W^s(\Lambda)$ is an open set, since $U \subset W^s(\Lambda)$.

Assume that the flow of X is complete. Given an open set U we denote by $\Lambda_X(U)$ the *maximal invariant set inside* U, i.e $\Lambda_X(U) = \bigcap_{t\in\mathbb{R}} X_t(U)$.

Notice that for every attractor $\Lambda_X(W^s(\Lambda)) = \Lambda$.

Let Λ be a compact invariant set for the flow of X. We say that Λ is *Lyapunov stable* for X if for every neighborhood U of Λ there exists a smaller neighborhood V of Λ such that $\omega(x) \subset U$, for every $x \in V$. We say that Λ is *Lyapunov unstable* if it is Lyapunov stable for $-X$.

Lemma 2.2.1. *Let Λ be an invariant set for $X \in \mathfrak{X}^1(M)$. Assume that there exists $x \in M$ such that $\omega(x) \cap \Lambda \neq \emptyset$.*

1. *If Λ is Lyapunov stable, then $\omega(x) \subset \Lambda$.*

2. *If Λ is Lyapunov unstable, then $x \in \Lambda$.*

Proof of (1). Take U a neighborhood of Λ, and let V be the smaller neighborhood of Λ such that if $y \in V$ then $\omega(y) \subset U$. Since $\omega(x) \cap \Lambda \neq \emptyset$, there exist $T > 0$ such that $X_T(x) \in V$. As a consequence,

$$\omega(x) = \omega(X_T(x)) \subset U.$$

Thus, $\omega(x)$ is a subset of any neighborhood U of Λ, concluding. □

Proof of (2). The argument is similar, only one has to iterate backwards in order to use the Lyapunov instability. The details are left to the reader. □

2.3 Sard's Theorem

In Chapter 6 we shall use Sard's Theorem. The version of this result, usually taught in graduate courses, holds for C^∞ maps on manifolds. However, we shall need to apply this result in low regularity, and for this reason we state here the original (sharper) version, see [Sa].

Theorem 2.3.1 (Sard)**.** *Let $\varphi : M^d \to N^n$ be a map of class C^k, let $\mathcal{S} \subset M$ be the singular set of φ. Assume further that*

1. $d \leq n$

2. $d > n$ and $k \geq d - n + 1$.

Then, $\varphi(\mathcal{S})$ has Lebesgue measure zero

In the appendix we shall describe an example, due to H. Whitney, of a C^1 function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that $\varphi(S) = [0, 1]$, and thus the hypothesis $d > n$ and $k \geq d - n + 1$ cannot be weakened.

2.4 Holonomies

Let $Y \in \mathfrak{X}^1(M)$ be a non-vanishing vector field in some compact region U of M. A codimension one submanifold $\Sigma \subset U$ of M is said to be a *transverse section* if Σ is everywhere transverse to Y.

Let Σ_0 and Σ_1 be two transverse sections, either disjoint or with $\Sigma_1 \subset \Sigma_0$. Assume that there exists a function $\tau : \Sigma_0 \to (0, +\infty)$, as regular as Y, such that $Y_t(x) \in \Sigma_1$ with $t \in [0, \tau(x)]$ if and only if $t = \tau(x)$, for every $x \in \Sigma_0$. A function with this property is called a *transition time*. With the transition time we define a map $\mathcal{P} : \Sigma_0 \to \Sigma_1$, called *holonomy*, by

$$\mathcal{P}(x) = Y_{\tau(x)}(x).$$

In the case $\Sigma_1 \subset \Sigma_0$ the holonomy is called the *first return map* or the *Poincaré map* and τ is called the *first return time function.*

Now we shall state a very general and certainly classical lemma that gives the relation between the derivative of the holonomy map and the derivative of the flow. We insert the proof here for the comfort of the reader.

Lemma 2.4.1. *For every $x \in \Sigma_0$ and every $v \in T_x\Sigma_0$,*

$$D\mathcal{P}(x)v - DY_{\tau(x)}(x)v = D\tau(x)v.Y(\mathcal{P}(x)).$$

Proof. Since this is a local problem we shall use folw box coordinates around $\mathcal{P}(x)$. Let us denote by B the flow box around $\mathcal{P}(x)$. Take a curve $\gamma : (-\varepsilon, \varepsilon) \to \Sigma_0$ such that $\gamma(0) = x$

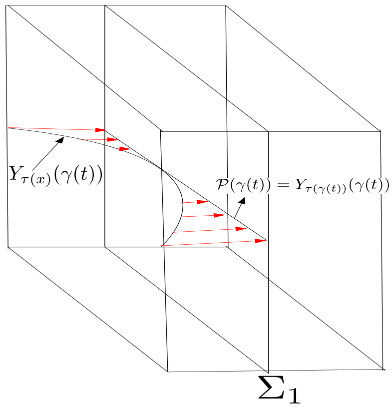


Figure 2.1: Flow box around $\mathcal{P}(x) \in \Sigma_1$ and the formula $Y_{\tau(\gamma(t))-\tau(x)} \circ Y_{\tau(x)}(\gamma(t)) = Y_{\tau(x)}(\gamma(t)) + (\tau(\gamma(t)) - \tau(x))Y(\mathcal{P}(x))$.

and $\dot{\gamma}(0) = v$, with ε small enough so that $\mathcal{P}(\gamma(t)) \in B$, for every $t \in (-\varepsilon, \varepsilon)$. Therefore, in the flow box coordinates, we have that

$$Y_{\tau(\gamma(t))-\tau(x)} \circ Y_{\tau(x)}(\gamma(t)) = Y_{\tau(x)}(\gamma(t)) + (\tau(\gamma(t)) - \tau(x))Y(\mathcal{P}(x)).$$

On the other hand, since $\mathcal{P}$ and $Y_{\tau(x)}$ are C^1 maps, we have for ε small enough that

$$\begin{aligned} \mathcal{P}(\gamma(t)) &= \mathcal{P}(x) + tD\mathcal{P}(x)v + R(t) \\ Y_{\tau(x)}(\gamma(t)) &= Y_{\tau(x)}(x) + tDY_{\tau(x)}(x)v + \overline{R}(t), \end{aligned}$$

Where the remainder maps have the property that

$$\frac{R(t)}{t}, \frac{\overline{R}(t)}{t} \to 0,$$

when $t \to 0$. Since, by definition, we can write

$$\mathcal{P}(\gamma(t)) = Y_{\tau(\gamma(t))-\tau(x)} \circ Y_{\tau(x)}(\gamma(t)),$$

it follows then

$$\begin{aligned} \mathcal{P}(x) + tD\mathcal{P}(x)v + R(t) &= Y_{\tau(x)}(p) + tDY_{\tau(x)}(x)v + \overline{R}(t) \\ &+ (\tau(\gamma(t)) - \tau(x))Y(\mathcal{P}(x)), \end{aligned}$$

and thus

$$DP(x)v - DY_{\tau(x)}(x)v = \frac{\tau(\gamma(t)) - \tau(x)}{t} Y(\mathcal{P}(x)) + \frac{R(t) + \overline{R}(t)}{t}, \tag{2.1}$$

because $Y_{\tau(x)}(x) = \mathcal{P}(x)$. Taking the limit as $t \to 0$ in the right-hand side of equation (2.1) the proof is complete. $\square$

Chapter 3

Topologically mixing homoclinic classes

Let $f : M^d \to M^d$ be a diffeomorphism of a closed manifold M. If p is a hyperbolic periodic point of f, then its *homoclinic class* $H(p)$ is the closure of the transversal intersections between the stable manifold and unstable manifold of the orbit of p:

$$H(p) = \overline{W^s(O(p)) \pitchfork W^u(O(p))}.$$

A homoclinic class $H(p)$ is said to be trival if it reduces to $O(p)$. We say that the homoclinic class $H(p)$ is isolated if there exists a neighborhood U of $H(p)$ such that $H(p) = \bigcap_{n\in\mathbb{Z}} f^n(U)$.

It is possible to give another definition (which justifies the word "class") by introducing an equivalence relation among periodic orbits. Indeed, we say that a hyperbolic periodic point q is *homoclinically related* to p if $W^s(O(p)) \pitchfork W^u(O(q)) \neq \emptyset$ and $W^u(O(p)) \pitchfork W^s(O(q)) \neq \emptyset$. For it is well known that a homoclinic class coincides with the closure of the hyperbolic periodic points homoclinically related to p.[1]

Moreover, the dynamics of $f|_{H(p)}$ is always transitive and in the case of an Axiom A diffeomorphism[2] f, homoclinic classes form the basic dynamical pieces. Furthermore, Bowen

[1]For a proof, see for instance the book of Newhouse [N], or Lema 3.3.3 in [San].

[2]A diffeomorphism f is said to be Axiom A if the non-wandering set $\Omega(f)$ is hyperbolic. Recall that

showed that a hyperbolic homoclinic class $H(p)$ admits a cyclic decomposition $H_1, ..., H_l$ i.e $f(H_j) = H_{j+1}$, with the convention that $l+1=1$, and such that $f^l|_{H_j}$ is topologically mixing.

Starting from this result of Bowen, Sigmund proved in the seventies [S2] the denseness of periodic points for a hyperbolic homoclinic class $H(p)$ at the ergodic level. His result can be summarized in the following equality

$$\overline{\{\mu_q; q \in \mathrm{Per}(f) \cap H(p)\}} = \mathcal{M}_f(H(p)),$$

where $\mathcal{M}_f(H(p))$ denotes the set of invariant measures supported in $H(p)$. Latter, in [S1], he treated the case where $H(p)$ is topologically mixing and proved that the Bernoulli measures are dense $\mathcal{M}_f(H(p))$.

These types of result rely on the *shadowing lemma*. One might wonder, following some ideas present the 1983 ICM addres of Mañé [Ma5], whether the perturbative tools available in C^1 topology, such as the *ergodic closing lemma* [Ma2] or the *connecting lemma* [H1] can play the same role to prove similar results for C^1-generic diffeomorphisms.

This line of research was initiated in the pioneer work of Abdenur, Bonatti and Crovisier. Let us state their result in precise terms.

Theorem 3.0.2 (Theorem 3.5, item (a), in [ABC])**.** *Let Λ be an isolated non-hyperbolic transitive set of a C^1-generic diffeomorphism f, then the set of periodic measures supported in Λ is a dense subset of the set $M_f(\Lambda)$ of invariant measures supported in Λ.*

In view of Theorem 3.0.2 it becomes natural to ask whether the denseness of Bernoulli measures would also be true in the C^1-generic scenario. In this chapter we shall answer affirmatively this question by proving the theorem below which is the first main result of this thesis.

Theorem A (Arbieto, Catalan, S-)**.** *For any generic diffeomorphism f, if the dynamics restricted to an isolated homoclinic class $H(p)$ is topologically mixing then the Bernoulli*

$\Omega(f) = \{x \in M; \forall \varepsilon > 0, \exists\, n > 0; f^n(B(x,\varepsilon) \cap B(x,\varepsilon) \neq \emptyset\}$. See the monograph [Bow1] of Bowen for a complete account on the properties of Axiom A diffeomorphism.

measures are dense in $\mathcal{M}_f(H(p))$. In particular, the set of weakly mixing measures contain a residual subset of $\mathcal{M}(H(p))$.

The chapter is organized in the following way. In Section 3.1 we introduce a property which implies mixing and we study conditions on a homoclinic class to ensure this property. In Section 3.2 we will discuss the period of a homoclinic class and the work of Abdenur and Crovisier [AC]. Then, we shall combine the results of [AC] with the large property to build topologically mixing horseshoes in Section 3.3. These horseshoes always support Bernoulli measures. Theorem A will be proved in Section 3.4.

3.1 Large Periods Property

The most common exapmples of topologically mixing transformations come from the uniformly hyperbolic world, and thus they have very very rich dynamical properties, such as denseness of periodic orbits. In this section we shall investigate a special class of topologically mixing transformations wich have dense periodic orbits.

Let $f : X \to X$ be a homeomorphism of a metric space. We say that f has the *large periods property* if for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for every $n \geq N_0$ there exists $p_n \in \text{Fix}(f^n)$, whose orbit under f is ε dense in X.

In words: f has the large periods property if, given any cover of the ambient space by small balls, f has periodic orbits, with *any* possible large enough period which visit every ball of the cover. Notice that the definition do not require existence of periodic orbits with all possible *minimal* periods.

A simple remark is that if X has an isolated point and f has the large period property then X is a singleton.

The large periods property can be used as a criterion to ensure mixing, as the next result shows.

Lemma 3.1.1. *Every homeomorphism of a metric space with the large periods property is topologically mixing*

Proof. Let $f : X \to X$ be a homeomorphism with the large periods property. Notice that f is transitive. Indeed, given U and V non-empty and disjoint open sets take $\varepsilon > 0$ and balls $B(u, \varepsilon) \subset U$ and $B(v, \varepsilon) \subset V$. By the large periods property, there exists a point $p \in \mathrm{Per}(f)$ whose orbit is ε dense in X. This implies that there exists a point $y \in B(v, \varepsilon)$ and $n > 0$ such that $f^n(y) \in B(u, \varepsilon)$. Thus f is transitive.

We now prove that f is topologically mixing. Let U and V be non-void and disjoint open sets. By the transitivity of f there exists a first iterate n_1 such that $f^{n_1}(U) \cap V \neq \emptyset$. In particular, $f^j(U) \cap V = \emptyset$ for every $j = 1, ..., n_1 - 1$. Take an open ball $B \subset U$, satisfying

$$f^{n_1}(B) \subset f^{n_1}(U) \cap V,$$

and $\varepsilon = diam(B)/2$. Let $N_0 = N_0(\varepsilon)$ be given by the large periods property.

We claim that $f^n(V) \cap U \neq \emptyset$, for every $n \geq N_0$. Indeed, we know that there exists $p \in \mathrm{Fix}(f^\tau)$, with $\tau = n + n_1$, whose orbit under f is ε dense in X. By the choice of ε, there is an iterate of p in B. Since p is periodic we shall assume for simplicity that p itself is in B. This implies that $f^{n_1}(p) \in V$, and therefore

$$f^n(f^{n_1}(p)) = f^{n+n_1}(p) = f^\tau(p) = p \in U.$$

This proves our claim, and establishes the lemma. □

The converse of this result is not true. There exists examples of topologically mixing diffeomorphisms which do not have dense periodic orbits. We may cite some research papers with interesting examples. Fayad [Fa] give an analalytic minimal (and thus without periodic orbits) and topologically mixing diffeomorphism of the five dimensional torus. Carvalho and Kwietniak [CK] give an example of a homeomorphism of a compact metric space with the two-sided limit shadowing property, but without periodic points. Theorem B in [CK] establishes that the two-sided limit shadowing property implies topological mixing. Nevertheless, it is possible to give an elementary example.

Example 3.1.2. We shall give an informal description of how to construct a topologically mixing flow on the two torus with one singularity and no periodic orbits. [3] Let $\widetilde{X}$ be a

[3] I thank Prof. Alejandro Kocsard who made me aware of this example.

constant vector field of $\mathbb{R}^2$ with integer coordinates and irrational slope. Let $X_t : \mathbb{T}^2 \to \mathbb{T}^2$ be the corresponding irrational flow, whose vector field is X, the projection of $\widetilde{X}$. Take $p \in \mathbb{T}^2$ and a smooth bump function $\psi : \mathbb{T}^2 \to [0,1]$ such that $\psi(x) = 0$ if and only if $x = p$. Let $Y(x) = \psi(x)X(x)$ and take Y_t the flow of Y. We claim that it has the following properties

1. Y_t is topologically mixing
2. $\mathrm{Zero}(Y) = \{p\}$ and $\mathrm{Per}(Y) = \emptyset$

By taking $f := Y_1$, the time one map, we obtain a topologically mixing diffeomorphism of $\mathbb{T}^2$ wich do not satisfy the large periods property.

Let us prove (1) and (2). By construction of Y it is easy to see that $\mathrm{Zero}(Y) = \{p\}$. Let $\mathcal{F}$ denote the foliation by orbits of X. Since Y is a repatrization of X, wich is zero only at p, we have that the foliation by orbits of Y is equal to $\mathcal{F}$. However, there exists one especial leaf, namely the one containing p. The points in the leaf $\mathcal{F}_p$ converge to p, in the past or in the future. Morevoer, if a point in $\mathcal{F}_p$ converge to p in the future, its backward orbit is dense in $\mathbb{T}^2$, and similarly, the points in $\mathcal{F}_p$ which converge to p in the past have a dense positive orbit. All leaves are dense, and all points in the other leaves have dense orbits, both positive and negative. As consequence, $\mathrm{Per}(Y) = \emptyset$.

It remains to see why Y_t is mixing. Take $B(x, \varepsilon)$ and $B(y, r)$ two small open balls in $\mathbb{T}^2$. By iterating (positively or negatively) we see that $B(x, \varepsilon)$ will intersect $\mathcal{F}_p$. For simplicity, let us assume that $B(x, \varepsilon) \cap \mathcal{F}_p \neq \emptyset$. Let I be a segment contained in $B(x, \varepsilon) \cap \mathcal{F}_p$. Either $Y_t(I) \to \{p\}$ as $t \to +\infty$ or as $t \to -\infty$. We shall assume the former case, the latter being treated with a similar argument. Since all points in $B(x, \varepsilon) \setminus I$ have a dense forward orbit but I converge to p in the future, we see that $Y_t(B(x, \varepsilon))$ will get streched and will become more and more dense in $\mathbb{T}^2$, as t grows. See figure 3.1. As a consequence, once $Y_t(B(x, \varepsilon))$ becomes dense enough to intersect $B(y, \delta)$, it will continue to intersect for every $T \geq t$. Thus Y_t is mixing.

Example 3.1.2 sugests a question. *Does every topologically mixing homeomorphism with* infinetely many *periodic orbits have the large periods property?*.

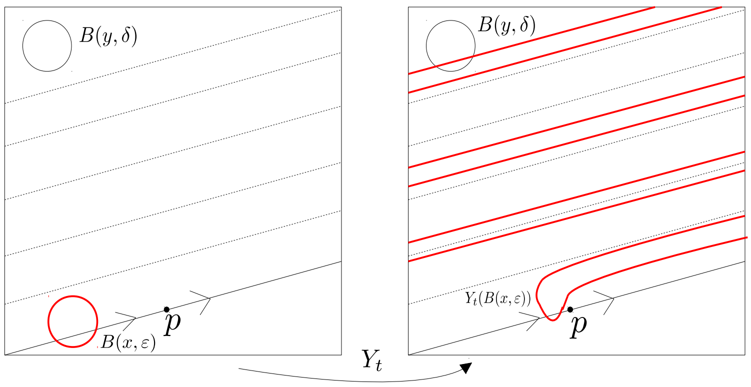


Figure 3.1: Pictorial proof that Y_t is mixing.

Morevoer, as we have commented in the begining of this section, it is easy to see that a topologically mixing uniformly hyperbolic transformation (such as Smale's horseshoe, a transitive Anosov map and mixing subshifts) has the large periods properties. Below we shall give a general lemma which will prove this. A natural question is then to investigate the relation between mixing and large periods outside the hyperbolic world. For instance,

Question 1. Let f be a topologically mixing C^1-generic diffeomorphism of a closed manifold. Does f have the large periods property?

In Section 3.1.8 we shall solve this question by constrcting large horseshoes with the large periods property. The first step in this construction will be given in this section. We will introduce below a version of the large periods property adapted to homoclinic classes.

Definition 3.1.3. Let $f : M \to M$ be a diffeomorphism and let $H(p)$ be a homoclinic class of f. We say that an invariant subset $\Lambda \subset H(p)$ has the *homoclinic large periods property* if for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for every $n \geq N_0$ it is possible to find a point $p_n \in \mathrm{Fix}(f^n)$ in Λ, and homoclinically related with p, whose orbit under f is ε dense in Λ.

In the sequel, we shall establish a result which produces hyperbolic horseshoes having the homoclinic large periods property when there exists a special type of homoclinic intersection.

For its proof we shall need the classical shadowing lemma.

Definition 3.1.4. Let $f : X \to X$ be a homeomorphism of a metric space X. Given $\delta > 0$ we say that a sequence $\{x_n\}$ is a ε-pseudo orbit if $d(f(x_n), x_{n+1}) < \varepsilon$, for every n. We say that the pseudo orbit is ε shadowed by a point $x \in X$, for $\varepsilon > 0$, if $d(f^n(x), x_n) < \varepsilon$, for every

n. The pseudo orbit is said to be periodic if there exists a minimum number τ such that $x_{n+\tau} = x_n$, for every n. The number τ is called the period of the pseudo orbit.

Theorem 3.1.5 (Shadowing Lemma [Rob]). *Let Λ be a locally maximal hyperbolic set. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that every periodic δ-pseudo orbit can be ε-shadowed by a periodic orbit. Moreover, if τ is the period of the pseudo orbit, then the periodic point is a fixed point for f^τ.*

To apply the shadowing lemma we need to find a hyperbolic set. Our main source of hyperbolic sets will be the classical Birkhoff-Smale's Theorem.

Theorem 3.1.6 (Birkhoff-Smale). *Let f be a diffeomorphisms with a hyperbolic periodic point p such that there exists a point of transverse intersection $q \in W^s(O(p)) \pitchfork W^u(O(p))$. Then, for any small enough neighborhood U of $O(p) \cup O(q)$ the maximal invariant set $\Lambda_U = \cap_{n\in\mathbb{Z}} f^n(U)$ is a hyperbolic set.*

For a proof, see Theorem 4.5, pg. 260 in [Rob]. In the sequel, we shall assume a special type of homoclinic intersection and will prove that, in this case, the horseshoe given by Theorem 3.1.6 will be topologically mixing.

The result below was taken from the paper [ACS] and is possibly a classical result. Nonetheless, since we could not find the proof in the literature we include it here for the sake of completeness.

Lemma 3.1.7. *Let f be a diffeomorphisms with a hyperbolic periodic point p such that there exists a point of transverse intersection $q \in W^s(p) \pitchfork W^u(f(p))$. Then, for any small enough neighborhood U of $O(p) \cup O(q)$, the restriction of f to the maximal invariant set $\Lambda_U = \cap_{n\in\mathbb{Z}} f^n(U)$ has the homoclinic large periods property.*

Proof. For this proof, we denote $\tau := \tau(p)$ the period of p.

Let U be a small enough neighborhood U of $O(p) \cup O(q)$ such that the maximal invariant set $\Lambda_U = \cap_{n\in\mathbb{Z}} f^n(U)$ is a hyperbolic set.

Take an arbitrary $\varepsilon > 0$ and $\delta > 0$ given by Theorem 3.1.5. We claim that there exists a number N_0 such that for every $n \geq N_0$ it is possible to construct a periodic δ-pseudo orbit inside U, with period exactly equal to n, and whose Hausdorff distance to $O(p) \cup O(q)$ is smaller than ε.

Once we have established this, the shadowing lemma will produce periodic orbits which are fixed points for f^n and whose Hausdorff distance to $O(p) \cup O(q)$ is 2ε. In particular, these orbits must be 3ε dense in Λ_U, with respect to the Hausdorff distance. Moreover, if ε is small enough, all of these periodic orbits will be homoclinically related by the hyperbolicity of Λ_U.

Thus, we are left to show our claim. With such goal in mind, we take a large iterate $x = f^{N\tau}(q)$ such that

$$f^{-r\tau}(x) \in B(p, \delta/2),$$

for every $r = 0, ..., \tau - 1$. Observe that $f^{-1}(x) \in W^u(p)$, since $q \in W^u(f(p))$. This implies that there exists a smallest positive integer $l \in \mathbb{N}$ such that

$$f^{-l\tau-1}(x) \in B(p, \delta/2).$$

Now, we can give the number N_0. For each $r = 1, ..., \tau - 1$, let $k_r = rl$ and take $L = \prod_{r=1}^{\tau-1} k_r$. We define $N_0 := L\tau$. Observe that if $n \geq N_0$ we can write

$$n = (a + L)\tau + r = (a + L - k_r)\tau + k_r\tau + r,$$

for some $r \in \{1, ..., \tau - 1\}$ and $a \in \mathbb{N}$.

To complete the proof, we shall give the pseudo orbit. It will be given by several strings of orbit, with jumps at specific points. For this reason, and for the sake of clarity, we divide the construction in several steps between each jump.

- *The first string:* Define $x_0 = f^{-(l+r)\tau-1}(x)$, $x_j = f^j(x_0)$, for every $j = 1, ..., l\tau$.
- *The second string:* Notice that $f(x_{l\tau}) = f^{-r\tau}(x) \in B(p, \delta/2)$. Put $x_{l\tau+1} = f^{-(l+r-1)\tau-1}(x) \in B(p, \delta/2)$, and $x_{l\tau+1+j} = f^j(x_{l\tau+1})$, for every $j = 1, ..., l\tau$.
- *The procedure continues inductively:* Notice again that $f(x_{2l\tau+1}) = f^{-(r-1)\tau}(x) \in B(p, \delta/2)$, and put $x_{2l\tau+2} = f^{-(l+r-2)\tau-1}(x)$. We proceed with the construction in

an analogous way, defining $x_{jl\tau+j} := f^{-(l+r-j)\tau-1}(x)$ and the next $l\tau$ terms of the sequence as simply the iterates of this point, for every $j < r$. In this manner we construct a sequence with $rl\tau + r - 1$ terms.

- *The last string:* Observe that $f(x_{rl\tau+r-1}) = f^{-\tau}(x) \in B(p, \delta/2)$. Hence, we can choose $x_{rl\tau+r} = x$ and the next $(a + L - k_r)\tau - 1$ terms of the sequence as simply the iterates of this point, all of which belongs to $B(p, \delta/2)$.

- *The last jump:* Finally, we close the pseudo orbit by putting $x_{(a+L-k_r)\tau+k_r\tau+r} = x_0$.

This gives a periodic δ-pseudo orbit with period n, as required. □

As an application, from Lemmas 3.1.1 and 3.1.7 we obtain the following result.

Proposition 3.1.8. *Let f be a diffeomorphisms with a hyperbolic periodic point p having a non empty transversal intersection between its stable manifold and the unstable manifold of $f(p)$, i.e. there exists $q \in W^s(p, f) \pitchfork W^u(f(p), f)$. Then, for any small enough neighborhood U of $O(p) \cup O(q)$, the maximal invariant set Λ_U in U is topologically mixing hyperbolic set.*

We notice that a mixing horseshoe such as Proposition 3.1.8 always carry a Bernoulli measure. Indeed, there exists the following result

Theorem 3.1.9 ([Bow2], Theorem 34)**.** *Let Λ be a topologically mixing isolated hyperbolic set. Then, there exists a Bernoulli measure supported in Λ.*

Remark 3.1.10. Actually Bowen constructs a measure such that $(f|_\Lambda, \mu_B)$ is a K-automorphism. But, in this case, (f_Λ, μ_B) is measure theoretically isomorphic to a mixing Markov chain and by [FO] it is isomorphic to a Bernoulli shift.

Thus, our strategy to prove denseness of Bernoulli measures can be summarized in the following way: by Theorem 3.0.2 it is enough to prove that a periodic measure μ_q can be approached by Bernoulli measures. For this we try to combine Proposition 3.1.8 and Theorem 3.1.9, noticing that if U is small enough then any measure supported in the horseshoe will be close to μ_q. To be able to apply Proposition 3.1.8 we need to ensure that the good intersection $W^s(q) \cap W^u(f(q)) \neq \emptyset$ will occur. However, this fact is a Corollary of the work of Abdenur and Crovisier [AC], as the next section shows.

3.2 The period of a homoclinic class

In this section we will describe a way of decomposing each homoclinic class $H(p)$ into a finite number of compact pieces, which are cyclically permuted and are topologically mixing in the returns. The key point in the proof is a number which describes all possible integers for which one has $W^u(f^n(p)) \pitchfork W^s(p) \neq \emptyset$. The results are due to Abdenur and Crovisier, in their work [AC]. Let us begin by defining this number.

Definition 3.2.1. Let $f : M^d \to M^d$ be a diffeomorphism of a closed manifold. Let $H(p)$ be a homoclinic class of f. The *period of the class*, denoted by $l = l(H(p))$, is the greatest common divisor of the periods of the periodic points q homoclinically related to p.

Concerning our strategy to find Bernoulli measures, the proposition below justifies our interest in the period of a homoclinic class.

Proposition 3.2.2. *If $\tilde{p}$ is homoclinically related to p, then $W^u(f^n(\tilde{p})) \pitchfork W^s(\tilde{p}) \neq \emptyset$ if, and only if, $n \in l(H(p))\mathbb{Z}$.*

Since our goal is to find the good type of intersection $W^s(q) \cap W^u(f(q)) \neq \emptyset$, we have to ensure that $l(H(p)) = 1$. For this, we shall see below that $l(H(p)) = 1$ *if and only if $H(p)$ is mixing.* Thus, for topologically mixing homoclinic classes, the good intersections will always occur. Let us proceed with the formal exposition.

Proposition 3.2.2 is a particular case of the following.

Theorem 3.2.3 (Proposition 1 in [AC])**.** *Let $q \in \mathrm{Per}(f)$ be a hyperbolic periodic point homoclinically related with p and satisfying also $W^u(p) \pitchfork W^s(q) \neq 0$. Then, $W^u(f^n(q)) \pitchfork W^s(p) \neq \emptyset$ if, and only if, $n \in l(H(p))\mathbb{Z}$. In particular, $W^u(q) \pitchfork W^s(p) \neq \emptyset$.*

Notice that if $\tilde{p}$ is homoclinically related with p, then $H(p) = H(\tilde{p})$. Thus, Proposition 3.2.2 follows from Theorem 3.2.3 with $q = p = \tilde{p}$. We will divide the proof of Theorem 3.2.3 into a series of lemmas.

In the sequel we fix $l = l(H(p))$ the period of $H(p)$ and fix q denoting a periodic point homoclinically related with p satisfying also $W^u(p) \pitchfork W^s(q) \neq 0$. Observe that Theorem 3.2.3

implies that the set

$$\mathcal{G}_{q,p} := \{n \in \mathbb{Z}; W^u(f^n(q) \pitchfork W^s(p) \neq \emptyset\}$$

is an additive subgroup of $\mathbb{Z}$, whose generator is l. So the argument has essentially three steps: we first use Birkhoff-Smale's Theorem to prove that $\mathcal{G}_{p,q} \subset l\mathbb{Z}$. In the second step we prove that $\mathcal{G}_{p,q}$ is an additive group. The end of the argument will be to show that $\mathcal{G}_{p,q}$ contains $l\mathbb{Z}$.

The lemma below starts the first step.

Lemma 3.2.4. *$\mathcal{G}_{p,p} \subset l\mathbb{Z}$.*

Proof. Take $n \in \mathcal{G}_{p,p}$ and let $x \in W^u(f^n(p)) \pitchfork W^s(p)$. By Birkhoff-Smale's Theorem there exists a neighborhood U of $O(p) \cup O(x)$ such that the maximal invariant set $\Lambda_U = \cap_{n\in\mathbb{Z}} f^n(U)$ is a hyperbolic set. We use the orbit of x to create a periodic pseudo orbit. Indeed, take a point $x_u = f^{-n-k_u\tau(p)}(x) \in W^u(p)$, with k_u a large integer. Similarly, take $x_s = f^{k_s\tau(p)}(x) \in W^s(p)$, with k_s another large integer. We build a periodic pseudo orbit in the following way: starts at x_u, follows its positive orbit until the point $f^{-1}(x_s)$ and jumps back to x_u. If $k_s + k_u$ is large enough then x_s and x_u are close and we can apply the shadowing lemma to find a periodic point q, homoclinically related with p. By construction, the period of q is of the form $\tau(q) = n + k\tau(p)$. Since, by definition $\tau(p) = li$ and $\tau(q) = lj$, with $i, j \in \mathbb{N}$ this proves that $n = l(j - ki) \in l\mathbb{Z}$ and establishes the lemma. $\square$

For the rest of the argument, the inclination lemma (see [dMP], Theorem 7.1, Chapter 2, pg. 82) will play a major role.

Lemma 3.2.5. *1. $\mathcal{G}_{p,q} \subset \mathcal{G}_{p,p}$*

2. if $m, n \in \mathcal{G}_{p,q}$ then $m + n \in \mathcal{G}_{p,q}$.

Proof. Let $n \in \mathcal{G}_{p,q}$. By definition one has $W^u(f^n(q)) \pitchfork W^s(p) \neq \emptyset$. Then, the inclination lemma implies that $W^s(p)$ accumulates on $W^s(f^n(q))$. On the other hand, since $W^u(p) \pitchfork W^s(q) \neq 0$, iterating by f^n, one concludes that $W^u(f^n(p)) \pitchfork W^s(f^n(q)) \neq \emptyset$. Thus, we obtain that $W^s(p) \pitchfork W^u(f^n(p)) \neq \emptyset$, and so $n \in \mathcal{G}_{p,p}$. This proves the first part of the lemma. Now, take also $m \in \mathcal{G}_{p,q}$. Iterating by f^n this implies that $W^u(f^{m+n}(q)) \pitchfork W^s(f^n(p)) \neq \emptyset$.

Moreover, since $W^s(p) \pitchfork W^u(f^n(p)) \neq \emptyset$, applying the inclination once more we obtain that $W^s(p)$ accumulates on $W^s(f^n(p))$ and thus it transversaly intersects $W^u(f^{m+n}(q))$. Therefore, $m+n \in \mathcal{G}_{p,q}$. The lemma is proved. $\square$

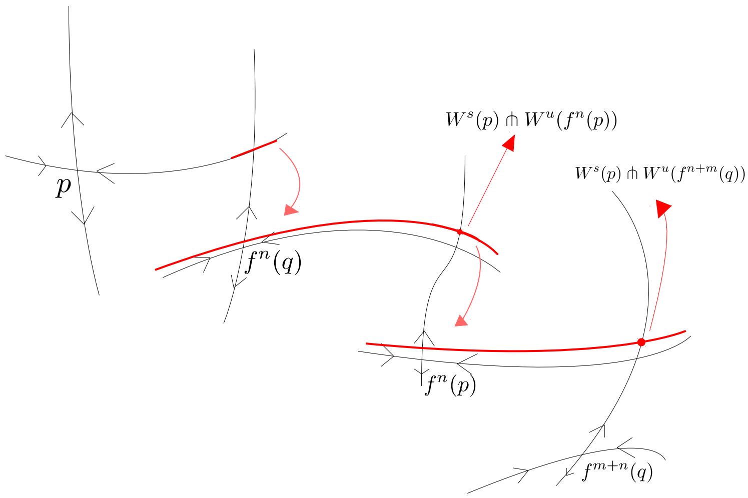


Figure 3.2: The inclination lemma applied twice: proof of Lemma 3.2.5.

With Lemma 3.2.5 we complete the first two steps of the proof.

Corollary 3.2.6. *1. $\mathcal{G}_{p,q} \subset l\mathbb{Z}$*

2. $\mathcal{G}_{p,q}$ is an additive subgroup of $\mathbb{Z}$.

3. $W^u(q) \pitchfork W^s(p) \neq \emptyset$.

4. $\mathcal{G}_{p,q} = \mathcal{G}_{q,p}$

Proof. Combining Lemmas 3.2.4 and 3.2.5 one concludes $\mathcal{G}_{p,q} \subset \mathcal{G}_{p,p} \subset l\mathbb{Z}$, which proves *1*. To prove *2* notice that if $m \in \mathcal{G}_{p,q}$ and $k \in \mathbb{Z}$ then $m - \tau(p)k \in \mathcal{G}_{p,q}$, simply because $f^{-\tau(p)k}(p) = p$. Therefore, if $n \in \mathcal{G}_{p,q}$ then $-n = (\tau(p)-1)n - \tau(p)n \in \mathcal{G}_{p,q}$. Together with Lemma 3.2.5, this establishes that $\mathcal{G}_{p,q}$ is a group. In particular, $0 \in \mathcal{G}_{p,q}$ and thus $W^u(q) \pitchfork W^s(p) \neq \emptyset$.

To prove the last item, we use the inclination lemma. Take $n \in \mathcal{G}_{p,q}$. By Lemma 3.2.5, $n \in \mathcal{G}_{p,p}$ and thus $W^u(f^n(p)) \pitchfork W^s(p) \neq \emptyset$. By the inclination lemma, $W^u(f^n(p))$ accumulates on $W^u(p)$. Since $W^u(p)$ transversaly intersects $W^s(q)$, we conclude that $W^u(f^n(p)) \pitchfork W^s(q) \neq$

$\emptyset$. Therefore, $n \in \mathcal{G}_{q,p}$. By reversing the roles of q and p in this argument we obtain the opposite inclusion $\mathcal{G}_{q,p} \subset \mathcal{G}_{p,q}$, concluding. $\square$

We are left to prove the inclusion $l\mathbb{Z} \subset \mathcal{G}_{p,q}$. For this, we will appeal to a little algebraic lemma.

Lemma 3.2.7. *Let $\mathcal{K} = \{k_n\}_{n=1}^{+\infty} \subset \mathbb{N}$ be an increasing sequence of positive integers. Let $\mathcal{G} \subset \mathbb{Z}$ be an additive subgroup and let k be the greatest common divisor of the numbers k_n. If $\mathcal{K} \subset \mathcal{G}$ then $k\mathbb{Z} \subset \mathcal{G}$.*

Proof. Since $\mathcal{G}$ is a group, it is enough to prove that $k \in \mathcal{G}$. For that, we first claim that it is possible to find two numbers $a = k_i$ and $b = k_j$ such that k is the greatest common divisor of a and b. Indeed, let $\{d_1 < d_2 < ... < d_r\}$ denote the divisors of k_1 which are bigger than k. If this set is empty, the claim is proved. Moreover, if $k_n \in \mathcal{K}$ is not divisible by any d_i, $i = 1, ..., r$ then we may take $a = k_1 < k_n = b$. Thus, we may assume that every k_n is divisible by some d_i. Let $\mathcal{K}_i$ denote the elements of $\mathcal{K}$ which are divisible by d_i. Then $\mathcal{K} = \cup_{i=1}^r \mathcal{K}_i$. Notice that there exists i such that $\mathcal{K}_i \setminus \cup_{m \neq i} \mathcal{K}_m \neq \emptyset$. In fact, if this is not true then $\mathcal{K}_1 \subset \cup_{m=2}^r \mathcal{K}_m$. Since $\mathcal{K}_2 \subset \cup_{m \neq 2}^r \mathcal{K}_m$, this implies that $\mathcal{K}_2 \subset \cup_{m=3}^r \mathcal{K}_m$. By induction, we would obtain that $\mathcal{K} = \mathcal{K}_r$, which would imply that the greatest common divisor of the numbers in $\mathcal{K}$ is $d_r > k$, a contradiction. Take $k_i \in \mathcal{K}_i \setminus \cup_{m \neq i} \mathcal{K}_m$. Now, we also have that $\cup_{m \neq i} \mathcal{K}_m \setminus \mathcal{K}_i \neq \emptyset$ because if not then $\mathcal{K} = \mathcal{K}_i$, which would imply that the greatest common divisor of the numbers in $\mathcal{K}$ is $d_i > k$ a contradiction again. Thus, there exists an element k_j in $\mathcal{K}$ which is not divisible by d_i. Let $a = k_i$ and $b = k_j$. Then, since a is divisible by d_i and not by any d_m, with $m \neq i$, while b is not divisible by d_i and since k is the greatest common divisor of $\mathcal{K} = \cup_{m=1}^r \mathcal{K}_m$ we conclude that k is the greatest common divisor of a and b. The claim is proved. It follows that there exists integers m, n such that $k = ma + nb$. Since $\mathcal{G}$ is a group and since $a, b \in \mathcal{G}$ this implies that $k \in \mathcal{G}$, concluding. $\square$

Recall that, for $q \in \mathrm{Per}(f)$, $\tau(q)$ denotes its period.

Proof of Theorem 3.2.3. Let $\mathcal{K} = \{\tau(q); q \text{ is homoclinically related with } p\}$. By definition, l is the greatest common divisor of the numbers in $\mathcal{K}$. Since $\mathcal{G}_{p,q}$ is a group, by Lemma 3.2.7 it suffices to show that for every periodic orbit γ, homoclinically related with p, one has

$\tau(\gamma) \in \mathcal{G}_{p,q}$. For this, we notice that by Corollary 3.2.6, $W^u(q) \pitchfork W^s(p) \neq \emptyset$. Thus, since $f^{\tau(q)}(q) = q$, we conclude that $\tau(q) \in \mathcal{G}_{p,q}$. Now, we claim that given γ, homoclinically related with p, there exists $q' \in \gamma$ such that $\mathcal{G}_{p,q} = \mathcal{G}_{p,q'}$. If such a claim is true, then we are done, since $\tau(q') \in \mathcal{G}_{p,q'}$.

Let us prove our claim. We may take $q' \in \gamma$ such that $W^u(q') \pitchfork W^s(p) \neq \emptyset$. By the inclination lemma, $W^u(q')$ accumulates on $W^u(p)$. As $W^u(p)$ transversely intersects $W^s(q)$ we have that $W^u(q')$ transvesely intersects $W^s(q)$. Thus, $W^u(q')$ accumulates on $W^u(q)$. As a concequence, if $W^u(f^n(q)) \pitchfork W^s(p)$ is not empty then $W^u(f^n(q') \pitchfork W^s(p)$ also is. Thus, $\mathcal{G}_{p,q} \subset \mathcal{G}_{p,q'}$

Moreover, since $\mathcal{G}_{p,q'} = \mathcal{G}_{q',p}$ one obtains that $W^u(p) \pitchfork W^s(q') \neq \emptyset$. Since $W^u(q)$ transversely intersects $W^s(p)$, by the inclination lemma it follows that $W^u(q)$ accumulates on $W^u(q')$. As before, this proves that $\mathcal{G}_{p,q'} \subset \mathcal{G}_{p,q}$. This completes the proof. □

The decomposition of $H(p)$ involves its pointwise homoclinic class.

Definition 3.2.8. Let $p \in \mathrm{Per}(f)$ be a hyperbolic periodic point. The *pointwise homoclinic class of* p is the set $h(p) = \overline{W^s(p) \pitchfork W^u(p)}$.

The pointwise homoclinic class is, in general, not invariant under f, excepet when $h(p) = H(p)$. This is the content of the next result.

Proposition 3.2.9 (Proposition 2 in [AC])**.** *Let $f : M^d \to M^d$ be a diffeomorphism of a closed manifold, let $p \in \mathrm{Per}(f)$ be hyperbolic and consider $H(p)$ its homoclinic class. Let $l = l(H(p))$ be its period. Then,*

1. *$H(p)$ is the union of the iterates $f^k(h(p))$*
2. *$f^l(h(p)) = h(p)$ and $f^l|_{h(p)}$ is topologically mixing*

It should be strongly emphasized that the iterates $f^k(h(p))$ may not be disjoint. Nevertheless, as a consequence of Proposition 3.2.9 we obtain the following.

Corollary 3.2.10. *$H(p)$ is topologically mixing if and only if $l(H(p)) = 1$.*

Proof. If $l(H(p)) = 1$ then $f(h(p)) = h(p)$ and thus $H(p) = h(p)$ and $H(p)$ is mixing. Conversely, assume by contraposition $l(H(p)) \geq 2$. Then, there exists a small ball $B \subset h(p) \setminus f(h(p))$. Then, for every $n > 1$, if $y \in B$ is such that $f^n(y) \in B$ then $f^{n+1}(y) \in f(h(p)$, and therefore $f^{n+1}(y) \notin B$. Thus f is not mixing. $\square$

To prove Proposition 3.2.9 we begin establishing the following lemma.

Lemma 3.2.11. *If $q \in \mathrm{Per}(f)$ is hyperbolic and homoclinically related with p so that $W^u(p)$ and $W^s(q)$ have a point of transverse intersection then*

$$h(p) = \overline{W^u(p) \pitchfork W^s(q)}.$$

In particular, $h(p) = h(q)$.

Proof. By Theorem 3.2.3 $W^u(q) \pitchfork W^s(p) \neq \emptyset$. Therefore p and q are homoclinically related fixed points of $f^{\tau(p)\tau(q)}$. Moreover, for $f^{\tau(p)\tau(q)}$, their homoclinic classes are $h(p)$ and $h(q)$, resp. Thus $h(p) = h(q)$. The inclination lemma implies the equality $h(p) = \overline{W^u(p) \pitchfork W^s(q)}$, concluding. $\square$

Proof of Proposition 3.2.9. Let k, m, n be three integers.

Claim 1. *$\overline{W^u(f^k(p)) \pitchfork W^s(f^m(p))}$ is either empty or concides with $f^{m+nl}(h(p))$.*

Proof. Notice that

$$\overline{W^u(f^k(p)) \pitchfork W^s(f^m(p))} = f^{m+nl}(\overline{W^u(f^{k-m-nl}(p)) \pitchfork W^s(f^{-nl}(p))}).$$

If $\overline{W^u(f^{k-m-nl}(p)) \pitchfork W^s(f^{-nl}(p))} \neq \emptyset$ then, iterating by f^{nl} one deduces that $k - m \in \mathcal{G}_{p,p}$, and by Theorem 3.2.3 we get that $k - m - nl \in l\mathbb{Z}$. Hence, again by Theorem 3.2.3 and the inclination lemma, we obtain that $W^u(f^{k-m-nl}(p))$ and $W^u(p)$ accumulates on each other. Similarly, $W^s(f^{-nl}(p))$ and $W^s(p)$ accumulate on each other. Thus,

$$\overline{W^u(f^{k-m-nl}(p)) \pitchfork W^s(f^{-nl}(p))} = h(p),$$

proving the claim. $\square$

The claim immediately implies that $h(p) = f^l(h(p)$ and $H(p)$ coincides with the union of the iterates $f^k(h(p))$. Let us prove that $f^l|_{h(p)}$ is mixing. Take U and V open sets which interset $h(p)$. Pick a point $x \in U \cap W^u(p) \pitchfork W^s(p)$. Let $D \subset W^u(p)$ be a small disk, containing x and contained in U. Take $D' \subset W^s(p)$ a small disk containing y and contained in V.

Since, by Theorem 3.2.3, $f^{nl}(W^u(p))$ are the sole iterates of $W^u(p)$ which transversely intersects $W^s(p)$ the inclination lemma implies that $f^{nl}(D)$ accumulates on $W^u(p)$, for every large n. Thus, $f^{nl}(D)$ intersects D' for every n large enough, proving the result. □

Abdenur and Crovisier [AC] developed a closing lemma with time control to show that it is possible to perturb the period of the class if the iterates $f^k(h(p))$ have non-empty intersection. Together with C^1-generic techniques they obtained the following result

Theorem 3.2.12 (Abdenur-Crovisier)**.** *Let f be a C^1-generic transitive diffeomorphism. Then, f is topologically mixing.*

Combining their result with the connecting lemma [H1], one obtain that the whole manifold is a topologically mixing homoclinic class.

3.3 Mixing and large periods

Let us give some applications of the results of Section 3.1.

Proposition 3.3.1. *Let f be a diffeomorphisms with a hyperbolic periodic point p such that the homoclinic class of p, $H(p)$, has the homoclinic large periods property. Then, $H(p(g))$ has the homoclinic large periods property for any diffeomorphism g close enough to f.*

Proof. Since $H(p)$ has the homoclinic large periods property the period of this homoclinic class has to be one, $l(O(p)) = 1$. Indeed, unless the class reduce itself to a fixed point, there will be two periodic points homoclinically related to p such that their periods are two distinct prime numbers. Hence, by Proposition 3.2.2 we have that $W^s(p) \pitchfork W^u(f(p)) \neq \emptyset$. Therefore, since this intersection is robust, we can conclude also by Proposition 3.2.2 that

$W^s(\tilde{p}) \pitchfork W^u(g(\tilde{p})) \neq \emptyset$ for every hyperbolic period point $\tilde{p}$ homoclinically related to $p(g)$, for every diffeomorphism g close enough to f.

So, take an arbitrary $\varepsilon > 0$. There exists a periodic point $\tilde{p} \in H(p(g))$, homoclinically related with $p(g)$ and whose orbit is $\varepsilon/2$ dense in $H(p(g))$. Now, Lemma 3.1.7 implies that there exists N_0 such that for every $n \geq N_0$ we can find a periodic orbit $\gamma = O(b)$ homoclinically related to $\tilde{p}$, $b \in \mathrm{Fix}(g^n)$, which contains a subset $\varepsilon/2$ close to $O(\tilde{p})$ in the Hausdorff distance. In particular, γ is an ε dense orbit inside $H(p(g))$. This establishes that $H(p(g))$ has the homoclinic large periods property, and completes the proof. □

Observe that the above proof establishes indeed that if a homoclinic class $H(p)$ of a diffemorphism f is such that $W^s(p) \pitchfork W^u(f(p)) \neq \emptyset$ then $H(p)$ has the homoclinic large periods property. Thus, combining these facts and Corollary 3.2.10 we have the following corollary.

Corollary 3.3.2. *Let f be a generic diffeomorphism. An isolated homoclinic class of f is topologically mixing if, and only if, it has homoclinic large periods property robustly.*

With the aid of Corollary 3.3.2 we can answer Question 1. Indeed, we have the following

Corollary 3.3.3. *Let f be a C^1-generic transitive diffeomorphism. Then, f has the large periods property*

Proof. By the result of Abdenur-Crovisier, f is mixing and M is a homoclinic class. In particular, M is an isolated homoclinic class. The conclusion follows from Corollary 3.3.2 □

3.4 Proof of Theorem A

In this section we put together the results we have seen so far in order to establish Theorem A. Let us briefly recall our strategy. By Theorem 3.0.2 it suffices to approach a periodic measure μ_q, for $q \in H(p)$ by a Bernoulli measure. By genericity $H(p) = H(q)$ and thus we have a good transverse intersection between $W^u(f(q))$ and $W^s(q)$. We will apply Proposition 3.1.7

together with Theorem 3.1.9, to find a Bernoulli measure supported in a horseshoe "close" to $O(p)$.

Therefore, one thing we need to prove is that the Bernoulli measure we will find is indeed close to the periodic measure μ_p. This is implied by the lemma below, which says that the points in the horseshoe spent arbitrarily large portions of their orbit shadowing the orbit of p. Its proof is completely elementary but is a worthwile exercise.

Lemma 3.4.1. *Let p be a hyperbolic periodic point and let $x \in W^u(O(p)) \pitchfork W^s(O(p))$. Then, for each $\varepsilon > 0$ there exists U a neighborhood of $O(p) \cup O(x)$ such that for every ergodic measure μ supported in the maximal invariant set Λ_U one has $d(\mu, \mu_p) < \varepsilon$.*

Proof. By definition of the weak star topology in the spae of Borel measures, we know that there exists a finite set of continuous functions $\{\varphi_1, ..., \varphi_r\}$ and a positive number β such that if $\left|\int \varphi_j d\mu - \int \varphi_j d\mu_p\right| \leq \beta$, for every $j = 1, ..., r$ then $d(\mu, \mu_p) < \varepsilon$. By uniform continuity there exists $\alpha > 0$ such that if $d(a, b) < \alpha$ then $|\varphi_j(a) - \varphi_j(b)| < \beta$, for every $j = 1, ..., r$.

Fix $B = B(O(p), \alpha)$ and take integers $n_0, n_1 > 0$ with the following property:

- $f^n(x) \in B$ if, and only if $n \leq -n_0$ or $n \geq n_1$.

In other words, $f^{-n_0}(x), f^{-n_0+1}(x), ..., f^{n_1}(x)$ are the only points in the orbit of x which lies outside B. Denote $m = n_0 + n_1$. For every N large, take $\delta > 0$ small enough such that if $B_i = B(f^{-n_0+i}(x), \delta)$, for $i = 0, ..., m$, then $y \in B_i$ for some i implies that $f^l(y) \in B$, for every $l \in [m, N + m]$. Given N large, such δ exists by continuity of f. Let $c = \max\{\|\varphi_j\|_{C^0}; j = 1, ..., r\}$. Take N large enoug so that

$$\frac{m}{N} < \frac{\beta}{8c}.$$

Let $V = \cup_{i=0}^m B_i$ and take $U = B \cup V$.

Let μ be an ergodic measure supported in Λ_U. Then, there exists $y \in \Lambda_U$ such that if $\mu_n = \frac{1}{n}\sum_{l=0}^{n-1} \delta_{f^l(y)}$ then

$$\int \varphi_j d\mu_n \to \int \varphi_j d\mu, \text{ for every } j = 1, ..., r.$$

Take n large enough so that $\left|\int \varphi_j d\mu - \int \varphi_j d\mu_n\right| < \frac{\beta}{2}$. Since $O(y) \subset U$, for every n large enough, the string of orbit $\{y, ..., f^n(y)\}$ can be divided in two parts, the points inside B

and the ones outside. Namely, consider the sets

$$J = \{l \in [0, n]; f^l(y) \in B\}$$

and $I = [0, n] \setminus J$. By our choice of U we have that for each m elements of I there are at least N elements in J. Thus, $\mathrm{card}(J) \geq (N/m)\,\mathrm{card}(I)$. This enables us to estimate, for each j,

$$\begin{aligned}
\left|\int \varphi_j d\mu - \int \varphi_j d\mu_p\right| &\leq \left|\int \varphi_j d\mu - \int \varphi_j d\mu_n\right| + \int |\varphi_j - \varphi_j(p)| d\mu_n \\
&\leq \frac{\beta}{2} + \frac{1}{n}\sum_{l\in J} |\varphi_j(f^l(y)) - \varphi_j(p)| + \frac{1}{n}\sum_{l\in I} |\varphi_j(f^l(y)) - \varphi_j(p)| \\
&\leq \frac{\varepsilon}{2} + \frac{\mathrm{card}(J)\beta}{4n} + \frac{2c\,\mathrm{card}(I)}{n}.
\end{aligned}$$

Since $n = \mathrm{card}(J) + \mathrm{card}(I)$, we have that $\mathrm{card}(J)/n < 1$ and

$$\frac{\mathrm{card}(I)}{n} \leq \frac{\frac{m}{N}}{\frac{\mathrm{card}(I)}{\mathrm{card}(J)} + 1} < \frac{m}{N} < \frac{\beta}{8c}.$$

Thus, for each j,

$$\left|\int \varphi_j d\mu - \int \varphi_j d\mu_p\right| \leq \frac{\beta}{2} + \frac{\beta}{4} + \frac{\beta}{4} \leq \beta.$$

This proves that $d(\mu, \mu_p) < \varepsilon$ and establishes the result. □

Now, we give the proof of Theorem A.

Proof of Theorem A. Let $H(p)$ be an isolated topologically mixing homoclinic class of a C^1 generic diffeomorphism f. Let μ be an invariant measure supported in $H(p)$ and let $\varepsilon > 0$ be arbitrarily chosen. By Theorem 3.0.2 there exists a measure $\mu_{\tilde{p}}$, supported on a hyperbolic periodic orbit $O(\tilde{p})$, with $\tilde{p} \in H(p)$, which is $\varepsilon/2$ close to μ.

Since f is C^1 generic, Theorem 4.2.5 implies that $H(\tilde{p}) = H(p)$. In particular, we have that $H(\tilde{p})$ is topologically mixing.

From Corollary 3.2.10 we know that there exists a point $q \in W^s(\tilde{p}) \pitchfork W^u(f(\tilde{p}))$. For every small neighborhood U of $O(\tilde{p}) \cup O(q)$, Proposition 3.1.8 tells us that the maximal invariant set $\Lambda_U = \cap_{n\in\mathbb{Z}} f^n(U)$ is a topologically mixing hyperbolic set. Moreover, by Lemma 3.4.1 we may choose U small so that every ergodic measure ν supported in Λ_U is $\varepsilon/2$ close to $\mu_{\tilde{p}}$.

Thus, if we take ν the Bernoulli measure supported in Λ_U, whose existence is ensured by Theorem 3.1.9, then ν is ε close to μ. This establishes the result. $\square$

Remark 3.4.2. The techniques employed above can be used to give a more geometric approach to Sigmund's result on the denseness of Bernoulli measures for hyperbolic topologically mixing basic sets [S1]. Indeed, our use of the large periods property gives an alternative to the symbolic approach of Sigmund and a proof of his result using our techniques would proceed by the same argument as above, in the proof of Theorem A. The only difference is to use Sigmund's result on denseness of periodic measures in a hyperbolic basic set, [S2] instead of Theorem 3.0.2.

Chapter 4

Robust transitivity far from tangencies

4.1 Statements

We say that a non-transversal intersection between $W^s(O(p))$ and $W^u(O(p))$ is a *homoclinic tangency*. We denote by $\mathcal{HT}(M)$ the set of diffeomorphisms exhibiting a homoclinic tangency. We will say that a diffeomorphism f is *far from homoclinic tangencies* if $f \notin cl(\mathcal{HT}(M))$.

Given p and q hyperbolic periodic points with $I(p) < I(q)$ we say that they form a *heterodimensional cycle* if there exists $x \in W^s(O(p)) \cap W^u(O(q))$, with

$$\dim\left(T_xW^s(O(p)) \cap T_xW^u(O(q))\right) = .0$$

and $W^u(O(p)) \pitchfork W^u(O(q)) \neq \emptyset$.

We recall the statements of the results in [ACS], concerning robustly transitive diffeomorphisms.

Theorem B. *There exists an open and dense subset among robustly transitive diffeomorphisms far from homoclinic tangencies formed by diffeomorphisms such that the whole manifold is a homoclinic class.*

Theorem C. *There is an open and dense subset among robustly transitive diffeomorphisms far from homoclinic tangencies formed by robustly topologically mixing diffeomorphisms.*

4.2 Some Tools

In this section, we collect some results that will be used in the proofs of the main results.

4.2.1 Perturbative Tools

We start with Franks' lemma [F]. This lemma enable us to deal with some non-linear problems using linear arguments.

Theorem 4.2.1 (Franks lemma)**.** *Let $f \in \mathrm{Diff}^1(M)$ and $\mathcal{U}$ be a C^1-neighborhood of f in $\mathrm{Diff}^1(M)$. Then, there exist a neighborhood $\mathcal{U}_0 \subset \mathcal{U}$ of f and $\delta > 0$ such that if $g \in \mathcal{U}_0(f)$, $S = \{p_1, \dots, p_m\} \subset M$ and $\{L_i : T_{p_i}M \to T_{g(p_i)}M\}_{i=1}^m$ are linear maps satisfying $\|L_i - Dg(p_i)\| \leq \delta$ for $i = 1, \dots m$ then there exists $h \in \mathcal{U}(f)$ coinciding with g outside any prescribed neighborhood of S and such that $h(p_i) = g(p_i)$ and $Dh(p_i) = L_i$.*

One of the main applications of Franks lemma is to change the index of a periodic orbit, after a perturbation, if the Lyapunov exponents of the orbit is weak enough. More precisely, we can prove the following:

Lemma 4.2.2. *Let $f \in \mathrm{Diff}^1(M)$ having a sequence of hyperbolic periodic points p_n with some index $s+1$, having negative Lyapunov exponents arbitrarily close to zero. Then, there exists g arbitrarily close to f having hyperbolic periodic points of indices s and $s+1$.*

Proof: Given a neighborhood $\mathcal{U}$ of f let us consider $\delta > 0$ given for this neighborhood and U_0 another small enough neighborhood of f. We will suppose that the sequence of periodic points p_n is such that the smallest eigenvalue λ_{p_n} of $Df^{\tau(p_n)}$ with absolute value smaller than 1, has multiplicity one. The argument is similar in the other cases.

Our hypothesis says that

$$\frac{1}{\tau(p_n)} \log \|Df^{\tau(p_n)}|E^s(p_n)\| = \frac{1}{\tau(p_n)} \log |\lambda_{p_n}|$$

approaches zero as n grows. Now, let us consider E_n as the eigenspace of the eigenvalues λ_{p_n}, and $\{E_l\}$ the other eigenspaces. We can define linear maps $L_i : T_{f^i(p)}M \to T_{f^{i+1}(p)}M$, equal to $Df(f^i(p))$ in all subspaces $Df^i(p)E_l$, but in $Df^i(p)E_n$ we choose L_i satisfying $\|L_i|\ Df^iE_n\| = (1+\alpha)\|Df(f^i(p))|Df^iE_n\|$, where $\alpha > 0$ depends on $\delta > 0$. Then, L_i is $\delta-$close to $Df(f^i(p))$, and also preserves the eigenspace $Df^i(p)E_n$.

Hence, using Franks lemma we can find $g \in \mathcal{U}$ such that p_n still is a hyperbolic periodic point and moreover $Dg(f^i(p)) = L_i$, where g depends on the periodic point p_n. In particular, E_n is a invariant subspace of $T_{p_n}M$ for $Dg^{\tau(p_n)}$ and moreover:

$$\|Dg^{\tau(p)}(p_n)|E_n\| = (1+\alpha)^{\tau(p_n)}\lambda_n.$$

Hence, by hypothesis, we can choose p equal some p_n, in order to have, after the above perturbation:

$$\frac{1}{\tau(p)}\log\|Dg^{\tau(p)}|E_n(p)\| > 0.$$

Since L_i can be chosen such that the other Lyapunov exponents of p keep unchanged, we have that p has index s. To finish the proof, we just observe that, Franks lemma changes the initial diffeomorphism only in a arbitrary neighborhood of the orbit of p, therefore the neighborhood $\mathcal{U}$ could be chosen such that the hyperbolic periodic point p_1 of f has a continuation, which implies that $p_1(g)$ is also a hyperbolic periodic point of g with index $s+1$. □

Another result that we shall use is Hayashi's connecting lemma [H1]. This will be helpful to create some heterodimensional cycles.

Theorem 4.2.3 (C^1-connecting lemma)**.** *Let $f \in \mathrm{Diff}^1(M)$ and p_1, p_2 hyperbolic periodic points of f, such that there exist sequences $y_n \in M$ and positive integers k_n such that:*

- *$y_n \to y \in W^u_{loc}(p_1, f))$, $y \neq p_1$; and*
- *$f^{k_n}(y_n) \to x \in W^s_{loc}(p_2, f))$, $x \neq p_2$.*

Then, there exists a C^1 diffeomorphism g C^1-close to f such that $W^u(p_1, g)$ and $W^s(p_2, g)$ have a non empty intersection close to y.

As it is well known, this result implies that if f is a generic diffeomorphism having a non-hyperbolic homoclinic class which contains two periodic points p and q with different indices then there exist arbitrarily small perturbations of f such that p and q belongs to a heterodimensional cycle.

4.2.2 Generic Results

The result below, of Bonatti and Crovisier [BC], proves that a large class of transitive diffeomorphism have the property that the whole manifold coincides with a homoclinic class.

Theorem 4.2.4 (Bonatti and Crovisier)**.** *There exists a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that for every transitive diffeomorphism $f \in \mathcal{R}$ if p is a hyperbolic periodic point of f then $M = H(p, f)$.*

Another generic result is the following

Theorem 4.2.5 (Theorem A, item (1), [CMP])**.** *There exists a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that for every $f \in \mathcal{R}$ if two homoclinic classes $H(p_1, f)$ and $H(p_2, f)$ are either equal or disjoint.*

The next result, from [ABCDW], says that generically, homoclinic classes are index complete.

Theorem 4.2.6 (Theorem 1 in [ABCDW])**.** *There is a residual subset $\mathcal{R} \in \mathrm{Diff}^1(M)$ of diffeomorphisms f such that, every $f \in \mathcal{R}$ and any homoclinic class containing hyperbolic periodic points of indices i and j, also contains hyperbolic periodic points of index k for every $i \leq k \leq j$.*

Crovisier, Sambarino and Yang in [CSY] showed that for any diffeomorphism f in an open and dense subset far from homoclinic tangencies, every homoclinic class of f has a kind of strong partial hyperbolicity. More precisely, the difference is that the "partially hyperbolic splitting" found by them could have either one or both trivial extremal bundles. In this last scenario, by our definition the diffeomorphism would not be partially hyperbolic. However, by an abuse of notation, we will continue calling it partially hyperbolic as in [CSY].

Their result gives other important properties. Like, information of the minimal and maximal indices of periodic points inside the homoclinic class. More precisely:

Theorem 4.2.7 (Theorem 1.1(2) in [CSY])**.** *There is an open and dense subset* $\mathcal{A} \subset \mathrm{Diff}^1(M) - \{cl(\mathcal{HT})\}$ *such that for every* $f \in \mathcal{A}$*, any homoclinic class* $H(p)$ *is a partially hyperbolic set of* f

$$T_{H(p)}M = E^s \oplus E^c_1 \oplus \dots E^c_k \oplus E^u,$$

with dim $E^c_i = 1$*,* $i = 1, \dots, k$*, and moreover the minimal stable dimension of the periodic points of* $H(p)$ *is* $dim(E^s)$ *or* $dim(E^s) + 1$*. Similarly the maximal stable dimension of the periodic orbits of* $H(p)$ *is* $dim(E^s) + k$ *or* $dim(E^s) + k - 1$*. For every* i*,* $1 \leq i \leq k$ *there exists periodic points in* $H(p)$ *whose Lyapunov exponent along* E^c_i*, is arbitrary close to* 0.

4.3 Robustly large Homoclinic class

In this section we shall prove Theorem B as a consequence of the following result:

Theorem 4.3.1. *Let* $f \in \mathrm{Diff}^1(M)$ *be a robustly transitive strong partially hyperbolic diffeomorphism, with* $TM = E^s \oplus E^c_1 \oplus \dots E^c_k \oplus E^u$*, having hyperbolic periodic points* p_s *and* p_u *with index* s *and* $d - u$*, respectively, where* $s = dim\ E^s$ *and* $u = dim\ E^u$*. Then, there exists an open subset* $\mathcal{V}_f$ *whose closure contains* f*, such that* $M = H(p_s(g)) = H(p_u(g))$ *for every* $g \in \mathcal{V}_f$*.*

Before we prove Theorem 4.3.1, let us see how it implies Theorem B.

Proof of Theorem B. First we observe that it suffices to deal with the interior of non-hyperbolic robustly transitive diffeomorphisms, since in the Anosov case the whole manifold is robustly a homoclinic class, which is a consequence of the shadowing lemma.

Recall that $\mathcal{T}_{NH}(M) \subset \mathcal{T}(M)$ denotes the interior of non-hyperbolic robustly transitive diffeomorphisms far from homoclinic tangencies. Hence, by Theorem 4.2.4 and Theorem 4.2.7 there exists a residual subset $\mathcal{R}$ in $\mathcal{T}_{NH}(M)$ such that if $f \in \mathcal{R}$ then:

a) M coincides with a homoclinic class;

b) f is partially hyperbolic, with the central bundle admitting a splitting in one dimension sub bundles. I.e., $TM = E^s \oplus E_1^c \oplus \ldots \oplus E_k^c \oplus E^u$;

c) either there exist a hyperbolic periodic point with index s, or there exists hyperbolic periodic points with index $s+1$ whose the $(s+1)-$Lyapunov exponent is arbitrarilly close to zero. Where $s = dim\ E^s$.

d) either there exist a hyperbolic periodic point with index $d-u$, or there exists hyperbolic periodic points with index $d-u-1$ whose the $(d-u-1)-$Lyapunov exponent is arbitrary close to zero. Where $u = dim\ E^u$.

According to Theorem 4.2.7, E^s and/or E^u could be trivial. However, this cannot happen in our situation. Indeed, we claim that both E^s and E^u are non-trivial. In particular, f is strongly partially hyperbolic. To see this, suppose by contradiction the existence of $f \in \mathcal{R}$ with E^s trivial. Hence, by item c) above, f should have either a source or hyperbolic periodic points with index one, with the only one Lyapunov negative exponent being arbitrary close to zero. In the last case, we can use Lemma 4.2.2 to perturb f in order to find also a source. Therefore, if E^s is trivial, then we can find a diffeomorphism g close to f, having a source, which is a contradiction with the transitivity of g. Similarly we conclude that E^u is also non-trivial. Henceforth, item b) above can be replaced by:

b') every $f \in \mathcal{R}$ is strongly partially hyperbolic.

Moreover, by the same argument above using Lemma 4.2.2, after a perturbation we can assume that f has hyperbolic periodic points of indices s and $d-u$. Thus, we can find a dense subset $\mathcal{R}_1$ inside $\mathcal{T}_{NH}(M)$ formed by robustly transitive strong partially hyperbolic diffeomorphisms f satisfying the hypothesis of Theorem 4.3.1. Then, considering $\mathcal{V}_f$ given by Theorem 4.3.1 for every $f \in \mathcal{R}_1$ we have that

$$\mathcal{A} = \bigcup_{f \in \mathcal{R}_1} \mathcal{V}_f,$$

is an open and dense subset of $\mathcal{T}_{NH}(M) \subset \mathcal{T}(M)$. By Theorem 4.3.1, for every diffeomorphism in $\mathcal{A}$ the whole manifold M coincides with a homoclinic class. This ends the proof $\square$

In the sequence we prove some technical results which are key steps in the proof of Theorem 4.3.1.

The following result allows to find open sets of diffeomorphisms for which the topological dimension of stable (and unstable manifold) of hyperbolic periodic points is larger than the differentiable dimension.

Lemma 4.3.2. *Let $f \in \mathrm{Diff}^1(M)$ be a robustly transitive strong partially hyperbolic diffeomorphism. Suppose there are hyperbolic periodic points p_j, $j = i,\ i+1, \ldots, k$, with indices $I(p_j) = j$ for f. Hence, given any small enough neighborhood $\mathcal{U}$ of f, where is defined the continuation of the hyperbolic periodic points p_j, there exists an open set $\mathcal{V} \subset \mathcal{U}$ such that for every $g \in \mathcal{V}$:*

$$W^s(p_k(g)) \subset cl(W^s(p_{k-1}(g))) \subset \ldots \subset cl(W^s(p_{i+1}(g))) \subset cl(W^s(p_i(g))), and$$

$$W^u(p_i(g)) \subset cl(W^u(p_{i+1}(g))) \subset \ldots \subset cl(W^u(p_{k-1}(g))) \subset cl(W^u(p_k(g))).$$

To prove the above lemma we will use the following result which is a consequence of Proposition 6.14 and Lemma 6.12 in [BDV], which are results of Diaz and Rocha [DR]. It is worth to point out that this result is a consequence of the well known blender technique, which appears by means of unfolding a heterodimensional co-dimensional one cycle far from homoclinic tangencies.

Proposition 4.3.3. *Let f be a C^1 diffeomorphism with a heterodimensional cycle associated to saddles p and $\tilde{p}$ with indices i and $i+1$, respectively. Suppose that the cycle is C^1-far from homoclinic tangencies. Then there exists an open set $\mathcal{V} \subset \mathrm{Diff}^1(M)$ whose closure contains f such that for every $g \in \mathcal{V}$*

$$W^s(\tilde{p}(g)) \subset cl(W^s(p(g))) \text{ and } W^u(p(g)) \subset cl(W^u(\tilde{p}(g))).$$

Proof of Lemma 4.3.2. Since f is a robustly transitive strong partially hyperbolic diffeomorphism, we can assume that every diffeomorphism $g \in \mathcal{U}$ is transitive and is strong partially hyperbolic, reducing $\mathcal{U}$ if necessary. In particular, $\mathcal{U}$ is far from homoclinic tangencies, $\mathcal{U} \subset (cl(\mathcal{HT}(M)))^c$. Now, using the transitivity of f, there are points x_n converging to the stable manifold of p_{i+1} whose a sequence of iterates $f^{m_n}(x_n)$ is converging to the unstable manifold of p_i. Hence, we can use Hayashi's connecting lemma, to perturb the diffeomorphism f to $\tilde{f}$ such that $W^u(p_i(\tilde{f}))$ intersects $W^s(p_{i+1}(\tilde{f}))$, which one we could assume be transversal after a perturbation, if necessary, since dim $W^u(p_i(\tilde{f}))$ + dim $W^s(p_{i+1}(\tilde{f})) > d$. Hence, we can use once more the connecting lemma to find $f_1 \in \mathcal{U}$ close to $\tilde{f}$ exhibiting a heterodimensional cycle between $p_i(f_1)$ and $p_{i+1}(f_1)$, since $\tilde{f}$ is also transitive. Moreover, and in fact this is needed to apply Proposition 4.3.3, the intersection between $W^s(p_i(f_1))$ and $W^u(p_{i+1}(f_1))$ could be assumed quasi-transversal in the sense that $T_qW^s(p_i(f_1)) \cap T_qW^u(p_{i+1}(f_1)) = \{0\}$. If this is not true, we can do a perturbation of the diffeomorphism using Franks lemma, to get such property.

Thus, since f_1 is far from homoclinic tangencies, we can use Proposition 4.3.3 to find an open set $\mathcal{V}_1 \subset \mathcal{U}$ such that

$$W^s(p_{i+1}(g)) \subset cl(W^s(p_i(g))) \text{ and } W^u(p_i(g)) \subset cl(W^u(p_{i+1}(g))),$$

for every $g \in \mathcal{V}_1$.

Now, since f_1 is also robustly transitive we can repeat the above argument to find $f_2 \in \mathcal{V}_1$ exhibiting a heterodimensional cycle between p_{i+1} and p_{i+2}. Thus, by Proposition 4.3.3 there exists an open set $\mathcal{V}_2 \subset \mathcal{V}_1$, such that

$$W^s(p_{i+2}(g)) \subset cl(W^s(p_{i+1}(g))) \text{ and } W^u(p_{i+1}(g)) \subset cl(W^u(p_{i+2}(g))),$$

for every $g \in \mathcal{V}_2$.

Repeating this argument finitely many times we will find open sets $\mathcal{V}_{k-i} \subset \mathcal{V}_{k-i-1} \subset \ldots \subset \mathcal{V}_1$ such that

$$W^s(p_{i+j}(g)) \subset cl(W^s(p_{i+j-1}(g))) \text{ and } W^u(p_{i+j-1}(g)) \subset cl(W^u(p_{i+j}(g))),$$

for every $g \in \mathcal{V}_j$, and $j = 1, \ldots k - i$.

Taking $\mathcal{V} = \mathcal{V}_{k-i}$ the result follows. $\square$

The next result use properties of a partially hyperbolic splitting to guarantee that some special kind of dense sub-manifolds in M should intersect each other transversally and densely in the whole manifold.

Lemma 4.3.4. *Let f be a partially hyperbolic diffeomorphism on M with non trivial stable bundle E^s, and having a hyperbolic periodic point p with index $s = dim\ E^s$. If $W^s(O(p))$ and $W^u(O(p))$ are dense in M, then $M = H(p)$.*

Proof. Let $E^s \oplus E^c \oplus E^u$ be the partially hyperbolic splitting. Using Remark 2.1.1 we know that the local strong stable manifolds have uniform size.

For any $x \in M$, since $W^u(O(p))$ is dense, there exists $q \in W^u(O(p))$ arbitrarily close to x. Also, by hypothesis of the index of p, and the partially hyperbolic structure, it should be true that $T_qW^u(O(p)) = E^c \oplus E^u$. Hence, by the continuity of the local strong stable manifold, $W^{ss}_{loc}(y)$ should intersect transversally $W^u(O(p))$ in a point close to q, for any point y close enough to q. In particular, since $W^s(O(p))$ is also dense, there exists $\tilde{q} \in W^s(O(p))$ such that $W^{ss}_{loc}(\tilde{q})$ intersects transversally $W^u(O(p))$. However, $W^{ss}_{loc}(\tilde{q})$ is contained in $W^s(O(p))$, which implies there is a transversal intersection between $W^s(O(p))$ and $W^u(O(p))$ close to q, in particular, close to x. $\square$

Finally, using the above lemmas we give a proof of Theorem 4.3.1.

Proof Theorem 4.3.1. Since p_s and p_u are hyperbolic periodic points, we take $\mathcal{U}$ small enough such that every diffeomorphism $g \in \mathcal{U}$ has defined the continuations $p_s(g)$ and $p_u(g)$. Reducing $\mathcal{U}$ if necessary, we could also assume that every $g \in \mathcal{U}$ is a strong partially hyperbolic diffeomorphism with same extremal bundles dimension as in the partially hyperbolic decomposition of TM as f, which follows by the continuity of the partially hyperbolicity and the existence of p_s and p_u robustly.

Now, using Theorem 4.2.4 together with Theorem 4.2.6 we can find a residual subset $\mathcal{R}$ in $\mathcal{U}$ such that M coincides with a homoclinic class for every $g \in \mathcal{R}$, and moreover g has hyperbolic periodic points of any index in $[s, d-u] \cap \mathbb{N}$.

We fix $g \in \mathcal{R}$, and let $p_s = p_s(g)$, p_{s+1}, …, $p_{d-u} = p_u(g)$ be hyperbolic periodic points of g with indices s, $s+1$, …, $d-u$, respectively. Also, for all $n \in \mathbb{N}$, let $\mathcal{V}_n \subset \mathcal{U}$ small neighborhoods of g, such that if $g_n \in \mathcal{V}_n$, then g_n converges to g in the C^1-topology, when n goes to infinity.

Now, since g is still a robustly transitive strong partially hyperbolic diffeomorphism having hyperbolic periodic points of all possible indices, we denote by $\tilde{\mathcal{V}}_n \subset \mathcal{V}_n$ the open sets given for g and $\mathcal{V}_n$ by Lemma 4.3.2. Hence, using the invariance of the stable manifold of hyperbolic periodic points, by Lemma 4.3.2 we have the following:

$$cl(W^s(O(p_{d-u}(r)))) \subset cl(W^s(O(p_{d-u-1}(r)))) \subset \ldots \subset cl(W^s(O(p_s(r)))), \tag{4.1}$$

for every $r \in \tilde{\mathcal{V}}_n$.

Claim: $W^u(O(p_s(r)))$ and $W^s(O(p_{d-u}(r)))$ are dense in M, for every $r \in \tilde{\mathcal{V}}_n$.

Since r is transitive, there exist $x \in M$ such that the forward orbit of x is dense in M. Now, since r is partially hyperbolic, for Remark 2.1.1 there exists the strong stable foliation that integrates the direction E^s. Moreover, these leafs have local uniform length. Hence, as done in the proof of Lemma 4.3.4, we can take $r^j(x)$ close enough to $p_s(r)$ such that $W^{ss}(x)$, the strong stable leaf containing x, intersects the local unstable manifold of $p_s(r)$, $W^u_{loc}(p_s(r))$. Therefore, since points in the same strong stable leaf have the same omega limit set, we have that $W^u(O(p_s(r)))$ is dense in the whole manifold M. We can repeat this argument using also the existence of a point y having a dense backward orbit, and the existence of the strong unstable foliation to conclude that $W^s(O(p_{d-u}(r)))$ is also dense in M.

Thus, by equation (4.1) and the Claim, we have that $W^s(O(p_s(r))))$ is dense in M. Similarly, we can show that $W^u(O(p_{d-u}(r))))$ is also dense in M.

Provided that r is strong partially hyperbolic, and that $W^s(O(p_i(r))))$ and $W^u(O(p_i(r))))$ are dense in M, for $i = s$ and $d-u$, we can apply Lemma 4.3.4 for f and f^{-1} to conclude that

$$M = H(p_s(r)) = H(p_{d-u}(r)),$$

for every $r \in \tilde{\mathcal{V}}_n$.

Hence, the proof is finished defining $\tilde{\mathcal{V}}_g = \cup \tilde{\mathcal{V}}_n$, and

$$\mathcal{V}_f = \bigcup_{g \in \mathcal{R}} \tilde{\mathcal{V}}_g,$$

which is an open and dense subset of $\mathcal{U}$, and hence contains f in its closure. $\square$

Chapter 5

The Poincaré-Hopf index

In this chapter we introduce the index of a vector field in a compact region which will play, in Chapters 6 and 7 a role similar to the Euler characteristic for existence of fixed points.

5.1 The index of an isolated zero

In all this section we fix M^d a d-dimension Riemannnian manifold, and X a C^1 vector field over M.

The Long Tubular Flow Theorem (Proposition 1.1, pag 93 in [dMP]) asserts that near a regular orbit wich is not periodic the topological behaviour of the vector field is very simple: it is conjugated to a translation. The complicated topological behaviour occurs in periodic orbits and zeros.

In this section, we shall describe an invariant which measures in some, this complicated topological behaviour near the zeros.

Let $x \in M$ be an isolated zero of X. The *Poincaré-Hopf index* $\mathrm{Ind}(X, x)$ is defined as follows: consider local coordinates $\varphi\colon U \to \mathbb{R}^d$ defined in a neighborhood U of x. Up to shrink U one may assume that x is the unique zero of X in U. Thus for $y \in U \setminus \{x\}$, $X(y)$ expressed in that coordinates is a non vanishing vector of $\mathbb{R}^d$, and $\frac{1}{\|X(y)\|}X(y)$ is a unit vector hence belongs to the sphere $\mathbb{S}^{d-1}$. Consider a small ball B centered at x. The map

$y \mapsto \frac{1}{\|X(y)\|}X(y)$ induces a continuous map from the boundary ∂B to $\mathbb{S}^{d-1}$. The Poincaré-Hopf index $\text{Ind}(X, x)$ is the topological degree of this map.

An intuitive description of the index goes as follows: at every point of ∂B the vector field X is pointing at some direction, which can be intified as a point in $\mathbb{S}^{d-1}$. The index measures how many turns X is given over ∂B.

It turns out, however, that studing the local topological behaviour of a vector field at an isolated zer can lead to deep consequences. Indeed, we have the following result.

Theorem 5.1.1 (Poincaré-Hopf)**.** *Let M be a closed manifold, and let $X \in \mathfrak{X}^1(M)$ be a vector fields with only finitely many zeros. Then,*

$$\sum_{x \in \text{Zero}(X)} \text{Ind}(X, x) = \chi(M),$$

where $\chi(M)$ is the Euler characteristic

Poincaré-Hopf Theorem is a remarkable result: at one hand, one has a dynamical object, the flow of a vector field. On the other hand, one has an important topological invariant of the manifold, the Euler characteristic. The theorem relates these two objects through an elegant and simple formula, in which the key concept is the Poincaé-hopf index. Through this formula, topological informations of the manifold can lead to dynamical consequences and dynamical properties can establish topological results. A simple ilustration is the following

Corollary 5.1.2. *Let M be a closed manifold with $\chi(M) \neq 0$. If $X \in \mathfrak{X}^1(M)$ then* $\text{Zero}(X) \neq \emptyset$.

For a proof of Theorem 5.1.1, we recommend [Mi], Chapter 6. Looking at this proof one notice that the properties of the index are actually related with the behaviour of X in a neighborhood of some *set* of zeros, and not just a single isolated zero. In the sequel we shall describe how to perform this generalization.

5.2 The index of a vector field in a compact region

So we fix M^d a manifold and $X \in \mathfrak{X}^1(M)$. Assume that $U \subset M$ is a compact region and that X does not vanish on the boundary ∂U. The *Poincaré-Hopf index* $\mathrm{Ind}(X, U)$ is defined as follows: consider a small perturbation Y of X so that the set of zeros of Y in U is finite. Poincaré-Hopf index $\mathrm{Ind}(X, U)$ is the sum of the indices of the zeros of Y in U.

Clearly, one has to check that this sum does not depend on the perturbation Y of X. This is one of the goals of this section.

Proposition 5.2.1. *If* $\{X^t\}_{t\in[0,1]}$ *is a continuous family of* C^1 *vector fields so that* $\mathrm{Zero}(X^t)\cap \partial U = \emptyset$*, then* $\mathrm{Ind}(X^t, U)$ *does not depend on* $t \in [0, 1]$.

This proposition will be proved in the next section, where we shall give a very useful tool for index calculations.

5.3 Trivializations of the tangent bundle and the index

Let $U \subset M$ be a compact region such that $\mathrm{Zero}(X) \cap \partial U = \emptyset$.

We shall assume in this section that ∂U is a codimension one submanifold and U is endowed with d continous vector fields $X^1 \dots X^d$ so that, at every point $z \in U$, $(X^1(z), \dots, X^d(z))$ is a basis of the tangent space T_zM. When one has such a basis the index can be calculated in terms of the topological behaviour of $X|_{\partial U}$ relative to this basis.

More precisely, the basis $(X^1, ..., X^d)$ endows U with an orientation. One can express the vector field X in this basis so that the vector $X(y)$, for $y \in U$, can be considered as a vector of $\mathbb{R}^d$. One defines in such a way a map $\mathcal{X}\colon \partial U \to \mathbb{S}^{d-1}$ by $y \mapsto \mathcal{X}(y) = \frac{1}{\|X(y)\|}X(y)$.

As ∂U has dimension $d-1$, and is oriented as the boundary of U, this map has a topological degree. We have the following

Lemma 5.3.1. *With the notations above the topological degre of* $\mathcal{X}$ *is* $\mathrm{Ind}(X, U)$*, i.e. for* *every vector field Y close to X and with finitely many zeros inside U, we have*

$$\sum_{y\in \mathrm{Zero}(Y)} \mathrm{Ind}(Y,y) = \deg(\mathcal{X})$$

Proof. Let Y be a vector field with with finitely many zeros inside U and denote by $\{\sigma_1,...,\sigma_n\}$ the set of zeros of Y inside U. Consider $B_1,...,B_n$ small closed balls centered at each σ_j such that the balls are disjoint and contained in $\mathrm{int}(U)$. If $x\in U$ Write $Y(x)=\sum_{l=1}^{d}\alpha_l(x)X^l(x)$. If $x\notin\{\sigma_1,...,\sigma_n\}$, we can define

$$\mathcal{Y}(x)=\frac{1}{\sqrt{\sum_{l=1}^{d}\alpha_l(x)^2}}(\alpha_1(x),...,\alpha_d(x))\in\mathbb{S}^{d-1}.$$

Consider the manifold with boundary $N=U\setminus\cup_{j=1}^{n}B_j$. Then, the map restriction $\mathcal{Y}|_{\partial N}$ has

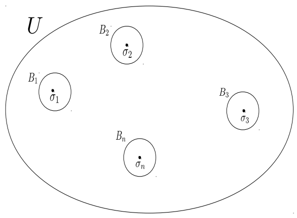


Figure 5.1: $N=U\setminus\cup_{j=1}^{n}B_j$

zero topological degree (see Lemma 1, pg. 28 in [Mi]) and thus

$$\sum_{j=1}^{n}\deg(\mathcal{Y}|_{\partial B_j})=\deg(\mathcal{Y}|_{\partial U}).$$

Notice that we can choose the balls B_j small enough such that, by definition, $\deg(\mathcal{Y}|_{\partial B_j})=\mathrm{Ind}(Y,\sigma_j)$. One concludes that

$$\sum_{j=1}^{n}\mathrm{Ind}(Y,\sigma_j)=\deg(\mathcal{Y}|_{\partial U}).$$

It remains to prove that if Y is close enough to X then $\deg(\mathcal{Y}|_{\partial U})=\deg(\mathcal{X}|_{\partial U})$. For this, let $0<\varepsilon<\min\{\|X(x)\|;x\in\partial U\}$. Such number ε exists due to the compactness of U, and the assummption that X is non vanishing at the boundary of U.

Assume that $\|Y(x)-X(x)\|<\varepsilon$, for every $x\in U$. Consider the family of vector fields $Z^t=X+t(Y-X)$, for $t\in[0,1]$. Notice that $\mathrm{Zero}(Z^t)\cap\partial U=\emptyset$, since if $Z^t(x)=0$ then $\|X(x)\|=t\|Y(x)-X(x)\|\leq\varepsilon$, and thus $x\notin\partial U$, due to our choice of ε.

Write $Z^t(x) = \sum_{l=1}^{d} \alpha_l^t(x) X^l(x)$, and consider the family of maps $\mathcal{Z}^t : \partial U \to \mathbb{S}^{d-1}$, defined by $\mathcal{Z}^t(x) = \frac{1}{\sqrt{\sum_{l=1}^{d} \alpha_l^t(x)^2}} (\alpha_1^t(x), ..., \alpha_d^t(x))$. Since the family of vector fields Z^t is continuous with t, the family $\mathcal{Z}^t$ is a homotopy between $\mathcal{X}|_{\partial U}$ and $\mathcal{Y}|_{\partial U}$. This proves that $\deg(\mathcal{Y}|_{\partial U}) = \deg(\mathcal{X}|_{\partial U})$ and establishes the result. $\square$

Using Lemma 5.3.1 we shall give a proof of Proposition 5.2.1

Proof of Proposition 5.2.1. We begin proving the result in the special case where $\text{Zero}(X^t)$ is finite for every t. For this we prove the

Claim 2. *Let Y be a vector field with $\text{Zero}(Y) \cap U$ finite and disjoint from ∂U. Then, for every vector field Z which is C^1 close enough to Y with $\text{Zero}(Z)$ finite,*

$$\text{Ind}(Y, U) = \text{Ind}(Z, U).$$

Proof. Fix $\sigma \in \text{Zero}(Y) \cap U$ Consider a small ball B centered at σ. Shrinking B, we may assume that there exists a trivialization for the tngent bundle TB. With this trivialization, since $\text{Ind}(Y, B) = \text{Ind}(Y, \sigma)$, we apply Lemma 5.3.1 and conclude that $\text{Ind}(Y, \sigma) = \sum_{z \in \text{Zero}(Z) \cap B} \text{Ind}(Z, z)$, for every Z close enough to Y. Summimng-up over all the zeros of Y inside U, one obtains the result. $\square$

Now, let X^t be a continuous family of C^1 vector fields such that $\text{Zero}(X^t)$ is finite for every $t \in [0, 1]$. Fix $c \in \mathbb{N}$. By Claim 2 we have that $\{t \in [0, 1]; \text{Ind}(X^t, U) = c\}$ is open and closed in $[0, 1]$. Indeed, openess is direct from Claim 2. For closedness, take $t_n \to t$ such that $\text{Ind}(X^{t_n}, U) = c$. By Claim 2, since the family is continuous, $\text{Ind}(X^{t_n}, U) = \text{Ind}(X^t, U)$, for every n large, and so $\text{Ind}(X^t, U) = c$. This proves that $\text{Ind}(X^t, U) = \text{Ind}(X^o, U)$, for every $t \in [0, 1]$.

Claim 3. *For every t there exists ε_t such that if Y and Z have only finitely many zeros inside U and are ε_t close to X in the C^1 topology then $\sum_{y \in \text{Zero}(Y)} \text{Ind}(Y, y) = \sum_{z \in \text{Zero}(Z)} \text{Ind}(Z, z)$.*

Proof. Since the space of vector fields is locally convex, there exists, if Y and Z are close enough to X and since the set of vector fields with only finitely many zeros is open and dense there exists a continuous path of vector fields, each one of them with only finitely many zeros, joinning Y to Z. Since the index sum is constant along the path, by the above, the claim is proved. $\square$

To complete the proof one has to notice that, since the set of vector fields with finitely many zeros in U is open and dense in the C^1 topology, given a continuous family X^t it is possible to find a family Y^t, arbitrarilly C^1 close of the original family, such that Y^t has finitely many zeros for every t. To complete the proof, one has to notice that by choosing the family Y^t close enough, one may require that $\mathrm{Ind}(Y^t, U) = \mathrm{Ind}(X^t, U)$, for every t. Indeed, by Claim 3 we can cover the path X^t with finitely many balls such that the index sum does not chang on each ball. Then, one just has to choose Y^t close enough so that it is contained in the union of these balls. $\square$

Let us also give a simple application of Lemma 5.3.1.

Proposition 5.3.2. *Let D be a topological disk (homeomorphic to $\mathbb{D}^2$), and let X be a vector field, with no zeros in the boundary. If $\mathrm{Ind}(X, D) \neq 0$ then X has a zero in the interior of D.*

Proof. Assume on the contrary that X has no zeros in D. Take a small flow box $[-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon]$. Take a smooth function $\varphi : \mathbb{R} \to [0, 1]$ such that $\varphi(x) = 0$ if and only if $x = 0$ and $\varphi \equiv 1$ outside $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$. Then, one may reparametrize $Y(x, y) = \varphi(x)X(x, y)$, for every $(x, y) \in [-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon]$. Then, the new vector field is everywhere collinear with X, vanishes only at the origin and equals to X outside $[-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon]$. By Lemma 5.3.1, $\mathrm{Ind}(X, D) = \mathrm{Ind}(Y, (0, 0))$. However, since Y (in the flow box) is collinear with the coordinate field ∂_x, one conludes that $\mathrm{Ind}(Y, (0.0)) = 0$. $\square$

5.4 The index of an isolated compact set of zeros

We say that a compact subset $K \subset \mathrm{Zero}(X)$ is *isolated* if there is a compact neighborhood U of K so that $K = \mathrm{Zero}(X) \cap U$; the neighborhood U is called an *isolating neighborhood of K*. The index $\mathrm{Ind}(X, U)$ does not depend of the isolating neighborhood V of K. Thus $\mathrm{Ind}(X, U)$ is called *the index of K* and denoted $\mathrm{Ind}(X, K)$.

Example 5.4.1. Let M be a closed manifold and $K = \mathrm{Zero}(X)$. Then, Poincaré-Hopf Theorem gives $\mathrm{Ind}(X, K) = \chi(M)$.

5.5 Topological degree of a map from $\mathbb{T}^2$ to $\mathbb{S}^2$

In this section we prove a lemma which will be used in Chapter 7 as a tool for index calculations.

We consider the sphere $\mathbb{S}^2$ (unit sphere of $\mathbb{R}^3$) endowed with the north and south poles denoted $N = (0, 0, 1)$ and $S = (0, 0, -1)$ respectively.

We denote by $\mathbb{S}^1 \subset \mathbb{S}^2$ the equator, oriented as the unit circle of $\mathbb{R}^2 \times \{0\}$. For $p = (x, y, z) \in \mathbb{S}^2 \setminus \{N, S\}$ we call *projection of p on $\mathbb{S}^1$ along the meridians* the point $\frac{1}{\sqrt{x^2+y^2}}(x, y, 0)$, which is intersection of $\mathbb{S}^1$ with the unique half meridian containing p.

We consider the torus $\mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. The result of this section if the following, which corresponds to Corollary 2.5 in [BS].

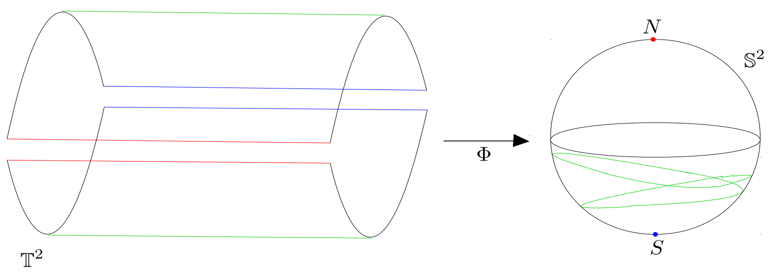


Figure 5.2: Proposition 5.5.1

Proposition 5.5.1. *Let $\Phi\colon \mathbb{T}^2 \to \mathbb{S}^2$ be a continuous map so that $\Phi^{-1}(N) = \{0\} \times \mathbb{R}/\mathbb{Z}$ and $\Phi^{-1}(S) = \{\frac{1}{2}\} \times \mathbb{R}/\mathbb{Z}$.*

Let $\varphi_+\colon \{\frac{1}{4}\} \times \mathbb{R}/\mathbb{Z} \to \mathbb{S}^1$ (resp. $\varphi_-\colon \{\frac{3}{4}\} \times \mathbb{R}/\mathbb{Z} \to \mathbb{S}^1$) be defined as follows: the point $\varphi_+(p)$ (resp. $\varphi_-(p)$) is the projection of $\Phi(p) \in \mathbb{S}^2 \setminus \{N, S\}$ on $\mathbb{S}^1$ along the meridians of $\mathbb{S}^2$.

Then

$$|\deg(\Phi)| = |\deg(\varphi_+) - \deg(\varphi_-)|$$

where $\deg()$ *denotes the topological degree, and $\{\frac{1}{4}\} \times \mathbb{R}/\mathbb{Z}$ and $\{\frac{3}{4}\} \times \mathbb{R}/\mathbb{Z}$ are endowed with* *the positive orientation of* $\mathbb{R}/\mathbb{Z}$*.*

Proof. Denote $d^+ = \deg(\varphi_+)$ and similarly $d^- = \deg(\varphi_-)$. Consider polar coordinates (θ, ϕ) on $\mathbb{S}^2$. For each $\phi \in [0, \pi]$ consider the map $F_t^\phi : \mathbb{S}^2 \setminus \{S, N\} \to \mathbb{S}^2 \setminus \{N, S\}$ given by

$$F_t^\phi(\cos\theta \sin\alpha, \sin\theta \sin\alpha, \cos\alpha) = (\cos\theta \sin((1-t)\alpha + t\phi), \sin\theta \sin((1-t)\alpha + t\phi), \cos((1-t)\alpha + t\phi)).$$

Then, F_t^ϕ is a homotopy between Id and the projection onto the ϕ-equator. Take $\psi : [0, 1] \to$

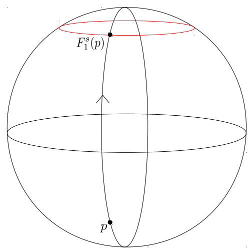


Figure 5.3: The projection onto the ϕ-equator of $\mathbb{S}^2 \setminus \{N, S\}$.

$[0, \pi]$ given by $\psi(r) = 4\pi r(1 - r)$ and consider the continuous map $f : [0, 1] \times \mathbb{T}^2 \to \mathbb{S}^2$ given by

$$f_t(r, s) = \begin{cases} F_t^{\psi(r)}(\Phi(r, s)) & \text{if } r \neq 0, \frac{1}{2} \\ \Phi(r, s) & \text{if } r = 0, \frac{1}{2}. \end{cases} .$$

Then f_t is a homotopy between Φ an the map $f := f_1$ which send each circle $\{r\} \times \mathbb{R}/\mathbb{Z}$ onto the $\psi(r)$-equator, for $r \neq \ 0, 1/2$, by projecting down $\Phi(r, s)$. Notice that for $r \in (0, 1/2)$, $f|_{\{r\} \times \mathbb{R}/\mathbb{Z}}$ is a degree d^+ covering of the circle, while for $r \in (1/2, 1)$ $f|_{\{r\} \times \mathbb{R}/\mathbb{Z}}$ is a degree d^- covering of the circle. Thus, we are left to prove that $|\deg(f)| = |d^+ - d^-|$.

For this, take a regular value of f in a $\psi(r)$-equator, for $r \in (0, 1/2)$. Then, this point has exactly d^+ pre-images in the circle $\{r\} \times \mathbb{R}/\mathbb{Z}$. Similarly, a regular value of f in a $\psi(r)$-equator, for $r \in (1/2, 1)$, has exactly d^- preimages in the circle $\{r\} \times \mathbb{R}/\mathbb{Z}$. Fix s, and let v be a vector tangent to the arc of circle $(0, 1/2) \times \{s\}$ in $\mathbb{T}^2$. Let u be a vector tangent to the arc $(1/2, 1) \times \{s\}$. Assume that the base points of v and u lies in the same level $f^{-1}(p)$ of f. Then, on has that $f_* u = \lambda v$, for some $\lambda < 0$. With these facts it is easy to conlude that the Brower degree of f is $\pm(d^+ - d^-)$. $\square$

Chapter 6

Commuting Vector Fields

In this chapter we introduce a very rich dynamical object: commuting vector fields. As we shall see below, two commuting vector fields over a manifold M are the same as an action of $\mathbb{R}^2$, and n-commuting vector fields are the same as an action of $\mathbb{R}^n$. The main problem about this object which shall concern this part of the thesis is the existence of fixed points.

Let us formally introduce commuting vector fields. Let M be a (not necessarily compact) manifold and X, Y be C^1-vector fields on M. Picard's Theorem asserts that the flow of X and Y are locally defined but they may not be complete.

Definition 6.0.2 (Commuting Vector Fields). We say that X *and* Y *commute* if for every point x there is $t(x) > 0$ such that for every $s, t \in [-t(x), t(x)]$ the compositions $X_t \circ Y_s(x)$ and $Y_s \circ X_t(x)$ are defined and coincide.

The existence of two commuting vector fields over a closed manifold M is equivalent to the existence of an action of $\mathbb{R}^2$ over M. Indeed, let X and Y be two commuting vector fields. Define a map $\varphi : \mathbb{R}^2 \times M \to M$ by $\varphi_{(s,t)}(p) = X_s \circ Y_t(p)$. We claim that φ is an action. Indeed, the group property follows from the commutation as follows:

$$\begin{aligned}
\varphi_{(s,t)+(r,l)}(p) &= \varphi_{(s+r,t+l)}(p) = X_{s+r} \circ Y_{t+l}(p) \\
&= X_s \circ X_r \circ Y_t \circ Y_l(p) = X_s \circ Y_t \circ X_r \circ Y_l(p) \\
&= \varphi_{(s,t)} \circ \varphi_{(r,l)}(p)
\end{aligned}$$

Conversely, given an action $\varphi : \mathbb{R}^2 \times M \to M$, we have flows $X_t(p) = \varphi_{(t,0)}(p)$ and

$Y_t(p) = \varphi_{(0,t)}$, which commute, since $\mathbb{R}^2$ is abelian. By taking the velocity of these flows one obtains two commuting vector fields. Thus, in the same way that the flow of a single vector field is a $\mathbb{R}$-action over M, the combined flow of two commuting vector fields is a $\mathbb{R}^2$-action over M. For more detailed informations, see [CN].

We have seen in Corollary 5.1.2 that the Euler charateristic is a good topological invariant to indicate the existence of zeros for actions of $\mathbb{R}$. It is natural to look for actions of $\mathbb{R}^2$ is and ask if the same assumption about non-vanishing of the Euler characteristic can lead to the same conclusions about existence of singularities. Notice that for actions of $\mathbb{R}^2$, for instance, there are two types of singularities: those which generate one-dimensional orbits and the fixed points, which correspond to common zeros for the vector fields generating the action.

Concerning the existence of fixed points, there exists a satisfactory solution on surfaces. It is a result of Elon Lima [Li2] in the sixties.

In this Chapter, we shall relate existence of common zeros for commuting vector fields with the Poincaré-Hopf index, defined in Chapter 2, and see applications of this connection. In particular, we will present a proof of Lima's Theorem.

In Chapter 7 we shall give our contribution to this question in dimension three.

This Chapter is organized in the following way. In Section 6.1 we study further commuting vector fields and prove some basic relations. In Section 6.2 we show Lima's beautiful proof [Li1] in $\mathbb{S}^2$, based on Poincaré-Bendixon's Theorem. Section 6.3 is devoted to the relation between the Poincaré-Hopf index and common zeros for commuting vector fields. In particular, we use this relation to obtain Lima's Theorem on any closed surface with non-vanishing Euler characteristic. In Sections 6.4 and 6.5 we discuss some results in dimension three that motivates the approach we shall use in the next chapter.

6.1 Commuting vector fields: basic properties

In all this section, unless otherwise stated, $X, Y \in \mathfrak{X}^1(M)$ are two commuting vector fields. We shall prove here some very basic relations. They reflect the heuristic idea that the flow of X is a symmetry of the flow of Y, and vice-versa. This heuristic idea can be formulated in more precise terms by saying that the flow of Y is invariant under conjugation by the diffeomorphism X_t, for every t for which the flow of X is defined. Nevertheless, we shall give an elementary presentation of these basic relations.

6.1.1 Equivalent definitions

The first property we shall present is the invariance of X under the tangent flow of Y. This can be seen as an infinitesimal version of the commutation relation and in fact this infinetesimal version implies the commutation.

Lemma 6.1.1. *For every $x \in M$, and any $t \in \mathbb{R}$ for which Y_t is defined, one has*

$$DY_t(x)X(x) = X(Y_t(x)).$$

Conversely, given $X, Y \in \mathfrak{X}^1(M)$ (not necessarily commuting) if $DY_t(x)X(x) = X(Y_t(x))$, for every $x \in M$, and any $t \in \mathbb{R}$ for which Y_t is defined, then X and Y commute.

Proof. Assume that X and Y commute. Let $t \in \mathbb{R}$ be such that the flow of Y at x is defined until time t. We shall prove that $DY_t(x)X(x) = X(Y_t(x))$. The only thing one has to take care is that the commutation is defined only for small times. Nonetheless, this can be overcomed by a standard connectedness argument.

Indeed, notice that there is a small neighborhood V of x in which the local diffeomorphism Y_t is defined. In particular, for every point $y \in V$, the integral curve $Y_{[0,t]}(y)$ is defined.

Consider now a small X-integral curve $X_{[-s_0,s_0]}(x)$, for some $s_0 > 0$, contained in V. We require that s_0 is so small that the integral curve $X_{[-s_0,s_0]}(Y_t(x))$ is defined, which is ensured by the Picard's Theorem. We claim that $Y_t \circ X_s(x) = X_s \circ Y_t(x)$, for every $s \in [-s_0, s_0]$. Indeed, consider the set $T = \{r \in [0,t]; Y_r \circ X_s(x) = X_s \circ Y_r(x), \text{ for every } s \in [-s_0, s_0]\}$. Then, T is obviously closed and since X and Y commute T is open. Thus $T = [0,t]$, which proves our claim.

Now, $X_s(x)$ is tangent to $X(x)$ at $s = 0$ and the integral curve $X_s(Y_t(x))$ is tangent to $X(Y_t(x))$ at $s = 0$. The above claim proves that, for s small, the local diffeomorphism Y_t

carries the integral curve $X_s(x)$ into the integral curve $X_s(Y_t(x))$, and thus $DY_t(x)X(x) = X(Y_t(x))$.

Conversely, assume that $DY_t(x)X(x) = X(Y_t(x))$, for every $x \in M$, and any $t \in \mathbb{R}$ for which Y_t is defined. Take $\varepsilon > 0$ and U a neighborhood of x, both of them small enough such that the flows X_s and Y_t are defined on U, for every $s, t \in [-\varepsilon, \varepsilon]$. Fix $t \in [-\varepsilon, \varepsilon]$ and consider the curve $\gamma : s \mapsto Y_t \circ X_s(x)$. By our assumption, for every $s \in [-\varepsilon, \varepsilon]$ the velocity of this curve at s is

$$DY_t(X_s(x))X(X_s(x)) = X(Y_t \circ X_s(x)).$$

Thus, γ is the integral curve of X through $Y_t(x)$. This proves that $Y_t \circ X_s(x) = X_s \circ Y_t(x)$, for every $s, t \in [-\varepsilon, \varepsilon]$ and establishes the lemma. $\square$

Remark 6.1.2. The connectedness argument that we employed in the proof of Lemma 6.1.1, if pushed a little further, gives the following: for every $x \in M$, for every $(s,t) \in \mathbb{R}^2$ such that the compositions $X_s \circ Y_t(x)$ and $Y_t \circ X_s(x)$ are defined, we have the equality $X_s \circ Y_t(x) = Y_t \circ X_s(x)$.

The infinitesimal version given by Lemma 6.1.1 can be used to prove a more popular definition of commutation: the vanishing of the Lie bracket.

Lemma 6.1.3. *$X, Y \in \mathfrak{X}^1(M)$ commute if, and only if, $[X,Y] = 0$.*

Before giving the proof, we quote a formula for the Lie bracket.

Lemma 6.1.4. *Let $X, Y \in \mathfrak{X}^1(M)$ be two vector fields and $x \in M$. Then,*

$$[X,Y](x) = \lim_{t \to 0} \frac{1}{t} \left\{ Y(X_t(p)) - DX_t(p)Y(p) \right\}.$$

For a proof see for instance Proposition 5.4, Chapter 0, in [dC].

Proof of Lemma 6.1.3. Assume that $X, Y \in \mathfrak{X}^1(M)$ commute, fix $x \in M$. By Lemma 6.1.1 we have that

$$\begin{aligned} [X,Y](x) &= \lim_{t \to 0} \frac{1}{t} \left\{ Y(X_t(p)) - DX_t(p)Y(p) \right\} \\ &= \lim_{t \to 0} \frac{1}{t} \left\{ Y(X_t(p)) - Y(X_t(p)) \right\} = 0. \end{aligned}$$

Conversely, assume that $[X, Y] = 0$. We shall apply Lemma 6.1.1 to prove that X and Y commute. Take $x \in M$ and $s \in \mathbb{R}$ so that the flow X_s is defined on a neighborhood U of x. Assume by contradiction that

$$DX_s(x)Y(x) \neq Y(X_s(x)).$$

By reducing U, we may assume that $X_s(U)$ is the domain of some local trivialization for the tangent bundle and we may use the norm $\|.\|$ induced by this local trivialization. Notice that the topology induced in the tangent bundle of $X_s(U)$ by this norm is equivalent to the restriction of the topology of TM to $X_s(U)$

Take V and W compact subsets of the tangent bundle, whose projection over M lies within $X_s(U)$, and $r > 0$ satisfying the following properties:

- $\mathrm{int}(V)$ is a neighborhood of $(X_s(x), DX_s(x)Y(x))$ in the tangent bundle,
- $\mathrm{int}(W)$ is a neighborhood of $(X_s(x), Y(X_s(x)))$ in the tangent bundle
- and $d(V, W) \geq r$

By continuity and Lemma 6.1.4 there exists $\varepsilon > 0$ such that for every $|t| < \varepsilon$ we have

$$DX_{s+t}(x)Y(x) \in \mathrm{int}(V) \text{ and } Y(X_{s+t}(x)) \in \mathrm{int}(W),$$

and

$$\left\| \frac{1}{t} \left\{ Y(X_{t+s}(p)) - DX_{t+s}(p)Y(p) \right\} \right\| < r.$$

However, this last inequality implies that $\|Y(X_{t+s}(p)) - DX_{t+s}(p)Y(p)\| < |t|r < \varepsilon r$, showing that $r \leq d(V, W) < r$, which is absurd. Thus $DX_s(x)Y(x) = Y(X_s(x))$, for every $x \in M$ and every $s \in \mathbb{R}$ such that the flow X_s is defined at x. By Lemma 6.1.1 this completes the proof. □

6.1.2 Invariant sets

As an immediate consequence of Lemma 6.1.1, if $X(x) = 0$ then $X(Y_t(x)) = 0$, for every $t \in \mathbb{R}$ such that the flow Y_t is defined. More generally, if $X(x) = \lambda Y(x)$ for some $\lambda \in \mathbb{R}$ then

$X(Y_t(x)) = \lambda Y(Y_t(x))$, for every $t \in \mathbb{R}$ for which the flow Y_t is defined. Though very easy to prove, these facts are important since they impose a strong rigidity on the dynamics of X and Y. We put them as a corollary below for further reference.

Corollary 6.1.5. *If $X(x) = \lambda Y(x)$ for some $\lambda \in \mathbb{R}$ then $X(Y_t(x)) = \lambda Y(Y_t(x))$, for every $t \in \mathbb{R}$ for which the flow Y_t is defined. In particular,*

- $\mathrm{Zero}(X)$ *is invariant under the flow of Y: if $x \in \mathrm{Zero}(X)$, then $Y_t(x) \in \mathrm{Zero}(X)$ for any $t \in \mathbb{R}$ for which Y_t is defined.*
- $\mathrm{Col}(X, Y)$ *is invariant under the flow of Y: if $x \in \mathrm{Col}(X, Y)$ and if $Y_t(x)$ is defined, then $Y_t(x) \in \mathrm{Col}(X, Y)$.*

Corollary 6.1.5 is one of the reasons because the commutation relation in general do not survive under perturbation of the vector fields: if you perturb X to $\tilde{X}$ and Y to $\tilde{Y}$ then the perturbed colinearity locus $\mathrm{Col}(\tilde{X}, \tilde{Y})$ has to be invariant under the flows of $\tilde{X}$ and $\tilde{Y}$. However, there exists one special type of perturbation which gives rise to new comuting vector fields: those giving by linear combinations between the vector fields.

Lemma 6.1.6. *For every $a, b, c, d \in \mathbb{R}$, $aX + bY$ commutes with $cX + dY$*

Proof. Just notice that $[aX + bY, cX + dY] = 0$ due to the $\mathbb{R}$-linearity of the Lie bracket, and then apply Lemma 6.1.3. □

Combining Corollary 6.1.5 with Lemma 6.1.6 one obtains

Corollary 6.1.7. $\mathrm{Col}(X, Y)$ *is invariant under the flow of $aX + bY$ for any $a, b \in \mathbb{R}$.*

6.1.3 Periodic orbits

Now, we assume that γ is a periodic orbit of Y, with period τ. Take $t \in \mathbb{R}$ such that the flow X_t is defined on a neighborhood of γ. Such t exists by the compactness of γ. We know that $X_t(\gamma)$ is an integral curve of Y. Morevover, if $x \in \gamma$ then

$$Y_\tau(X_t(x)) = X_t(Y_\tau(x)) = X_t(x),$$

and thus $X_t(\gamma)$ is a periodic orbit of Y, of period τ. This discussion proves the following

Lemma 6.1.8. *If γ is isolated among the periodic orbits of X of the same period τ, then γ is invariant under the flow of Y; as a consequence, $\gamma \subset \mathrm{Col}(X, Y)$.*

Since, as we mentioned before, the flow of Y is invariant under conjugation by the diffeomorphism X_t it is natural that the stable set $W^s(\gamma)$ of a periodic orbit will be invariant under the flow of X. We shall prove this by an elementary topological argument.

Lemma 6.1.9. *Let X and Y be two commuting vector fields over M. Let γ be a periodic orbit of Y. Assume that γ is invariant by the flow of X. Then, $W^s(\gamma)$ is invariant by the flow of X.*

Proof. We have to show that for any point $p \in W^s(\gamma)$ and any $t \in \mathbb{R}$ such that $X_t(p)$ is defined one has $\omega_Y(X_t(p)) = \gamma$. To do so, take $\sigma \in \gamma$. Since γ is invariant under t flow of X there exists a sequence $t_k \to +\infty$ such that

$$Y_{t_k}(p) \to X_{-t}(\sigma).$$

Since X and Y commute,

$$Y_{t_k}(X_t(p)) = X_t(Y_{t_k}(p)) \to X_t(X_{-t}(\sigma)) = \sigma,$$

therefore $\sigma \in \omega_Y(X_t(p))$, and we conclude that $\gamma \subset \omega_Y(X_t(p))$.

On the other hand, assume by contradiction that we can find some $q \in \omega_Y(X_t(p))$ such that $q \notin \gamma$. Then, we can choose $\varepsilon > 0$ satisfying $d(q, \gamma) > 4\varepsilon$. Take $t_k \to +\infty$ with $Y_{t_k}(X_t(p)) \to q$. Since $p \in W^s(\gamma)$, up to pass to a subsequence, we have that $Y_{t_k}(p) \to \sigma \in \gamma$. This ensures that we can find t_k large enough so that

$$d(Y_{t_k}(X_t(p)), q) < \varepsilon \quad \text{and} \quad d(X_t(Y_{t_k}(p)), X_t(\sigma)) < \varepsilon,$$

which, due to the commutation of X and Y, implies that

$$d(q, X_t(q)) < 3\varepsilon.$$

But this contradicts our choice of ε. This establishes that $\omega_Y(X_t(p)) \subset \gamma$ and completes the proof. □

In particular, if γ is a hyperbolic periodic orbit of Y, Lemma 6.1.9 says that the flow $X_t(x)$ of a point $x \in W^s(\gamma)$ remains in the stable manifold. Therefore, X has to be tangent to the stable manifold.

6.1.4 Normal component and ratio function

In this subsection we consider X and Y such that Y has no zeros in some compact region. More precisely, let $U \subset M$ be a compact region of M where Y is non-vanishing, i.e $\mathrm{Zero}(Y) \cap U = \emptyset$. We shall use the notations of Section 2.4

Consider a hyperplane field P everywhere transverse to Y over U. Let N be a vector field defined as the projection of X onto P, parallel to Y. In particular, there exists a function $\mu : U \to \mathbb{R}$, with the same regularity as X and Y, such that

$$X(x) = N(x) + \mu(x)Y(x). \tag{6.1}$$

Definition 6.1.10 (Normal component and ratio function). In the notations above, the vector field N is called *a normal component of X in the direction of Y*, or simply a *normal component* when there is no risk of confusion. The function μ is called a *ratio function* of X with respect to Y.

Notice that the ratio function and the normal componment are not uniquely defined. Nonetheless, $\mathrm{Zero}(N) = \mathrm{Col}(X, Y)$ and $x \in \mathrm{Col}(X, Y)$ if and only if $X(x) = \mu(x)Y(x)$ and these properties are determined by X and Y.

Assume now that there exists two transverse sections $\Sigma_0 \subset U$ and $\Sigma_1 \subset U$, tangent to P, with a transition time $\tau : \Sigma_0 \to (0, +\infty)$ and a holonomy $\mathcal{P} : \Sigma_0 \to \Sigma_1$ well defined. Recall that the existence of Σ_0 and Σ_1 with these properties can always be obtained locally.

The following lemma gives two formulas relating holonomies, normal components and the ratio functions.

Lemma 6.1.11. *1. Any normal component is invariant under the holonomy:*

$$D\mathcal{P}(x)N(x) = N(\mathcal{P}(x)).$$

2. $-D\tau(x)N(x) = \mu(\mathcal{P}(x)) - \mu(x)$.

Proof. Recall that $DY_{\tau(x)}(x)Y(x) = Y(\mathcal{P}(x))$ by definition of the flow, and $DY_{\tau(x)}(x)X(x) = X(\mathcal{P}(x))$ by Lemma 6.1.1. With this in mind, apply the linear map $DY_{\tau(x)}(x)$ on both sides of Equation 6.1 to obtain

$$X(\mathcal{P}(x)) = DY_{\tau(x)}(x)N(x) + \mu(x)Y(\mathcal{P}(x)).$$

Using Lemma 2.4.1 we may replace $DY_{\tau(x)}(x)N(x)$ by $D\mathcal{P}(x)N(x) - (D\tau(x)N(x))Y(\mathcal{P}(x))$. Since $X(\mathcal{P}(x)) = N(\mathcal{P}(x)) + \mu(\mathcal{P}(x))Y(\mathcal{P}(x))$, we get that

$$N(\mathcal{P}(x)) + \mu(\mathcal{P}(x))Y(\mathcal{P}(x)) = D\mathcal{P}(x)N(x) + (\mu(x) - D\tau(x)N(x))Y(\mathcal{P}(x)).$$

Since $N(\mathcal{P}(x)) - D\mathcal{P}(x)N(x) \in P(\mathcal{P}(x))$ and P is transverse to Y, and Y is non-vanishing in U, we conclude that $N(\mathcal{P}(x)) - D\mathcal{P}(x)N(x) = 0$ and $\mu(\mathcal{P}(x)) - \mu(x) + D\tau(x)N(x) = 0$. This establishes the result. □

Corollary 6.1.12. *The restriction of any ratio function to* $\mathrm{Col}(X, Y)$ *is invariant under the flow of* Y*: if* $x \in \mathrm{Col}(X, Y)$ *belongs to the domain of* Y_t *for some* $t \in \mathbb{R}$ *then* $\mu(Y_t(x)) = \mu(x)$.

Proof. If $x \in \mathrm{Col}(X, Y)$ then $N(x) = 0$. The result follows now from the second item of Lemma 6.1.11. □

6.2 Lima's Theorem

In this section we shall study the existence of fixed points for actions of $\mathbb{R}^2$ on surfaces. First, it is clear that there exist two commuting and everywhere non-vanishing vector fields on the two torus: the coordinate fields ∂_x and ∂_y are never zero and their bracket vanishes. So, on the torus there are $\mathbb{R}^2$ actions with no fixed points.

In 1962, Elon Lima proved in [Li1] that an action of $\mathbb{R}^2$ on the sphere $\mathbb{S}^2$ always have a fixed point. Later, he proved in [Li2] that the Euler characteristic is a good topological invariant to indicate the existence of fixed points for actions of $\mathbb{R}^2$ on closed surfaces.

Theorem 6.2.1 (Lima)**.** *Let* S *be a closed surface with* $\chi(S) \neq 0$*. Let* $X, Y \in \mathfrak{X}^1(M)$ *be two commuting vector fields. Then,*

$$\mathrm{Zero}(X) \cap \mathrm{Zero}(Y) \neq \emptyset.$$

One nice feature about the proof of Lima is that it is a dynamical proof: it is a proof by contradiction that uses the Poincaré-Bendixon Theorem to create a nested sequence $\{D_n\}$ of disks, whose boundaries are contained in $\mathrm{Zero}(X)$, for n odd, and in $\mathrm{Zero}(Y)$ for n even. The assumption that $\mathrm{Zero}(X) \cap \mathrm{Zero}(Y) = \emptyset$ leads then to a contradiction. In particular, the spirit of the proof is to use dynamics to show that the non-existence of common zeros leads to a contradiction.

For the reader to appretiate better this approach, we give below the proof for $S = \mathbb{S}^2$ in full detail. Latter we shall use the relation between existence of common zeros and the Poincaré-Hopf index to prove the full statement.

6.2.1 Proof of Lima's Theorem on $\mathbb{S}^2$

Let $X, Y \in \mathfrak{X}^1(\mathbb{S}^2)$ be two commuting vector fields. Assume by contradiction that

$$\mathrm{Zero}(X) \cap \mathrm{Zero}(Y) = \emptyset.$$

Take $r := d(\mathrm{Zero}(X), \mathrm{Zero}(Y)) > 0$. By Corollary 5.1.2, there exists $\sigma_1^X \in \mathrm{Zero}(X)$. By Lemma 6.1.5, $O_Y(\sigma_1^X) \subset \mathrm{Zero}(X)$. In particular, $\omega_Y(\sigma_1^X) \subset \mathrm{Zero}(X)$. Poincaré-Bendixon's Theorem [PdM] implies that $\omega_Y(\sigma_1^X)$ is a periodic orbit for Y. Thus, there exists a topological disk D_1^X whose boundary is precisely $\omega_Y(\sigma_1^X)$.

Now, since Y has no zeros in the boundary of D_1^X, we may apply Proposition 5.3.2 and find $\sigma_1^Y \in \mathrm{Zero}(Y)$ in the interior of D_1^X. By Poincaré-Bendixon again $\omega_X(\sigma_1^Y)$ is a periodic orbit for X. Since $O_X(\sigma_1^Y) \subset \mathrm{Zero}(Y)$ we have that this orbit cannot get too close to the boundary of D_1^X. In particular, it never leaves D_1^X. As a consequence $\omega_X(\sigma_1^Y)$ bounds a smaller disk $D_1^Y \subset D_1^X$ such that $d(\partial D_1^X, \partial D_1^Y) \geq r$.

Corollary 5.1.2 gives a zero σ_2^X of X contained in $\mathrm{int}(D_1^Y)$, whose ω-limit under Y is a periodic orbit which bounds a smaller disk D_2^X contained in D_1^Y, and the distance between the boundaries is bigger than r. Proceeding by induction, we find a nested sequence of disks D_n^X and D_n^Y satisfying

- $\partial D_n^* \subset \mathrm{Zero}(*)$, $* = X, Y$

- $D_n^Y \subset D_n^X$
- $d(\partial D_n^X, \partial D_n^Y) \geq r$.

These properties imply that each set $D_n^X - D_n^Y$ contains a disk with diameter at least $\frac{r}{2}$, and therefore

$$\text{Area}(D_n^X - D_n^Y) \geq \frac{\pi r^2}{4},$$

from which one concludes that $\text{Area}(\mathbb{S}^2) = \infty$. This contradiction proves that $\text{Zero}(X) \cap \text{Zero}(Y) \neq \emptyset$.

6.3 The Poincaré-Hopf index and common zeros

There exists an immediate difficulty to generalize Lima's Theorem to higher dimensions: every closed $2n+1$-manifold has zero Euler characteristic. To bypass this difficulty, we may replace the Euler characteristic by the Poincaré-Hopf index. In [Bo1] C. Bonatti proposed the following local problem

Problem 1. Let M^d be a manifold, and U be a compact set. Let $X, Y \in \mathfrak{X}^1(M)$ be two commuting vector fields. Assume that $\text{Zero}(X) \subset \text{int}(U)$ so that $\text{Ind}(X, U) \neq 0$. Is it true that $\text{Zero}(X) \cap \text{Zero}(Y) \cap U \neq \emptyset$?

Problem 1 is the first main theme of this thesis. The answer is known to be yes in dimension two. Indeed, there exists the following result.

Proposition 6.3.1. *Let M be a surface and $U \subset M$ be a compact set. Assume that $X, Y \in \mathfrak{X}^1(M)$ are two commuting vector fields, and that the following two properties are satisfied*

- $\text{Zero}(X) \cap \partial U = \emptyset$,
- $\text{Zero}(Y) \cap U = \emptyset$

Then, $\text{Ind}(X, U) = 0$.

Let us see how this proposition answers affirmatively Problem 1 on surfaces

Corollary 6.3.2. *Let M be a surface, and U be a compact set. Let $X, Y \in \mathfrak{X}^1(M)$ be two commuting vector fields. Assume that* $\mathrm{Zero}(X) \subset \mathrm{int}(U)$ *so that* $\mathrm{Ind}(X, U) \neq 0$*. Then,* $\mathrm{Zero}(X) \cap \mathrm{Zero}(Y) \cap U \neq \emptyset$

Proof. Assume on the contrary that $\mathrm{Zero}(X) \cap \mathrm{Zero}(Y) \cap U = \emptyset$. Then, there is a smaller compact set $V \subset U$ such that $\mathrm{Zero}(X) \subset \mathrm{int}(V)$ and $\mathrm{Zero}(Y) \cap V = \emptyset$.

Since $\mathrm{Ind}(X, V) = \mathrm{Ind}(X, U)$, we can apply Proposition 6.3.1 to (X, Y, V) and conclude that $\mathrm{Ind}(X, U) = 0$. $\square$

Proposition 6.3.1 also implies Lima's Theorem.

Proof of Theorem 6.2.1. Assume by contradiction that $\mathrm{Zero}(X) \cap \mathrm{Zero}(Y) = \emptyset$. Then, there exists a compact neighborhood U of $\mathrm{Zero}(X)$ which is disjoint from $\mathrm{Zero}(Y)$. Thus, (X, Y, U) satisfies all the hypothesis of Proposition 6.3.1 and therefore $\mathrm{Ind}(X, U) = 0$. However, as in Example 5.4.1, we have by Poincaré-Hopf Theorem that $\mathrm{Ind}(X, U) = \chi(M)$, proving that $\chi(M) = 0$. This proves the result. $\square$

It is worth to note that none of the above proofs have nothing special of dimension two: they show indeed that Problem 1 generalizes Lima's result, and it gives a relation between the index and existence of common zeros.

6.3.1 Dynamics of first return maps and Lima's Theorem

As we have seen, Proposition 6.3.1 implies Theorem 6.2.1. For this reason, we shall now explain how to prove Proposition 6.3.1. As in the proof of Lima's Theorem on $\mathbb{S}^2$, the main point is to study the dynamical properties of X and Y to reach a contradiction if the conclusion does not hold. Our goal in this section is to make an informal presentation of this study, focusing on the ideas and in the structure of the argument.

To have a more clear picture and to avoid clumsy technical details in a first presentation we shall assume that X and Y are C^2 vector fields. Later we explain how to remove this extra assumption.

Let X and Y be C^2 commuting vector fields on a surface M. Let U be a compact set satisfying $\mathrm{Zero}(X) \cap \partial U = \mathrm{Zero}(Y) \cap U = \emptyset$. Assume by contradiction that $\mathrm{Ind}(X, U) \neq 0$.

Consider the function $g : U \to \mathbb{R}$, defined by $g(x) = \frac{\|X(x)\|^2}{\|Y(x)\|^2}$. By Sard's Theorem, since g is C^2, there exists $\varepsilon > 0$ as small as we please such that ε^2 is a regular value of g. In particular, $g^{-1}(\varepsilon^2)$ is a finite union of circles and segments.

Notice that $\mathrm{Zero}(X - \varepsilon Y) \subset g^{-1}(\varepsilon^2)$. Since $\mathrm{Zero}(X - \varepsilon Y)$ is invariant under the flow of Y, the connected components of $g^{-1}(\varepsilon^2)$ which contains $\mathrm{Zero}(X - \varepsilon Y)$ are circles and in fact are periodic orbits of Y. Thus, $\mathrm{Zero}(X - \varepsilon Y) = \gamma_1 \cup ... \cup \gamma_n$, where each γ_i is a periodic orbit of Y. Since

$$0 \neq \mathrm{Ind}(X - \varepsilon Y, U) = \sum_{i=1}^{n} \mathrm{Ind}(X - \varepsilon Y, \gamma_i),$$

there exists γ_i such that $\mathrm{Ind}(X - \varepsilon Y, \gamma_i) \neq 0$. Therefore, up to replace X by $X - \varepsilon Y$, we may assume (and we do it) that $\mathrm{Zero}(X)$ is a periodic orbit γ of Y, with $\mathrm{Ind}(X, \gamma) \neq 0$.

Thus there exists an annulus $\mathcal{A}$ containing γ in its interior and such that $\mathrm{Zero}(X) \cap \mathcal{A} = \gamma$. Let $\partial\mathcal{A}^+$ and $\partial\mathcal{A}^-$ be the two circles which are the connected components of $\partial\mathcal{A}$. We can take a basis for the tangent bundle $T\mathcal{A}$ formed by Y and some vector field $Y^\perp$ everywhere transverse to Y. With this basis we can write $X(x) = \alpha(x)Y(x) + \beta(x)Y^\perp(x)$. If $x \notin \gamma$ we can define

$$\mathcal{X}(x) = \frac{1}{\sqrt{\alpha(x)^2 + \beta(x)^2}}(\alpha(x), \beta(x)) \in \mathbb{S}^1.$$

Take any circle C^+ (resp. C^-) homotopic in $\mathcal{A} \setminus \gamma$ to $\partial\mathcal{A}^+$ (resp. $\partial\mathcal{A}^-$).

Then, Lemma 5.3.1 implies that

$$\mathrm{Ind}(X, \gamma) = \deg(\mathcal{X}|_{C^+}) + \deg(\mathcal{X}|_{C^-}).$$

Since $\mathrm{Ind}(X, \gamma) \neq 0$ we may assume without lost of generality that $\deg(\mathcal{X}|_{C^+}) \neq 0$, the other case being treated simmilarly.

Let Σ be a small transverse section to the flow of Y (for instance, a small integral curve of $Y^\perp$). Denote by $\mathcal{P}$ the first return map to Σ. Let Σ^+ be the component of $\Sigma \setminus \gamma$ which intersect $\partial\mathcal{A}^+$. Take $p \in \Sigma^+$ and let $[\mathcal{P}(p), p]$ be a segment in Σ^+.

We claim the there exists $q \in [\mathcal{P}(p), p]$ such that $X(q) = \lambda Y(q)$, for some real number $\lambda \neq 0$. Indeed, the closed curve C^+ formed by the concatenation between the Y-orbit segment joinning p to $\mathcal{P}(p)$ with the segment $[\mathcal{P}(p), p]$ is homotopic in $\mathcal{A} \setminus \gamma$ to $\partial\mathcal{A}^+$. Thus, $\deg(\mathcal{X}|_{C^+}) \neq 0$ and so $\mathcal{X}|_{C^+}$ is surjective. In particular, there must exists a point $q \in C^+$ such that $\mathcal{X}(q) = (1, 0)$, which implies that $X(q) = \lambda Y(q)$, for some $\lambda > 0$. Either, q belongs to the orbit segment, which imples that $p \in \mathrm{Col}(X, Y)$, or $q \in (\mathcal{P}(p), p)$, and so the claim is proved.

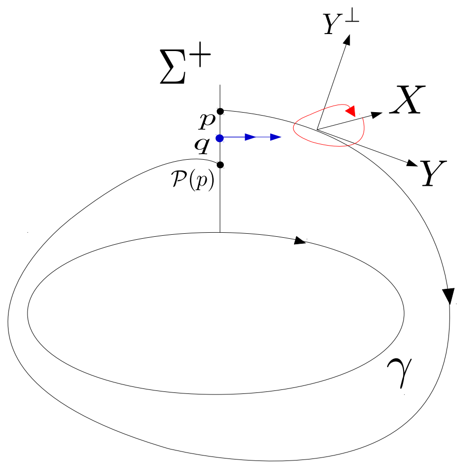


Figure 6.1: X has to turn in all directions, with respect to the basis $\{Y, Y^\perp\}$. In particular, it most be collinear with Y in somewhere.

By Poincaré-Bendixon's Theorem, the orbit of q accumulates on a periodic orbit γ'. If $\gamma' = \gamma$, we have a contradiction because then $0 = X|_\gamma = \lambda Y|_\gamma$. Thus, γ' is homotopic in $\mathcal{A} \setminus \gamma$ to $\partial\mathcal{A}^+$, and so $\deg(\mathcal{X}|_{\gamma'}) \neq 0$. However, since $X|_{\gamma'} = \lambda Y|_{\gamma'}$ we have that $\mathcal{X}|_{\gamma'} \equiv \frac{\lambda}{|\lambda|}$ and thus $\deg(\mathcal{X}|_{\gamma'}) = 0$. Therefore, we have reached a contradiction.

The above argument suggests the following structure:

1. Replace X by $X + \varepsilon Y$ to simplify $\mathrm{Zero}(X)$, making it become a periodic orbit γ of Y. This step uses Sard's Theorem to choose ε

2. To put a basis on the tangent bundle of a neighborhood of $\mathrm{Zero}(X)$, with Y the first vector of the basis, and use Lemma 5.3.1 to show that X has to turn in all directions, in a non-trivial way, at the boundary of a neighborhood of γ. In particular, it has to be be collinear with Y at a some point q in a transverse section to γ.

3. Use the dynamics of the first return map around γ to show that (2) leads to a contradiction. Since Y is non-vanishing and X is invariant under the tangent flow of Y, it cannot be that X turns in all directions: it does not turn at all!

This structure, in turn, suggests an approach to Problem 1 in dimension three. As it is always the case in mathematics, before trying to push further an approach one has to look very carefuly to the simple cases that it can solve. In our case, notice that the C^2 regularity was used only to apply Sard's Theorem. In fact, we shall see below that this not only can be done in C^1 regularity but is actually enough to fully prove Proposition 6.3.1.

The point of giving the above simpler proof, with two more steps, was to ilustrate how the dynamics of the first return map and the colinearity locus play a role in the problem. In dimension three, as we shall see in the next chapter, the simplification given by Sard's Theorem is not enough and the dynamical properties of the first return map became fundamental.

6.3.2 Another proof of Lima's Theorem

In the sequel, we shall present the complete argument for proving Proposition 6.3.1, which in particular, as we saw above, will prove Theorem 6.2.1.

The key step is the lemma below, which is a more intricate application of Sard's Theorem, whose aim is to replace the triple (X, Y, U) by a new one $(\tilde{X}, \tilde{Y}, \tilde{U})$ where $\tilde{U}$ is an annulus foliated by periodic orbits of Y, and each periodic orbit coincides with $\mathrm{Zero}(\tilde{X} - t\tilde{Y})$. In particular, the boundary of a neighborhood of $\mathrm{Zero}(\tilde{X})$ consists in two circles at which X is every where collinear with Y. In the argument we gave before, we obtained this using the dynamics of the first return map, and then showed that X cannot turn.

The technical difficulty of the lemma is due to the fact that Sard's Theorem may fail for a C^1 function from a surface to $\mathbb{R}$ (see Section 2.3 and the Appendix)

In order to apply Sard's Theorem in C^1 regularity we replace the map g of the argument of Section 6.3.1 by a ratio function and use that ratio functions are invariant when restricted to the colinearity locus. Since Y is non-vanishing this will enable us to consider the restriction of this ratio function to a transverse section of Y, of dimension one. Then, we apply Sard to this restriction. The lemma below is taken from [BS].

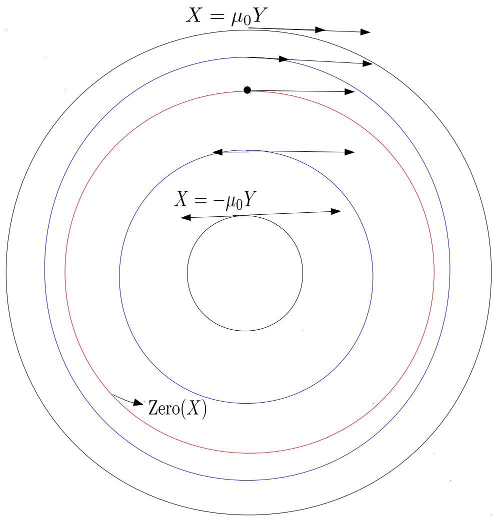


Figure 6.2: $\mathrm{Col}(X, Y, U)$ can be turned into an annulus foliated by periodic orbits of Y

Lemma 6.3.3. *Let S be a surface. Let X and Y two commuting vector fields over S and U a compact region of S such that $\mathrm{Zero}(Y) \cap U = \mathrm{Zero}(X) \cap \partial U = \emptyset$ and $\mathrm{Ind}(X, U) \neq 0$. Then, there exists $\tilde{X}$ and $\tilde{Y}$ two commuting vector fields over S, $\tilde{U}$ a compact region of S and $\mu_0 > 0$ with the following property:*

- $\mathrm{Zero}(\tilde{Y}) \cap \tilde{U} = \mathrm{Zero}(\tilde{X}) \cap \partial\tilde{U} = \emptyset$ *and* $\mathrm{Ind}(\tilde{X}, \tilde{U}) \neq 0$
- *for any $t \in [-\mu_0, \mu_0]$, the set of zeros of $\tilde{X} - t\tilde{Y}$ in $\tilde{U}$ consists precisely in 1 periodic orbit γ_t of $\tilde{Y}$;*
- *for any $t \notin [-\mu_0, \mu_0]$, the set of zeros of $\tilde{X} - t\tilde{Y}$ in $\tilde{U}$ is empty;*
- $\mathrm{Col}(\tilde{X}, \tilde{Y}, \tilde{U})$ *is a C^1 annulus;*
- *there is a C^1-diffeomorphism $\varphi\colon \mathbb{R}/\mathbb{Z} \times [-\mu_0, \mu_0] \to \mathrm{Col}(\tilde{X}, \tilde{Y}, \tilde{U})$ so that, for every $t \in [-\mu_0, \mu_0]$, one has*
$$\varphi(\mathbb{R}/\mathbb{Z} \times \{t\}) = \gamma_t.$$

Proof. Notice that there is $\mu_1 > 0$ so that for any $t \in [-\mu_1, \mu_1]$ one has $\mathrm{Zero}(X - tY) \cap \partial U = \emptyset$ and $\mathrm{Ind}(X - tY, U) \neq 0$.

As $X - tY$ and Y commute, $\mathrm{Zero}(X - tY, U)$ is invariant under the flow of Y. Futhermore, as $\mathrm{Zero}(X - tY)$ does not intersect ∂U the Y-orbit of a point $x \in \mathrm{Zero}(X - tY, U)$ remains in the compact set U hence is complete.

Consider now a ratio function $\mu\colon S \to \mathbb{R}$ and recall that the restriction $\mu|_{\mathrm{Col}(X,Y)}$ is invariant under the flows of X and Y (see Corollary 6.1.12).

Let $\mathcal{L} = \bigcup_{t\in[-\mu_1,\mu_1]} \mathrm{Zero}(X - tY) \cap U$. Then $\mathcal{L}$ is a compact set, contained in S disjoint from the boundary of U and invariant under Y: it is a compact lamination of S.

Let $\sigma \subset S$ be a union of finitely many compact segments with end points out of $\mathcal{L}$ and so that the interior of σ cuts transversely every orbit of Y contained in $\mathcal{L}$.

Claim 4. *Lebesgue almost every $t \in [-\mu_1, \mu_1]$ is a regular value of the restriction of μ to σ.*

Proof. Recall that Sard's theorem requires a regularity $n - m + 1$ if one consider maps from an m-manifold to an n-manifold. As μ is C^1 and $\dim \sigma = 1$ we can apply Sard's theorem to the restriction of μ to σ, concluding. □

Consider now a regular value $t \in (-\mu_1, \mu_1)$ of the restriction of μ to σ. Then $\mu^{-1}(t) \cap \sigma$ consists in finitely many points. Furthermore, $\mu^{-1}(t) \cap \sigma$ contains $\mathrm{Zero}(X - tY) \cap \sigma$.

Claim 5. *For $t \in [-\mu_1, \mu_1]$, regular value of the restriction of μ to σ, the compact set $\mathrm{Zero}(X - tY) \cap U$ consists in finitely many periodic orbits γ_i, $i \in \{1, \dots, n\}$ of Y.*

Proof. $\mathrm{Zero}(X - tY) \cap U$ is a compact sub lamination of $\mathcal{L} \subset S$ consisting of orbits of Y, and contained in $\mu^{-1}(t)$. Now, σ cuts transversely each orbit of this lamination and $\sigma \cap \mu^{-1}(t)$ is finite. One deduces that $\mathrm{Zero}(X - tY) \cap U$ consists in finitely many compact leaves, concluding. □

Notice that $\mathrm{Ind}(X - tY, U) = \sum_{i=1}^{n} \mathrm{Ind}(X - tY, \gamma_i)$. Thus there is i so that

$$\mathrm{Ind}(X - tY, \gamma_i) \neq 0.$$

Claim 6. *There is a neighborhood Γ_i of γ_i in S which is contained in $\mathrm{Col}(X, Y, U)$ and which consists of periodic orbits of Y.*

Proof. Let p be a point in $\sigma \cap \gamma_i$. As p is a regular point of the restriction of μ to σ there is a segment $I \subset \sigma$ centered at p so that the restriction of μ to I is injective and the derivative of μ does not vanish.

As γ_i has non-zero index for any s close enough to t, $\mathrm{Zero}(X - sY)$ contains an isolated compact subset K_s contained in a small neigborhood of γ_i, and hence in U, thus in $\mathrm{Col}(X, Y, U)$ and thus in a small neighborhood of γ_i in $\mathcal{L} \subset S$. This implies that each orbit of Y contained in K_s cuts I. However, μ is constant equal to s on K_s and thus $\mu^{-1}(s) \cap I$ consist in a unique point. One deduces that K_s is a compact orbit of Y.

Since this holds for any s close to t, one obtain that any point q of I close to p is the intersection point of $K_{\mu(q)} \cap I$. In other words, a neighborhood of p in I is contained in $\mathrm{Col}(X, Y, U)$ and the corresponding leaf of $\mathcal{L}$ is a periodic orbit of Y, concluding. $\square$

Notice that Γ_i is contained in $\mathrm{Col}(X, Y, U)$ so that the function μ is invariant under Y on Γ_i. As the derivative of μ is non vanishing (by construction) on $\Gamma_i \cap I$ one gets that the derivative of the restriction of μ to Γ_i is non-vanishing. One deduces that Γ_i is diffeomorphic to an annulus: it is foliated by circles and these circles admit a transverse orientation.

For concluding the proof it remains to choose a compact neighborhood $\tilde{U}$ of γ_i in M, which is a manifold with boundary, whose boundary is transverse to S and so that $\tilde{U} \cap S = \Gamma_i$. $\square$

We are now in position to give the complete proof of Proposition 6.3.1.

Proof of Prosposition 6.3.1. Assume the a triple (X, Y, U) satisfies the assumptions of the Proposition, but not its conclusion. Then, Lemma 6.3.3 ensures a new triple $(U, \tilde{X}, \tilde{Y})$ for which $\mathrm{Zero}(\tilde{X})$ is a periodic orbit of $\tilde{Y}$ and $\tilde{U}$ is an annulus whose boundary is the union of two periodic orbits γ and ξ of Y. Moreover, $\tilde{X}|_\gamma = \mu_0 Y$ and $\tilde{X}|_\xi = -\mu_0 Y$.

We can take a basis for the tangent bundle $T\tilde{U}$ formed by Y and some vector field $Y^\perp$ everywhere transverse to Y. With this basis we can write $\tilde{X}(x) = \alpha(x)\tilde{Y}(x) + \beta(x)Y^\perp(x)$. If $x \notin \mathrm{Zero}(\tilde{X})$ we can define

$$\tilde{\mathcal{X}}(x) = \frac{1}{\sqrt{\alpha(x)^2 + \beta(x)^2}}(\alpha(x), \beta(x)) \in \mathbb{S}^1.$$

Then, Lemma 5.3.1 imples that

$$\operatorname{Ind}(\tilde{X}, \operatorname{Zero}(\tilde{X})) = \deg(\tilde{\mathcal{X}}|_\gamma + \deg(\tilde{\mathcal{X}}|_\xi)).$$

However, since $\tilde{X}|_\gamma = \mu_0 Y$ and $\tilde{X}|_\xi = -\mu_0 Y$, we have that $\tilde{\mathcal{X}}|_\gamma \equiv (1,0)$ and $\tilde{\mathcal{X}}|_\xi \equiv (-1,0)$. Therefore, $\deg(\tilde{\mathcal{X}}|_\gamma = 0$ and $\deg(\tilde{\mathcal{X}}|_\xi) = 0$, which implies that $\operatorname{Ind}(\tilde{X}, \operatorname{Zero}(\tilde{X})) = 0$, a contradiction. $\square$

6.4 The case of suspensions

The approach we effectively employed to solve Problem 1 in dimension two (see Proposition 6.3.1) was to replace X by $X - \varepsilon Y$, with the choice of ε giving by an appliction of Sard's Theorem (Lemma 6.3.3), and show that the new X cannot turn in all directions on a neighborhood of $\operatorname{Zero}(X)$ because ε and U can be chosen so that X is everywhere collinear with Y in U.

In this section we start to push further this approach to dimension three. A very simple case to "test" Problem 1 in dimension three is the case where Y is the suspension of a diffeomorphism f of some closed manifold, with constant roof function. In this case, the map f is a global return map and every point returns at the same time. This permits us to choose a normal component and a ratio function with very special properties and apply the above approach in an annalogus way. The result below emerged in a collaboration with Christian Bonatti and Sébasitien Alvarez which is still going on.

Theorem 6.4.1 (Alvarez-Bonatti-S.). *Let S be a closed surface and $f : S \to S$ a C^2 diffeomorphism. Let M_f be the suspension manifold and Y^f the suspension vector field. Assume that X is a C^2 vector fields which commutes with Y^f and that there exists a compact region $U \subset M_f$ such that* $\operatorname{Zero}(X) \cap \partial U = \emptyset$. *Then,* $\operatorname{Ind}(X, U) = 0$

Proof. The manifold M_f is a fiber bundle, with fiber S, over $\mathbb{S}^1$. Denote the fibers by S_t, for $t \in \mathbb{S}^1$. If $x \in S_t$ we consider the projection $N(x)$ of $X(x)$, parallel to $Y^f(x)$, over the fiber. In other other words, there exists a normal component N, tangent to the fibers, and a ration function $\mu : M_f \to \mathbb{R}$, both of class C^2. In particular, Equation 6.1 holds. The key point in the proof is the following.

Claim 7. *The levels sets $\mu^{-1}(c)$ are invariant under the flow of Y^f, for every $c \in \mathbb{R}$.*

Proof. Indeed, take $x \in \mu^{-1}(c)$ and fix $s \in \mathbb{R}$. We shall use Lemma 6.1.11 to prove that $Y_s^f(x) \in \mu^{-1}(c)$. Consider the fibers S_t and S_r, such that $x \in S_t$ and $Y_s^f(x) \in S_r$. Since Y^f is a suspension the transition time between to fibers S_t and S_r is constant over the fiber, and therefore Lemma 6.1.11 proves that $\mu(x) = \mu(\mathcal{P}(x))$, where $\mathcal{P}$ is the holonomy map from S_t to S_r. By noticing that $Y_s^f(x) = \mathcal{P}^n(x)$, for some integer n, the proof is complete. □

Apply Sard's Theorem to $\mu|_S$. Then, there exists $b > a > 0$, with b arbitrarily small, such that every $\varepsilon \in [a, b]$ is a regular value of $\mu|_S$. Up to replace X by $X - \frac{a+b}{2}Y$ we may assume that 0 is a regular value of $\mu|_S$. In particular, if $|\varepsilon| < \frac{a+b}{2}$ is fixed then $(\mu|_S)^{-1}(\varepsilon)$ is a union of finitely many C^2 curves, say $\gamma_1, ..., \gamma_m$.

Since S is closed, these curves are circles. By Claim 7 these circles are invariant under f and thus their saturation under the flow of Y^f is a disjoint union of finitely many tori $T_1^\varepsilon, ..., T_n^\varepsilon$, where n is independent of ε. Now, take any point $x \in \mu^{-1}(\varepsilon)$. Since Y^f is a suspension, there is an iterate $Y_t^f(x) \in S_0$. By Claim 7, $Y_t^f(x)$ belongs to one of the circles γ_j, $j = 1, ...m$. Therefore,

$$\mu^{-1}(\varepsilon) = \bigcup_{l=1}^{n} T_l^\varepsilon.$$

Notice that $T_l = \bigcup_{|\varepsilon|<\frac{a+b}{2}} T_l^\varepsilon$ is diffeomorphic to the product $[0, 1] \times \mathbb{T}^2$ and is a compact region containing $\mathrm{Zero}(X) \cap T_l^0$, whose boundary is the disjoint union $T_l^{-\frac{a+b}{2}} \cup T_l^{\frac{a+b}{2}}$. Since $\mathrm{Ind}(X, U) = \sum_{l=1}^n \mathrm{Ind}(X, T_l)$, it only remains to prove that $\mathrm{Ind}(X, T_l) = 0$ for every $l = 1, ...n$.

For this, we shall apply Lemma 5.3.1. Indeed, there exists a basis $\{e_1, e_2, e_3\}$ for the tangent bundle of T_l such that $e_3 = Y$ everywhere, e_2 is everywhere tangent to the levels and transverse to Y and e_1 is everywhere transverse to the the foliation of T_l by tori T_l^ε. We can write $X(x) = \alpha(x)e_1(x) + \beta(x)e_2(x) + \mu(x)e_3(x)$. For every $x \in T_l \setminus T_l^0$, we can define

$$\mathcal{X}(x) = \frac{1}{\sqrt{\alpha(x)^2 + \beta(x)^2 + \mu(x)^2}}(\alpha(x), \beta(x), \mu(x)) \in \mathbb{S}^2.$$

Lemma 5.3.1 shows that $\deg(\mathcal{X}|_{\partial T_l}) = \mathrm{Ind}(X, T_l)$. Finally, notice that $\mathcal{X}|_{T_l^{\frac{a+b}{2}}}$ is not surjective: since $\mu|_{T^{\frac{a+b}{2}}_l} \equiv \frac{a+b}{2}$, we have, in particular, that X cannot be equal to λY on $T_l^{\frac{a+b}{2}}$, for $\lambda < 0$ and thus the south pole $(-1, 0, 0)$ of $\mathbb{S}^2$ is not achieved by $\mathcal{X}|_{T_l^{\frac{a+b}{2}}}$. By the same argument, $\mathcal{X}|_{T_l^{-\frac{a+b}{2}}}$ is not surjective either. Thus, $\deg(\mathcal{X}|_{\partial T_l}) = 0$, which completes the proof. $\square$

Despite its proof being very easy, Theorem 6.4.1 reveals the main difficulties in trying to push the approach we have being adopting to dimension three. Sard's Theorem can give some control on the geometry of the colinearity locus, but to go further and determine $\mathrm{Ind}(X, U)$ one has invariably to control also the dynamics of Y.

Another possibility, still within this approach, is try to use improved versions of Sard's Theorem, if higher regularity is assumed, in order to obtain further control on the geometry of the colinearity locus. We shall discuss a little about this in the sequel.

6.5 Analytic commuting vector fields on 3-manifolds

As we mentioned in the introudction, Bonatti [Bo1] proved existence of common zeros for analytic commuting vector fields on manifolds of dimensions less or equal than four under the same topological assumptions of Corollary 6.3.2. Indeed, the main result of [Bo1] is the following

Theorem 6.5.1 (Bonatti)**.** *Let X and Y be two* analytic *commuting vector fields on M^d, for $d = 3, 4$. Let U be a compact region of M such that*

1. $\mathrm{Zero}(X) \cap \partial U = \emptyset$;
2. $\mathrm{Zero}(Y) \cap U = \emptyset$.

Then, $\mathrm{Ind}(X, U) = 0$.

As in Corollary 6.3.2, this result gives the existence of common zeros. In this section we shall comment the main ideas of the proof. We shall not enter in any detail since the techniques are very specific for analytic vector fields and our goal is actually try to improve them for make them work in low regularity. Thus, what shall concern us is try to point out the main difficulty of the paper [Bo1].

The first step of the proof is the lemma below, which does not assume that the vector fields are analytic (but the proof we present is the same found in the paper, see Lemme 1.c.1 of [Bo1]).

Lemma 6.5.2. *Let X and Y be two commuting vector fields over a d-dimensional manifold M. Assum that X and Y are of class C^d and that X and Y are everywhere collinear. Let $U \subset M$ be a compact region such that*

1. $\mathrm{Zero}(X) \cap \partial U = \emptyset$;
2. $\mathrm{Zero}(Y) \cap U = \emptyset$.

Then, $\mathrm{Ind}(X, U) = 0$.

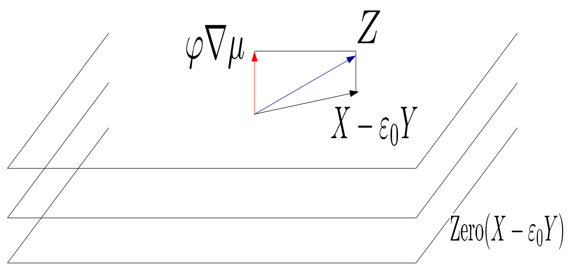


Figure 6.3: Creating a vector field close to X with no zeros

Proof. Let $\mu : U \to \mathbb{R}$ be a ratio function. Since X and Y are everywhere collinear the levels $\mu^{-1}(\varepsilon)$ are equal to $\mathrm{Zero}(X - \varepsilon Y)$ and thus are preserved by the flow of Y. Applying Sard's Theorem, which can be done since μ is C^d, one finds $b > a > 0$, with b as small as we please such that $\mu^{-1}(\varepsilon)$ is a compact codimension one submanifold of M, for every $\varepsilon \in [a, b]$. Moreover, since $\mathrm{Zero}(X) \cap \partial U = \emptyset$, by taking b sufficiently small, the levels $\mu^{-1}(\varepsilon)$, for $\varepsilon \in [a, b]$, are disjoint from ∂U. Let $\varepsilon_0 = \frac{a+b}{2}$. Consider the gradient[1] $\nabla\mu$ vector field and take a bump function $\varphi : U \to \mathbb{R}$ such that φ is zero outside a very small neighborhood V of $\mu^{-1}(\varepsilon_0)$ and $\varphi \equiv \delta$ in $\mu^{-1}(\varepsilon_0)$, for some $\delta > 0$ small. Let $Z = X - \varepsilon_0 Y + \varphi\nabla\mu$. Since $X - \varepsilon_0 Y$ is tangent to $\mu^{-1}(\varepsilon_0)$ and $\nabla\mu$ is transverse to all the levels, if V is small enough then Z has no zeros in U. Moreover, $Z = X - \varepsilon_0 Y$ at the boundary of U. Thus, choosing also δ very small, one obtains a vector field Z, close to X, with no zeros inside U, proving that $\mathrm{Ind}(X, U) = 0$. $\square$

Remark 6.5.3. We do not know how to prove Lemma 6.5.2 in C^1 regularity, though this should certainly be true.

Let us now survey the rest of the proof of Theorem 6.5.1 in dimension three. Lemma 6.5.2, enables Bonatti to assume that the colinearity locus is an analytic set of dimension at most two. In this case, using techniques of stratification for analytic sets, he proves that it suffices to consider the case where the colinearity locus is an annulus $[-\mu_0, \mu_0] \times \mathbb{S}^1$ and each circle $\{t\} \times \mathbb{S}^1$ corresponds to $\mathrm{Zero}(X - tY)$ and is a periodic orbit of Y, precisely as in the conclusion of Lemma 6.3.3.

In the final step of the proof, Bonatti uses once more techniques from complex analysis to show that the plane field defined by X and Y, outside the colinearity locus can be extended to a neighborhood of $\mathrm{Zero}(X)$, excluding at most finitely many circles $\{t\} \times \mathbb{S}^1$. With this plane field, he performs a construction of a vector field Z, close to X with no zeros on a compact region U containing $\mathrm{Zero}(X)$ in its interior, and thus concluding that $\mathrm{Ind}(X, U) = 0$. The construction of Z is very similar to the construction we presented above in the proof of Lemma 6.5.2.

In view of this, a natural attempt is try to improve Bonatti's paper by showing that its main difficulty can be solved in the C^1 setting. This is the content of next chapter.

[1]with respect to the some riemannian metric of M.

Chapter 7

Existence of common zeros for commuting vector fields on 3-manifolds

What is the difficulty to solve Problem 1 in dimension three? As we saw in Section 6.3 what we have to prove is a statement analogous to that of Proposition 6.3.1, that is

Conjecture 1. *Let M be a 3-manifold. Let $X, Y \in \mathfrak{X}^1(M)$ be two commuting vector fields and $U \subset M$ a compact region such that $\mathrm{Zero}(Y) \cap U = \mathrm{Zero}(X) \cap \partial U = \emptyset$. Then,*

$$\mathrm{Ind}(X, U) = 0.$$

Let us approach Conjecture 1 by contradiction. Take $X, Y \in \mathfrak{X}^1(M)$ two commuting vector fields and $U \subset M$ a compact region of the 3-manifold M and assume that $\mathrm{Zero}(Y) \cap U = \mathrm{Zero}(X) \cap \partial U = \emptyset$, but $\mathrm{Ind}(X, U) \neq 0$.

We would like to use the dynamics of Y, and its holonomies, to reach a contradiction with $\mathrm{Ind}(X, U) \neq 0$, inspired by the proof of Proposistion 6.3.1 that we gave in Section 6.3.1.

So, take a plane field P over U, everywhere transverse to Y and let N and μ be, respectively, the associated normal component ratio function. Assume that X and Y are C^3, so that we can apply Sard's Theorem to μ, as we did in the proof of Theorem 6.4.1. Thus, one can find an interval $[a, b] \subset \mathbb{R}$, with $b > a > 0$ and b as small as we please so that every

$\varepsilon \in [a, b]$ is a regular value of μ. Then, the level sets $\mu^{-1}(\varepsilon)$ are smooth surfaces, embedded in M and they comprise a foliation of some compact region V of M. Moreover, since μ is a ratio function

$$\text{Zero}(X - \varepsilon Y) \subset \mu^{-1}(\varepsilon).$$

In Theorem 6.4.1, the levels where tangent to Y, and this allowed us to quickly conclude that X has index zero on V, just using that it is invariant under the tangent flow of Y.

However, in general, the levels are not invariant by Y. Therefore, one of the difficulties to prove Conjecture 1 is to analyze the dynamics of Y near to $\text{Zero}(X - \varepsilon Y)$, when these sets are (invariant) subsets of a non-invariant foliation by (non-invariant) surfaces, possibly with boundary.

In the analytic case, as we saw in Section 6.5 there is a more powerful version of Sard's Theorem which makes it possible to assume that, though the levels are not invariant by Y, there exists a surface S, everywhere transverse to the levels and containing $\cup_{\varepsilon\in[a,b]} \text{Zero}(X - \varepsilon Y)$. To prove Conjecture 1 under this extra assumption was the main difficulty in the analytic case [Bo1].

Thus, a good (and non trival) "test" for Conjecture 1 is to tackle the particular case when X and Y are not analytic, but this extra assumption is satisfied. Even though to assume that the colinearity locus is confined to a closed submanifold of M is not a realistic assumption when X and Y are C^1, it can be seen as a first step towards Conjecture 1 since the techniques used in the analitic case are not at all available in the C^1 setting.

In this chapter we shall see how we can effectively use the dynamics of Y and its holonomies to show that X cannot, at the same time, have a non-zero index and be invariant under the tangent flow of Y. The result, which was a collaboration with Christian Bonatti [BS], is the following.

Theorem D (Bonatti, S)**.** *Let M be a 3-manifold and X and Y be two C^1 commuting vector fields over M. Let U be a compact subset of M such that $\text{Zero}(Y) \cap U = \text{Zero}(X) \cap \partial U = \emptyset$. Assume that $\text{Col}(X, Y, U)$ is contained in a C^1-surface which is a closed submanifold of M. Then,*

$$\text{Ind}(X, U) = 0.$$

The rest of this chapter is devoted to the proof of Theorem D.

7.1 Prepared counter examples

Our proof is a long proof by contradiction. The idea is to try to push forward the approach we presented in Section 6.3.1. The first step is to symplify the colinearity locus using Sard's Theorem.

This simplification involves making successive replacements of counter examples until we find a counter example with special geometric configurations (the prepared counter examples that we shall introduce below) and then, in this special counter examples, we analyse the dynamics of Y and how it influences the topological behaviour of X.

For the sake of clarity, it will be convinent to formally introduce the notion of counter examples.

Definition 7.1.1. Let M be a 3-manifold, U a compact subset of M and X, Y be C^1 vector fields on M. We say that (U, X, Y) is a *counter example to Theorem D* if

- X and Y commute
- $\mathrm{Zero}(Y) \cap U = \emptyset$
- $\mathrm{Zero}(X) \cap \partial U = \emptyset$
- $\mathrm{Ind}(X, U) \neq 0$.
- the collinearity locus, $\mathrm{Col}(X, Y, U)$, is contained in a C^1 surface which is a closed submanifold of M.

Let us illustrate our simplifying procedure by a simple argument:

Remark 7.1.2. If M is a 3-manifold carrying a counter example (U, X, Y) to Theorem D, then there is an orientable manifold carrying a counter example to Theorem D. Indeed, consider the orientation double cover $\tilde{M} \to M$ and $\tilde{U}, \tilde{X}, \tilde{Y}$ the lifts of U, X, Y on $\tilde{M}$. Then the Poincaré-Hopf index of $\tilde{X}$ on $\tilde{U}$ is twice the one of X on U, and $(\tilde{U}, \tilde{X}, \tilde{Y})$ is a counter example to Theorem D.

Thus we can assume (and we do it) without loss of generality that M is orientable.

Most of our simplifying strategy will now consist in combinations of the following remarks

Remark 7.1.3. If (U, X, Y) is a counter example to Theorem D, then there is $\varepsilon > 0$ so that $(U, aX + bY, cX + dY)$ is also a counter example to Theorem D, for every a, b, c, d with $|a - 1| < \varepsilon$, $|b| < \varepsilon$, $|c| < \varepsilon$ and $|d - 1| < \varepsilon$.

Remark 7.1.4. If (U, X, Y) is a counter example to Theorem D, then (V, X, Y) is also a counter example to Theorem D for any compact set $V \subset U$ containing $\mathrm{Zero}(X, U)$ in its interior.

Remark 7.1.5. Let (U, X, Y) be a counter example to Theorem D and assume that $\mathrm{Zero}(X, U) = K_1 \cup \cdots \cup K_n$, where the K_i are pairwise disjoint compact sets. Let $U_i \subset U$ be compact neighborhood of K_i so that the U_i, $i = 1, \ldots, n$, are pairwise disjoint.

Then there is $i \in \{1, \ldots, n\}$ so that (U_i, X, Y) is a counter example to Theorem D.

We introduce now the counter examples with the special configuration that will permit us to relate the topological behaviour of X with the dynamics of Y and its holonomies.

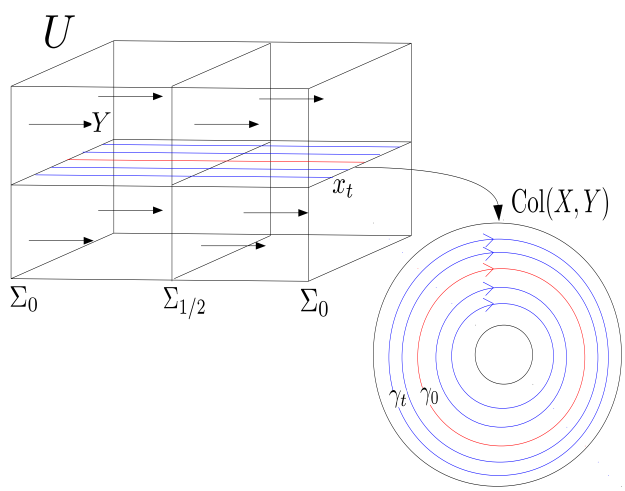


Figure 7.1: A prepared counter example to Theorem D. The red line is $\mathrm{Zero}(X)$ and the blue lines are $\mathrm{Zero}(X - tY)$, with $t \neq 0$. The left and the right side of the cube are identified.

Definition 7.1.6. We say that $(U, X, Y, \Sigma, \mathcal{B})$ is a *prepared counter example to Theorem D* if

1. (U, X, Y) is a counter example to Theorem D

2. There is $\mu_0 > 0$ so that (U, X, Y) satisfies the conclusion of Lemma 6.3.3:

   - for any $t \in [-\mu_0, \mu_0]$, the set of zeros of $X - tY$ in U consists precisely in 1 periodic orbit γ_t of Y;
   - for any $t \notin [-\mu_0, \mu_0]$, the set of zeros of $X - tY$ in U is empty;
   - $\mathrm{Col}(X, Y, U)$ is a C^1 annulus;
   - there is a C^1-diffeomorphism $\varphi\colon \mathbb{R}/\mathbb{Z} \times [-\mu_0, \mu_0] \to \mathrm{Col}(X, Y, U)$ so that, for every $t \in [-\mu_0, \mu_0]$, one has
   $$\varphi(\mathbb{R}/\mathbb{Z} \times \{t\}) = \gamma_t.$$

3. U is endowed with a foliation by discs; more precisely there is a smooth submersion $\Sigma\colon U \to \mathbb{R}/\mathbb{Z}$ whose fibers $\Sigma_t = \Sigma^{-1}(t)$ are discs; furthermore, the vector field Y is transverse to the fibers Σ_t.

4. Each periodic orbit γ_s, $s \in [-\mu_0, \mu_0]$, of Y cuts every disc Σ_t in exactly one point. In particular the period of γ_s coincides with its return time on Σ_0 and is denoted $\tau(s)$, for $s \in [-\mu_0, \mu_0]$.

   Thus $s \mapsto \tau(s)$ is a C^1-map on $[-\mu_0, \mu_0]$. We require that the derivative of τ does not vanish on $[-\mu_0, \mu_0]$.

5. $\mathcal{B}$ is a triple (e_1, e_2, e_3) of C^0 vector fields on U so that

   - for any $x \in U$ $\mathcal{B}(x) = (e_1(x), e_2(x), e_3(x))$ is a basis of T_xU.
   - $e_3 = Y$ everywhere
   - the vectors e_1, e_2 are tangent to the fibers Σ_t, $t \in \mathbb{R}/\mathbb{Z}$. In other words, $D\Sigma(e_1) = D\Sigma(e_2) = 0$
   - The vector e_1 is tangent to $\mathrm{Col}(X, Y)$ at each point of $\mathrm{Col}(X, Y)$.

Lemma 7.1.7. *If there exists a counter example (U, X, Y) to Theorem D then there is a prepared counter example $(\tilde{U}, \tilde{X}, \tilde{Y}, \Sigma, \mathcal{B})$ to Theorem D.*

Proof. The two first items of the definition of a prepared counter example to Theorem D are given by Lemma 6.3.3. For getting the third item, it is enough to shrink U. For getting item (4), one replace Y by $Y + bX$ for some $b \in \mathbb{R}$, with $|b|$ small enough. This does not change the orbits γ_t, as X and Y are both tangent to γ_t, but it changes its period. Thus, this allows us to change the derivative of the period τ at $s = 0$. Then one shrink again U and μ_0 so that the derivative of τ will not vanish on $\mathrm{Col}\,(X, Y, U)$. $\square$

Remark 7.1.8. If $(U, X, Y, \Sigma, \mathcal{B})$ is a prepared counter example to Theorem D, then for every $t \in (-\mu_0, \mu_0)$, $(U, X - tY, Y, \Sigma, \mathcal{B})$ is a prepared counter example to Theorem D.

Whenever $(U, X, Y, \Sigma, \mathcal{B})$ is a prepared counter example to Theorem D, we shall denote by $\mathcal{P}$ the *first return map*, defined on a neighborhood of $\mathrm{Col}(X, Y) \cap \Sigma_0$ in Σ_0.

Remark 7.1.9. As the ambient manifold is assumed to be orientable (see Remark 7.1.2), the vector field Y is normally oriented so that the Poincaré map $\mathcal{P}$ preserves the orientation.

7.1.1 Counting the index of a prepared counter example

Definition 7.1.10. Let $(U, X, Y, \Sigma, \mathcal{B})$ be a prepared counter example to Theorem D. In particular, U is a solid torus (C^1-diffeomorphic to $\mathbb{D}^2 \times \mathbb{R}/\mathbb{Z}$) and $\mathrm{Zero}(X - tY)$, $t \in (-\mu_0, \mu_0)$, is an essential simple curve γ_t isotopic to $\{0\} \times \mathbb{R}/\mathbb{Z}$. An *essential torus* T is the image of a continuous map from the torus $\mathbb{T}^2$ in the interior of U, disjoint from $\gamma_0 = \mathrm{Zero}(X)$ and homotopic, in $U \setminus \gamma_0$, to the boundary of a tubular neighborhood of γ_0.

In other words, $H_2(U \setminus \gamma_0, \mathbb{Z}) = \mathbb{Z}$, and T is essential if it is the generator of this second homology group.

We shall now describe how we use the basis $\mathcal{B}$, which comes with a prepared counter example, and an essential torus T to calculate the index.

Consider, for each point $x \in U$, the expression of X in the basis $\mathcal{B}$:

$$X(x) = \alpha(x)e_1(x) + \beta(x)e_2(x) + \mu(x)e_3(x). \tag{7.1}$$

For $x \notin \gamma_0$ one considers the vector

$$\mathcal{X}(x) = \frac{1}{\sqrt{\alpha(x)^2 + \beta(x)^2 + \mu(x)^2}} \left(\alpha(x), \beta(x), \mu(x)\right) \in \mathbb{S}^2. \tag{7.2}$$

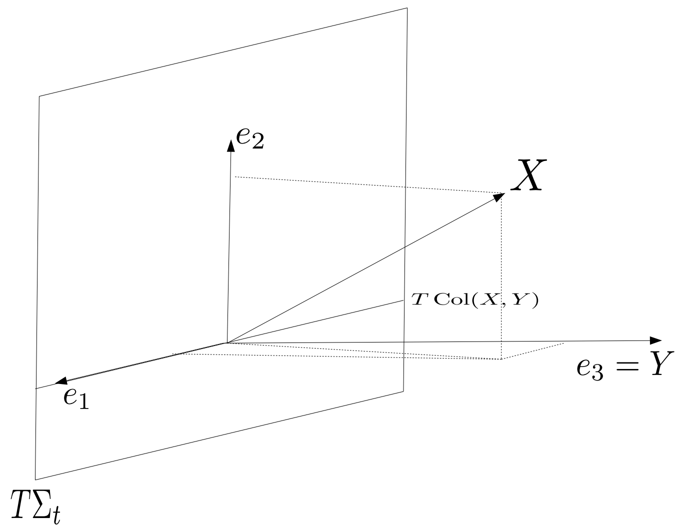


Figure 7.2: $X(x) = \alpha(x)e_1(x) + \beta(x)e_2(x) + \mu(x)e_3(x)$.

The map restriction $\mathcal{X}|_T : T \to \mathbb{S}^2$ has a topological degree, which, by Lemma 5.3.1, coincides with $\mathrm{Ind}(X, U)$, for some choice of an orientation on T. In particular, we have the following.

Lemma 7.1.11. *Let $(U, X, Y, \Sigma, \mathcal{B})$ be a prepared counter example to Theorem D. Let T be an essential torus in U. Then,*

$$\deg(\mathcal{X}|_T) \neq 0.$$

Our goal is to show that Lemma 7.1.11 leads to a contradiction.

The search for this contradiction is splitted in three different cases, according to the derivative of the first return map $\mathcal{P} : \Sigma_0 \to \Sigma_0$. Indeed, notice that $\mathcal{P}$ has a line of fixed points, namely $x_t = \gamma_t \cap \Sigma_0$, for $t \in [-\mu_0, \mu_0]$. Thus $1 \in \mathrm{spec}(D\mathcal{P}(x_t))$, for every $t \in (-\mu_0, \mu_0)$, where $\mathrm{spec}(L)$ stands for the set of eigenvalues of a linear map L. We can distinguish three cases.

1. **Partially Hyperbolic Case:** there exists $t \in (-\mu_0, \mu_0)$ such that $\mathrm{spec}(D\mathcal{P}(x_t)) = \{1, \lambda\}$, where $\lambda \in \mathbb{R} \setminus \{1\}$. We will show in Lemma 7.3.1 that this case cannot occur.

2. **Shear Case:** there exists $t \in (-\mu_0, \mu_0)$ such that $\mathrm{spec}(D\mathcal{P}(x_t)) = \{1\}$, but $D\mathcal{P}(x_t) \neq Id$. We prove that this case is not possible in Proposistion 7.5.2

3. **Identity Case:** $D\mathcal{P}(x_t) = Id$, for every $t \in (-\mu_0, \mu_0)$. The final contradiction will be achieved in Lemma 7.6.10.

Our main tool an all of the above three cases is a formula (see Proposistion 7.2.6) that will reduce the calculation of $\deg(\mathcal{X}|_T)$ to an understanding the topological behaviour of the normal component, which is a two dimensional vector field. This dimension reduction will be performed in the next section.

Remark 7.1.12. In a prepared counter example, the tangent planes of foliation by discs Σ_t, being everywhere transverse to Y, induces a prefered normal component and a ratio function (see Section 6.1.4 for the definitions). Since $e_3 = Y$ everywhere, the e_3-coordinate μ of X is precisely this ratio function.

7.2 Linking numbers

In the whole section, $(U, X, Y, \Sigma, \mathcal{B})$ is a prepared counter example to Theorem D.

7.2.1 Notations

In this section we fix our notation for holonomies and transition times in a prepared counter example.

Recall that $\Sigma_t = \Sigma^{-1}(t)$, $t \in \mathbb{R}/\mathbb{Z}$, is a family of cross section, each Σ_t is diffeomorphic to a disc, and we identify Σ_0 with the unit disc $\mathbb{D}^2$.

Definition 7.2.1. Consider $t \in \mathbb{R}$. Consider $x \in \Sigma_0$ and $y \in \Sigma_t$. We say that y is *the image by holonomy of Y over the segment* $[0, t]$, and we denote $y = P_t(x)$, if there exists a continuous path $x_r \in U$, $r \in [0, t]$, so that $\Sigma(x_r) = r$, $x_0 = x$, $x_t = y$, and for every $r \in [0, t]$ the point x_r belongs to the Y-orbit of x.

The holonomy map P_t is well defined in a neighborhood of $\mathrm{Col}(X, Y, U) \cap \Sigma_0$ and is a C^1 local diffeomorphism.

If $t = 1$ then P_1 is the first return map $\mathcal{P}$ (defined before Remark 7.1.9) of the flow of Y on the cross section Σ_0.

Remark 7.2.2. With the notation of Definition 7.2.1, there is a unique continuous function $\tau_x\colon [0,t] \to \mathbb{R}$ so that $\tau_x(0) = 0$ and $x_r = Y_{\tau_x(r)}(x)$ for every $r \in [0,t]$.

We denote $\tau_t(x) = \tau_x(t)$ and we call it the *transition time from* Σ_0 *to* Σ_t. The map $\tau_t\colon \Sigma_0 \to \mathbb{R}$ is a C^1 map and by definition one has

$$P_t(x) = Y_{\tau_t(x)}(x) \tag{7.3}$$

We denote $\tau = \tau_1$ and we call it the first return time of Y on Σ_0.

Remark 7.2.3. In Definition 7.1.6 item 4 we defined $\tau(s)$ as the period of γ_s; in the notation above, it coincides with $\tau(x_s)$ where $x_s = \gamma_s \cap \Sigma_0$.

In this case, Equation 7.3 takes the special form

$$\mathcal{P}(x) = Y_{\tau(x)}(x) \tag{7.4}$$

7.2.2 The normal conponent

We introduce a normal component for the prepared counter example.

Definition 7.2.4. For every t and every $x \in \Sigma_t$ we define *the normal component of* X, which we denote by $N(x)$, the projection of $X(x)$ on $T_x\Sigma_t$ parallel to $Y(x)$.

Thus $x \mapsto N(x)$ is a C^1-vector field tangent to the fibers of Σ and which vanishes precisely on $\mathrm{Col}(X, Y, U)$.

Moreover, in the basis $\mathcal{B}$, $N(x) = \alpha(x)e_1(x) + \beta(x)e_2(x)$ (see Equation 7.1), and we have the following formula

$$X(x) = N(x) + \mu(x)Y(x),$$

for every $x \in U$.

7.2.3 The normal component of X and the index of X

Let $(U, X, Y, \Sigma, \mathcal{B})$ be a prepared counter example to Theorem D and N be the normal component of X.

Let us define, for $x \in U \setminus \mathrm{Col}\,(X, Y, U)$,

$$\mathcal{N}(x) = \frac{1}{\sqrt{\alpha(x)^2 + \beta(x)^2}}\,(\alpha(x), \beta(x)) \in \mathbb{S}^1 \subset \mathbb{S}^2,$$

where $\mathbb{S}^1$ is the unit circle of the plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$.

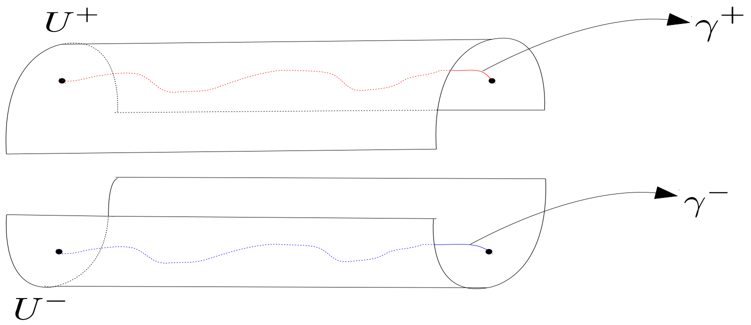


Figure 7.3: Linking numbers

Recall that U is homeomorphic to the solid torus so that its first homology group $H_1(U, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$, by an isomorphism sending the class of γ_0, oriented by Y, on 1. Let U^+ and U^- be the two connected components of $U \setminus \mathrm{Col}\,(X, Y, U)$. These are also solid tori, and the inclusion in U induces isomorphisms of the first homology groups which allows us to identify $H_1(U^\pm, \mathbb{Z})$ with $\mathbb{Z}$.

The continuous map $\mathcal{N}\colon U^\pm \to \mathbb{S}^1$ induces morphisms on the homology groups $H_1(U^\pm, \mathbb{Z}) \to H_1(\mathbb{S}^1, \mathbb{Z})$. As these groups are all identified with $\mathbb{Z}$, these morphisms consist in the multiplication by an integer $\ell^\pm(X, Y)$. In other words, consider a closed curve $\gamma^\pm \subset U^\pm$ homotopic in U to γ_0. Then $\ell^\pm(X, Y)$ is the topological degree of the restriction of $\mathcal{N}$ to $\gamma^\pm$. See Figure 7.3.

Definition 7.2.5. The integer $\ell^+(X, Y)$ is called *linking number* of X with respect to Y in U^+, and the integer $\ell^-(X, Y)$ is the linking number of X with respect to Y in U^-.

Proposition 7.2.6. *Let $(U, X, Y, \Sigma, \mathcal{B})$ be a prepared counter example to Theorem D. Then*

$$|\operatorname{Ind}(X, U)| = |\ell^+(X, Y) - \ell^-(X, Y)|$$

Proof. Consider a tubular neighborhood of γ_0 whose boundary is an essential torus T which cuts $\mathrm{Col}\,(X, Y, U)$ transversely and along exactly two curves σ_+ and σ_-. Then the map $\mathcal{X}$ on T takes the value $N \in \mathbb{S}^2$ (resp. $S \in \mathbb{S}^2$) exactly on σ_+ (resp. σ_-), where N and S are the points on $\mathbb{S}^2$ corresponding to $e_3 = Y$ and $-e_3$.

We identify T with $\mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ so that σ_- and σ_+ correspond to $\{\frac{1}{2}\} \times \mathbb{R}/\mathbb{Z}$ and $\{0\} \times \mathbb{R}/\mathbb{Z}$ respectively. It remains to apply Proposition 5.5.1 to $\Phi = \mathcal{X}$ and $\varphi = \mathcal{N}$, noticing that $\mathcal{N}$ is the projection of $\mathcal{X}$ on $\mathbb{S}^1$ along the meridians. This gives the announced formula. $\square$

As a direct consequence of Proposition 7.2.6 one gets

Corollary 7.2.7. *Let $(U, X, Y, \Sigma, \mathcal{B})$ be a prepared counter example to Theorem D, then*

$$(\ell^+(X, Y), \ell^-(X, Y)) \neq (0, 0)$$

Proposistion 7.2.6 is our main thechnique to perform index calculations. In fact, it is one of the main novelties in our work [BS]. Moreover, Corollary 7.2.7 gives information about the topological behaviour of the normal component. Our strategy in the next section will be to combine this information with dynamical information about the first return map $\mathcal{P}$. These dynamical data have, of course, a different nature in each one of the three cases in which we splitted the proof.

7.3 The Partially Hyperbolic Case

In this section we deal with in the Partially Hyperbolic Case. As we shall see in the proof, this case enables us to assume that $\mathrm{Col}(X, Y, U)$ is normally hyperbolic for the flow of Y. The main idea is that X has to preserve the stable manifolds and thus the normal component is not permited to turn, making both linking numbers $\ell^\pm$ vanish, a contradiction with Corollary 7.2.7.

The main result of this section is the following.

Lemma 7.3.1. *Let* $(U, X, Y, \Sigma, \mathcal{B})$ *be a prepared counter example to Theorem D. Then, the first return map* $\mathcal{P}\colon \Sigma_0 \to \Sigma_0$ *of the flow of* Y *satisfies: for every point* x_t *(of* $\mathrm{Col}(X, Y, U) \cap \Sigma_0$*) the unique eigenvalue of the derivative of* $\mathcal{P}$ *at* x_t *is* 1.

Proof. The argument is by contradiction. We assume that the derivative of $\mathcal{P}$ at some point of $\mathrm{Col}(X, Y) \cap \Sigma_0$ has some eigenvalue of different from 1.

Recall that we denote $x_t = \gamma_t \cap \Sigma_0$, $t \in [-\mu_0, \mu_0]$, where $\gamma_t = \mathrm{Zero}(X - tY))$. Notice that the first return map $\mathcal{P}$ is the identity map in restriction to the segment $\{x_t\}_{t\in(-\mu_0,\mu_0)} = \mathrm{Col}(X, Y) \cap \Sigma_0$. In particular, the derivative of $\mathcal{P}$ at x_t admits 1 as an eigenvalue. Since $\mathcal{P}$ preserves the orientation (see Remark 7.1.9), the other eigenvalue is positive.

Claim 8. *There exists* $\tilde{U} \subset U$, $\tilde{X} = X - tY$ *and a prepared counter example to Theorem D* $(\tilde{U}, \tilde{X}, Y, \Sigma, \mathcal{B})$ *for which the surface* $\mathrm{Col}(\tilde{X}, Y, \tilde{U})$ *is normally hyperbolic.*

Proof. As the property of having a eingenvalue of mudulus different from 1 is an open condition, there exists an interval $[\mu_1, \mu_2] \subset [-\mu_0, \mu_0]$ on which the condition holds. Consider $t = \frac{\mu_1+\mu_2}{2}$ and $\tilde{X} = X - tY$.

Then, one obtains a new prepared counter example to Theorem D by replacing X by $\tilde{X}$; Now, by shrinking U one gets a tubular neighborhood $\tilde{U}$ of γ_t so that $\mathrm{Col}(\tilde{X}, Y, \tilde{U}) = \bigcup_{s\in[\mu_1,\mu_2]} \gamma_s$.

Moreover, the derivative of $\mathcal{P}$ at each point x_s, $s \in [\mu_1, \mu_2]$, has an eigenvalue different from 1 in a direction tranverse to $\mathrm{Col}(\tilde{X}, Y, \tilde{U}) \cap \Sigma_0$. By compactness and continuity these eigenvalues are uniformly far from 1 so that $\mathrm{Col}(\tilde{X}, Y, \tilde{U}) \cap \Sigma_0$ is normally hyperbolic for $\mathcal{P}$.

Thus $\mathrm{Col}(\tilde{X}, Y, \tilde{U})$ is an invariant normally hyperbolic annulus for the flow of Y. □

By virtue of the above claim (up to change X by $\tilde{X}$ and U by $\tilde{U}$) one may assume that $\mathrm{Col}(X, Y, U)$ is normally hyperbolic, and (up change Y by $-Y$) one may assume that $\mathrm{Col}(X, Y, U)$ is normally contracting.

This implies that every periodic orbit γ_t has a local stable manifold $W^s(\gamma_t)$ which is a C^1-surface depending continuously on t for the C^1-topology and the collection of these surfaces build a $\mathcal{C}^0$-foliation $\mathcal{F}^s$ tangent to a continuous plane field E^s, in a neighborhood of $\mathrm{Col}(X, Y, U)$. Furthermore, E^s is tangent to Y, and hence is tranverse to the fibers of Σ.

Up to shrink U, one may assume that $\mathcal{F}^s$ and E^s are defined on U.

Claim 9. *There is a basis $\tilde{\mathcal{B}} = (\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ so that $(U, X, Y, \Sigma, \tilde{\mathcal{B}})$ is a prepared counter example to Theorem D and $\tilde{e}_2$ is tangent to E^s.*

Proof. Choose $\tilde{e}_2$ as being a unit vector tangent to the intersection of E^s with the tangent plane of the fibers of Σ. It remains to choose $\tilde{e}_1$ tranverse to e_2 and tangent to the fibers of Σ and tangent to $\mathrm{Col}(X, Y)$ at every point of $\mathrm{Col}(X, Y)$. □

Up to change $\mathcal{B}$ by the basis $\tilde{\mathcal{B}}$ given by the claim above, we will now assume that e_2 is tangent to E^s.

Claim 10. *The vector field X is tangent to E^s.*

Proof. The flow of the vector field X leaves invariant the periodic orbit γ_t of Y, since X is collinear with Y along γ_t, and X commutes with Y. As a consequence, it preserves the stable manifold $W^s(\gamma_t)$ (see Lemma 6.1.9) for every t. This implies that X is tangent to the foliation $\mathcal{F}^s$ and therefore to E^s. □

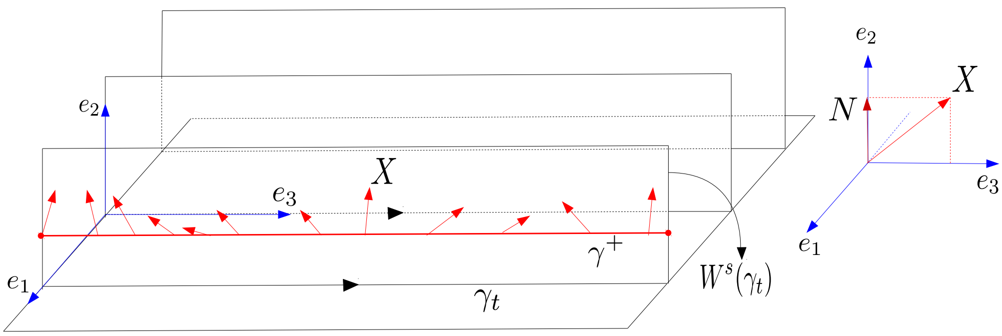


Figure 7.4: X is confined within the stable manifolds of the periodic orbits, and thus cannot turn.

To complete the argument we shall prove that both linking numbers $\ell^{\pm}(X, Y)$ vanish. Take a closed curve γ^+, disjoint of γ_0 and contained in $U^+ \cap W^s(\gamma_t)$, for some t. Combining Claims 10 and 9 we conclude that the e_1 coordinate of X is zero along γ^+. Since X is never collinear with $e_3 = Y$ along γ^+, we conclude that the map $\mathcal{N}|_{\gamma^+}$ is constant and thus has zero topological degree. This proves that $\ell^+(X,Y) = 0$. A similar argument proves that $\ell^-(X,Y) = 0$. This finishes the proof. □

7.4 Angular variation of the normal component N

In this section we shall establish an important thechnical result about the angular variation of the normal component along some straight lines segments in Σ_0. We will prove that N has to turn at least once on $\mathbb{S}^1$ along segemtns of form $[x, \mathcal{P}^2(x)]$. As we consequence, for every prescribed direction in $T\Sigma_0$ there will be a point q in $[x, \mathcal{P}^2(x)]$ such that $N(q)$ is pointing in this direction.

We will combine latter this information with the local dynamics of $\mathcal{P}$ to reach a contradiction. This will be done by different arguments in the Shear Case and in the Identity Case.

It is important to remark that the results in this section are independent of Lemma 7.3.1. We prefer to present them after Lemma 7.3.1 because the Partially Hyperbolic Case, being the simplest one, also serves as an illustration of our approach.

Recall that we denote $\{x_t\} = \gamma_t \cap \Sigma_0$. For every pair of points $x, y \in \Sigma_0$, we denote the segment of straight line joinning x and y and contained in Σ_0 by $[x, y]$ (for some choice of coordinates on Σ_0).

Lemma 7.4.1. *For any $K > 0$ there is a neighborhood O_K of γ_0 with the following property.*

Consider $x \in O_K \cap \Sigma_0$ and $u \in T_x\Sigma_0$ a unit vector, and write $u = u_1 e_1(x) + u_2 e_2(x)$. Consider $t \in [0, K]$ and $v = DP_t(u) \in T_{P_t(x)}\Sigma_t$ the image of u by the derivative of the holonomy. Write $v = v_1 e_1(P_t(x)) + v_2 e_2(P_t(x))$.

Then

$$\left(\frac{u_1}{\sqrt{u_1^2+u_2^2}}, \frac{u_2}{\sqrt{u_1^2+u_2^2}}\right) \neq -\left(\frac{v_1}{\sqrt{v_1^2+v_2^2}}, \frac{v_2}{\sqrt{v_1^2+v_2^2}}\right)$$

Proof. Assuming, by contradiction, that the conclusion does not hold we get $y_n \in \Sigma_0$, unit vectors $u_n \in T_{x_n}\Sigma_0$ and $t_n \in [0, K]$ so that y_n tends to $x_0 = \gamma_0 \cap \Sigma_0$, u_n tends to a unit vector u in $T_{x_0}\Sigma_0$, t_n tends to $t \in [0, K]$ and the image v_n of the vector u_n, expressed in the basis $\mathcal{B}$, is collinear to u_n with the opposite direction.

Then $DP_t(x_0)u$ is a vector that, expressed in the basis $\mathcal{B}$, is collinear to u with the opposit direction. In other words, u is an eigenvector of $DP_t(x_0)$, with a negative eigenvalue.

However, for every $t \in \mathbb{R}$ the vector e_1 is an eigenvector of $DP_t(x_0)$, with a positive eigenvalue, and $DP_t(x_0)$ preserves the orientation, leading to a contradiction. $\square$

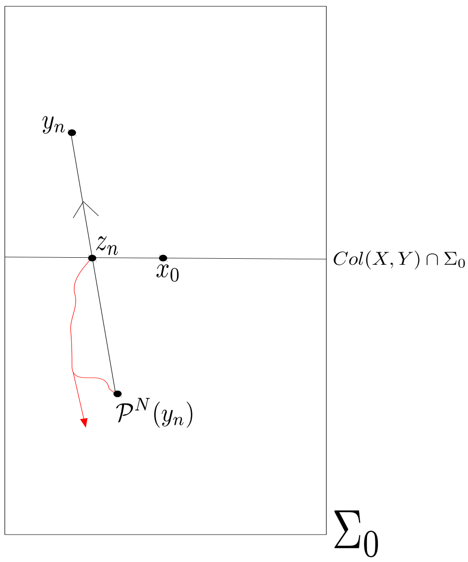


Figure 7.5: Proof of Lemma 7.4.2: the red curve is the image of $[z_n, y_n]$ under $\mathcal{P}^N$. By the mean value theorem there must be a tangent vector to this red curve parallel to the segment $[z_n, \mathcal{P}^N(y_n)]$

Lemma 7.4.2. *For every $N \in \mathbb{N}$ there exists a neighborhood $O_N \subset \Sigma_0$ of x_0 so that if $x \in O_N \setminus \mathrm{Col}(X, Y, U)$ then the segment of straight line $[x, \mathcal{P}^N(x)]$ is disjoint from $\mathrm{Col}(X, Y, U)$.*

Proof. Assume that there is a sequence of points $y_n \to x_0$, $y_n \notin \mathrm{Col}(X, Y, U)$ so that the segment $[y_n, \mathcal{P}^N(y_n)]$ intersect $\mathrm{Col}(X, Y, U) \cap \Sigma_0$ at some point z_n. Recall that $\mathrm{Col}(X, Y, U) \cap \Sigma_0$ consists in fixed points of the first return map $\mathcal{P}$. In particular, z_n is a fixed point of $\mathcal{P}^N$.

The image of the segment $[z_n, y_n]$ is a C^1 curve joining z_n to $\mathcal{P}^N(y_n)$. Notice that the segments $[z_n, y_n]$ and $[z_n, \mathcal{P}^N(y_n)]$ are contained in the segment $[y_n, \mathcal{P}^N(y_n)]$ and oriented in opposite direction. One deduces that there is a point w_n in $[y_n, z_n]$ so that the image under the derivative $D\mathcal{P}^N$ of the unit vector u_n directing the segment $[z_n, y_n]$ is of form $\lambda_n u_n$ with $\lambda_n < 0$. See Figure 7.5 below.

Since y_n tends to x_0, one deduces that $D\mathcal{P}^N(x_0)$ has a negative eigenvalue. This contradicts the fact that both eigenvalues of $D\mathcal{P}^N(x_0)$ are positive and completes the proof. $\square$

Recall that U^+ and U^- are the connected components of $U \setminus \mathrm{Col}\,(X, Y, U)$.

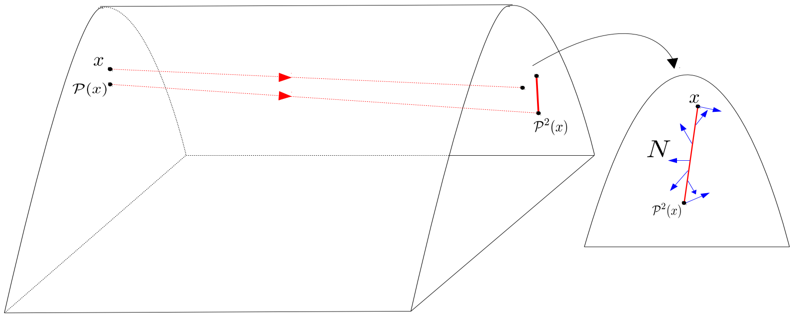


Figure 7.6: Corollary 7.4.3.

Corollary 7.4.3. *If $x \in U^{\pm} \cap \Sigma_0 \cap O_3$ let $\theta_x : \mathbb{R}/\mathbb{Z} \to U$ be the curve obtained by concatenation of the Y-orbit segment from x to $\mathcal{P}^2(x)$ and the straight line segment $[\mathcal{P}^2(x), x]$, that is:*

- *for $t \in [0, \frac{1}{2}]$, $\theta_x(t) = Y_{2t.\tau_2(x)}(x)$ where τ_2 is the transition time from x to $\mathcal{P}^2(x)$*
- *for $t \in [\frac{1}{2}, 1]$, $\theta_x(t) = (2 - 2t)\mathcal{P}^2(x) + (2t - 1)x$.*

Then θ_x is a closed curved contained in $U^{\pm}$ and whose homology class in $H_1(U^{\pm}, \mathbb{Z}) = \mathbb{Z}$ is 2.

Proof. The unique difficulty here is that the curve don't cross $\mathrm{Col}\,(X, Y, U)$ and that is given by Lemma 7.4.2. $\square$

The next corollary achieves the thechnical result we announced at the beginning of this section. See Figure 7.7

Corollary 7.4.4. *Let $(U, X, Y, \Sigma, \mathcal{B})$ be a prepared counter example to Theorem D, and assume that $\ell^+(X, Y) \neq 0$.*

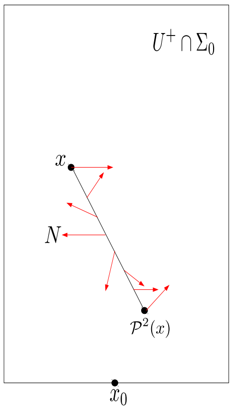


Figure 7.7: Topological behaviour of the normal component.

Consider $x \in U^+ \cap \Sigma_0 \cap O_3$. Then the angular variation of the vector $\mathcal{N}(y)$ for $y \in [x, \mathcal{P}^2(x)]$ is strictly larger than 2π in absolute value. In particular,

$$\mathcal{N}([x, \mathcal{P}^2(x)]) = \mathbb{S}^1.$$

The same statement holds in U^- if $\ell^-(X,Y) \neq 0$.

Proof. Lemma 7.4.1 implies that the angular variation of $\mathcal{N}(\theta_x(t))$ is contained in $(-\pi, \pi)$ for $t \in [0, \frac{1}{2}]$. However, the topological degree of the map $\mathcal{N}\colon \theta_x \to \mathbb{S}^1$ is $2\ell^+(X,Y)$ which has absolute value at least 2. Thus the angular variation of $\mathcal{N}$ on the segment $[x, \mathcal{P}^2(x)]$ is (in absolute value) at least 3π concluding. $\square$

As an immediate consequence one gets

Corollary 7.4.5. *If $\ell^+(X,Y) \neq 0$, then $\mathcal{P}^2$ has no fixed points in $O_3 \cap U^+$.*

7.4.1 The return map at points where N is pointing in opposite directions.

We have seen in the proof of Corollary 7.4.4 that the vector $\mathcal{N}$ has an angular variation larger than 3π along the segment $[x, \mathcal{P}^2(x)]$, as $x \in \Sigma_0$ approaches x_0. In this section we use the large angular variation of $\mathcal{N}$ for establishing a relation between the return map $\mathcal{P}$, the return time τ, and the coordinate μ of X in the Y direction.

Recall that, for every $x \in U \setminus \mathrm{Col}(X, Y, U)$, $\mathcal{N}(x)$ is a unit vector contained in $\mathbb{S}^1$, unit circle of $\mathbb{R}^2$.

Lemma 7.4.6. *Assume that there exists sequences $q_n, \overline{q}_n \in \Sigma_0 \setminus \mathrm{Col}(X, Y, U)$ converging to x_0, such that the following two conditions are satisfied:*

1. *$\mathcal{N}(q_n) \to (1, 0)$ and $\mathcal{N}(\overline{q}_n) \to (-1, 0)$, as $n \to +\infty$,*

2. *$(\mu(\mathcal{P}(q_n)) - \mu(q_n))(\mu(\mathcal{P}(\overline{q}_n)) - \mu(\overline{q}_n)) \geq 0$, for every n.*

Then, $D\tau(x_0)e_1(x_0) = 0$.

Proof. Recall that $N(x) = \alpha(x)e_1(x) + \beta(x)e_2(x)$ and $\mathcal{N}(x) = \frac{1}{\sqrt{\alpha(x)^2+\beta(x)^2}}(\alpha(x), \beta(x))$ for $x \in U \setminus \mathrm{Col}(X, Y, U)$.

By Corollary 6.1.11, we have

$$-D\tau(q_n)\frac{N(q_n)}{\sqrt{\alpha(q_n)^2 + \beta(q_n)^2}} = \frac{\mu(\mathcal{P}(q_n)) - \mu(q_n)}{\sqrt{\alpha(q_n)^2 + \beta(q_n)^2}}, \tag{7.5}$$

and

$$-D\tau(\overline{q}_n)\frac{N(\overline{q}_n)}{\sqrt{\alpha(\overline{q}_n)^2 + \beta(\overline{q}_n)^2}} = \frac{\mu(\mathcal{P}(\overline{q}_n)) - \mu(\overline{q}_n)}{\sqrt{\alpha(\overline{q}_n)^2 + \beta(\overline{q}_n)^2}}. \tag{7.6}$$

Multiplying side by side Equations 7.5 and 7.6 and using the second assumption of the lemma, we get

$$D\tau(q_n)\frac{N(q_n)}{\sqrt{\alpha(q_n)^2 + \beta(q_n)^2}}D\tau(\overline{q}_n)\frac{N(\overline{q}_n)}{\sqrt{\alpha(\overline{q}_n)^2 + \beta(\overline{q}_n)^2}} \geq 0.$$

Notice that the first assumption of the lemma is equivalent to $\frac{N(q_n)}{\sqrt{\alpha(q_n)^2+\beta(q_n)^2}} \to e_1(x_0)$ and $\frac{N(\overline{q}_n)}{\sqrt{\alpha(\overline{q}_n)^2+\beta(\overline{q}_n)^2}} \to -e_1(x_0)$. Since $q_n, \overline{q}_n \to x_0$, from the continuity of $D\tau$, we conclude that

$$0 \geq -\left(D\tau(x_0)(e_1(x_0))\right)^2 \geq 0,$$

which completes the proof. □

Since we assumed $(D\tau(x_0)(e_1(x_0)) \neq 0$ (item (4) of the definition of a prepared counter example to Theorem D), one gets the following corollary:

Corollary 7.4.7. *Let $(U, X, Y, \Sigma, \mathcal{B})$ is a prepared counter example to Theorem D. Assume that there exists sequences $q_n, \overline{q}_n \in \Sigma_0 \setminus \mathrm{Col}(X, Y, U)$ converging to x_0, such that $\mathcal{N}(q_n) \to (1, 0)$ and $\mathcal{N}(\bar{q}_n) \to (-1, 0)$, as n tends to $+\infty$. Then*

$$(\mu(\mathcal{P}(q_n)) - \mu(q_n))(\mu(\mathcal{P}(\overline{q}_n)) - \mu(\overline{q}_n)) < 0,$$

for every n large enough.

7.5 The Shear Case

In this section, $(U, X, Y, \Sigma, \mathcal{B})$ is a prepared counter example to Theorem D so that the derivative of the first return map $\mathcal{P}$ at some point $x_t = \Sigma_t \cap \gamma_t$ is not the identity map. By Remark 7.1.8 we may assume that $D\mathcal{P}(x_0)$ is not the identity map. Recall that $D\mathcal{P}(x_0)$ admits an eigenvalue equal to 1 directed by $e_1(x_0)$, has no eigenvalues different from 1 by Lemma 7.3.1, and is orientation preserving.

The main idea in this section is the following: since $D\mathcal{P}(x_0) \neq Id$, it has to be a Shear Matrix i.e $D\mathcal{P}(x_0)(x, y) = (x, y + cx)$, where $c \in \mathbb{R} \setminus \{0\}$. Thus the local dynamics of $\mathcal{P}$ close to x_0 acts as a translation in the horizontal coordinate. This will contradict Corollary 7.4.7.

Recall that $\mathrm{Col}\,(X, Y, U)$ cuts the solid torus U in two components U^+ and U^-. Let us denote $\Sigma_+ = \Sigma_0 \cap U^+$ and $\Sigma_- = \Sigma_0 \cap U^-$.

Lemma 7.5.1. *Let $(U, X, Y, \theta, \mathcal{B})$ be a prepared counter example to Theorem D so that the derivative of the first return map $\mathcal{P}$ at the point $0 = \Sigma_0 \cap \gamma_0$ is not the identity map.*

Then, there is a neighborhood W of 0 in Σ_0 so that the map $x \mapsto f(x) = \mu(\mathcal{P}(x)) - \mu(x)$ restricted to W vanishes only on $\mathrm{Col}\,(X, Y, U)$.

More precisely,

- $(\mu(\mathcal{P}(x)) - \mu(x))(\mu(\mathcal{P}(y)) - \mu(y)) > 0$ *if x and $y \in W \cap \Sigma_+$ and if x and $y \in W \cap \Sigma_-$*

- $(\mu(\mathcal{P}(x)) - \mu(x))(\mu(\mathcal{P}(y)) - \mu(y)) < 0$ *if* $x \in W \cap \Sigma_+$ *and* $y \in W \cap \Sigma_-$ *and if* $x \in W \cap \Sigma_-$ *and* $y \in W \cap \Sigma_+$

- $(\mu(\mathcal{P}(x)) - \mu(x)) = 0$ *and* $x \in W$ *if and only if* $x \in \mathrm{Col}\,(X, Y, U)$.

Proof. The derivative of f is given by

$$Df(x)v = D\mu(\mathcal{P}(x))D\mathcal{P}(x)v - D\mu(x),$$

for every $x \in \Sigma_0$ close to 0 and every $v \in T_x\Sigma_0$. We claim that $Df(0) \neq 0$. Indeed, notice that $D\mu(0)e_1 \neq 0$ and $Df(0)e_1 = 0$. It follows that $\ker(D\mu(0))$ is a line transverse to e_1 and thus every vector $v \in T_0\Sigma_0$ can be uniquely written as $v = \lambda e_1 + k$, for some $k \in \ker(D\mu(0))$ and some scalar λ. It suffices now to show that $k \in \ker(D\mu(0))$ implies $Df(0)k \neq 0$. Assume on the contrary that $Df(0)k = 0$. Then, $D\mu(0)D\mathcal{P}(0)k = D\mu(0)k = 0$ and therefore $D\mathcal{P}(0)k \in \ker(D\mu(0))$ which implies that $\ker(D\mu(0))$ contains an eigenvector of $D\mathcal{P}(0)$ transversal to e_1. By Lemma 7.3.1 this implies that $D\mathcal{P}(0) = Id$, contradicting the assumption of the lemma.

Since f is C^1, there is a neighborhood W of 0 such that $Df(x) \neq 0$ for every $x \in W$. This neighborhood is foliated by the level sets $f^{-1}(t)$, t close enough to 0. To complete the proof, one simply observe that as f vanishes on $\mathrm{Col}(X, Y, U)$, we have that $\mathrm{Col}(X, Y, U) \cap W = (f|_W)^{-1}(0)$. □

We are now ready to prove the following proposition:

Proposition 7.5.2. *If* $(U, X, Y, \Sigma, \mathcal{B})$ *is a prepared counter example to Theorem D then* $D\mathcal{P}(x)$ *is the identity map for every* $x \in \mathrm{Col}\,(X, Y, U) \cap \Sigma_0$.

Proof. Up to exchange $+$ by $-$, we assume that $\ell^+(X, Y) \neq 0$. Then Corollary 7.4.4 implies that there are sequences $q_n, \bar{q}_n \in \Sigma_+$ tending to x_0 and so that $\mathcal{N}(q_n) = (1, 0)$ and $\mathcal{N}(-q_n) = (-1, 0)$. More precisely Corollary 7.4.4 implies that for any $x \in \Sigma_+$ close enough to x_0 the segment $[x, \mathcal{P}^2(x)]$ contains points $q, \bar{q}$ with $\mathcal{N}(q) = (1, 0)$ and $\mathcal{N}(\bar{q}) = (-1, 0)$. Now Lemma 7.4.2 implies that the segment is contained in Σ_+, concluding.

Now Corollary 7.4.7 implies that, for n large enough, the sign of the map $(\mu(\mathcal{P}(x))-\mu(x))$ is different on q_n and on $\bar{q}_n$. One concludes with Lemma 7.5.1 which says that this sign cannot change if the derivative $D\mathcal{P}(x_0)$ is not the identity. Thus we proved $D\mathcal{P}(x_0) = Id$.

Now if $x \in \mathrm{Col}\,(X, Y, U)$ then x is of the form $x = x_t = \gamma_t \cap \Sigma_0$. According to Remark 7.1.8 $(U, X-tY, Y, \Sigma, \mathcal{B})$ is also a prepared counter example to Theorem D, and the linking number of $X - tY$ with Y is the same as the linking number of X with Y. Therefore the argument above establishes that $D\mathcal{P}(x) = Id$. □

7.6 The Identity Case

In the whole section, $(U, X, Y, \Sigma, \mathcal{B})$ is a prepared counter example to Theorem D. According to Corollary 7.2.7 one of the linking numbers $\ell^+(X, Y), \ell^-(X, Y)$ does not vanish, so that, up to exchange $+$ with $-$, one may assume $\ell^+(X, Y) \neq 0$. According to Proposition 7.5.2 the derivative $D\mathcal{P}(x)$ is the identity map for every $x \in \mathrm{Col}\,(X, Y, U) \cap \Sigma_0$.

This means that, in a neighborhood of $\mathrm{Col}\,(X, Y, U)$, the diffeomorphism $\mathcal{P}$ is C^1 close to the identity map. This case is by far the most difficulty one. Corollary 7.4.7 and Corollary 7.4.4 will play a fundamental role in the argument.

Let us give a brief sketch: the first step is to give a complete dynamical description of the first return map near $\mathrm{Col}(X, Y, U) \cap \Sigma_0$. We will first show in Lemma 7.6.1 that, in the neighborhood of $\mathrm{Col}\,(X, Y, U)$ the vectors $\mathcal{P}(x) - x$ are almost tangent to the kernel of $D\mu$. As the fibers of $D\mu$ are tranverse to $\mathrm{Col}\,(X, Y, U)$ we shall get in Lemma 7.6.4 a topological dynamics of $\mathcal{P}$ similar to the Partially Hyperbolic Case of Section 7.3. The second step is to find a segment of an integral curver of N, contained in $\Sigma_0 \cap U^+$ and very close to $\mathrm{Col}(X, Y, U)$, which is invariant under $\mathcal{P}$. To find this curve we quotient a domain in $\Sigma_0 \cap U^+$ by the dynamics of $\mathcal{P}$ and apply Poincaré-Bendixson. The final step is to show that the angular variation of such an integral curve of N joining x to $\mathcal{P}^2(x)$ is very small. We then use this to contradict Corollary 7.4.4.

7.6.1 Quasi invariance of the map μ by the first return map.

Recall that μ is the coordinate of X in the Y direction: $X(x) = N(x) + \mu(x)Y(x)$. The aim of this section is to prove

Lemma 7.6.1. *If $x_n \in \Sigma_+$ is a sequence of points tending to $x \in \mathrm{Col}\,(X, Y, U)$ and if $v_n \in T_{x_n}\Sigma_+$ is the unit tangent vector directing the segment $[x_n, \mathcal{P}(x_n)]$ then $D\mu(x_n)(v_n)$ tends to* 0.

According to Remark 7.1.8, it is enough to prove Lemma 7.6.1 in the case $x = x_0 = \gamma_0 \cap \Sigma_0$. Lemma 7.6.1 is now a straighforward consequence of the following lemma

Lemma 7.6.2.

$$\lim_{x \to x_0, x \in \Sigma_+} \frac{\mu(\mathcal{P}(x)) - \mu(x)}{d(\mathcal{P}(x), x)} = 0,$$

where $d(\mathcal{P}(x), x)$ denotes the distance between x and $\mathcal{P}(x)$.

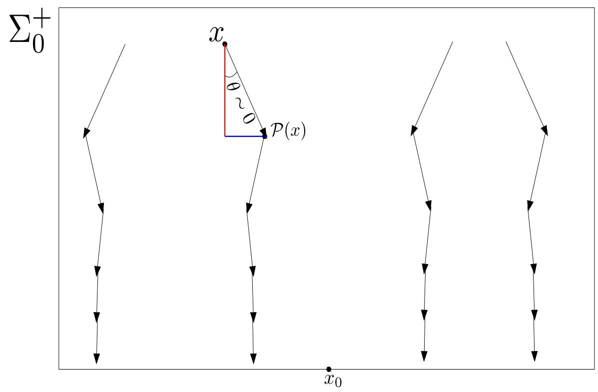


Figure 7.8: The equality $\lim_{x \to x_0, x \in \Sigma_+} \frac{\mu(\mathcal{P}(x)) - \mu(x)}{d(\mathcal{P}(x), x)} = 0$ means that the vectors $\mathcal{P}(x) - x$ are almost vertical (tangent to the fibers of μ.

Proof. Fix $\varepsilon > 0$ and let us prove that $\frac{|\mu(\mathcal{P}(x)) - \mu(x)|}{d(\mathcal{P}(x), x)}$ is smaller than ε for every x close to x_0 in Σ_+. Recall that, according to Lemma 7.4.2, there is a neighborhood O_2 of γ_0 so that if $x \in O_2^+ = O_2 \cap \Sigma_+$ then the segment of straight line $[x, \mathcal{P}^2(x)]$ is contained in Σ_+. Furthermore, Corollary 7.4.4 says that $\mathcal{N}|_{[x, \mathcal{P}^2(x)]}$ is surjective onto $\mathbb{S}^1$ (unit circle in $\mathbb{R}^2$). In particular, there are points $q_x, \bar{q}_x \in [x, \mathcal{P}^2(x)]$ so that $\mathcal{N}(q_x) = (1, 0)$ and $\mathcal{N}(\bar{q}_x) = (-1, 0)$.

According to Corollary 7.4.7 one gets

$$(\mu(\mathcal{P}(q_x)) - \mu(q_x))(\mu(\mathcal{P}(\overline{q}_x)) - \mu(\overline{q}_x)) < 0, \tag{7.7}$$

for every $x \in O_2^+$.

The diffeomorphism $\mathcal{P}$ is C^1-close to the identity in a small neighborhood of x_0 Now [Bo2] (see also [Bo3]) implies that there is a neigborhood V_1 of x_0 in Σ_0 so that if $x \in V_1$ then

$$\|(\mathcal{P}(x) - x) - (\mathcal{P}(y) - y)\| < \frac{1}{2}\|\mathcal{P}(x) - x\|,$$

for every y with $d(x, y) < 3\|\mathcal{P}(x) - x\|$. In particular, $\|\mathcal{P}^2(x) - \mathcal{P}(x)\| < 2\|\mathcal{P}(x) - x\|$, and thus $\|\mathcal{P}^2(x) - x\| < 3\|\mathcal{P}(x) - x\|$.

Consider the function $f(x) = \mu(\mathcal{P}(x)) - \mu(x)$. Since $D\mathcal{P}(x_0) = Id$, we have that $Df(x_0) = 0$. As a consequence, there exists a neighborhood $V_2 \subset V_1$ of x_0 such that $|Df(x)| < \frac{\varepsilon}{9}$, for every $x \in V_2$.

Since $\mathcal{P}(x_0) = x_0$, we can choose a smaller neighborhood V_3 such that $\mathcal{P}(x), \mathcal{P}^2(x) \in V_2$, for every $x \in V_3$. This ensures that

$$\frac{|f(q_x) - f(x)|}{d(x, \mathcal{P}(x))} < \frac{\varepsilon}{9}\frac{d(q_x, x)}{d(x, \mathcal{P}(x))} \leq \frac{\varepsilon}{3}.$$

Similar estimates hold with $\overline{q}_x$ in place of q_x and in place of x, respectively.

By Inequality (7.7) we see that $f(q_x)$ and $f(\bar{q}_x)$ have opposite signs and thus

$$\frac{|f(q_x) + f(\bar{q}_x)|}{d(x, \mathcal{P}(x))} \leq \frac{|f(q_x) - f(\bar{q}_x)|}{d(x, \mathcal{P}(x))} \leq \frac{\varepsilon}{3}.$$

We deduce

$$\begin{aligned}
\left|\frac{2f(x)}{d(x, \mathcal{P}(x))}\right| &= \frac{|f(x) - f(q_x) + f(q_x) + f(\bar{q}_x) + f(x) - f(\bar{q}_x)|}{d(x, \mathcal{P}(x))} \\
&\leq \frac{|f(x) - f(q_x)|}{d(x, \mathcal{P}(x))} + \frac{|f(\bar{q}_x) + (q_x)|}{d(x, \mathcal{P}(x))}\frac{|f(x) - f(\bar{q}_x)|}{d(x, \mathcal{P}(x))} \\
&\leq \varepsilon.
\end{aligned}$$

This establishes that $\frac{|\mu(\mathcal{P}(x)) - \mu(x)|}{d(\mathcal{P}(x), x)}$ is smaller than ε for every $x \in V_3 \cap \Sigma_+$ and completes the proof. $\square$

Remark 7.6.3. The Lemmas 7.6.1 and 7.6.2 depend a priori on the choice of coordinate on Σ_0 since they are formulated in terms of segments of straight line $[x, \mathcal{P}(x)]$, and vectors $\mathcal{P}(x) - x$. Nevertheless, the choice of coordinates on Σ_0 was arbitrary (see first paragraph of Section 7.4) so that it holds indeed for any choice of C^1 coordinates on Σ_0 (on a neighborhood of x_0 depending on the choice of the coordinates).

7.6.2 Local dynamics of the first return map $\mathcal{P}$

In this section $(U, X, Y, \Sigma, \mathcal{B})$ is a prepared counter example to Theorem D, and we assume $\ell^+(X, Y) \neq 0$.

Recall that Σ_0 is a disc endowed with an arbitrary (but fixed) choice of coordinates. Also, $\mu\colon \Sigma_0 \to \mathbb{R}$ is a C^1-map whose derivative do not vanish along $\mathrm{Col}\,(X, Y, U)$ and $\mathrm{Col}\,(X, Y, U) \cap \Sigma_0$ is a C^1-curve.

Therefore, one can choose a C^1-map $\nu\colon \Sigma_0 \to \mathbb{R}$ so that

- there is a neighborhood O of x_0 in Σ_0 so that $(\mu, \nu)\colon O \to \mathbb{R}^2$ is C^1 diffeomorphism,
- $\mathrm{Col}\,(X, Y, U) \cap O = \nu^{-1}(\{0\})$
- $\nu > 0$ on Σ_+

We denote by $(\mu(x), \nu(x))$ the image of x by (μ, ν).

Notice that $(\mu(x), \nu(x))$ are local coordinates on Σ_0 in a neighborhood of x_0. Remark 7.6.3 allows us to use Lemma 7.6.1 and Lemma 7.6.2 in the coordinates (μ, ν).

As a consequence, there exist $\varepsilon > 0$ so that for any point $x \in \Sigma_+$ with $(\mu(x), \nu(x)) \in [-\varepsilon, \varepsilon] \times [0, \varepsilon]$, one has

$$|\mu(\mathcal{P}(x)) - \mu(x)| < \frac{1}{100}|\nu(\mathcal{P}(x)) - \nu(x)|. \tag{7.8}$$

In particular, since $\mathcal{P}$ has no fixed point in Σ_+ (Corollary 7.4.5), one gets that $\nu(\mathcal{P}(x)) - \nu(x)$ does not vanish for $(\mu(x), \nu(x)) \in [-\varepsilon, \varepsilon] \times (0, \varepsilon]$, and in particular it has a constant sign. Up to change $\mathcal{P}$ by its inverse $\mathcal{P}^{-1}$ (which is equivalent to replace Y by $-Y$), one may assume

$$\nu(\mathcal{P}(x)) - \nu(x) < 0, \text{ for } (\mu(x), \nu(x)) \in [-\varepsilon, \varepsilon] \times (0, \varepsilon].$$

Next lemma allows us to define stable sets for the points in $\mathrm{Col}\,(X, Y, U)$ and shows that every point in Σ_+ close to x_0 belongs to such a stable set.

Lemma 7.6.4. *Let $x \in \Sigma_+$ be such that $(\mu(x), \nu(x)) \in [-\frac{9}{10}\varepsilon, \frac{9}{10}\varepsilon] \times (0, \varepsilon]$. Then, for any integer $n \geq 0$, $\mathcal{P}^n(x)$ satisfies*

$$(\mu(\mathcal{P}^n(x)), \nu(\mathcal{P}^n(x))) \in [-\varepsilon, \varepsilon] \times (0, \varepsilon]$$

Furthermore the sequence $\mathcal{P}^n(x)$ converges to a point $x_\infty \in \mathrm{Col}\,(X, Y, U) \cap \Sigma_0$ and we have

- $\nu(x_\infty) = 0$
- $\mu(x_\infty) - \mu(x) \leq \frac{\nu(x)}{100} \leq \frac{\varepsilon}{100}$.

The map $x \mapsto x_\infty$ is continuous.

Proof. Consider the trapezium D (in the (μ, ν) coordinates) whose vertices are $(-\varepsilon, 0)$, $(\varepsilon, 0)$, $(-\frac{9\varepsilon}{10}, \varepsilon)$, and $(\frac{9\varepsilon}{10}, \varepsilon)$. This trapezium D is contained in $[-\varepsilon, \varepsilon] \times [0, \varepsilon]$ and contains $[-\frac{9}{10}\varepsilon, \frac{9}{10}\varepsilon] \times [0, \varepsilon]$. Thus for proving the first item it is enough to check that D is invariant under $\mathcal{P}$. For that notice that, for any x with $(\mu(x), \nu(x)) \in [-\varepsilon, \varepsilon] \times [0, \varepsilon]$ one has that $(\mu(\mathcal{P}(x)), \nu(\mathcal{P}(x))$ belongs to the triangle $\delta(x)$ whose vertices are $(\mu(x), \nu(x))$, $(\mu(x) - \frac{\nu(x)}{100}, 0)$, $(\mu(x) + \frac{\nu(x)}{100}, 0)$ (according to Equation 7.8); one conclude by noticing that, if $(\nu(x), \mu(x))$ belongs to D then $\delta(x) \subset D$.

Let us show that $\mathcal{P}^n(x)$ converges.

The sequence $\nu(\mathcal{P}^n(x))$ is positive and decreasing, hence converges, and

$$\sum |\nu(\mathcal{P}^{n+1}(x)) - \nu(\mathcal{P}^n(x))|$$

converges. As $|\mu(\mathcal{P}^{n+1}(x)) - \mu(\mathcal{P}^n(x))| < \frac{1}{100}|\nu(\mathcal{P}^{n+1}(x)) - \nu(\mathcal{P}^n(x))|$ one deduces that the sequence $\{\mu(\mathcal{P}^n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence, hence converges.

The continuity of $x \mapsto x_\infty$ follows from the inequality $\mu(x_\infty) - \mu(x) \leq \frac{\nu(x)}{100}$ applied to $\mathcal{P}^n(x)$ with n large, so that $\nu(\mathcal{P}^n(x))$ is very small, and from the continuity of $x \mapsto \mathcal{P}^n(x)$. $\square$

For any point y with $(\mu(y), \nu(y)) \in [-\frac{8}{10}\varepsilon, \frac{8}{10}\varepsilon] \times \{0\}$ the *stable set of* y, which we denote by $S(y)$, is the the union of $\{y\}$ with the set of points $x \in \Sigma_+$ with $(\mu(x), \nu(x)) \in [-\frac{9}{10}\varepsilon, \frac{9}{10}\varepsilon] \times (0, \varepsilon]$ so that $x_\infty = y$. The continuity of the map $x \mapsto x_\infty$ implies the following remark:

Remark 7.6.5. For any point y with $(\mu(y), \nu(y)) \in [-\frac{8}{10}\varepsilon, \frac{8}{10}\varepsilon] \times \{0\}$, $S(y)$ is a compact set which has a non-empty intersection with the horizontal lines $\{x, \nu(x) = t\}$ for every $t \in (0, \varepsilon]$.

If E is a subset of $\mathrm{Col}\,(X, Y, U) \cap \Sigma_0$ so that $\mu(y) \in [-\frac{8}{10}\varepsilon, \frac{8}{10}\varepsilon]$ for $y \in E$ one denotes

$$S(E) = \bigcup_{y \in E} (S(y)).$$

Lemma 7.6.6. *Let $I \subset \mathrm{Col}\,(X, Y, U)$ be the open interval $(\mu(x), \nu(x)) \in (-\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon) \times \{0\}$. Consider the quotient space Γ of $S(I) \setminus I$ by the dynamics. Then Γ is a C^1-connected surface diffeomorphic to a cylinder $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$.*

Proof. Consider the compact triangle whose end points are $(-\frac{\varepsilon}{2}, 0)$, $(0, \frac{\varepsilon}{10})$ and $(+\frac{\varepsilon}{2}, 0)$. Let $\bar{\Delta}$ be its preimage by (μ, ν).

$\bar{\Delta}$ is a triangle with one side on $\mathrm{Col}(X, Y, U)$. Let $\Delta = \bar{\Delta} \setminus \mathrm{Col}(X, Y, U)$. We denote by $\partial\Delta$ the union of the two other sides.

As the vectors directing the two other sides have a first coordinated larger than the second, one deduces that Δ is a trapping region for $\mathcal{P}$:

$$x \in \Delta \Longrightarrow \mathcal{P}(x) \in \Delta.$$

Now

$$\mathcal{P}(\partial\Delta)$$

is a curve contained in the interior of Δ and joining the vertex $(-\frac{\varepsilon}{2}, 0)$ to the vertex $(+\frac{\varepsilon}{2}, 0)$. Thus $\partial\Delta$ and $\mathcal{P}(\partial\Delta)$ bound a strip diffeomorphic to $[0, 1] \times \mathbb{R}$ in Δ.

Let Γ be the cylinder obtained from this strip by gluing $\partial\Delta$ with $\mathcal{P}(\partial\Delta)$ along $\mathcal{P}$.

It remains to check that Γ is the quotient space of $S(I) \setminus I$ by $\mathcal{P}$. For that, one just remark that the orbit of every point in $S(I) \setminus I$ has a unique point in the strip, unless in the case where the orbits meets $\partial\Delta$: in that case the orbits meets the strip twice, the first time on $\partial\Delta$, the second on $\mathcal{P}(\partial\Delta)$. $\square$

Let us end this section by stating important straighforward consequences of the invariance of the vector field N under $\mathcal{P}$.

Remark 7.6.7.
- For every $y \in \mathrm{Col}(X, Y, U)$ with $\mu(y) \in [-\frac{8}{10}\varepsilon, \frac{8}{10}\varepsilon]$, the stable set $S(y)$ is invariant under the flow of N.
- The vector field N induces a vector field, denoted by N_Γ, on the quotient space Γ. As Γ is a C^1 surface, N_γ is only C^0. However, it defines a flow on Γ which is the quotient by $\mathcal{P}$ of the flow of N.
- The continuous map $x \mapsto x_\infty$ in invariant under $\mathcal{P}$ and therefore induces on Γ a continuous map $\Gamma \to (-\varepsilon/2, \varepsilon/2)$, and x_∞ tends to $-\varepsilon/2$ when x tends to one end of the cyclinder Γ and to $\varepsilon/2$ when x tends to the other end. This implies that, for every $t \in (-\varepsilon/2, \varepsilon/2)$ the set of points $x \in \Gamma$ for which $x_\infty = t$ is compact. Recall that this set is precisely the projection on Γ of $S(y)$ where $y \in \mathrm{Col}(X, Y, U)$ satisfies $(\mu(y), \nu(y)) = (t, 0)$.
- The vector field N_Γ on Γ leaves invariant the levels of the map $x \mapsto x_\infty$. As a consequence, every orbit of N_Γ is bounded in Γ.

One deduces

Lemma 7.6.8. *For every $y \in \mathrm{Col}(X, Y, U)$ with $\mu(y) \in [-\frac{8}{10}\varepsilon, \frac{8}{10}\varepsilon]$, the stable set $S(y) \setminus \{y\}$ contains an orbit of N which is invariant under $\mathcal{P}$. In particular, there exists x in $\Delta \cap S(x_0)$ whose orbit by N is invariant under $\mathcal{P}$.*

Proof. A Poincaré Bendixson argument implies that, for every flow on the cylinder $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$ without fixed points, for every bounded orbit the ω-limit set is a periodic orbit. Furthermore this periodic orbit is not homotopic to a point (otherwise it bounds a disc containing a zero).

Since N_Γ has no zeros, one just applies this argument to the flow of it restricted to the a level of the map $x \mapsto x_\infty$. The level contains a periodic orbit which is not homotopic to 0, hence corresponds to an orbit of N joining a point in $S(y)$ to is image under $\mathcal{P}$. This orbit of N is invariant under $\mathcal{P}$, concluding. $\square$

7.6.3 End of the proof of Theorem D: the vector field N does not rotate along a $\mathcal{P}$-invariant orbit of N

From now on, $(U, X, Y, \Sigma, \mathcal{B})$ is a prepared counter example to Theorem D with $\ell^+(X, Y) \neq 0$. According to Proposition 7.5.2 the derivative $D\mathcal{P}(x)$ is the identity map for every $x \in \mathrm{Col}\,(X, Y, U) \cap \Sigma_0$.

According to Lemma 7.6.8, there is a point x in the stable set $S(x_0)$ whose N-orbit is invariant under $\mathcal{P}$.

Lemma 7.6.9. *The angular variation of the vector $N(y)$, for $y \in [\mathcal{P}^n(x), \mathcal{P}^{n+1}(y)]$, tends to 0 when $n \to +\infty$.*

Before proving Lemma 7.6.9 let us conclude the proof of Theorem D.

Proof of Theorem D. Lemma 7.6.9 is in contradiction with Corollary 7.4.4, which asserts that the angular variation of the vector N along any segment $[z, \mathcal{P}^2(z)]$ for $z \in \Sigma_+$ close enough to x_0 is larger than 2π. This contradiction ends the proof of Theorem D. $\square$

It remains to prove Lemma 7.6.9. First, notice that the angular variation of N along a segment of curve is invariant under homotopies of the curve preserving the ends points. Therefore Lemma 7.6.9 is a straighforward consequence of next lemma:

Lemma 7.6.10. *The angular variation of the vector $N(y)$ along the N-orbit segment joining $\mathcal{P}^n(x)$ to $\mathcal{P}^{n+1}(x)$ tends to 0 when $n \to +\infty$.*

As N is (by definition) tangent to the N-orbit segment joining $\mathcal{P}^n(x)$ to $\mathcal{P}^{n+1}(x)$, its angular variation is equal to the angular variation of the unit tangent vector to this orbit segment.

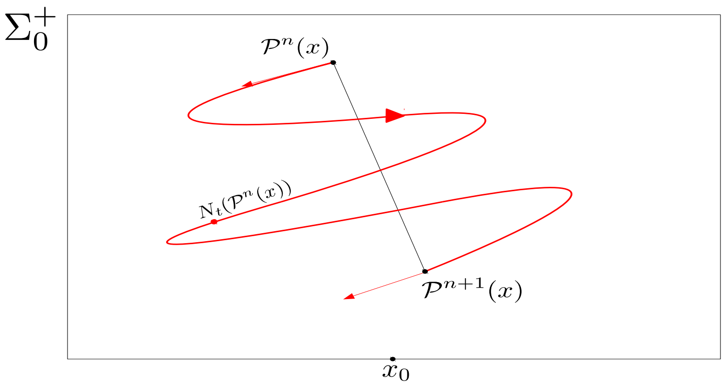


Figure 7.9: Lemma 7.6.10: an integral curve of N along wich the angular of N is small.

7.6.4 Proof of Lemma 7.6.10: the tangent vector to a $\mathcal{P}$-invariant embedded curve do not rotate

Remark 7.6.11. For n large enough the point $\mathcal{P}^n(x)$ belongs to the region Δ defined in the previous section, and whose quotient by the dynamics $\mathcal{P}$ is the cyclinder Γ. Then,

- any continuous curve γ in Δ joining $\mathcal{P}^n(x)$ to $\mathcal{P}^{n+1}(x)$ induces on Γ a closed curve, homotopic to the curve induced by the N-orbit segment joining $P^n(x)$ to $\mathcal{P}^{n+1}(x)$.
- The curve induced by γ is a simple curve if and only if γ is simple and disjoint from $\mathcal{P}^i(\gamma)$ for any $i > 0$.
- If the curve γ is of class C^1, the projection will be of class C^1 if and only if the image by $\mathcal{P}$ of the unit vector tangent to γ at $\mathcal{P}^n(x)$ is tangent to γ (at $\mathcal{P}^{n+1}(x)$).
- on the cylinder, any two C^1-embbedding σ_1, σ_2 of the circle, so that $\sigma_1(0) = \sigma_2(0)$ are isotopic through C^1-embeddings σ_t with $\sigma_t(0) = \sigma_i(0)$.

We consider $\mathcal{I}(\Delta, \mathcal{P})$ as being the set of C^1-immersed segment I in $\Delta \setminus \partial\Delta$, so that:

- if y, z are the initial and end points of I then $z = \mathcal{P}(y)$
- if u is a vector tangent to I at y then $\mathcal{P}_*(u)$ is tangent to I (and with the same orientation.

In other words, $I \in \mathcal{I}(\Delta, \mathcal{P})$ if the projection of I on Γ is a C^1 immersion of the circle, generating the fundamental group of the cylinder. We endow $\mathcal{I}(\Delta, \mathcal{P})$ with the C^1-topology.

We denote by $Var(I) \in \mathbb{R}$ the angular variation of the unit tangent vector to I along I. In other words, for $I\colon [0,1] \to \Delta$, consider the unit vector

$$\mathring{I}(t) = \frac{dI(t)/dt}{\|dI(t)/dt\|} \in \mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}.$$

One can lift $\mathring{I}$ is a continuous map $\dot{I}\colon [0,1] \to \mathbb{R}$. Then

$$Var(I) = \dot{I}(1) - \dot{I}(0),$$

this difference does not depend on the lift.

The map $Var\colon \mathcal{I}(\Delta, \mathcal{P}) \to \mathbb{R}$ is continuous. Let $\widetilde{Var}(I) \in \mathbb{R}/2\pi\mathbb{Z}$ be the projection of $Var(I)$. In other words, $\widetilde{Var}(I)$ is the angular variation modulo 2π.

Remark 7.6.12. Let $I_n \in \mathcal{I}(\Delta, \mathcal{P})$ be a sequence of immersed segments such that $I_n(0)$ tends to $x_0 \in \Sigma$. Then $\widetilde{Var}(I_n)$ tends to 0.

Indeed, since $D\mathcal{P}(I_n(0))$ tends to the identity map, the angle between the tangent vectors to I_n at $I_n(1)$ and $I_n(0)$ tends to 0.

As a consequence of Remark 7.6.12, we get the following lemma:

Lemma 7.6.13. *There is a neighborhood O of x_0 in Σ so that to any $I \in \mathcal{I}(\Delta, \mathcal{P})$ with $I(0) \in O$ there is a (unique) integer $[var](I) \in \mathbb{Z}$ so that*

$$Var(I) - 2\pi[var](I) \in \left[-\frac{1}{100}, \frac{1}{100}\right].$$

Furthermore, the map $I \mapsto [var](I)$ is locally constant in $\mathcal{I}(\Delta, \mathcal{P})$, hence constant under homotopies in $\mathcal{I}(\Delta, \mathcal{P})$ keeping the initial point in O.

As a consequence we get

Lemma 7.6.14. *If I and J are segments in $\mathcal{I}(\Delta, \mathcal{P})$ with the same intial point in O and whose projections on the cyclinder Γ are simple closed curves, then*

$$[var](I) = [var](J).$$

Proof. Since the projection of I and J are simple curves which are not homotopic to a point in the cylinder Γ, the projections of I and J are isotopic on Γ by an isotopy keeping the initial point. One deduces that I and J are homotopic through elements $I_t \in \mathcal{I}(\Delta, \mathcal{P})$ with the same initial point. Indeed, the isotopy on Γ between the projection of I and J can be lifted to the universal cover of Γ. This universal cover is diffeomorphic to a plane $\mathbb{R}^2$, in which $\Delta \setminus \partial\Delta$ is an half plane (bounded by two half lines). There is a diffeomorphism of $\mathbb{R}^2$ to $\Delta \setminus \partial\Delta$ which is the indentity on $I \cup J$. The image of the lifted isotopy induces the announced isotopy through elements in $\mathcal{I}(\Delta, \mathcal{P})$. Now, as $[var](I_t)$ is independent of t, one concludes that $[var](I) = [var](J)$. $\square$

Now Lemma 7.6.10 is a consequence of Lemma 7.6.14 and of the folowing lemma:

Lemma 7.6.15. *For any $n > 0$ large enough there is a curve $I_n \in \mathcal{I}(\Delta, \mathcal{P})$ whose initial point is $\mathcal{P}^n(x)$ and such that:*

- *the projection of I_n on Γ is a simple curve*
- $[var](I_n) = 0$.

End of the proof of Lemma 7.6.10. The N-orbit segment J_n joining $\mathcal{P}^n(x)$ to $\mathcal{P}^{n+1}(x)$ belongs to $\mathcal{I}(\Delta, \mathcal{P})$ and its projection on Γ is a simple curve. Hence Lemma 7.6.14 asserts that, for n large enough, $[var](J_n) = [var](I_n) = 0$ where I_n is given by Lemma 7.6.15.

Now, when n tends to infinity, $Var(J_n) - 2\pi[var](J_n)$ tends to 0 (according to Remark 7.6.12), that is, $Var(J_n)$ tends to 0. This is precisely the statement of Lemma 7.6.10. $\square$

Proof of Lemma 7.6.15. As n tends to $+\infty$ the derivative of $\mathcal{P}$ at $\mathcal{P}^n(x)$ tends to the identity map. Thus, the segment $[\mathcal{P}^n(x), \mathcal{P}^{n+1}(x)]$ may fail to belong to $\mathcal{I}(\Delta, \mathcal{P})$ only by a very small angle between v_n and $D\mathcal{P}(v_n)$, where v_n is the unit vector directing $[\mathcal{P}^n(x), \mathcal{P}^{n+1}(x)]$. Therefore, one easily builds a segment I_n joining $\mathcal{P}^n(x)$ to $\mathcal{P}^{n+1}(x)$, whose derivative at $\mathcal{P}^n(x)$ is v_n and its derivative at $\mathcal{P}^{n+1}(x)$ is $D\mathcal{P}(v_n)$ and whose derivative at any point of I_n belongs to an arbitrarily small neighborhood of v_n. In particular $I_n \in \mathcal{I}(\Delta, \mathcal{P})$ and $[var](I_n) = 0$ for n large. In order to complete the proof, it remains to show that

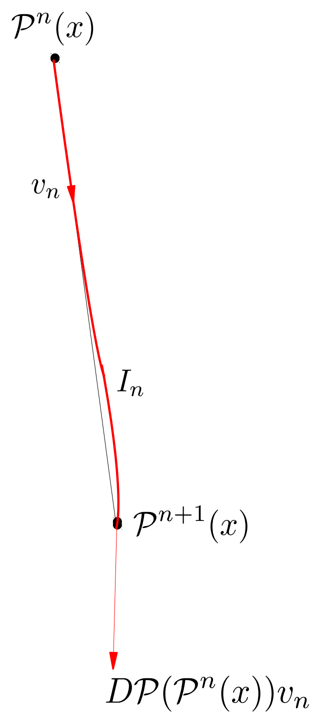


Figure 7.10: A segment I_n built by interpolating v_n and $\mathcal{P}_* v_n$ along $[\mathcal{P}^n(x), \mathcal{P}^{n+1}(x)]$.

Claim 11. *For n large enough I_n projects on Γ as a simple curve.*

Proof. We need to prove that for n large enough and for any $i > 1$, $I_n \cap \mathcal{P}^i(I_n) = \emptyset$ and $I_n \cap \mathcal{P}(I_n)$ is a singleton (the endpoint of I_n which is the image of its intial point).

Indeed, it is enough to prove that, for any $y \in I_n$ different from the initial point,

$$\nu(\mathcal{P}(y)) < \inf_{z \in I_n} \nu(z).$$

As the action of $\mathcal{P}$ consists in lowing down the value of ν, the further iterates cannot cross I_n.

For proving that, notice that the vectors tangent to I_n are very close to v_n which is uniformly (in n large) transverse to the levels of ν. As $D\mathcal{P}(y)$ tends to the identity map when y tends to 0, for n large, the vectors tangent to $\mathcal{P}(I_n)$ are also transverse to the levels of ν. Hence $\mathcal{P}(I_n)$ is a segment starting at the end point of I_n (which realizes the infimum of ν on I_n) and ν is strictly decreasing along $\mathcal{P}(I_n)$, concluding. $\square$

This ends the proof of Lemma 7.6.15 (and so of Theorem D). $\square$

Chapter 8

Existence of attractors for non-singular flows on 3-manifolds

Let M be a three manifold. The goal of this chapter is to present the proof that inside the open subset of $\mathfrak{X}^1(M)$ formed by non-singular vector fields there exists a residual set $\mathcal{R}$ such that the flow of every $x \in \mathcal{R}$ possess an attractor. More precisely, the result we shall present in this chapter is the following.

Theorem E (Arbieto, Morales, S.). *There exists a residual subset $\mathcal{R}$ of $\mathfrak{X}^1_{NS}(M)$ (the space of non-singular vector fields) such that for every $X \in \mathcal{R}$ one of the following assertions is true*

1. *X has infinitely many sinks*
2. *X has a finite number of hyperbolic attractors, whose topological basins cover a full Lebesgue measure subset of M.*

The proof given below is the same of the paper [AMS1], though with a different presentation. The first section of this chapter is dvoted to describe the adaptation of techinique for diffeomorphisms to flows. This technique enables one to prove an ergodic property about the divergence of a vector field, which is very useful to study existence of attractors.

8.1 J-weak orbits

The result we shall describe now is the flow adaptation of diffeomorphism result due to Araújo in his thesis [A]. We presented the flow version in our paper [AMS1]. Even though the adaptation is not difficult, at the time we were thinking about Araúsjo's result for non-singular flows it was not clear if such an adaptation was possible and, in fact, the search for this adaptation was my first contact with mathematical research.

8.1.1 J-weak orbits for diffeomorphisms

For the sake of completeness and for the comfort of the reader we shall give Araújo's proof.

Let δ_p be the Dirac measure supported on a point p. Consider a diffeomorphism $f : M^d \to M^d$, of a d-dimensional closed manifold M.

Definition 8.1.1. The orbit $O(x)$ of a point $x \in M$ is said to be *J-weak* if there exists $N > 0$ such that $n \geq N$ implies

$$|\det Df^n(x)| < (1+\delta)^n.$$

We denote by $\Lambda(\delta, f)$ the set of all J-weak orbits. Recall that m denotes the normalized Lebesgue measure of M.

Lemma 8.1.2 (Araújo)**.** $m\left(\Lambda(\delta, f)\right) = 1$.

Proof. Take $\epsilon > 0$ and for $n \in \mathbb{N}$, consider $\Lambda_n = \{x \in M; |\det Df^n(x)| \geq (1+\delta)^n\}$. Notice that $N > 0$,

$$\bigcap_{n=N}^{+\infty} (M - \Lambda_n) \subset \Lambda(\delta, f).$$

Thus

$$m\left(\Lambda(\delta, f)\right) \geq 1 - m\left(\bigcup_{n=N}^{+\infty} \Lambda_n\right) \geq 1 - \sum_{n=N}^{+\infty} m\left(\Lambda_n\right). \tag{8.1}$$

On the other hand, since f^n is a diffeomorphism, by change of variables we have

$$1 = \int_M |\det Df^n| dm \geq \int_{\Lambda_n} |\det Df^n| dm \geq (1+\delta)^n m\left(\Lambda_n\right).$$

It follows that

$$m(\Lambda_n) \leq \frac{1}{(1+\delta)^n}.$$

Therefore, if N is so large that $\sum_{n=N}^{\infty} m(\Lambda_n) < \epsilon$ one obtains from 8.1 that

$$m(\Lambda(\delta, f)) \geq 1 - \sum_{n=N}^{+\infty} m(\Lambda_n) \geq 1 - \epsilon.$$

Since $\epsilon > 0$ was chosen arbitrarily, the lemma is proved. □

8.1.2 J-weak orbits for flows

Let X be a vector field on a d-dimensional closed manifold M.

Definition 8.1.3. We say that a point x generates a *J-weak orbit* (for the flow of X) if there exists an integer $N > 0$ such that if $t \geq N$ we have

$$\det DX_t(x) < (1+\delta)^t.$$

The set of all points generating J-weak orbits is denoted by $\Lambda(\delta, X)$.

As in Lemma 8.1.2 our goal is to prove that $m(\Lambda(\delta, X)) = 1$. The proof of Lemma 8.1.2 is based on a very elegant idea: a diffeomorphism cannot expand volume along large strings of orbits in a positive measure set. However, implicity in the proof is a countability argument which takes advantage that the time is given by integers. This is the reason why neither the result nor the proof of Araújo can be immediately translated for flows: the time is given by an uncountable set!

To handle this difficult we shall apply Lemma 8.1.2 to X_t, for small values of t, and try to "propagate" the J-weakness for all times nearby nt, with $n \in \mathbb{N}$.

More precisely, first, we notice that for any $t \in \mathbb{R}$ by applying Lemma 8.1.2 to the diffeomorphism X_t one obtains $m(\Lambda(\delta, X_t)) = 1$. Besides that, we have the following

Lemma 8.1.4. *Given $\varepsilon > 0$ there exists $q \in \mathbb{N}$ such that if $|t| < 1/q$ then for any $y \in M$ we have*

$$|\det DX_t(y) - 1| < \varepsilon.$$

Proof. Let $\varepsilon > 0$ be given. Since $X_0 = Id$, notice that $\det DX_0(y) = 1$, for every $y \in M$. Since the map $\Phi : \mathbb{R} \times M \to \mathbb{R}$, $\Phi(t, x) = \det DX_t(x)$ is continuous and M is compact, there exists a $\delta > 0$ such that $|t| < \delta$ implies $|\det DX_t(y) - 1| < \varepsilon$, for every $y \in M$. Now, just take q large enough. $\square$

We are now in position to prove

Lemma 8.1.5. $m\,(\Lambda(\delta, X)) = 1.$

Proof. Take $\delta > \delta' > 0$ and $\varepsilon > 0$ such that

$$1 + \varepsilon < \frac{1+\delta}{1+\delta'}.$$

With this ε by Lemma 8.1.4 one obtains a large integer $q \in \mathbb{N}$. Lemma 8.1.2 implies that $m\left(\Lambda(\delta', X_{1/q})\right) = 1$. To complete the proof it suffices to establish that $\Lambda(\delta', X_{1/q}) \subset \Lambda(\delta, X)$.

For this, notice that for any $x \in \Lambda(\delta', X_{1/q})$ there exists $N > 0$ such that

$$\det DX_{n/q}(x) < (1+\delta')^{n/q} \text{ for every } n \geq N.$$

Take $t \geq N$. There exist $n \in \mathbb{N}$ such that

$$\frac{n}{q} \leq t \leq \frac{n+1}{q}.$$

Thus, using Lemma 8.1.4 we obtain

$$\begin{aligned} |\det DX_t(x) - \det DX_{n/q}(x)| &= |(\det DX_{t-\frac{n}{q}}(X_{\frac{n}{q}}(x)) - 1)| \det DX_{\frac{n}{q}}(x) \\ &< \varepsilon(1+\delta')^{n/q} < \varepsilon(1+\delta')^t. \end{aligned}$$

Thus, for any $x \in \Lambda(\delta', X_{1/q})$ there exist N such that for every $t \geq N$ we have

$$\det DX_t(x) < (1+\varepsilon)(1+\delta')^t < (1+\delta)^t,$$

due to our choice of ε and δ'. This completes the proof. $\square$

By considering accumulations of measures uniformly distributed along the orbit of a point in $\Lambda(\delta, X)$, we shall obtain below a large set in M generating invariant measures with zero average dissipation (Lemma 8.1.6).

Indeed, given a three-dimensional flow X and $t > 0$ we define the Borel probability measure

$$\mu_{p,t} = \frac{1}{t}\int_0^t \delta_{X_s(p)} ds.$$

(Sometimes we write $\mu^X_{p,t}$ to indicate the dependence on X.)

Denote by $\mathcal{M}(p, X)$ as the set of Borel probability measures $\mu = \lim_{k\to\infty} \mu_{p,t_k}$ for some sequence $t_k \to \infty$. Recall that a Borel probability measure μ on M is *invariant* under X if $\mu \circ X_{-t} = \mu$ for every $t \geq 0$. Notice that every measure in $\mathcal{M}(p, X)$ is invariant. With these notations we have the following lemma.

Lemma 8.1.6. *For every three-dimensional flow X there is a full Lebesgue measure set L_X of points x satisfying*

$$\int \operatorname{div} X d\mu \leq 0, \qquad \forall \mu \in \mathcal{M}(x, X).$$

Proof. Define

$$L_X = \bigcap_{k\in\mathbb{N}^+} \Lambda(1/k, X).$$

By Lemma 8.1.5, $m(L_X) = 1$.

Take $x \in L_X$, $\mu \in \mathcal{M}(x, X)$ and $\epsilon > 0$. Fix $k > 0$ with $\log\left(1 + \frac{1}{k}\right) < \epsilon$.

By definition we have $x \in \Lambda(1/k, X)$. and so there is $N \in \mathbb{N}$ such that

$$|\det DX_t(x)|^{\frac{1}{t}} < 1 + \frac{1}{k}, \qquad \forall t \geq N.$$

Take a sequence $\mu_{x,t_i} \to \mu$ with $t_i \to \infty$. Then, we can assume $t_i \geq N$ for all i. From this and the Liouville's Formula [Ma1] we obtain,

$$\int \operatorname{div} X d\mu = \lim_{i\to\infty} \int \operatorname{div} X d\mu_{x,t_i} = \lim_{i\to\infty} \frac{1}{t_i}\int_0^{t_i} \operatorname{div} X(X_s(x))ds =$$

$$\lim_{i\to\infty} \frac{1}{t_i} \log|\det DX_{t_i}(x)| \leq \log\left(1 + \frac{1}{k}\right) < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we obtain the result. □

8.1.3 Lyapunov stable sets and J-weak orbits

Lemma 8.1.7. *Let $X \in \mathfrak{X}^1(M)$ and let Λ be a Lyapunov stable set for X. Then, there exists an ergodic measure μ with* $\operatorname{supp}(\mu) \subset \Lambda$ *and such that* $\int \operatorname{div}(X)d\mu \leq 0$

Proof. First note that, by definition of a Lyapunov stable set, given any neighborhood U of Λ there exists a neighborhood $\Lambda \subset V \subset U$ such that for any $x \in V$, $X_t(x) \in U$ for all $t \geq 0$. In particular, for every $x \in V$ and for every $\nu \in \mathcal{M}(x, X)$, one has $\text{supp}(\nu) \subset \overline{U}$. By Lemma 8.1.6 there exists some point $x \in V$ such that for any $\nu \in \mathcal{M}(x, X)$ one has

$$\int \text{div}(X) d\nu \leq 0. \tag{8.2}$$

This gives a sequence of meausres ν_n, satisfying equation (8.2) and supported on the open sets

$$U_n = \{x \in M;\ d(x, \Lambda) < \frac{1}{n}\}.$$

Taking a subsequence, if necessary, we can suppose that ν_n converge in the wak topolgy for some measure μ. Since the convergence is weak, μ also satisfies equation (8.2).

We claim that $\text{supp}(\mu) \subset \Lambda$. To see this, we argue by contradiction. If there is some point $y \in \text{supp}(\mu) \cap (M - \Lambda)$, then, by compactness of Λ there exists $r > 0$ such that for every sufficiently large n,

$$B(x, 2r) \cap U_n = \emptyset.$$

We can take a continuous function $\phi : M \to \mathbb{R}$ that is identiclly 1 in the ball $B(x, r)$ and that is zero outside the ball $B(x, 2r)$. Since $x \in \text{supp}(\mu)$,

$$\int \phi d\mu \geq \mu(B(x, r)) > 0.$$

However, for every large n,

$$\int \phi d\mu_n = 0,$$

and this contradicts $\mu_n \to \mu$ in the weak topology.

Therefore, μ is an invariant measure supported on Λ and satisfing equation (8.2). By ergodic decomposition, the conclusion follows. $\square$

8.2 Structure of the proof

In this section we state some C^1-generic results that we shall combine together to obtain Theorem E. Some of these results are straightforward applications of known results. For those, we present a proof in this section. The other ones, which require longer proofs (known or new), we shall proof in the remaining parts of the chapter.

Lebesgue measure of stable sets

Bowen proved [Bow1] that, for a C^2 vector field, a hyperbolic set attracts a positive measure subset of M if and only if it is a topological attractor. Since C^2 vector fields form a dense subset of $\mathfrak{X}^1(M)$ and since hyperbolic sets are stable it is natural to conjecture that the same statement will be true for C^1-generic vector fields. As we could not found a proof in the literature, we shall present a proof of this here, for isolated hyperbolic sets.

Proposition 8.2.1. *Let X be a C^1 generic vector field over M, and let Γ be an isolated hyperbolic set. If Γ is not an attractor then $m\left(W^s(\Gamma)\right) = 0$.*

The proof is postponed to Section 8.3

Dominated splitting over dissipative orbits

In order to benefit from the previous result, we shall study dissipative periodic orbits and show that, when X has finitely many sinks robustly, the closure of the set of dissipative periodic orbits admits a dominated splitting for the Linear Poincaré Flow.

The precise definition is the following.

Definition 8.2.2. A vector field X is said to have finitely many sinks robustly if it has a neighborhood $\mathcal{U} \subset \mathfrak{X}^1(M)$ such that if $Y \in \mathcal{U}$ then Y has finitely many sinks and $\mathrm{card}(\mathrm{Sink}(Y)) = \mathrm{card}(\mathrm{Sink}(X))$. We denote by $S(M)$ the set of vector fields with finitely many sinks robustly.

The lemma below motivates our interest in the class $S(M)$.

Lemma 8.2.3. *If a C^1-generic vector field X has finitely many sinks then $X \in S(M)$.*

Proof. By classical hyperbolic theory the map $X \in \mathfrak{X}^1(M) \mapsto \overline{\mathrm{Sink}(X)} \in \mathcal{K}(M)$, where $\mathcal{K}(M)$ is the space of compact subsets of M endowed with the Hausdorff topology, is lower semicontinuous. By a classical topological result, there existis a residual subset of $\mathfrak{X}^1(M)$ where this map is continuous. So, take $X \in \mathfrak{X}^1(M)$ which is a continuity point of the map $X \in \mathfrak{X}^1(M) \mapsto \overline{\text{Sink}(X)} \in \mathcal{K}(M)$. If X has finitely many sinks, say

$$\text{Sink}(X) = \{\gamma_1, ..., \gamma_n\},$$

it follows that for every Y sufficiently close to X, $\text{Sink}(Y) = \{\gamma_1^Y, ..., \gamma_n^Y\}$ where each γ_i^Y is the unique continuation of γ_i. This proves the lemma. $\square$

Recall the definition of the so-called linear Poincaré Flow. Given a non-singular point $x \in M$, denote by N_x the subspace of T_xM orthogonal to $\langle X \rangle$, we define the Linear Poincaré Flow by

$$P_t(x)v := \pi(X_t(x))DX_t(x)v,$$

where $\pi(y) : T_yM \to N_y$ is the orthogonal projection in the direction of the vector field. Notice that the Linear Poincaré Flow is the derivative of the holonomies of orthogonal sections to the flow (see Lemma 2.4.1).

When x is a hyperbolic periodic point, the hyperbolic subspaces are projected into subspaces invariant by the Linear Poincaré Flow, and we call these subspaces the hyperbolic decomposition with respect to the Linear Poincaré Flow.

Definition 8.2.4. Given $\delta > 0$ we say that a periodic point p, with period $\pi(p)$ belongs to the set $\text{P}(\delta, X)$ if

$$\frac{1}{\pi(p)} \log |\det P_{\pi(p)}(p)| < \delta.$$

The heuristic idea behind this definition is that these points have almost more contraction than expansion. In particular, there are no sources in $\text{P}(\delta, X)$.

Definition 8.2.5. Let Λ be an invariant set. Let $\Lambda_0 := \Lambda - \text{Zero}(X)$ and consider the normal bundle N defined in Λ_0. We say that Λ admits a *dominated splitting* for the Linear Poincaré Flow if there exists a decomposition $N_x = E_x \oplus F_x$, for every $x \in \Lambda_0$ and constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|P_t(x)|_E\| \|P_{-t}(X_t(x))|_F\| \leq c\lambda^t,$$

for every $t \geq 0$.

Proposition 8.2.6 implies the existence of a dominated splitting $E^s \oplus E^u$ for the Linear Poincaré Flow over $\overline{\mathrm{P}(\delta, X)} - \mathrm{Zero}(X)$.

Proposition 8.2.6. *Given $X \in S(M)$ there is a C^1 neighborhood $\mathcal{U}$ of X, and constants $C > 0$, $0 < \lambda < 1$ and $\delta_0 > 0$ such that for every $Y \in \mathcal{U}$ and $p \in \mathrm{P}(\delta, Y)$, with $0 < \delta \leq \delta_0$ if $E^s \oplus E^u$ is the hyperbolic decomposition for the Linear Poincaré Flow over the orbit of p, then*

$$\left\| P_t^Y(p)|_{E^s} \right\| \left\| P_{-t}^Y(Y_t(p))|_{E^u} \right\| \leq C\lambda^t,$$

for every $t \geq 0$.

The proof is a small adaptation of the original argument of Araújo. The ideas is to use the arguments from Mañé's paper [Ma2], making sure that the same perturbations used there can be performed within the dissipative periodic orbits. For completeness, we shall present a proof in Section 8.4.

Domination and hyperbolicity

In [AH] Arroyo and Hertz extended to non singular three dimensional flows a deep result for surface diffeomorphisms due to Pujals and Sambarino [PS]. They proved that for a C^2 nonsingular flow domination over a transitive invariant set implies hyperbolicity. Using a clever trick of [BGY] we shall prove that for C^1 generic three dimensional flows domination, over *any invariant compact set*, implies hyperbolicity

Proposition 8.2.7. *Let X be a C^1 generic vector field over M. Assume that Λ is a compact invariant, admitting a dominated splitting for the Linear Poincaré Flow and with* $\mathrm{Zero}(X) \cap \Lambda = \emptyset$. *Then, Λ is a hyperbolic set for X.*

This proposition is an improvement of Lemma 3.1 in [BGY], and will be proved in Section 8.5.

Neutral homoclinic classes

A result of Carballo, Morales and Pacífico [CMP] says that, for C^1-generic vector fields, homoclinic classes are neutral: the intersection of a Lyapunov stable set for X with a Lyapunov stable set for $-X$. Using this, one easily obtains the following

Lemma 8.2.8. *Let X be a C^1 generic vector field over M. Let $H(p)$ be a homoclinic class of X and $x \in M$. If $\omega(x) \cap H(p) \neq \emptyset$ then $\omega(x) \subset H(p)$.*

Proof. Let X be a C^1 generic vector field over M and let $H(p)$ be a homoclinic class of a periodic orbit of X. Assume that there exists $x \in M$ such that $\omega(x) \cap H(p) \neq \emptyset$. By Theorem 3.1 in [CMP] one knows that there exists a Lyapunov stable set for Λ and a Lyapunov stable unstable Γ, such that $H(p) = \Lambda \cap \Gamma$. Applying Lemma 2.2.1, one obtains that $\omega(x) \subset \Lambda \cap \Gamma = H(p)$. $\square$

Omega-limit sets of J-weak orbits

To obtain that the basin of the hyperbolic attractors cover a full Lebesgue measure subset of M we shall prove that they contain the sets L_X (see Lemma 8.1.6). The key result is the following.

Lemma 8.2.9. *For every $\delta > 0$, for every C^1 generic non-singular vector field X and for every $x \in L_X$, we have $\omega(x) \cap \overline{\mathrm{P}(\delta, X)} \neq \emptyset$.*

Proof. Lemma 8.1.6 implies that every $\mu \in \mathcal{M}(x, X)$ satisfies $\int \mathrm{div}(X) d\mu \leq 0$. By ergodic decomposition one obtains an ergodic invariant measure for X, whose suport is contained in $\omega(x)$ and such that $\int \mathrm{div}(X) d\mu \leq 0$. By the Ergodic Closing Lemma [Ma1] (see also Theorem 3.5 in [AS] for the C^1 generic version for flows) there exists periodic points σ_n (since $X \in \mathfrak{X}^1_{NS}(M)$) such that the measures $\mu_n := \int \delta_{X_t(\sigma_n)} dt$ satisfy

1. μ_n converge to μ in the weak topology
2. $\mathrm{supp}(\mu_n) \to \mathrm{supp}(\mu)$ in the Hausdorff topology.

In particular, $\sigma_n \in \mathrm{P}(\delta, X)$ for every n large. Since $\mathrm{supp}(\mu) \subset \omega(x)$, the proof is complete. $\square$

Lyapunov stable sets and dissipative orbits

To profit from the previous results, we need to relate attactors with the dissipative periodic orbits $\mathrm{P}(\delta, X)$.

Lemma 8.2.10. *Let X be a C^1 generic vector non-singular vector field, let $\delta > 0$ and Λ be a Lyapunov stable set. Then, $\Lambda \cap \overline{\mathrm{P}(\delta, X)} \neq \emptyset$.*

Proof. By Lemma 8.1.7, there exists an ergodic measure μ with $\mathrm{supp}(\mu) \subset \Lambda$ and such that $\int \mathrm{div}(X) d\mu \leq 0$. The conclusion follows once more from the Ergodic Closing Lemma. $\square$

Proof of Theorem E

Let X be a C^1 generic, non-singular, vector field over the 3-manifold M. Assume that X has finitely many sinks. Then, by Lemma 8.2.3 $X \in S(M)$. Take $\delta > 0$ sufficiently small. Then, Proposition 8.2.6 implies that $\mathrm{P}(\delta, X)$ admits a dominated splitting. By Proposition 8.2.7, $\overline{\mathrm{P}(\delta, X)}$ is a hyperbolic set. In particular, there exists $p_1, .., p_n \in \mathrm{P}(\delta, X)$ such that

$$\overline{\mathrm{P}(\delta, X)} = \bigcup_{l=1}^{n} H(p_l) \cup \mathrm{Sink}(X),$$

and each $H(p_l)$ is a hyperbolic isolated set.

Claim 12. *All attractors of X are subsets of $\overline{\mathrm{P}(\delta, X)}$. More precesely, let Λ be an attractor of X. Then, either Λ is a periodic sink or $\Lambda = H(p_l)$, for some $l \in \{1, .., n\}$.*

Proof. Assume that there exists Λ an attractor for X, which is not a sink. In particular, Λ is a Lyapunov stable set. By Lemma 8.2.10, $\Lambda \cap \overline{\mathrm{P}(\delta, X)} \neq \emptyset$. It follows from Lemma 2.2.1 that $\Lambda = H(p_l)$, for some $l \in \{1, .., n\}$. $\square$

In particular, X has finitely many attractors.

Let $I \subset \{1, .., n\}$ be such that $i \in I$ if, and only if $H(p_i)$ is not an attractor.

Claim 13. $m\left(\{x \in L_X; \omega(x) \cap H(p_i) \neq \emptyset,\ \textit{for some}\, i \in I\}\right) = 0.$

Proof. Take $x \in L_X$ such that $\omega(x) \cap H(p_i) \neq \emptyset$, for some $I \in I$. Since X is C^1 generic, Lemma 8.2.8 proves that $x \in W^s(H(p_i))$. It remains to notice that

$$m\left(\bigcup_{i \in I} W^s(H(p_i))\right) = 0,$$

by Proposition 8.2.1. □

Consider the set $B_X = L_X \setminus \{x \in L_X; \omega(x) \cap H(p_i) \neq \emptyset,\ \text{for some}\, i \in I\}$. Claim 13 and Lemma 8.1.6 pove that $m(B_X) = 1$. Take a point $x \in B_X$. By Lemma 8.2.9, either $x \in W^s(\mathrm{Sink}(X))$ or $x \in W^s(H(p_l))$, and $H(p_l)$ is a hyperbolic attractor.

This proves that X has finitely many attractors whose basins cover a full Lebesgue measure subset of M, and establishes Theorem E. □

8.3 Lebesgue measure of stable sets

Proposition 8.2.1 is possibly a folklore result. We shall present a proof here for the sake of completeness.

Proof of Proposition 8.2.1. Let $\mathcal{B}_0$ be a countable base of open sets for the topology of M. Let $\mathcal{B}$ the set whose elements are finite unions of elements in $\mathcal{B}_0$. For each $U \in \mathcal{B}$, recall that $\Lambda_X(U)$ denotes the maximal invariant for the flow of X in U. Consider the set

$$\mathcal{O}_U = \{X \in \mathfrak{X}^1(M); \Lambda_X(U) \text{ is hyperbolic and is not an attractor}\}.$$

Claim 14. *$\mathcal{O}_U$ is open in $\mathfrak{X}^1(M)$.*

Proof. Take $X \in \mathcal{O}_U$. By continuation of hyperbolic sets, there exists a neighborhood $\mathcal{U}$ of X such that $\Lambda_Y(U)$ is a hyperbolic set, for every $Y \in \mathcal{U}$. Moreover, there exists a homeomorhism $h : \Lambda_Y(U) \to \Lambda_X(U)$, which conjugates the dynamics. We shall prove that

$\mathcal{U} \subset \mathcal{O}_U$. Assume that this is not true. Then, there exists $Y \in \mathcal{U}$ such that $\Lambda_Y(U)$ is an attractor. In particular, $W^u_Y(y) \subset \Lambda_Y(U)$, for every $y \in \Lambda_Y(U)$. As a consequence,

$$W^u_X(h(y)) = h\left(W^u(y)\right) \subset \Lambda_X(U),$$

for every $y \in \Lambda_Y(U)$. This implies that $\Lambda_X(U)$ contains the unstable manifolds of all its points, and thus must be an attractor, a contradiction. This proves the claim. □

Denote by $\Lambda^-_X(U) = \cap_{t\leq 0} X_t(U)$. Consider now $\mathcal{O}_{U,n} = \{X \in \mathcal{O}_U; m(\Lambda^-_X(U)) < 1/n\}$. Each $\mathcal{O}_{U,n}$ is open, by Lemma 17 of [AO].

Claim 15. *$\mathcal{O}_{U,n}$ is dense in $\mathcal{O}_U$*

Proof. Take $x \in \Lambda^-_X(U)$. Then, by definition, $X_t(x) \in U$, for every $t > 0$. In particular, $\omega(x) \subset \Lambda_X(U)$. Therefore, $\Lambda^-_X(U) \subset W^s(\Lambda_X(U))$ and thus, if X is C^2, we have $m\left(\Lambda^-_X(U)\right) = 0$. Since, $\mathcal{O}_U$ is open in $\mathfrak{X}^1(M)$ and the set of C^2 vector fields is dense in $\mathfrak{X}^1(M)$, the claim is proved. □

Consider $\mathcal{R} = \bigcap_{U\in\mathcal{B}, n\in\mathbb{N}} \mathcal{O}_{U,n} \cup (\mathfrak{X}^1(M) \setminus \overline{\mathcal{O}_U})$. Then, $\mathcal{R}$ is a residual subset of $\mathfrak{X}^1(M)$. Take $X \in \mathcal{R}$. Let Γ be an isolated hyperbolic set which is not an attractor. Since Γ is isolated, there exists $U \in \mathcal{B}$ such that $\Gamma = \Lambda_X(U)$. Since Γ is not an attractor, $X \in \mathcal{O}_{U,n}$, for every $n \in \mathbb{N}$. Thus, $m\left(\Lambda^-_X(U)\right) = 0$ It reamains to notice the following

Claim 16. $W^s(\Lambda_X(U)) = \bigcup_{n\in\mathbb{N}} X_{-n}(\Lambda^-_X(U))$

Proof. If $x \in W^s(\Lambda_X(U))$ then there exists an integer $N > 0$ such that $X_N(x) \in U$ and $X_t(X_N(x)) \in U$, for every $t \geq 0$. Thus, $X_N(x) \in \cap_{t\leq 0} X_t(U)$, and therefore $x \in X_{-N}(\Lambda^-_X(U))$. Conversely, if $x \in X_{-n}(\Lambda^-_X(U))$ then $X_n(x) \in \Lambda^-_X(U)$ and thus $X_{n+t}(x) \in U$, for every $t \geq 0$, and thus $\omega(x) \subset \Lambda_X(U)$. Onde deduces that $X_{-n}(\Lambda^-_X(U)) \subset W^s(\Lambda_X(U))$, for every n, concluding. □

This completes the proof of Proposition 8.2.1. □

8.4 Dominated splitting over dissipative orbits

To prove Proposition 8.2.6 we shall need the following lemmas.

Lemma 8.4.1. *Let $X \in S(M)$. There exist $\delta_1 > 0$, $c > 0$ and a neighborhood $\mathcal{U}$ of X such that if $Y \in \mathcal{U}$, $x \in \mathrm{P}(\delta, Y)$ with $\delta < \delta_1$ is not a sink and has period T then the characteristic multipliers λ and μ of p are real and satisfy*

$$\log|\lambda| < -cT < 0 < cT < \log|\mu|.$$

Proof. During this proof we shall assume that $|\lambda| \leq |\mu|$. The hypotesis $p \in \mathrm{P}(\delta, Y)$ implies that $|\lambda\mu| < e^{T\delta}$. Thus, we cannot have a complex eigenvalue. Indeed, since the linear map $P_T^Y : N_p \to N_p$ is two dimensional, if λ is the complex eigenvalue then $|\lambda| < e^{T\delta}$. So, performimg a linear perturbation and using Frank's lemma, we can turn this periodic orbit into a sink, a contradiction with $X \in S(M)$. So, we have two real eigenvalues $|\lambda| \leq 1 \leq |\mu|$.

Now, assume by contradiction that for every $\delta > 0$ small and every $c > 0$ we can find $p \in \mathrm{P}(\delta, Y)$, for some Y as close as we want to X, with

$$|\mu| < e^{Tc}.$$

Then, we perform a linear perturbation of the form

$$A|_{E^u} = e^{-Tc} P_T^Y,$$

and by virtue of Frank's Lemma again we obtain a sink for some Z close to X, breaking down our hypotesis. Therefore, one of the inequalities we are looking for is true.

On the other hand, if the inequality with λ is not true for some Y close to X and $p \in \mathrm{P}(\delta, Y)$, taking a small δ and using that $|\lambda\mu| < e^{T\delta}$, we can violate the other inequality that we had just proved, an absurd. This proves the lemma. □

Lemma 8.4.2. *Let $X \in S(M)$. Then, there exists $\delta > 0$ a C^1 neighborhood $\mathcal{U}$ of X and a constant $\alpha > 0$ such that for every $Y \in \mathcal{U}$ and $p \in \mathrm{P}(\delta, Y)$ we have* $\mathrm{Angle}(E_p^s, E_p^u) > \alpha$.

Proof. Let us suppose by contradiction that for some Y arbitrarily close to X, $\mathrm{Angle}(E_p^s, E_p^u) := \gamma$ is close to zero at some $p \in \mathrm{P}(\delta, Y)$. Consider a orthonormal basis of N_p whose first vector is some unitary vector $s \in E_p^s$, and the second vector is $s^\perp \in [E_p^s]^\perp$. In this basis

$$P^Y_{\pi(p)}(p) = \begin{bmatrix} \lambda & \frac{\mu-\lambda}{\gamma} \\ 0 & \mu \end{bmatrix}.$$

Let me explain how one sees this. Observe that

$$s + \gamma s^{\perp} = u \in E^u_p,$$

thus

$$\lambda s + \gamma P^Y_{\pi(p)} s^{\perp} = \mu u.$$

Rearranging things one easily gets that

$$P^Y_{\pi(p)} s^{\perp} = \frac{\mu - \lambda}{\gamma} s + \mu s^{\perp}.$$

Consider a matrix of the form

$$A(\beta) = \begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix},$$

in this basis, and note that we can choose an arbitrary small number β such that $B(\beta) = A(\beta)P^Y_{\pi(p)}(p)$ has two eigenvalues with modulus $\sqrt{|\lambda\mu|}$, wich is either biger then 1, or smaller or equal to 1.

Nevertheless, using Franks' Lemma we perturb Y to Z in such a way that $p \in \mathrm{Per}(Z)$, and

$$DZ_{\pi(p)}(p) = A(\beta) DY_{\pi(p)}(p).$$

Note that

$$\frac{1}{\pi(p)} \log|\det DZ_{\pi(p)}(p)| = \frac{1}{\pi(p)} \log|\det DY_{\pi(p)}(p)| < \delta,$$

and thus $p \in \mathrm{P}(\delta, Z)$. Since $DZ_{\pi(p)}(p)$ has two eigenvalues with the same modulus, we get a contradiction with lemma 8.4.1. $\square$

With these preparatory lemmas we can conclude the proof of Proposition 8.2.6. The argument is a simple adaptation of the argument in Araújo's thesis [A].

Proof of Proposition 8.2.6. It's enough to prove the following statement: there exists a C^1-neighborhood $\mathcal{U}$ of X and a number $T > 0$ such that for every $Y \in \mathcal{U}$ and every $p \in \mathrm{P}(\delta, Y)$ there exists $0 \leq t \leq T$ such that

$$\left\|P_t^Y(p)|_{E^s}\right\| \left\|P_{-t}^Y(X_t(p))|_{E^u}\right\| < \frac{1}{2}.$$

Suppose this is not true. Then, for every $T > 0$ there exists a vector field Y, as close as we want to X, and a periodic point $p \in \mathrm{P}(\delta, Y)$ such that

$$\left\|P_t^Y(p)|_{E^s}\right\| \left\|P_{-t}^Y(X_t(p))|_{E^u}\right\| \geq \frac{1}{2}, \tag{8.3}$$

for every $0 \leq t \leq T$. We can assume that $\pi(p) \to \infty$ as $Y \to X$. Indeed, by lemma 8.4.1

$$\begin{aligned} \left\|P_{k\pi(p)}^Y(p)|_{E^s}\right\| \left\|P_{-k\pi(p)}^Y(p)|_{E^u}\right\| &\leq e^{-2ck\pi(p)} \\ &\to 0 \text{ when } k \to \infty, \end{aligned}$$

and thus there exists k_0, wich depends only upon c, such that

$$\left\|P_{k_0\pi(p)}^Y(p)|_{E^s}\right\| \left\|P_{-k_0\pi(p)}^Y(p)|_{E^u}\right\| < \frac{1}{2}.$$

This implies that $\pi(p) \geq \frac{T}{k_0}$, and since T is large, $\pi(p)$ is also large.

Take $v \in E_p^s$ and $u \in E_p^u$ unitary vectors. By (8.3) we have that $\frac{\|P_t^Y(p)v\|}{\|P_t^Y(p)u\|} \geq \frac{1}{2}$, for every $0 \leq t \leq T$. Now, define $L : E_p^u \to E_p^s$ by $L(u) = \alpha v$, for a small $\alpha > 0$ and $\overline{L} : E_p^u \to E_p^s$ by

$$\overline{L} := (1+\alpha)^{\pi(p)} P_{\pi(p)}^Y(p)|_{E^s} \circ L \circ P_{-\pi(p)}^Y(p)|_{E^u}.$$

If α is small enough then

$$\|\overline{L}\| \leq (1+\alpha)^{\pi(p)} e^{-2c\pi(p)} \|L\| \leq \alpha.$$

Consider the maps $P, S : N_p \to N_p$ given by

- $Pu = Lu$ and $Pv = 0$
- $Sv = 0$ and $S(u + \overline{L}u) = -\overline{L}u.$

This maps have the following four properties

$$P|_{E_p^s} = 0,\ (Id + P)E_p^u = G,\ S|_{E_p^s} = 0 \text{ e } (Id + S)\overline{G} = E_p^u,$$

where $G := \{u + Lu; u \in E_p^u\}$ e $\overline{G} := \{u + \overline{L}u; u \in E_p^u\}$. Moreover, observe that

$$\det(Id + S) = 1, \text{ and } \det(Id + P) \leq 1 + \alpha.$$

Using lemma 8.4.2 and Lemma II.10 of [Ma2] one can see that

$$||P|| \leq \left(\frac{1+\gamma}{\gamma}\right)\alpha \text{ e } ||S|| \leq \left(\frac{1+\gamma}{\gamma}\right)\alpha.$$

Finally, we define $T_j : T_{Y_j(p)}M \to T_{Y_{j+1}(p)}M$ by $T_j|_{E^s_{Y_j(p)}} = \alpha Id$ and $T_j|_{E^u_{Y_j(p)}} = 0$, for every integer $0 \leq j \leq [\pi(p)]$, and $T_{\pi(p)} : T_{Y_{[\pi(p)]}(p)}M \to T_pM$ defined in the same way, where $[\pi(p)]$ means the integer part of $\pi(p)$. It's easy to see that for every $j = 1, ...\pi(p)$, $\det(Id + T_j) = 1 + \alpha$.

Now, we are in conditions to define the perturbations. The first one is built up in order to slightly rotate [1] E_p^u, sending it to G, and then we go along the orbit of p through P_1^Y and in the end we make a small stretch. In formulas:

$$L_0 = (1 + \alpha)^{-1}(Id + T_1) \circ P_1^Y(p) \circ (Id + P) : T_pM \to T_{P_1^Y(p)}M.$$

The stretch made by the map $Id + T_1$ is devised make G lay down (a little bit) to the subspace E^s. In order to use the absence of domination to continue this and produce a small angle between G and E^s we define the maps

$$L_j = (Id + T_{j+1}) \circ P_1^Y(Y_j(p)) : T_{Y_j(p)}M \to T_{Y_{j+1}(p)}M,$$

for $1 \leq j \leq [pi(p)]$ and, in the final steps, we also use the map $Id + S$ to send $\overline{G}$ back to E_p^u via

$$L_{\pi(p)-[\pi(p)]} = (Id + S) \circ (Id + T_0) \circ P^Y_{\pi(p)-[\pi(p)]}(Y_{[\pi(p)]}) : T_{g^{\pi(p)-1}(p)}M \to T_pM.$$

It can be easily seen that $\|L_j - P_1^Y(Y_j(p))\|$ is as small as we please, provided that α is small enough. By Franks' Lemma, we can find a vector field Z close to Y, and therefore close to X, such that $O_Y(p)$ is a periodic orbit for Z, moreover, $P_1^Z(Y_j(p)) = L_j$, for every

[1]Even though its matrix is triangular, not a rotation!

j. Note that

$$\begin{aligned}
\frac{1}{\pi(p)} \log |\det P^Z_{\pi(p)}(p)| &= \frac{1}{\pi(p)} \log \left| \prod_{j=0}^{\pi(p)-1} \det L_j \right| \\
&\le \frac{1}{\pi(p)} \log (1+\alpha)^{\pi(p)} \det P^Y_{\pi(p)}(p) \\
&= \log(1+\alpha) + \frac{1}{\pi(p)} \log |\det P^Y_{\pi(p)}(p) < \delta,
\end{aligned}$$

if α is small enough, and thus $p \in \mathrm{P}(\delta, Z)$. Also, by shrinking α once again, one obtains that E^s_p is the stable space of p. By lemma 8.4.1 p is hyperbolic and thus, since E^u_p is invariant under $P^Z_{\pi(p)}(p)$ we have that it is the unstable space of p. But from this we can derive a contradiction as follows. Take m big, and consider $E^\sigma_m = P^Z_m(p)E^\sigma_p$, $\sigma = s, u$. Let $\beta := \mathrm{Angle}(E^s_m, E^u_m)$ and consider the vectors

$$u_1 = L_m(u) = P^Z_m(p)u + \delta(1+\delta)^m P^Z_m(p)v,$$

and

$$u_2 = L_m(\delta v) = \delta(1+\delta)^m P^Z_m(p)v.$$

Using Lemma II.10 of [Ma2] again it follows that

$$\begin{aligned}
||P^Z_m(p)u|| &= ||u_1 - u_2|| \ge \left(\frac{\beta}{1+\beta}\right) ||u_1|| \\
&\ge \frac{\beta}{1+\beta} \left(\delta(1+\delta)^m ||P^Z_m(p)v|| - ||P^Z_m(p)u||\right).
\end{aligned}$$

Thus,

$$1 \ge \frac{\beta}{1+\beta} \left(\frac{\delta(1+\delta)^m}{2} - 1\right),$$

since

$$\frac{1}{2} ||P^Z_m(p)u|| \le ||P^Z_m(p)v||.$$

Hence

$$\beta \le \frac{2}{\delta(1+\delta)^m - 4} \le \gamma,$$

provided that m is big enough to $\delta(1+\delta)^m \ge 4 + \frac{2}{\gamma}$., wich can be done because $\pi(p) \to \infty$ as $Y \to X$. This contradicts Proposition 8.4.2, and completes the proof. $\square$

8.5 Domination and hyperbolicity

We now give the proof of Proposition 8.2.7.

Proof of Proposition 8.2.7. By Lemma 3.1 in [BGY] we have that there is a residual subset $\mathcal{Q}_1$ of three-dimensional flows for which every transitive set without singularities but with a LPF-dominated splitting is hyperbolic. Fix $X \in \mathcal{Q}_1$ and a compact invariant set Λ without singularities but with a LPF-dominated splitting $N^X_\Lambda = N^{s,X}_\Lambda \oplus N^{u,X}_\Lambda$. Suppose by contradiction that Λ is not hyperbolic. Then, by Zorn's Lemma, there is a minimally nonhyperbolic set $\Lambda_0 \subset \Lambda$ (c.f. p.983 in [PS]). Assume for a while that Λ_0 is not transitive. Then, $\omega(x)$ and $\alpha(x) = \omega_{-X}(x)$ are proper subsets of Λ_0, for every $x \in \Lambda_0$. Therefore, both sets are hyperbolic and then we have

$$\lim_{t\to\infty} \|P^X_t(x)/N^{s,X}_x\| = \lim_{t\to\infty} \|P^X_{-t}(x)/N^{u,X}_x\| = 0, \qquad \text{for all } x \in \Lambda_0,$$

which easily implies that Λ_0 is hyperbolic (see [ASS]). Since this is a contradiction, we conclude that Λ_0 is transitive. As $X \in \mathcal{Q}_1$ and Λ_0 has a LPF-dominated splitting (by restriction), we conclude that Λ_0 is hyperbolic, a contradiction once more proving the result.

□

Chapter 9

Lyapunov Stability and Sectional Hyperbolicity for Higher Dimensional Flows

In this chapter we present proof of our result about generic vector fields without points accumulated by periodic orbits of different indices, namely we prove the following.

Theorem F. *Let $X \in \mathcal{X}^1$ be a C^1-generic vector field without points accumulated by hyperbolic periodic orbits of different Morse indices. Then, X has finitely many sinks and sectional-hyperbolic transitive Lyapunov stable sets for which the union of the basins is residual in M.*

The presentation has no substantial changes from the original article.

9.1 Tools

Previously we state some basic results. The first one is the main result in [MP1].

Lemma 9.1.1. *For every C^1-generic vector field $X \in \mathcal{X}^1$ there is a residual subset R_X of M such that $\omega(x)$ is a Lyapunov stable set, $\forall x \in R_X$.*

With the same methods as in [CMP] and [MP1] it is possible to prove the following variation of this lemma. We shall use the standard stable and unstable manifold operations $W^s(\cdot), W^u(\cdot)$ (c.f. [HPS]).

Lemma 9.1.2. *For every C^1-generic vector field $X \in \mathcal{X}^1$ and every hyperbolic closed orbit O of X the set $\{x \in W^u(O) \setminus O : \omega(x)$ is Lyapunov stable$\}$ is nonempty (it is indeed residual in $W^u(O)$).*

An extension to higher dimensions of the three-dimensional arguments in [M2] (e.g. [Lo]) imply the following lemma.

Lemma 9.1.3. *A sectional-hyperbolic set Λ of $X \in \mathcal{X}^1$ contains only finitely many attractors, i.e., the collection $\{A \subset \Lambda : A$ is an attractor of $X\}$ is finite.*

The following concept comes from [GaLiW].

Definition 9.1.4. We say that a compact invariant set Λ of $X \in \mathcal{X}^1$ has a definite index $0 \leq Ind(\Lambda) \leq n-1$ if there are a neighborhood $\mathcal{U}$ of X in $\mathcal{X}^1$ and a neighborhood U of Λ in M such that $I(O) = Ind(\Lambda)$ for every hyperbolic periodic orbit $O \subset U$ of every vector field $Y \in \mathcal{U}$. In such a case we say that Λ is *strongly homogeneous (of index $Ind(\Lambda)$)*.

The importance of the strongly homogeneous property is given by the following result proved in [GaLiW]: If a strongly homogeneous sets Λ with singularities (all hyperbolic) of $X \in \mathcal{X}^1$ is C^1 robustly transitive, then it is partially hyperbolic for either X or $-X$ depending on whether

$$I(\sigma) > Ind(\Lambda), \qquad \forall \sigma \in Sing_X(\Lambda) \tag{9.1}$$

or

$$I(\sigma) \leq Ind(\Lambda), \qquad \forall \sigma \in Sing_X(\Lambda) \tag{9.2}$$

holds. This result was completed in [MeM] by proving that all such sets are in fact sectional-hyperbolic for either X or $-X$ depending on whether (9.1) or (9.2) holds. Another proof of this completion can be found in [GaWZ].

On the other hand, [AM] observed that the completion in [MeM] (or [GaWZ]) is also valid for transitive sets with singularities (all hyperbolic of Morse index 1 or $n-1$) as soon as $n \geq 4$ and $1 \leq Ind(\Lambda) \leq n-2$. The proof is the same as [GaLiW] and [MeM] but with the preperiodic set playing the role of the natural continuation of a C^1 robustly transitive set.

Now we observe that such a completion is still valid for limit cycles or when the periodic orbits are dense. In other words, we have the following result.

Lemma 9.1.5. *If a strongly homogeneous set Λ with singularities (all hyperbolic) of $X \in \mathcal{X}^1$ satisfying $1 \leq Ind(\Lambda) \leq n-2$ is a limit cycle or has dense periodic orbits, then it is sectional-hyperbolic for either X or $-X$ depending on whether (9.1) or (9.2) holds.*

This lemma motivates the problem whether a strongly homogeneous set with hyperbolic singularities which is a limit cycle or has dense periodic orbits satisfies either (9.1) or (9.2). For instance, Lemma 3.3 of [GaWZ] proved this is the case for all C^1 robustly transitive strongly homogeneous sets. Similarly for strongly homogeneous limit cycles with singularities (all hyperbolic of Morse index 1 or $n-1$) satisfying $n \geq 4$ and $1 \leq Ind(\Lambda) \leq n-2$ (e.g. Proposition 7 in [AM]). Consequently, all such sets are sectional-hyperbolic for either X or $-X$. See Theorem A in [GaWZ] and Corollary 8 in [AM] respectively.

Unfortunately, (9.1) or (9.2) need not be valid for general strongly homogeneous sets with dense periodic orbits even if $1 \leq Ind(\Lambda) \leq n-2$. A counterexample is the nonwandering set of the vector field in S^3 obtained by gluing a Lorenz attractor and a Lorenz repeller as in p. 1576 of [MP2]. Despite, it is still possible to analyze the singularities of a strongly homogeneous set with dense periodic orbits even if (9.1) or (9.2) does not hold. For instance, adapting the proof of Lemma 2.2 in [GaWZ] (or the sequence of lemmas 4.1, 4.2 and 4.3 in [GaLiW]) we obtain the following result.

Lemma 9.1.6. *If Λ is a strongly homogeneous set with singularities (all hyperbolic) and dense periodic orbits of $X \in \mathcal{X}^1$, then every $\sigma \in Sing_X(\Lambda)$ satisfying $I(\sigma) \leq Ind(\Lambda)$ exhibits a dominated splitting $\hat{E}^u_\sigma = E^{uu}_\sigma \oplus E^c_\sigma$ with $dim(E^{uu}_\sigma) = n - Ind(\Lambda) - 1$ over σ such that the strong unstable manifold $W^{uu}(\sigma)$ tangent to E^{uu}_σ at σ (c.f. [HPS]) satisfies $\Lambda \cap W^{uu}(\sigma) = \{\sigma\}$.*

9.2 Proof

Now we can prove our result.

Proof of the Theorem. Let $X \in \mathcal{X}^1$ be a C^1-generic vector field without points accumulated by hyperbolic periodic orbits of different Morse indices. By [AM], since X is C^1 generic, it follows that if $\mathrm{Per}_i(X)$ denotes the union of the periodic orbits with Morse index i, then the closure $\mathrm{Cl}(\mathrm{Per}_i(X))$ is strongly homogeneous of index $Ind(\mathrm{Cl}(\mathrm{Per}_i(X))) = i$, $\forall 0 \leq i \leq n-1$. Moreover, X is a star flow and so it has finitely many singularities and also finitely many sinks and sources (c.f. [Liao], [?]).

Let us prove that $\omega(x)$ is sectional-hyperbolic for all $x \in R_X$ where $R_X \subset M$ is the residual subset in Lemma 9.1.1. We can assume that $\omega(x)$ is nontrivial and has singularities for, otherwise, $\omega(x)$ is hyperbolic by Theorem B in [GaW] and the Pugh's closing lemma [Pu1] and [Pu2].

Since X is C^1 generic we can further assume that $\omega(x) \subset \mathrm{Cl}(\mathrm{Per}_i(X))$ for some $0 \leq i \leq n-1$ by the closing lemma once more. Since X has finitely many singularities sinks and sources we have $1 \leq i \leq n-2$ (otherwise $\omega(x)$ will be reduced to a singleton which is absurd). Since $\mathrm{Cl}(\mathrm{Per}_i(X))$ is strongly homogeneous of index i we have that $\omega(x)$ also does so $1 \leq Ind(\omega(x)) \leq n-2$. Then, since $\omega(x)$ is a limit cycle, we only need to prove by Lemma 9.1.5 that (9.1) holds for $\Lambda = \omega(x)$. To prove it we proceed as in Corollary B in [GaLiW], namely, suppose by contradiction that (9.1) does not hold. Then, there is $\sigma \in Sing_X(\Lambda)$ such that $I(\sigma) \leq Ind(\omega(x))$. Since $\omega(x) \subset \mathrm{Cl}(\mathrm{Per}_i(X))$ and $\mathrm{Cl}(\mathrm{Per}_i(X))$ is a strongly homogeneous set with singularities, all hyperbolic, in $\Omega(X)$ we have by Lemma 9.1.6 that there is a dominated splitting $\hat{E}^u_\sigma = E^{uu}_\sigma \oplus E^c_\sigma$ for which the associated strong unstable manifold $W^{uu}(\sigma)$ satisfies $\mathrm{Cl}(\mathrm{Per}_i(X)) \cap W^{uu}(\sigma) = \{\sigma\}$. However $W^{uu}(\sigma) \subset \omega(x)$ since $\sigma \in \omega(x)$ and $\omega(x)$ is Lyapunov stable. As $\omega(x) \subset \mathrm{Cl}(\mathrm{Per}_i(X))$ we conclude that $W^{uu}(\sigma) = \{\sigma\}$ so $\dim(E^{uu}_\sigma) = 0$. But $\dim(E^{uu}_\sigma) = n-i-1$ by Lemma 9.1.6 so $\dim(E^{uu}_\sigma) \geq n-n+2-1 = 1$ a contradiction. We conclude that (9.1) holds so $\omega(x)$ is sectional-hyperbolic for all $x \in R_X$.

Next we prove that $\omega(x)$ is transitive for $x \in R_X$. If $\omega(x)$ has no singularities, then it is hyperbolic and so a hyperbolic attractor of X. Otherwise, there is $\sigma \in Sing_X(\Lambda)$. By

Lemma 9.1.2 we can select $y \in W^u(\sigma) \setminus \{\sigma\}$ with Lyapunov stable ω-limit set. On the other hand, $\omega(x)$ is Lyapunov stable and so $W^u(\sigma) \subset \omega(x)$. Then, we obtain $y \in \omega(x)$ satisfying $\omega(x) = \omega(y)$ thus $\omega(x)$ is transitive.

It remains to prove that X has only finitely many sectional-hyperbolic transitive Lyapunov stable sets. Suppose by absurd that there is an infinite sequence A_k of sectional-hyperbolic transitive Lyapunov stable sets. Clearly the members in this sequence must be disjoint, so, since there are finitely many singularities, we can assume that none of them have singularities. It follows that all these sets are hyperbolic and then they are all nontrivial hyperbolic attractors of X. In particular, every A_k has dense periodic orbits by the Anosov closing lemma. We can assume that there is $1 \leq i \leq n-2$ such that each Λ_k belong to $Cl(Per_i(X))$. Define

$$\Lambda = Cl\left(\bigcup_{k\in\mathbb{N}} A_k\right).$$

Notice that Λ contains infinitely many attractors (the A_k's say). Moreover, Λ is a strongly homogeneous set of index $Ind(\Lambda) = i$ with dense periodic orbits (since each A_k does).

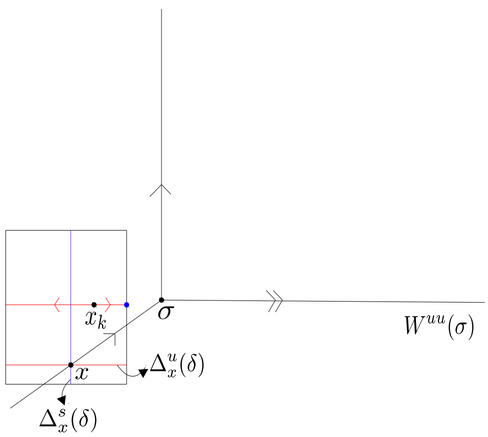


Figure 9.1: The blue points will converge to the strong unstable manifold of σ, a contradiction since $\Lambda \cap W^{uu}(\sigma) = \emptyset$.

Let us prove that Λ satisfies (9.1). Indeed, suppose by contradiction that it does not, i.e., there is $\sigma \in Sing_X(\Lambda)$ such that $I(\sigma) \leq Ind(\Lambda)$. By Lemma 9.1.6 there is a dominated splitting $\hat{E}^u_\sigma = E^{uu}_\sigma \oplus E^c_\sigma$ for which the associated strong unstable manifold $W^{uu}(\sigma)$ satisfies $\Lambda \cap W^{uu}(\sigma) = \{\sigma\}$.

Take a sequence $x_k \in A_k$ converging to some point $x \in W^s(\sigma) \setminus \{\sigma\}$. By Corollary 1 p. 949 in [GaWZ] there is a dominated splitting $D = \Delta^s \oplus \Delta^u$ for the linear Poincaré flow ψ_t which, in virtue of Lemma 2.2 in [GaWZ], satisfies $\lim_{t\to\infty} \psi_t(\Delta^u_x) = E^{uu}_\sigma$. Using exponential maps we can take a codimension one submanifold Σ orthogonal to X of the form $\Sigma = \Delta^s_x(\delta) \times \Delta^u_x(\delta)$ where $\Delta^*_x(\delta)$ indicates the closed δ-ball around x in Δ^*_x $(* = s, u)$. Since $\psi_t(\Delta^u_x) \to E^{uu}_\sigma$ as $t \to \infty$ we can assume by replacing x by $X_t(x)$ with $t > 0$ large if necessary that $\Delta^u_x(\delta)$ is almost parallel to E^{uu}_σ. In particular, since $\Lambda \cap W^{uu}(\sigma) = \{\sigma\}$, one has $(\partial\Delta^s_x(\delta) \times \Delta^u_x(\delta)) \cap \Lambda = \emptyset$ where $\partial(\cdot)$ indicates the boundary operation. Since both $\partial\Delta^s_x(\delta) \times \Delta^u_x(\delta)$ and Λ are closed we can arrange a neighborhood U of $\partial\Delta^s_x(\delta) \times \Delta^u_x(\delta)$ in Σ such that $U \cap \Lambda = \emptyset$.

Now we consider k large in a way that x_k is close to x. Replacing x_k by $X_t(x_k)$ with suitable t we can assume that $x_k \in \Sigma$. Since $x_k \in A_k$ and A_k is a hyperbolic attractor we can consider the intersection $S = W^u(x_k) \cap \Sigma$ of the unstable manifold of x_k and Σ. It turns out that S is the graph of a C^1 map $S : \Delta^u_x(\rho) \to \Delta^s_x(\delta)$ for some $0 < \rho \leq \delta$ whose tangent space T_yS is almost parallel to Δ^u_x. We assert that $\rho = \delta$. Otherwise, it would exist some boundary point $z \in \partial S$ in the interior of Σ. Since A_k is a hyperbolic set and $z \in A_k$ we could consider as in [M3] the unstable manifold $W^u(z)$ which will overlap $W^u(x)$. Since $z \in Int(W^u(z))$ and $W^u(z) \subset A_k$ (for Λ_k is an attractor) we would obtain that z is not a boundary point of S, a contradiction which proves the assertion. It follows from the assertion that A_k (and so Λ) would intersect U (see the esquematic picture for this argument in Figure 9.1) which is absurd since $U \cap \Lambda = \emptyset$. Thus (9.1) holds.

Then, Lemma 9.1.5 implies that Λ is sectional-hyperbolic for X and so Λ has finitely many attractors by Lemma 9.1.3. But, as we already observed, Λ contains infinitely many attractors so we obtain a contradiction. This contradiction proves the finiteness of sectional-hyperbolic transitive Lyapunov stable sets for X thus ending the proof of the theorem. $\square$

Appendix A

Appendix: Whitney's example

In this chapter we shall describe an example where Sard's Theorem may fail if its regularity assumption is not satisfied. We shall construct a C^1 function $F : \mathbb{R}^2 \to \mathbb{R}$ and a continuum (connected and compact) K such that $DF(x) = 0$, for every $x \in K$, but $F(K) = [0, 1]$. In particular, the set of singular values of F has positive measure. The example is due to Whitney [Wi1].

The construction goes through a fractal procedure which is also behind the Cantor staircase function. In fact, Whitney's function is very similar to the Cantor staircase, but its differentiability properties improve when it is considered as function on $\mathbb{R}^2$. For this reason we shall first recall, in Section A.1 the Cantor staircase and see why it is not everywhere differentiable. In Section A.2 we shall construct a non rectifiable path, parametrized by the unit interval, and define a modified Cantor staircase on this path with good differentiability properties. In the final section we shall use an extension result and will obtain Whitney's function from this modified Cantor staircase.

Since these constructions are fairly elementary, despite the fact that Whitney's example is important, this chapter is aimed to be more pedagogical than the others. For this reason, we shall adopt a less formal style in this chapter, in particular inserting informal discussions within the proofs.

A.1 The Devil Staircase

We shall now recall the construction of the Cantor function, elsewhere also known as the "Devil Staircase". The pictorial way of giving the construction is the following: on the middle third of $[0,1]$ put $f \equiv 1/2$, on the middle thirds of the two remaining intervals, put, respectively, $f \equiv 1/4$ and $f \equiv 3/4$. On the middle thirds of the four remaining intervals, put, respectively, $f \equiv 1/8$, $f \equiv 3/8$, $f \equiv 5/8$ and $f \equiv 7/8$. Proceed inductively. See Figure A.1.

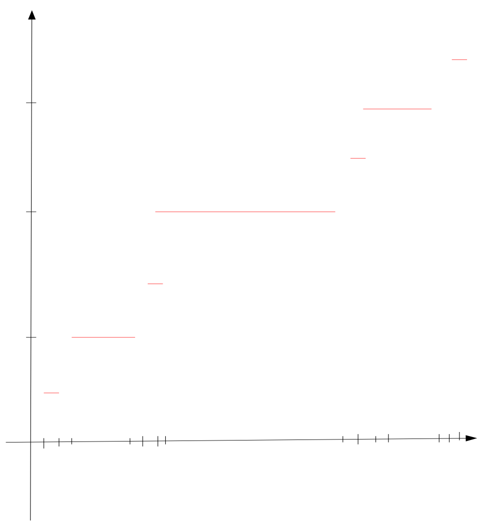


Figure A.1: The first three steps of construction of the Devil Staircase

The reader not familiar with the Cantor function should have this pictorial procedure in mind while reading the formal construction.

Base p expansions

Let $p \geq 2$ be an integer. Expansions in base p for numbers in $[0,1]$ are an important tool in the examples of Cantor and Whitney. For this reason, we shall describe them in a geometrical sense, which will give us an easy way of speaking about such expansions.

Pick a number $x \in [0,1]$. Divide $[0,1]$ in p intervals of equal lenght, of the form

$$[n/p, (n+1)/p).$$

Label them with the digits $\{0, 1, ..., p-1\}$. Since these intervals form a partition of $[0, 1]$, there exists only one which contains x. Let x_1 be the label of such an interval. Now, divide $[x_1/p, (x_1+1)/p)$ into p intervals of the same lenght and of the form $[\frac{x_1}{p} + \frac{n}{p^2}, \frac{x_1}{p} + \frac{n+1}{p^2})$. Label them with the digits $\{0, 1, ..., p\}$. Let x_2 be the label of the interval in this second division which contains x. Proceed by induction. In this way, we find a nested sequence of compact intervals (the closure of the intervals in the partition), and a sequence of digits $\{x_l\}$, all of the intervals containing p and with lenght p^{-n}. Their intersection must be x. Also, by the form of these intervals, it is clear that $x = \sum_{n=1}^{\infty} x_l p^{-l}$.

The only possibility for $x \in [0, 1]$ to have more than one expansion is $x = 0.x_1...x_n100...$ and $x = 0.x_1...x_n y_1 y_2...$, with $y_l = p-1$, for every l, since $\sum_{n\in\mathbb{N}} \frac{p-1}{p^n} = 1$.

Also, by this geometrical way of interpreting the base p expansion, if x and y have the same first n digits, then their firts n labels are the same and thus they are in the same intervals until the n^{th} step. As a consequence, $|x - y| < p^{-n}$.

The formal contruction

Theorem A.1.1. *There exists a continuous surjective map $f : [0, 1] \to [0, 1]$, with vanishing derivative on a dense open set of full measure (the complement of the Cantor set).*

Proof. Set $f(0) = 0$ and $f(1) = 1$. For $x \in (0, 1)$, consider its base 3 expansion: $x = \sum_{l=1}^{\infty} x_l 3^{-l}$, where $x_l \in \{0, 1, 2\}$. Consider the integer $p(x) \in \mathbb{N} \cup \{\infty\}$ with the property that $x_l \in \{0, 2\}$, for every $l = 1, ..., p(x) - 1$ and $x_{p(x)} = 1$. That is, $p(x)$ is the first position in the base 3 expansion of x in which the digit 1 appears. Define

$$f(x) = \sum_{l=1}^{p(x)-1} \left(\frac{x_l}{2}\right) 2^{-l} + 2^{-p(x)}. \tag{A.1}$$

In the definition of f, we use the convention that $2^{-\infty} = 0$, which is to say that if $p(x) = \infty$ then $f(x) = \sum_{l=1}^{\infty} \left(\frac{x_l}{2}\right) 2^{-l}$. For instance, if $x = (0.020221)_3$ then

$$f(x) = 0.2^{-1} + 1.2^{-2} + 0.2^{-3} + 1.2^{-4} + 1.2^{-5} + 2^{-6} = (0.010111)_2.$$

So, in words, the rule which defines f is: take the base 3 expansion of x, let $p(x)$ the first position in which an odd digit appears. Truncate it and consider $(0.x_1...x_{p(x)-1}1)$. Divide all the first $p(x)$ digits by 2, except the last digit (which is 1) and interpret the remaining digits as the base 2 expansion of $f(x)$.

Let us check the f is well defined. It is clear that this definition is compatible with $f(0) = 0$. Notice that it is also compatible with $f(1) = 1$: since $1 = \sum_{n\in\mathbb{N}} 2/3^n$ we have $f(1) = \sum_{n\in\mathbb{N}} 1/2^n = 1$. Finally, let $\{x_l\}$ and $\{y_l\}$ be two different base 3 expansions of x. Assume that $\{x_l\} = (0.x_1...x_n1000...)_3$ and $\{y_l\} = (0.x_1...x_n022222...)$. Then,

$$\begin{aligned} f(\{x_l\}) &= \sum_{l=1}^{n} \left(\frac{x_l}{2}\right) 2^{-l} + 2^{-n-1} \\ &= \sum_{l=1}^{n} \left(\frac{x_l}{2}\right) 2^{-l} + 2^{-n-1} \sum_{m=1}^{\infty} 2^{-m} \\ &= \sum_{l=1}^{n} \left(\frac{x_l}{2}\right) 2^{-l} + \sum_{m=n+2}^{\infty} 2^{-m} \\ &= f(\{y_l\}). \end{aligned}$$

The other possibility of non-uniqueness is $\{x_l\} = (0.x_1...x_n2000...)_3$ and $\{y_l\} = (0.x_1...x_n122222...)$. But in this case, by the very definition of f it is automatic that $f(\{x_l\}) = f(\{y_l\})$. This proves that the non-uniqueness of base 3 expansion does not affect the definition of f.

Now observe that the integer levels of the map $p : [0,1] \to \mathbb{N} \cup \{\infty\}$ correspond precisely to the removed intervals in the construction of the Cantor set: $p^{-1}(1)$ is the middle third of $[0,1]$, $p^{-1}(2)$ are the middle thirds of $[0,1/3]$ and $[2/3,1]$ and so on. By definition, f is constant on each of this intervals. This proves that f has vanishing derivative in the complement of the Cantor set.

We now claim that f is continuous. To see this, take $x = (0.x_1...x_nx_{n+1}...)_3$ and fix $\varepsilon > 0$. The key observation is that for each n, if $y = (0.y_1...y_ny_{n+1}...)_3$ is close enough to x then $x_l = y_l$, for every $l = 1, ..., n$. If $p(x) < n$ then $f(x) = f(y)$. If $p(x) \geq n$, then we can ensure that $f(x)$ and $f(y)$ have its first n digits in the base 2 expansion equal. Thus, taking n so large that $2^{-n} < \varepsilon$, one has that for every y close enough to x

$$|f(x) - f(y)| \leq 2^{-n} < \varepsilon.$$

Since f fixes 0 and 1, by the intermediate value property f is surjective. This completes the proof. $\square$

Let us discuss a bit on the differentiability properties of the function f of Theorem A.1.1. Let $x \in [0,1]$ such that $p(x) = \infty$ and take y such that the first n digits in their base 3 expansion are equal but $x_{n+1} \neq y_{n+1}$. Since the first n digits are the same, $|x - y| < 3^{-n}$. Since $x_{n+1} \neq y_{n+1}$, $|f(x) - f(y)| \geq 2^{-n-1}$. Therefore,

$$\frac{|f(x) - f(y)|}{|x - y|} \geq \frac{2^{-n-1}}{3^{-n}} = \frac{1}{2}\left(\frac{3}{2}\right)^n \to \infty,$$

as $n \to \infty$. Since $p^{-1}(\infty)$ is precisely the Cantor, we see that f is not differentiable at the Cantor set.

A.2 The Devil Path

The key point which makes f non differentiable is the change of basis, since it implies that $|x - y|$ decreases at a rate 1/3, while $|f(x) - f(y)|$ decreases at a rate 1/2. Notice that there is no C^1 function with the same properties of f, due to the Fundamental Theorem of Calculus: if g is C^1 and has zero derivative almost everywhere then, for every $a > b$,

$$g(a) - g(b) = \int_a^b g'(x)dx = 0.$$

Essentialy, what enable us to use the Fundamental Theorem of Calculus is the fact that every path in $\mathbb{R}$ (being an interval) is rectifiable. Thus, if we want to construct a C^1 Cantor-like function in $\mathbb{R}^2$ the first thing we need is a non-rectifiable path.

Thus, let us construct a non-rectifiable path.

Take the unit square in $\mathbb{R}^2$. Choose any $0 < \varepsilon < 1$. Put inside of it, 4 smaller squares of side 1/3 such that the distance between any two of them is $\frac{1}{3+\varepsilon}$. See Figure A.2. Draw 5 straight line segments joining the middle point of the squares (the red segments of Figure A.2). This is the first step of the construction. Now, put inside each small saquare 4 even smaller squares, of side 1/9 such that the distance between any two of them is $\frac{1}{3(3+\varepsilon)}$. In each square of side 1/3, draw 5 straight line segments joining the middle points of the squares, in the same way as in the first step (these segments are the blue segments in Figure A.2). That is, scale the first step by a factor of 1/3 and put a copy of it in each little square.

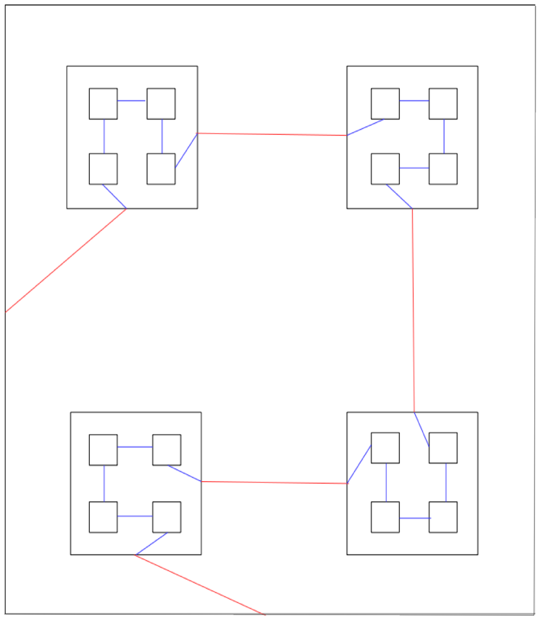


Figure A.2: The first two steps of the construction of the continuum K

Now, inside one of the 16 small squares of the second step, put 4 squares of size $1/27$ and $\frac{1}{3^2(3+\varepsilon)}$ far apart from each other. Inside each one of these 64 small squares, draw 5 red segments joining the middle points, as in the first step. Proceed inductively.

Let K be the unoin of all red and blue paths, defined on each step of the construction together with their limit points, which correspond precisely to the limit of all possible sequences of squares. It is easy to see that, K is a continuum.

Lemma A.2.1. *The total lenght of K is infinity*

Proof. Let r be the sum of the lenghts of the red paths in the first step of the construction of K. On the second step, there are 4 unions of blue paths, each one them a scaled copy by a factor of $1/3$ of the first union of red paths. Thus, the total lenght of the blue paths in the second step is $\frac{4r}{3}$. On the third step, there will be 16 unions of small red paths, each union with lenght $\frac{r}{9}$. By induction, we see that the total lenght of K is $\sum_{n\in\mathbb{N}} r\left(\frac{4}{3}\right)^n$, which diverges, concluding. $\square$

A.3 Whitney's Devil

We are now in position to define a function $W : K \to \mathbb{R}$, similar to f, but with good differentiability properties.

We first parametrize K using base 9 expansions: divide $[0, 1]$ into nine equal portions and label them with $\{0, 1, 2, ..., 8\}$. With the even intervals, parametrize the red segments of the first step in the construction of K. Divide each odd interval into nine equal portions, labeled in the same way. Use the even intervals to parametrize the blue paths of the second step. Proceed inductively. In this parametrization, the points which belong to some red or blue segment of some step in the construction of K correspond to a point in $[0, 1]$ whose base 9 expansion has an even digit. The fractal obtained as all the limits of all possible sequences of squares corresponds to the points in $[0, 1]$ whose base 9 expansion has only odd digits.

Consider the map $P : [0, 1] \to \mathbb{N} \cup \{\infty\}$ such that $P(x)$ is the first position in the base 9 expansion of x in which an even digit appears. Said otherwise, let $\{x_l\}$ be the base 9 expansion of x. Then, $x_l \in \{1, 3, 5, 7\}$, for every $l = 1, ..., P(x) - 1$ and $x_{P(x)} \in \{0, 2, 4, 6, 8\}$, or $x_l \in \{1, 3, 5, 7\}$ for every $l \geq 1$, and $P(x) = \infty$.

For $x = \sum_{n\in\mathbb{N}} x_l 9^{-l}$, we define

$$W(x) = \sum_{l=1}^{P(x)-1} \left(\frac{x_l - 1}{2}\right) 4^{-l} + \left(\frac{x_{P(x)}}{2}\right) 4^{-P(x)}.$$

To facilitate the anology between W and f, we want to interpret $W(x)$ as sequence of digits in base 4. Since x may have the digit 8 in the position $P(x)$, we appeal to the standard convention

$$(0.x_1...x_n4000...)_4 \mapsto (0.x_1...x_{n-1}(x_n + 1)000...)_4. \tag{A.2}$$

For instance, $(0.34)_4 = (1.0)_4 = 1$, and for this reason

$$W((0.78x_3x_4...)_4) = W((0.8x_2x_3...)_4) = 1.$$

Let us prove that the non uniquenes of the base 9 expansion does not affect the definition of W, in the same way with f. Let $\{x_l\}$ and $\{y_l\}$ be two base 9 expansions of x. Then there exists n such, $x_l = 8$, for every $l \geq n+1$, and $x_n < 8$, while $y_l = x_l$, for $l = 1, ..., n-1$, $y_n =$

x_n+1, and $y_l = 0$, for $l \geq n+1$. If $P(\{x_l\}) \leq n$, it is automatic that $W(\{x_l\}) = W(\{y_l\})$. Then, $P(\{x_l\}) = n+1$ while $P(\{y_l\}) = n$. Therefore, by definition of W we get

$$W(\{y_l\}) = \sum_{l=1}^{n-1} \left(\frac{x_l - 1}{2}\right) 4^{-l} + \left(\frac{x_n+1}{2}\right) 4^{-n}.$$

On the other hand, since $x_{n+1} = 8$,

$$W(\{x_l\}) = (0.\left(\frac{x_1-1}{2}\right)...\left(\frac{x_n-1}{2}+1\right)00...)_4.$$

Since $\frac{x_n-1}{2} + 1 = \frac{x_n+1}{2}$, we get the conclusion that $W(\{y_l\}) = W(\{x_l\})$.

Using the parametrization of K, we can consider $W : K \to \mathbb{R}$.

The key point is that, by considering W defined in K we get the following differentiability property.

Lemma A.3.1.

$$\frac{|W(x) - W(y)|}{d(x,y)} \to 0,$$

uniformly for $x, y \in K$

The idea of the proof is the following: if W is considered as a function on $[0,1]$, then, while $|x-y|$ decreases at a rate $1/9$, $|W(x) - W(y)|$ decreases at a rate $1/4$. However, for $x, y \in K$, by the way K is constructed, $|x-y|$ decreases at a rate $1/3$, thus $|W(x) - W(y)|$ is uniformly faster than $|x-y|$, for $x, y \in K$

Proof of Lemma A.3.1. Let $x, y \in K$ be such that $W(x) \neq W(y)$. This implies that there exists $n \in \mathbb{N}$ such that at the n^{th} step of the construction of K the points x and y must be in different parts. Said otherwise, the first $n-1$ digits of x and y in their base 9 expansion are equal, but their n^{th} digits are different. Geometrically, this means that either (A) they belong to different squares in the n^{th} step; (B) or one of them belong to a segment of n^{th} step and the other to a square. (C) or they belong to different red segments, See Figure A.3. The following claim rule out the possibility of $d(x,y)$ being much smaller than $|W(x) - W(y)$.

Claim 17. *If x and y belong to a pair of touching red and blue segments, in step n, then $W(x) = W(y)$.*

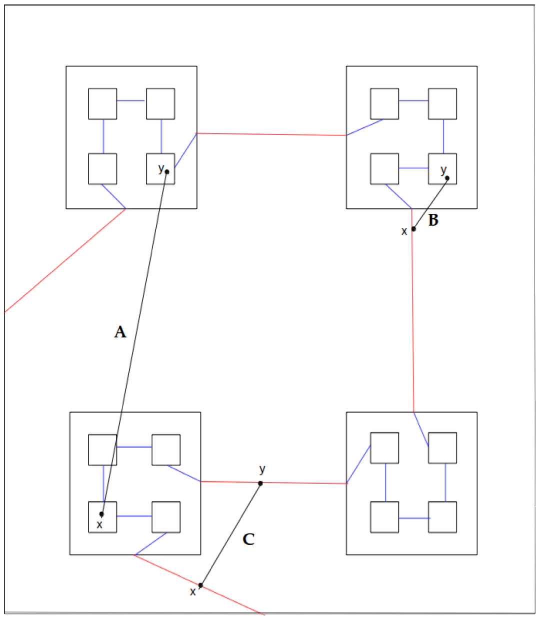


Figure A.3: Three possibilities (A, B and C) for x and y at step n such that $W(x) \neq W(y)$.

Proof. To fix ideas, let us assume that the segments in step n are red, while the segments in step $n+1$ are blu. Thus, the step n is a scaled copy of Figure A.2, by a factor $(1/3)^n$. Assume that x and y belong to a pair of touching red and blue segments, in step n. For simplicity we may also assume that x belongs to a red segment and y to small blue segment. Then, denoting by $\{x_l\}$ and $\{y_l\}$ the base 9 expansions of x and y, respectively, we conclude that x_n is even and there two possibilities for y_n and y_{n+1}: either $y_n = x_n - 1$ and $y_{n+1} = 8$ or $y_n = x_n + 1$ and $y_{n+1} = 0$. In the latter case, since we are assuming that $x_l = y_l$, for $l = 1, \ldots n-1$ and since x_n is even, it follows easily that $W(x) = W(y)$. In the former case, due to convention (A.2) we arrive at the same conclusion again. This establishes the claim. $\square$

Therefore, x and y do not belong to a pair of touching red and blue segments. Moreover, it is easy to check that in cases (A) and (B) the distance between x and y is bounded from below by a constant times $(3+\varepsilon)^{-n}$. In case (C), since x and y do not belong to a pair of touching red and blue segments, we can conclude that $d(x,y)$ is bounded from below by half of the distance between two small squares of step $n+1$. Thus,

$$d(x,y) \geq 2^{-1}(3+\varepsilon)^{-n-1}. \tag{A.3}$$

Since x and y separate only at the n^{th} step of the construction, $W(x)$ and $W(y)$ have the same $n-1$ first digits in their base 4 expansion, which implies that $|W(x)-W(y)| \leq 4^{-n+1}$. Thus,

$$\frac{|W(x)-W(y)|}{d(x,y)} \leq \frac{4^{-n+1}}{2^{-1}(3+\varepsilon)^{-n-1}} \leq D\left(\frac{3+\varepsilon}{4}\right)^n, \tag{A.4}$$

where D is a (fixed) positive number[1]. Let us now complete the proof. Take $\xi > 0$ arbitrary and $m \in \mathbb{N}$ such that $D\left(\frac{3+\varepsilon}{4}\right)^m < \xi$. Take $\delta = 2^{-1}(\frac{1}{3+\varepsilon})^m$. If $d(x,y) < \delta$, then the base 9 expansions of x and y must coincide until their n^{th} digit, for some n with $n > m$, due to the estimate (A.3). Therefore, we conclude from estimate (A.4) that

$$\frac{|W(x)-W(y)|}{d(x,y)} \leq \frac{4^{-n+1}}{2^{-1}(3+\varepsilon)^{-n-1}} \leq D\left(\frac{3+\varepsilon}{4}\right)^m < \xi.$$

This ends the proof. □

The last effort in order to obtain a C^1 counter example to Sard's Theorem is to extend the function W.

Theorem A.3.2. *There exists a C^1 function $F : \mathbb{R}^2 \to \mathbb{R}$ such that $DF(x) = 0$, for every $x \in K$, and $F(K) = [0,1]$.*

Proof. Due to Lemma A.3.1, the result is now an immediate application of Whitney's Extension Theorem [Wi2]. □

[1] In fact, $D = 24$

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