Ergodic Properties of C^1 Generic Diffeomorphisms

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Approximation of Invariant Measures by Periodic Orbits

The shadowing property, given by the ergodic closing lemma of Mañé [3] is very usefull and when combined with some other hypotesis can be very powerfull. For example, let us analise the case when f has an ergodic measure μ.

In this case, by Birkhoff Ergodic Theorem, for a generic point $x \in \mathrm{Supp}(\mu)$ the foward orbit is dense in the support of μ. Once we shadow this orbit by a periodic orbit, as long as the shadowing is sufficiently fine, we get that this periodic orbit is Hausdorff close of $\mathrm{Supp}(\mu)$. This alredy gives a slightly improvment in the ergodic closing lemma. A natural question is wheter we can also approximate the Lyapunov Exponents of μ by those of this periodic orbit. The answer is yes, but we have to work a little more in order to establish this.

0.1 Proposition. *Lef μ be an ergodic measure of a diffeomorphism f. Take a neighborhood $\mathcal{U}$ of f, a neighborhood V of μ in the space $\mathcal{M}_f(M)$ of invariant measures endowed with the weak-star topology, a neighborhood U of $\Lambda := \mathrm{Supp}(\mu)$ in the space C_M of compact subsets of M endowed with the Hausdorff topolgy, and a neighborhood O of the Lyapunov vector $L(\mu)$ of μ in $\mathbb{R}^d$. Then there exist $g \in \mathcal{U}$ having a periodic point x such that the periodic measure μ_x belongs to V, the orbit $O(x)$ belongs to U and the Lyapunov vector of μ_x belongs to O.*

Proof. Take an $\epsilon > 0$ small, and take a generic point $x \in \mathrm{Supp}(\mu)$. Suppose that $T_xM = \bigoplus_{l=1}^{k} E_l$ is the Oseledts splitting of μ at x, and let $n_0 > 0$ have the following properties:

1. for every $l = 1, \dots, k$, $v \in E_l$ implies $\left|\frac{1}{n}\log\left\|Df^n(x)v\right\| - \lambda_l\right| < \epsilon$, for every $n \geq n_0$.
2. the string $\{x, \dots, f^n(x)\}$ is ϵ-dense in the support of μ
3. if $V = V(\phi_1, \dots, \phi_r; \delta)$ and $g \in \mathcal{U}$ is such that $x \in \mathrm{Per}(g)$ with period $n \geq n_0$ and the g-orbit of x ϵ-shadows the f-orbit of x until the n-th iterate then
$$\left|\frac{1}{n}\sum_{j=0}^{n-1}\phi_l(g^j(x)) - \int \phi_l d\mu\right| < \delta,$$
for every $l = 1, \dots, r$.

The two first items follows, resp., from Osledts and Birkhoff theorems. The third one is obteined combining continuity of the functions ϕ_l, using the shadowing, and Birkhoff Ergodic Theorem.

Since x is gereric, by the ergodic closing lemma there exist $g \in \mathcal{U}$, such that x is a periodic point of g with period $n \geq n_0$ and the g-orbit of x ϵ-shadows the f-orbit of x until the n-th iterate. By items (2) and (3) above, we have that the periodic measure μ_x belongs to V and the orbit $O(x)$ belongs to U. We only have to perturb g a little bit to control the Lyapunov exponents.

By the shadowing property we can use local charts arround the f-orbit of x, and compare $Dg(g^j(x))$ with $Df(f^j(x))$. By shrinking $\mathcal{U}$ if necessary, we may assume that these maps are close. Using isometries $A_j : T_{g^j(x)}M \to T_{f^j(x)}M$, we have that the maps
$$L_j = A_{j+1}^{-1} Df(f^j(x)) A_j$$
are close to $Dg(g^j(x))$. Franks Lemma gives us a diffeomorphism $\overline{g} \in \mathcal{U}$, that prserves the g-orbit of x, and achieve the map L_j as derivative at the point $g^j(x)$. Moreover the exponets of the peridoic measure of $\overline{g}$ are given by
$$\frac{1}{n}\log|\sigma| = \frac{1}{n}\log\left\|Df^n(x)v\right\| \in (\lambda_i - \epsilon, \lambda_i + \epsilon),$$
where σ is an eigenvalue of $D\overline{g}^n(x)$, and λ_i is one of the Lyapunov exponents of μ. This shows that the Lyapunov vector of μ_x is ϵ-close to the Lyapunov vector of μ, and completes the proof. □

Once we have this perturbative result, we can make it generic and in this case we don't need to perturb the diffeomorphism to obtain the required periodic measures. The argument (in spirit) is the same that genarates the General Density Theorem of Phug from the closing lemma.

0.2 Theorem. *Given an ergodic measure μ of a C^1-generic diffeomorphism f, there is a sequence $x_n \in \mathrm{Per}(f)$ such that*

- *the measures μ_{x_n} converge to μ in the weak-star topology;*
- *the periodic orbits $O(x_n)$ converge to $\mathrm{Supp}(\mu)$ in the Hausdorff topology;*
- *the Lyapunov vectors $L(\mu_{x_n})$ converge to the Lyapunov vector $L(\mu)$.*

Proof. Take a contable basis $\mathcal{U}_n$ for the topology of $\mathrm{Diff}^1(M)$. In each of these open sets we shall find a residual subset with the required properties. Together, these local residuals gives rise to a global one as required.

This observation permit us to work locally, so lets set $\mathcal{U}_n = \mathcal{U}$ and prove the theorem in $\mathcal{U}$. Moreover, we can assume that for every $g \in \mathcal{U}$ the functions $\log\|Dg^{-1}\|$ and $\log\|Dg\|$ are bounded above and bellow by some constant $a > 0$, so they takes all its values in the compact interval $[-a, a]$ of the line. This implies that for every periodic measure ν of any $g \in \mathcal{U}$ the Lyapunov vector belongs to the compact subset $\mathcal{K} = [-a, a]^d$ of $\mathbb{R}$. Consider the product space $X = \mathcal{M}(M) \times C_M \times \mathcal{K}$, and, as usual, C_X the space of compacts subsets of $\mathcal{K}$ endowed with the Hausdorff topology. For any $g \in \mathcal{U}$, define
$$\Phi(g) = \overline{\{(\mu_x, O(x), L(\mu_x)) \in x;\ x \in \mathrm{Per}(g)\}} \in C_X.$$
Is not difficult to see that the function
$$\Phi : KS^1(\overline{\mathcal{U}}) \to C_X,$$
that assign to each Kupka-Smale diffeomorphisms $g \in \overline{\mathcal{U}}$ the compact subset of X, $\Phi(g)$, is lower seimicontinuous. In fact, this is just a standard consequence of the analitc continuation of hyperbolic periodic orbits (and this is why we restrict ourselfs to the set of Kupkca-Smale diffeomorphisms).

By the Semicontinuity Lemma, there is a residual subset $\mathcal{R}$ of $\mathcal{U}$, entirely formed by continuity points of the map Φ. We claim that if $f \in \mathcal{R}$, and μ is an ergodic measure of f, then
$$(\mu, \mathrm{Supp}(\mu), L(\mu)) \in \Phi(f).$$
If this is not the case, then there is a positive distance between $(\mu, \mathrm{Supp}(\mu), L(\mu))$ and $\Phi(f)$ in C_X. At one hand, since f is a continuity point of Φ, for every g sufficiently close to f, $\Phi(g)$ is close to $\Phi(f)$, say, for example, half of the distance between $(\mu, \mathrm{Supp}(\mu), L(\mu))$ and $\Phi(f)$ in C_X. At the other hand, Proposition 0.1 enable us to perturb f and obtain a map g as close to f as we want, such that, for some $x \in \mathrm{Per}(g)$, $(\mu_x, O(x), L(\mu_x))$ is close to $(\mu, \mathrm{Supp}(\mu), L(\mu))$. This explodes $\Phi(g)$ and leads to a contradiction, wich finishes the proof. □

Bonatti, Gourmelon and Vivier, [2], have a perturbation result wich allow us to alter the derivative of the diffeomorphism along a periodic orbit in order to make it have all eigenvalues equal in some direction of the tangent space, where lack of domination appears. Using this tool we can prove the following

0.3 Corollary. *Let f be a genric diffeomorphism and μ an ergodic measure for f. Define $\Lambda := \mathrm{Supp}(\mu)$ and let*
$$T_\Lambda M = F_1 \oplus \dots \oplus F_k$$
be the finest dominated splitting. Then, there exists a sequence of periodic orbits $x_n \in \mathrm{Per}(f)$, such that

- *$\mu_{x_n} \to \mu$ in the weak topology;*
- *$O(x_n) \to \Lambda$ in the Hausdorff topology;*
- *for each $i = 1, \dots, k$, the Lyapunov exponents of μ_{x_n} inside F_i converge to $\lambda_{F_i} = \frac{\int \log|\det Df|_{F_i}| d\mu}{\dim F_i}$.*

The results we have so far deal only with ergodic measures. For increase their worth, we must ensure that set of ergodic measure is big enough for the study of the whole space of invariant measures. In the case of an Axiom A diffeomorphism this is true, at least in the topological sense, by a result of Sigmund [4]. In our generic case we have the same conclusion for measures supported on compact, isolated and transitive invariant sets. Since every periodic measure is ergodic, it suffices to work with them.

0.4 Theorem. *Let Λ be an isoleted non-trivial transitive set of a C^1-generic diffeomorphism f. Then the set $\mathcal{P}_f(\Lambda)$ of periodic measures supported on Λ is dense in the set $\mathcal{M}_f(\Lambda)$ of invariant measures supported on Λ.*

Generic Measures for Generic Diffeomorphisms

Now we shall look for properties of generic invariant measures for a generic diffeomorphism. Since the results we have so far fits better when the invarant measure is also ergodic, is natural try to show that ergodicity is a generic property in the space of inavariant measures. It turns out that the set of ergodic measures of a continuous transformation on a compact space is always a G_δ

0.5 Proposition. *Let $f \in \mathrm{Diff}^1(M)$ and $\Lambda \subset M$ a compact invariant. Then, the set $\mathcal{M}_f^{erg}(\Lambda)$ of ergodic invariant measures supported on Λ is a G_δ subset of $\mathcal{M}_f(\Lambda)$.*

In virtue of theorem 0.4, if f and Λ, in the statement of theorem 0.4, are resp. generic and isoleted then $\mathcal{M}_f^{erg}(\Lambda)$ turns into a residual subset of $\mathcal{M}_f(\Lambda)$.

We also wish to look for measures that have a big support. This is true for generic measures, as the next result shows.

0.6 Proposition. *Let f and Λ be as in the proposition 0.5. Then, for every generic measure μ in $\mathcal{M}_f(\Lambda)$ we have*
$$\mathrm{Supp}(\mu) = \bigcup_{\nu \in \mathcal{M}_f(\Lambda)} \mathrm{Supp}(\nu).$$

Again, if f and Λ are resp. generic and isoleted then we have that generic invariant measures are fully supported in Λ. Using these results and the fact that for generic ergodic measures the Lyapunov exponents vary continuously, we can prove the following result

0.7 Theorem. *Let f and Λ be as in theorem 0.4. Then, every generic $\mu \in \mathcal{M}_f(\Lambda)$ is ergodic, fully supported, the Osledets Splitting is dominated and all Lyapunov exponents are non-zero.*

Proof. We fix arbitraly a subbundle F of the finest dominated splitting over Λ.

By the results above we have a residual subset $\mathcal{R}$ of $\mathcal{M}_f(\Lambda)$ such that every measure in $\mathcal{R}$ is ergodic, fully supported and is a continuity point of the map $L : \nu \mapsto L(\nu)$. We shall add one more property to this residual: the sums of the Lyapunov exponents of each of the measures inside F are non-zero.

For this prurpose, first note that by Franks Lemma, given any diffeomorphism f and any periodic point p of f, we can perturb f along the orbit of p to make all the sums $\lambda_i + \dots + \lambda_j$, of Lyapunov exponents of μ_p, $0 \leq i \leq j \leq d$, non-zero. Hence, for every integer n, the set
$$A_n = \left\{f \in KS^1;\ \forall p \in \mathrm{Per}(f),\ \pi(p) \leq n,\ I(\mu_p) \neq 0\right\},$$
where $I(\mu_p) = \int \log|\det Df|_F| d\mu_p$, is dense in $\mathrm{Diff}^1(M)$. Moreover, by the continuation of hyperbolic periodic orbits, and since the function $\log|\det D(.)|_F|$ is continuous, we have that for every g sufficiently close to f, all the continuations p_g of periodic points $p \in \mathrm{Per}(f)$, with $\pi(p) \leq n$ also satisfy $I(\mu_{p_g}) \neq 0$. Indeed, since f has olnly a finite number periodic points with period less then n, if g a little more sufficiently close to f, we garantee that all the periodic points of g with period less then n are continuations of those periodic points of f. This shows that A_n is also open, and so $\mathcal{A} = \cap A_n$ is residual.

Intersecting this residual with the residual of theorem 0.2 we obtain that for every periodic measure of f, $I(\mu_x) \neq 0$. By theorem 0.4, and since the integral $I(.)$ vary continously with the measure, there is an open and dense subset of $\mathcal{M}_f(\Lambda)$ entirely formed by measures ν such that $I(\nu) \neq 0$. Intersecting this open and dense subset with $\mathcal{R}$, we gain the residual with the property that we were seeking.

By corollary 0.3, there is a sequence $x_n \in \mathrm{Per}(f)$, such that the Lyapunov exponents inside F converge to $\lambda_F = \frac{I(\mu)}{\dim F}$ (wich is non-zero, since $I(\mu) \neq 0$) and $\mu_{x_n} \to \mu$. Since μ is a continuity point of the map $L\nu \mapsto L(\nu)$, we have that inside F there is only one Lyapunov exponent. Repeating the same argument for the other subbundles of the finest dominated splitting, we cnclude that the Oseledts splitting is equal to the finest dominated splitting, and moreover all the exponents are non-zero. So we are done. □

References

[1] Abdenur, F. Bonatti, C Crovisier, S. *Nonuniform hyperbolicity for C^1 generic diffeomorphisms.* Israel Journal of Mathematics 183, pp. 1-60, 2011

[2] Bonatti, C. Gourmelon, N. Vivier, T. *Perturbations of the derivative along periodic orbits.* Ergodic Theory Dynam. Systems 26 (2006), no. 5, 1307-1337.

[3] Mañé, R. *An ergodic closing lemma.* Ann. of Math (2), (1982), no. 3, v.116, 503-540.

[4] Sigmund, K. *Generic properties of invariant measures for Axiom A diffeomorphisms.* Invent. Math. 11 1970 99-109.