Weakly Hyperbolic Iterated Functions Systems on Compact Spaces

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Introduction

Iterated function systems were introduced in [H] as a unified way of generate a broad class of fractals. Nowadays, such systems occurs in many places in mathematics and other scientific areas. In [H], Hutchinson introduced and first studied hyperbolic Iterated function systems, i.e. a finite collection of contractions over a complete metric space. He was interested in construct attractors, both in the topological and measure-theoretic viewpoint.

After that, many authors proposed several generalizations of his result. One direction was to weaker the hyperbolic assumption, allowing some weak forms of contraction. For instance, we have the so called average contraction, studied by [BDEG]. Also, we have the ϕ-contractions studied by [JL] and [M]. In both cases, the existence of the attractors was proved.

In [E], Edalat defined the notion of weakly hyperbolic Iterated function systems (precise definition below) as a finite collection of maps on a compact metric space such that the diameter of the space by any combination of the maps goes to zero. In theory, this definition could allow some non-contractions, which was ruled out in the previous settings to obtain a topological attractor.

Another way to extend the results of Hutchinson is related with the parameter space. In Hutchinson's paper the parameter space is a finite set, since he deals with finitely many contractions. In [Ha], this theory was extended to the case when the parameter space is a infinite countable set. In [L] and [Me] the authors consider compact metric spaces as the parameter spaces. However, in this context, only uniform contractions and average contractions are studied.

One of the purposes of this work is to study these questions in the setting of weakly hyperbolic iterated function systems with compact parameter space, thus unifying some of the previous results.

Definitions

Let Λ and X be complete metric spaces and $w : \Lambda \times X \to X$ a continuous map. Such a map is called an *Iterated Function System* (IFS for short). The space Λ is called the *parameter space* and X is called the *phase space*. The space $\Lambda^{\mathbb{N}}$ of infinite words with alphabet in Λ will be denoted by $\Omega := \Lambda^{\mathbb{N}}$. Given a fixed parameter $\lambda \in \Lambda$, we will denote by $w_\lambda : X \to X$ the partial map generated by this parameter, which is defined by $w_\lambda(x) := w(\lambda, x)$. In this work we shall investigate Iterated Functions Systems with compact parameter and phase spaces. Let us denote the map $w_{\lambda_1 \dots \lambda_n} := w_{\lambda_1} \circ \dots \circ w_{\lambda_n}$, where $(\lambda_1, \dots, \lambda_n)$ is a word on Λ. Following [E] we have the

0.1 Definition. *If X is a compact metric space and $\Lambda = \{1, \dots, N\}$ then we say that an IFS $w : \Lambda \times X \to X$ is* Weakly Hyperbolic *if for every $\sigma \in \Omega$ we have:*

$$lim_{n\to\infty} Diam(w_{\sigma_1 \dots \sigma_n} X) = 0$$

The topological attractor

Let us denote by $\mathcal{K}(X)$ the family of all compact subsets of X endowed with the Hausdorff metric. The Hutchinson-Barnsley operator $\mathcal{F} : \mathcal{K}(X) \longrightarrow \mathcal{K}(X)$ is given by:

$$\mathcal{F}(A) := \bigcup_{\lambda \in \Lambda} w_\lambda(A) = w(\Lambda \times A) \text{ for } A \in \mathcal{K}(X).$$

0.2 Definition. *An IFS w has an* attractor *$A \in \mathcal{K}(X)$, if $\mathcal{F}^n(B) \to A$ in the Hausdorff topology for every $B \in \mathcal{K}(X)$. If $A \in \mathcal{K}(X)$ is a fixed point of $\mathcal{F}$ then we say that A is an* invariant set *by w.*

Theorem 1. *Let w be a weakly hyperbolic IFS on the compact metric space X and with a compact parameter space Λ. Then $\mathcal{F}$ has an attractor K that is also a compact invariant set. Furthermore, we have that $w_{\sigma_1} \circ \dots \circ w_{\sigma_n}$ has a unique contractive fixed point $\forall \sigma \in \Omega \, \forall n \geq 1$ and K is the closure of these fixed points.*

The measure-theoretical attractor

First, we recall the topologies on the measure space. Let (X, d) be a complete and separable metric space and consider the space

$$\text{Lip}_1(X; \mathbb{R}) = \{f : X \to \mathbb{R} : |f(x) - f(y)| \leq d(x, y) \text{ para todo } x, y \in X\}.$$

Let $\mathcal{M}(X)$ be the set of the borel probability measures μ such that $\mu(f) := \int_X f d\mu < +\infty$ for each $f \in \text{Lip}_1(X; \mathbb{R})$. Then we define the *Hutchinson metric* in $\mathcal{M}(X)$ by:

$$H(\nu, \mu) = \sup \left\{ \left| \int_X f d\nu - \int_X f d\mu \right| ; f \in \text{Lip}_1(X; \mathbb{R}) \right\}.$$

Under the measure-theoretical point of view we also have a notion of attractor, but before we need to define the transfer operator:

0.3 Definition. *Fix a probability $p \in \mathcal{M}(\Lambda)$. The* Transfer Operator *$T_p : \mathcal{M}(X) \to \mathcal{M}(X)$ is defined by the formula*

$$T_p(\mu)(B) := \int_\Lambda \mu(w_\lambda^{-1}(B)) dp(\lambda),$$

for every Borel set B and for each measure $\mu \in \mathcal{M}(X)$. If a measure $\mu \in \mathcal{M}(X)$ is a fixed point of the transfer operator we say that μ is an invariant measure *for w.*

0.4 Definition. *Let X be a complete metric space and $\mathcal{M}(X)$ as before. We say that a probability $\nu \in \mathcal{M}(X)$ is a* measure-theoretical attractor *for w if $T_p^n(\mu) \xrightarrow{n} \nu$ in the Hutchinson metric for all $\mu \in \mathcal{M}(X)$.*

0.5 Definition. *Fix $p \in \mathcal{M}(\Lambda)$ and $\mathbb{P}$ the product measure induced by p in Ω. We say that an invariant measure for w is ergodic if for all continuous function $f : X \to \mathbb{R}$, for all $x \in X$ and σ $\mathbb{P}$-a.e. we have:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} f(w_{\sigma_j} \circ \dots \circ w_{\sigma_1}(x)) = \int_X f d\mu.$$

Theorem 2. *If X is a compact metric space and w is a weakly hyperbolic IFS then w has a measure-theoretical attractor $\nu \in \mathcal{M}(X)$. Furthermore, ν is the unique fixed point of the transfer operator and is ergodic. If $p(U) > 0$ for every open set $U \subset \Lambda$ then we have that $supp(\nu) = K$, where K is the attractor given by theorem 1.*

Drawing the attractor

Here we take inspiration from [BV1] to give a result about ways visualizing the attractor through orbits of the *IFS* instead of computing the full Hutchinson-Barnsley operator.

Before stating the result, we give some definitions

0.6 Definition. *An* orbit of the IFS *starting at some point x is a sequence $\{x_k\}_{k=0}^\infty$ such that $x_0 = x$, $x_{k+1} = w_{\lambda_k}(x_k)$, for some sequence $\{\lambda_k\}_{k=1}^\infty \in \Omega$ in the parameter space.*

0.7 Definition. *Given an IFS $w : \Lambda \times X \to X$ with attractor A, we say that an orbit starting at x* draws *the attractor if $A = \lim_{k\to\infty} \{x_n\}_{n=k}^\infty$, in the Hausdorff metric.*

In order to study orbits that draws the atractor, we shall consider the following class of probability measures $p \in \mathcal{P}(\Lambda)$ in the parameter sapace:

0.8 Definition. *We say that a probability $p \in \mathcal{P}(\Lambda)$ is* fair *if there exists a positive function $f : (0, +\infty) \to (0, 1]$ such that $p(B(\lambda, \delta)) \geq f(\delta)$, for every $\lambda \in \Lambda$. In other words, we shall consider measures with a uniform lower bound for the measure of balls with a fixed radius.*

We will consider the product measure induced by p in the product space Ω, and will denote it by $\mathbb{P}$.

Theorem 3. *Let (X, d) be a compact metric space. Let w be a weakly hyperbolic IFS. Consider $p \in \mathcal{M}(\Lambda)$ a fair probability measure, and $\mathbb{P} \in \mathcal{M}(\Omega)$ the associetade product measure. Then, given $x \in X$, a $\mathbb{P}$-total probability set of orbits of x draws the attractor K of w.*

Sketch of some proofs

Sketch of proof of Theorem 1. The first main step is to prove that $Diam(w_{\sigma_1 \dots \sigma_n} X)$ goes to 0 uniformly in $\sigma \in \Omega$. We prove this by contradiction. Indeed, if this is not the case, then for some $\varepsilon_0 > 0$ we can find a sequence of finite words such that

$$Diam(w_{i_1^k \dots i_{n_k}^k}(X)) \geq \varepsilon_0 \text{ for any } k \in \mathbb{N}. \quad (1)$$

Using compactness and a diagonal argument, we can assume that $i_l^k \xrightarrow{k} \sigma_k \in \Lambda$. Then, by continuity of the IFS, one can prove that $Diam(w_{\sigma_1, \dots, \sigma_n} X) \geq \varepsilon_0$, and thus $\sigma = (\sigma_n)$ do not satisfies the definition of weak hyperbolicity, a contradiction. Having proved this, one can show that the limit

$$\Gamma(\sigma) := \lim_{n\to\infty} \Gamma(\sigma, n, x) := \lim_{n\to\infty} w_{\sigma_1 \dots \sigma_n}(x) \quad (2)$$

exists for every $\sigma \in \Omega$ and $x \in X$, does not depend on x and is uniform on σ. Moreover, the function $\Gamma : \Omega \to X$, defined by the above equation is continuous. Thus $\Gamma(\Omega)$ is a compact set. It reamains to show that $K := \Gamma(\Omega)$ is an attractor. This can be done, following [M], where this was proved in the case Λ is a finite set. □

Sketch of proof of Theorem 3. Fix a point $x \in X$. We first remark that it is enough to prove the following: for every $\varepsilon > 0$ there exists an integer $K_\varepsilon > 0$ and a set $\mathcal{B}_\varepsilon \subset \Omega$, with $\mathbb{P}(\mathcal{B}_\varepsilon) = 1$ such that every x-orbit $\{x_{k+1} = w_{\sigma_k}(x_k)\}$, generated by some sequence $\sigma = (\sigma_k) \in \mathcal{B}_\varepsilon$ satisfies $d_H(A, \{x_k\}_{k\geq L}) < \varepsilon$, for every $L \geq K_\varepsilon$.

To see this, take $\varepsilon_n = \frac{1}{n}$ and define $\mathcal{B} = \cap_n \mathcal{B}_{\varepsilon_n}$. Obvioiusly, $\mathbb{P}(\mathcal{B}) = 1$. Moreover, it is easy to see that $\mathcal{B} \subset \mathcal{A}(x)$. Indeed, take $\sigma \in \mathcal{B}$ and consider $\{x_k\}$ the x-orbit generated by σ. For any $\varepsilon > 0$ we can take a large n with $\varepsilon_n < \varepsilon$. Since $\sigma \in \mathcal{B}_{\varepsilon_n}$, we have that $L \geq K_{\varepsilon_n}$ implies $d_H(A, \{x_k\}_{k\geq L}) < \varepsilon_n < \varepsilon$. Thus, $A = \lim_{L\to\infty} \{x_k\}_{k\geq L}$, wich proves that $\mathcal{B} \subset \mathcal{A}(x)$.

Thus, we are are left to prove the above remark. We shall do this by showing that for each $\varepsilon > 0$, we can find $K_\varepsilon > 0$ such that for every $L \geq K_\varepsilon$ there exists $B_L \subset \Omega$ with $\mathbb{P}(B_L) = 1$, and such that if $\sigma \in B_L$ then the correspondent x-orbit satisfies $d_H(A, \{x_k\}_{k\geq L}) < \varepsilon$. If this is true, then $\mathcal{B}_\varepsilon = \cap_L B_L$ is the desired set.

So, let us fix $\varepsilon > 0$ and exhibit the integer K_ε. By definition of an attractor, there exists K_ε such that $k \geq K_\varepsilon$ implies that

$$d_H(\mathcal{F}^k(\{x\}), A) < \varepsilon, \quad (3)$$

in particular, given any sequence $\{\lambda_k\}_{j=1}^\infty \in \Omega$, the correspondent orbit satisfies $x_k \in \mathcal{F}^k(\{x\}) \subset B(A, \varepsilon)$, for every $k \geq K_\varepsilon$. Take $L \geq K_\varepsilon$ and let us construct the set B_L.

Since the sole obstruction for an orbit do not draw the attractor is it get stuck in some part, the key observation is that, by virtue of (3), for any point a in $B(A, \varepsilon)$, we can find a finite sequence of parameters that "corrects" the orbit of a, making it visit every portion of A. By continuity, this correcting sequence is robust. Since p is a fair probability measure, it can be shown that we have a uniform lower bound for the measure of every set of *finite* words that are corrected in some moment. This uniform lower bound is shown then to imply that with zero $\mathbb{P}$-measure an infinite word will never be corrected, again using that p is a fair measure. But once an orbit is corrected, it satisfies the desired. □

References

[BDEG] M. F. Barnsley, S. G.Demko, J. H. Elton and J. S. Geronimo Invariant measures for Markov processes arising from iterated function systems with place-dependent probabilities. Ann. Inst. H. Poincaré Probab. Statist. 24 (1988), no. 3, 367-394

[BV1] M. F. Barnsley and Andrew Vince The chaos game on a general iterated function system. *Ergodic Theory Dynam. Systems.*, 31 (2011), no. 4, 1073-1079.

[E] A. Edalat Power Domains and Iterated Function Systems. Inform. and Comput. 124 (1996), no. 2, 182-197.

[Ha] M. Hata On the structure of self-similar sets. . *Japan J. Appl. Math. 2* (1985), no. 2, 381-414.

[H] J. E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, 30 (1981), 713-747.

[JL] J. R. Jachymski and G. G. Lukawska. The Hutchinson-Barnsley Theory for infinite iterated function systems. *Bull. Austral. Math. Soc.*, 72 (2005), 441-454.

[L] Gary B. Lewellen. Self-similarity *Rocky Mountain J. Math.* , 23 (1993), no. 3 , 1023-1040.

[M] L. Máté. The Hutchinson-Barnsley theory for certain non-contraction mappings. *Period. Math. Hungar.*, 27 (1993), 21-33.

[Me] F. Mendivil, A generalization of ifs with probabilities to infinitely many maps. Rocky Mountain J. Math. 28 (1998), no. 3, 1043-1051.