APPROXIMATION OF ω-LIMIT SETS, BY MARIE-CLAUDE ARNAUD

BRUNO SANTIAGO

Abstract. Tranlation of Arnaud's paper.

1. Introduction

The aim of this note is to expose the work [1], where the following result is proved.

1.1. **Theorem** (Arnaud). *There exists a residual subset of* $\mathrm{Diff}^1(M)$ *such that for every* $x \in M$, $\omega(x)$ *is accumulated in the Haudorff topology by periodic orbits.*

The proof is solely based on the connecting lemma of Wen-Xia plus some beatiful, and yet simple, ideias. It gives a nice example of a good use this amazing tool that is the connecting lemma.

2. Proof

We shall use the Semicontinuity Lemma to produce the residual set that will give Theorem 1.1. For definitions and proofs, see [2]. Consider $\mathcal{K}(X)$ the space of compact subsets of X, endowed with the Hausdorff topology, where X is a compact metric space. Consider the function

$$P : KS^1(M) \to \mathcal{K}(\mathcal{K}(M)),$$

that assignes to each C^1 Kupka-Smale diffeomorphism f the closure of its periodic periodic orbits. By the analitic continuation of hyperbolic periodic orbits, this function is semicontinuos. By the semicontinuity lemma, there exists a residual

$$\mathcal{R}_P \subset \mathrm{Diff}^1(M),$$

where each $f \in \mathcal{R}_P$ is a continuity point of P.

Take a generic $f \in KS^1(M) \cap \mathcal{R}_P$, and let us suppose by absurd that Theorem 1.1 is not true for f. This implies that $\omega(x, f)$ is disjoint of $P(f)$. Since $\mathcal{K}(\mathcal{K}(M))$ is compact, we can take open sets U and V in this space, with disjoint closures, and such that $P(f) \in V$ and $\omega(x, f) \in U$. since f is a continuity point of P, for every g close enough to f, $P(g) \in V$.

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Now, we apply the following perturbative result.

2.1. **Lemma.** *Given a generic f, $x \in M$, neighborhods U of $\omega(x, f)$ in the Hausdorff topology and $\mathcal{U}$ of f in the C^1 topology, there exists $g \in \mathcal{U}$, $p \in \mathrm{Per}(g)$ such that $O(p) \in U$.*

This enables us to blow the continuity of P at f, leading to a contradiction. Thus, we are left to prove the above perturbative lemma, wich is the main part of Arnaud's proof.

2.1. **A Technical Lemma.** For the proof of Lemma 2.1 we shall need to stablish the denseness of non-periodic points for a non-trivial $\omega(x, f)$, where non trivial means more than one single periodic orbit.

For this, we first remark that every $\omega(x, f)$ cannot be decomposed as the union of two non-void invariant compact subsets. Indeed, suppose this is the case, say

$$\omega(x, f) = A \cup B.$$

Separate A and B by disjoint open sets U and V with $f(U) \cap V = \emptyset$. Let $N > 0$ be integer. Take a hit of x in U, and a hit in V, both of them larger than N. Take the minimum $n > N$ such that $f^n(x) \notin U$. By minimality, $f^n(x) \in f(U)$, and thus we have found $n > N$ such that

$$f^n(x) \notin U \cup V.$$

But this contradicts $\omega(x, f) = A \cup B$.

This shows that for each periodic point $y \in \omega(x, f)$, with $\omega(x, f)$ beeing non-trivial, we cannot have a disjoint decomposition

$$\omega(x, f) = O(y) \cup (\omega(x, f) - O(y)).$$

This shows that for every periodic point $y \in \omega(x, f)$, $\omega(x, f) - \{y\}$ is open and dense in $\omega(x, f)$. Since f is generic, there are only countably many periodic orbits, and thus $\omega(x, f) - \mathrm{Per}|_{\omega(x,f)}(f)$ is a dense G_δ.

This stablishes the denseness of non-periodic points, as desired.

2.2. **Proof of the Perturbative Result.** Take a neighborhood W of $\omega(x, f)$ and using compactness, consider a finite set $\{p_1, ..., p_n\} \subset \omega(x, f)$ with the following property: there exists neighborhoods $V_1, ..., V_n$, of $p_1, .., p_n$, respectively, all of them contained in W and such that any compact subset of M that intersect everey V_k and is contained in W belongs to U. Moreover, by the technical lemma we can assume that none of the points p_k are periodic.

Now, applying the *connecting lemma* of Wen-Xia we get integers $m_1, ..., m_n$, one for each $p_1, ..., p_n$, that gives the lengh of the orbit-tube inside wich we can connect orbits. Denote by m the maiximum of $m_1, ..., m_n$, and remove from the list $\{p_i\}$ those points that belongs to a same string of lenght m. In the end of this, we obtain that the strings $\{f^k(p_i)\}_{k=0}^{m-1}$ are pairwise disjoint for every $i \in \{1, ..., N\}$, for some $N \leq n$.

Then, we sharply shrink the neighborhoods $V_1, ..., V_N$ to obtain the following properties:

(1) The open sets $f^k(V_i)$ are pairwise disjoint, for every $k = 1, ..., m-1$ and every $i = 1, ..., N$, and all of them lies inside W.
(2) The integer m verifies the conclusion of the connecting lemma; moreover, we associate to each V_i a smaller neghborhood $V_i' \subset V_i$, given by the connecting lemma.
(3) Every compact inside W that intersect every $f^k(V_i)$ belongs to U.

Let us assume that $N \geq 2$, and prove the result in this case.

Choose $n_0 > 0$ such that $n \geq n_0$ implies $f^n(x) \in W$. Also choose integers $n_j \geq n_0$, $j = 1, ..., N$, such that $f^{n_j}(x) \in V_j'$, and consider $n^* > \max n_j$.

This first step is designed to garantee a hit in the past in the tube that will be used for create the periodic orbit. Our goal now is to produce a hit in the future, but **making sure** that such a future hit will happen after all the neighborhoods V_j has been visited.

To achieve this, Arnaud's ideia is to consider an interval $[l, i]$, for some $l \geq n^*$, with two properties

- All neighborhoods V_j are visited by some $f^r(x)$, with $r \in [l, i]$.
- The interval $[l, i]$ has **the minimal possible length** between all the interval satsfying the above item.

The minimality of the interval $[l, i]$ forces that $f^l(x) \in V_{j_0}$, for some j_0 (wich can be assumed to be 1). Moreover, and this is the core of the proof, the minimality gives the following beautifull property:

$$\text{para todo } r \in [l+1, i],\ f^r(x) \notin V_1.$$

Now, by definition of $\omega(x, f)$, there exists $r_1 > i$ such that $f^{r_1}(x) \in V_1'$. Thus, if we consider the point $p = f^{r_2}(x)$, where $r_2 \in [l+1, i]$ is such that $f^{r_2}(x) \in V_2$, we have that the past orbit of p contains the point $f^{n_1}(x) \in V_1'$, while its future orbit, after visit $V_2, ..., V_N$ (wich is guaranteed by the choise of $[l, i]$ with the minimal property described above) hits again V_1' in the point $f^{r_1}(x)$.

By virtue of the *connecting lemma* of Wen-Xia, we can turn p in a periodic point for some $g \in \mathcal{U}$, and such that $g = f$ outside the tube $\cup_{j=0}^{m} f^j(V_j)$. Thus, the string

$$\{f^{r_2}(x), ..., f^i(x)\},$$

wich has empty intersection with V_1, is containded in the g-orbit of p. This makes sure that the g-orbit of p visit every $V_2, ..., V_N$. Since the connection is made in the tube $\cup_{j=0}^{m} f^j(V_j)$, we have that the g-orbit of p also visits this tube, wich shows that $O_g(p) \in U$, as desired.

Now, we deal with the case $N = 1$.

We first shrink V_1 until we obtain a point $p_2 \in \omega(x, f) - \cup_{k=-m}^{k=m} f^k(\overline{V_j})$. In this stage, we choose n^* as in the previous case, and a minimal interval $[l, i]$ exactly as above. If $j_0 = 1$, the same argument works. If $j_0 = 2$, we just observe that the above argument argument has a simmetry, and we can use p_2 with the role of p_1 and apply the same argument.

We are done.

References

[1] Arnaud, Marie-Claude. *APPROXIMATION DES ENSEMBLES ω-LIMITES DES DIFFEOMORPHISMES PAR DES ORBITES PERIODIQUES* Ann. Scient. Éc. Norm. Sup. 4e série, t. 36, 2003, p. 173 à 190.

[2] Santiago, B. *The Semicontinuity Lemma.* Personal Lecture Notes

Bruno Santiago
Instituto de Matemática
Universidade Federal do Rio de Janeiro
P. O. Box 68530
21945-970 Rio de Janeiro, Brazil
E-mail: bruno_santiago@im.ufrj.br