ERGODIC DECOMPOSITION OF INVARIANT MEASURES

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ABSTRACT. The aim of this note is to present to myself a light version of the ergodic decomposition of invariant measures. In particular, in this note there will be no mention to Rokhlin's Desintegration Theorem. Nevertheless, the result we shall prove here is quite usefull for most purposes.

1. Introduction

Let $T : X \to X$ be a homeomorphism of a compact metric space. Denote by $\mathcal{P}(T)$ the space of T-invariant probability measures. The main result we are seeking in this note is the

1.1. **Theorem.** *Given $\mu \in \mathcal{P}(T)$ there exists a full measure set $N \subset X$ such that if $x \in N$, there exists a unique ergodic measure μ_x, with $x \in \mathrm{Supp}(\mu_x)$, such that for every $f \in L^1(\mu)$ the fucntion $x \mapsto \int f d\mu_x$ is well defined and μ-integrable and*

$$\int f d\mu = \int \left(\int f d\mu_x \right) d\mu.$$

The proof will be given through a detour of lemmas. The ideia is to prove it first for continuous functions and then use an approximation argument in order to pass for arbitrary integrable functions. Such procedure is possible due to the especial topological assumptions we put on the phase space. Moreover, the continuity of the dynamical system T will be also important.

2. The convergence of the asymptotic averages

In this section we shall prove (as an easy corolary of Birkhoff Ergodic Theorem) that for a typical point, the time averages

$$f_n(x) := \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x)$$

of $f \in C^0(X)$ converges. The main point here is the existence of a total probability set for which the time averages converges, regardless the function. This improves

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Birkhoff (in $C^0(X)$) since Birkhoff only gives that *for each function* there exists a full measure set on which the time averages converges.

2.1. **Lemma.** *Let* $\Sigma_0(T) = \{x \in X; \{f_n(x)\}$ *is a Cauchy sequence, for every* $f \in C^0(X)\}$. *Then, for every* $\mu \in \mathcal{P}(T)$, $\mu(\Sigma_0(T)) = 1$.

Proof. Let $\mathcal{D} \subset C^0(X)$ denote a countable dense subset. If $f \in \mathcal{D}$ is fixed, then, by Birkhoff's Theorem the set of points for wich f_n converges has full measure and therefore taking the intersection we obtain a total probility set for wich $\{f_n(x)\}$ is Cauchy, for every $f \in \mathcal{D}$. The only thing we have to prove is that if x belongs to such a set, and $g \in C^0(X)$, then $\{g_n(x)\}$ is a Cauchy sequence. This is achieved by a standard interpolation argument. You fix $\varepsilon > 0$, and take $f \in \mathcal{D}$ with $\|f - g\| < \varepsilon$. Then, $\|f_k - g_k\| < \varepsilon$, for every $k > 0$. This implies that

$$\|f_n - f_m\| \leq \|f_n - g_n\| + \|g_n - g_m\| + \|g_m - f_m\| < 3\varepsilon,$$

for m, n large. □

If $x \in \Sigma_0(T)$ then we have defined a positive linear functional over $C^0(X)$, given by $L_x(f) := \lim f_n(x)$. By Riesz, there exists a unique measure μ_x which represents this functional, and since $L_x(1) = 1$ this measure is a probability. Moreover, since T is assumed to be continuous, we have that μ_x is an invariant measure. This fact is a little bit tricky, because one would invoke that the asymptotic time average is an invariant function, which indeed is true, but cannot be applied here. The problem is that the averages only exists inside a full measure set wich *depends* upon the function. Once we replace f by $f \circ T$, we no longer know if the average exists at x, and this is exactly where the argument fails.

3. Poincaré Recurrence Revisited

Now, we shall prove that for a typical point $x \in \mathrm{Supp}(\mu_x)$. Observe this means that for every neighborhood U of x, $\mu_x(U) > 0$ and since μ_x measures the average number of times that the orbit of x visits U,[1] this can be viewed as a refinement of Poincaré Recurrence. The key point is the following tautological consequence of the ergodic theorem.

3.1. **Lemma.** *If* $f \in L^1(\mu)$ *and* $f \geq 0$ *then for* μ*-almost every* x *if* $f'(x) := L_x(f) = 0$ *then* $f(x) = 0$

Proof. Let $A = \{x \in X; f'(x) = 0\}$. Since f' is measurable, A is measurable. Let $g = \chi_A f$. If $x \in A$ then $g = f$, and since A is invariant one has $g' = f'$ in A,

[1] OK, with respect to continuous functions, but lets forget about it for a while

whenever both sides are defined. If $x \notin A$ then $g(x) = 0$. Since, $\int g d\mu = \int g' d\mu$, this implies that

$$\int_A f d\mu = \int_A f' d\mu = 0,$$

which gives the lemma. □

Applying lemma 3.1 with $f = \chi_U$ we obtain that if the average number of times that the orbit of x visits U is zero then $x \notin U$. This, in particular, proves Poincaré Recurrence. But, our interest in lemma 3.1 is the following.

3.2. **Proposition.** *Let $\Sigma_1(T) := \{x \in \Sigma_0(T); x \in \mathrm{Supp}(\mu_x)\}$. Then, $\Sigma_1(T)$ is a total probability set.*

Proof. Take a countable basis U_n of X and take $C^0(X)$ functions $\phi_n > 0$ inside U_n and zero outside, dominated by one. Then, by lemma 3.1,

$$A_n = \{x \in \Sigma_0(T); \phi_n'(x) = 0 \text{ implies } \phi_n(x) = 0\}$$

is a total probability set. Therefore, putting $\Sigma_1(T)$ as the intersection of the A_n's we obtain a total probability set. Take x there. Take U a neighborhood of x. There exists $x \in U_n \subset U$. Since $x \in A_n$, and $x \in U_n$ we have that $\phi_n(x) > 0$ which implies that $\phi_n'(x) > 0$, thus

$$\mu_x(U) \geq \mu_x(U_n) = \int \chi_{U_n} d\mu_x \geq \int \phi_n d\mu_x = \phi_n'(x) > 0.$$

□

4. The ergodicity of the asymptotic measures

This is the major section of the note, and the most beautiful one. The arguments here where taken from [1]. We shall first give an interesting lemma, that gives more accurate control on how the means in time k are converging to the asymptotic mean. More precisely, we shall prove that if one consider the asymptotic average quadratic error between the the mean of time k and the asymptotic mean, then, first it is well defined (i.e. the limit exists) and in average with respect to μ it goes to zero as k goes to infinity.

It is a reasonable result, and the proof is outstanding. It is an extremely simple application of the L^2 convergence of the means.

4.1. **Lemma.** *Let $f \in L^2(\mu)$. Then, for μ almost every $x \in X$ there exists*

$$\psi^k(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \left\{ f_k(T^j x) - f'(x) \right\}^2,$$

and

$$\int_X \psi^k d\mu \to 0,$$

when $k \to \infty$.

Proof. Since $f \in L^2(\mu)$, $f' \in L^2(\mu)$ and thus $(f_k - f')^2 \in L^1(\mu)$. By the Ergodic Theorem

$$\left\{(f_k - f')^2\right\}'$$

is defined almost everywhere and, since $f' \circ T = f'$ we have that

$$\left\{(f_k - f')^2\right\}' = \psi^k(x),$$

whenever the left-hand side exists. Moreover,

$$\int_X \left\{(f_k - f')^2\right\}' d\mu = \int_X (f_k - f')^2 d\mu,$$

and since $f_k \to f'$ in the $L^2(\mu)$ norm, it follows that

$$\int_X \psi^k d\mu \to 0,$$

when $k \to \infty$. □

Now comes the sunset on the beach. We shall apply the above lemma to prove the ergodicity of the asymptotic measure, for a typical point.

4.2. **Proposition.** *Let* $\Sigma_2(T) := \{x \in \Sigma_1(T); \mu_x \text{ is ergodic}\}$. *Then,* $\Sigma_2(T)$ *is a total probability set.*

Proof. Let μ be an invariant measure. We want to prove that $\mu(\Sigma_2(T)) = 1$. However, to prove that μ_x is ergodic requires to prove that f' is μ_x a.e constant for every $f \in L^1(\mu_x)$. The first observatoin is that it suffices to prove this for a dense subset of $L^1(\mu)$. Indeed, it is a general fact that a measure is ergodic if f' is a.e constant for a dense subset of L^1.

A sketch of proof goes as follows. Note that to prove that f' is a.e constant with respect to some measure ν, one only needs to prove the following equation

$$\int_X \left| f' - \int_X f d\nu \right| d\nu = 0. \tag{1}$$

Now, assume that (1) holds for every g in some dense subset of $L^1(\nu)$. Take $f \in L^1(\nu)$ and let g be is this dense subset, ε close to f. In equation (1), interpolate $+ - g'$, observing that $g' = \int_X g d\mu$, and then using triangle inequality you get that the integral is bounded by the sum of the L^1 distance between f and g, which is small, and the L^1 distance between f' and g'. But, to estimate the L^1 distance between f' and g', you use Fatou's lemma and discovers that it is bounded by the L^1 distance between f and g. This proves that (1) is less than 2ε, and gives the desired.

The second observation is that, since X is a compact metric space, one can take this dense subset to be a countable dense subset of $C^0(X)$. The third observation is that by definition of μ_x, for a continuous function one can replace (1) by

$$\int_X \{f' - f'(x)\}^2 \, d\mu_x = 0. \tag{2}$$

If we prove that given a fixed continuous function f, (2) holds for μ a.e $x \in X$ we obtain the existence of a full measure set (which depends upon f) for which μ_x is ergodic. Then taking the intersection of these sets with f varying inside a countable dense subset of $C^0(X)$, by the first observation above, we obtain the result.

Therefore, we are left to prove that, fixed a continuous function f, (2) holds for μ a.e x.

To prove this, one first note that $\|f_k\|_{C^0} \leq \|f\|_{C^0}$ and $f_k \to f$ a.e. Thus, by dominated convergence theorem,

$$\begin{aligned}
\int_X \{f' - f'(x)\}^2 \, d\mu_x &= \lim_{k\to\infty} \int_X \{f_k - f'\}^2 \, d\mu_x \\
&= \lim_{k\to\infty} \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \left\{ f_k(T^j x) - f'(x) \right\}^2 \\
&= \lim_{k\to\infty} \psi^k(x).
\end{aligned}$$

We have shown that for every $x \in \Sigma_0(T)$, the above limit exists and we have defined a function $x \in \Sigma_0(T) \mapsto \psi(x) := \int_X \{f' - f'(x)\}^2 \, d\mu_x$. The equality we proved just shows that

$$\int_X \psi d\mu = \int_X \lim_{k\to\infty} \psi^k d\mu.$$

Since $|f_k(T^j x) - f'(x)|^2 \leq (|f_k(T^j x)| + |f'(x)|)^2$, which is dominated by $4\|f\|^2_{C^0}$, we can apply the dominated convergence theorem and get

$$\int_X \psi d\mu = \lim_{k\to\infty} \int_X \psi^k d\mu = 0,$$

by lemma 4.1. This completes the proof. $\square$

5. The ergodic decompsition

Take $\mu \in \mathcal{P}(T)$. Then, for μ almost every point x, μ_x is an ergodic invariant measure. Let $f \in C^0(X)$. Then, by the ergodic theorem f' is a.e defined and is integrable. By definition of μ_x, for a.e $x \in X$,

$$f'(x) = \int_X f d\mu_x,$$

and since $\int_X f' d\mu = \int_X f d\mu$, we have proved that

$$\int f d\mu = \int \left(\int f d\mu_x \right) d\mu.$$

Now, take a positive $f \in L^1(\mu)$. We have that f is an increasing a.e limit of continuous functions, say f_n. This implies that

$$\int_X f d\mu_x = \lim_{n\to\infty} \int_X f_n d\mu_x = \lim_{n\to\infty} f_n'(x).$$

Since f_n increasing implies f_n' increasing and uniformly bounded (by f) we can apply again the monotone convergence theorem and obtain that $x \mapsto \int_X f d\mu_x$ is integrable, and also

$$\int_X \int_X f d\mu_x d\mu = \int_X \lim_{n\to\infty} f_n' d\mu = \lim_{n\to\infty} \int_X f_n' d\mu = \lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

This completes the proof of Theorem 1.1. □

References

[1] Oxtoby, John C. *Ergodic sets.* Bull. Amer. Math. Soc. 58, (1952). 116-136

[2] Mañé, Ricardo *Ergodic theory and differentiable dynamics*. Springer-Verlag, Berlin, 1987

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