NON-EXPLOSION OF HOMOCLINIC CLASSES

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Abstract. In this note we shall present a proof that robustly in a neighborhood of a hyperbolic set, the number of homoclinic classes is finite and uniformly bounded.

1. Introduction

The argument of this note arose from a discussion at IMPA with Alexander Arbieto, Andrés Lopez and Carlos Morales. I should thank them for that nice day of work. The idea was to prove that inside a neighborhood of a hyperbolic set the number of attractors is finite. For a single dyamical system this quite easy, due to the stable manifold theorem. Moreover, this can be generalized (with the same proof) for homoclinic classes. Since a hyperbolic set persists, this shows that robustly the number of homoclinic classes in a neighborhood is also finite. The problem is then to show that this number can not explode. The answer is yes, and the result is

1.1. **Lemma.** *Let Λ be a hyperbolic set. Then, there exists a neighborhood U of Λ and a neighborhood $\mathcal{U}$ of f with the following property: there exists $n \in \mathbb{N}$ such that for every $g \in \mathcal{U}$, the number of homoclinic classes of g which are contained in U is bounded by n.*

It turns out that the proof is quite the same, with an elegant adapatation which uses the pigeonhle principle.

2. proof

The starting point is an elementary lemma.

2.1. **Lemma.** *Let K be a compact metric space and take $\delta > 0$. Assume that there exists a sequence of finite sets $K_n = \{x_1, ..., x_{l_n}\} \subset K$, and that $l_n \to \infty$. Then, there exists $m \in \mathbb{N}$ and $x_i, x_j \in K_m$ with $d(x_i, x_j) < \delta$.*

Proof. Cover K with a finite number, say N, of balls with diameter δ. Let $m \in \mathbb{N}$ be such that $l_m > N$. We claim that the conclusion holds for K_m. Indeed, if this is not the case, then each ball in the cover has at most one point of K_m, which implies that $l_m \leq N$, a contradiction. This proves the lemma. □

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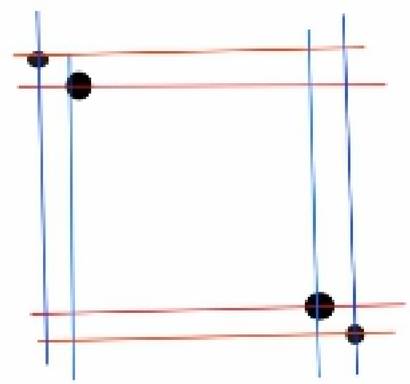

FIGURE 1. Heteroclinic intersection.

Proof of Lemma 1.1. By the hyperbolic theory, there exists $\varepsilon > 0, \mathcal{U}$ a neighborhood of f and U a neighborhood of Λ such that $\cap_{n\in\mathbb{Z}} g^n(U)$ is a hyperbolic set for every $g \in \mathcal{U}$ and the local invariant manifolds $W^s_\varepsilon(x, g)$ and $W^u_\varepsilon(x, g)$ have uniform size, ε. Moreover, there exists $\delta > 0$ such that any two points δ-close, the intersections

$$W^s_\varepsilon(x, g) \cap W^u_\varepsilon(y, g), \quad W^u_\varepsilon(x, g) \cap W^s_\varepsilon(y, g)$$

are non-empty. Let $m \in \mathbb{N}$ be given by lemma 2.1 with δ and $\overline{U}$. Assume that lemma 1.1 is not true if this n, U. Then, there exists $f_n \to f$, with the number l_n of homoclinic classes inside U going to infinity. For each n choose a unique point x_i in each homoclinic class of f_n inside U, and let $K_n = \{x_1, ..., x_{l_n}\}$. By lemma 2.1 we can find $x_i, x_j \in K_m$ δ-close.

But this implies that x_i and x_j are accumulated by periodic points heteroclinicaly related, and thus x_i and x_j are in the same homoclinic class, violating the definition of K_m. This contradiction completes the proof. □

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