MEASURABLE PARTITIONS, DISINTEGRATION AND CONDITIONAL MEASURES

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Abstract. In this short note we review Rokhlin Desintegration Theorem and give some applications.

1. Introduction

Consider a measure space $(M, \mathcal{A}, \mu)$. Suppose that we partition X in an arbitrary way. Is it possible to recover the measure μ from its restriction to the elements of the partition?

In this note, we shall address this question, giving some affirmative answer and applying this idea to obtain interesting results.

Let us start with a simple (positive) example.

Example 1.1. Consider the two torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, endowed with the Lebesgue measure m. The torus can be easily partitioned by the sets $\{y\} \times \mathbb{S}^1$.

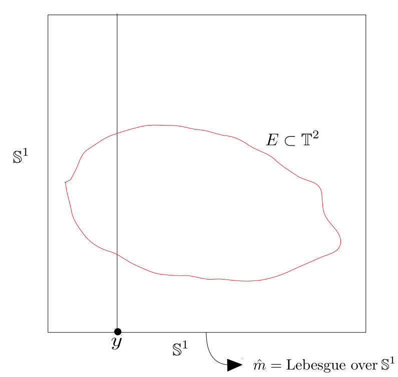

Figure 1.

Denote by m_y the Lebesgue measure over the circle $\{y\} \times \mathbb{S}^1$, and $\hat{m}$ the Lebesgue measure over $\mathbb{S}^1$.

If $E \subset \mathbb{T}^2$ is a measurable set we know from basic measure theory that

$$m(E) = \int m_y(E) d\hat{m}(y) \tag{1.1}$$

2010 *Mathematics Subject Classification.* Primary .

Thanks to Aaron Brown for helpful conversations and the organizing committee of the conference "Workshop for young researchers: groups acting on manifolds" for the opportunity of participating in this wonderful meeting.

We would like to have a disintegration like this of a measure with respect to a partition in more general situations. One would naively ask: does it always exist a disintegration for a given partition?

As we shall see later, for some simple, dynamically defined, partitions no disintegration exists at all. In the next sections we shall define formally the notion of a disintegration and try to explore a little bit this concept.

2. Disintegration and Conditional Measures

Notice that if we remove from $\mathbb{S}^1$ all $y \in Q \cap \mathbb{S}^1$, for some measurable set $Q \subset \mathbb{S}^1$ with $\hat{m}(Q) = 0$, equality (1.1) is not affected. Thus, it is natural to think of negligible "amounts of sets" in a partition.

More formally, let $(M, \mathcal{B}, \mu)$ be a probability space. Let $\mathcal{P}$ be a partition of M into measurable sets. Let $\pi : M \to \mathcal{P}$ be the natural projection:

$$\pi(x) \textit{ is the unique element of } \mathcal{P} \textit{ such that } x \in \pi(x).$$

We can turn $\mathcal{P}$ into a measure space $(\mathcal{P}, \hat{\mathcal{B}}, \hat{\mu})$, by saying

$$Q \in \hat{\mathcal{B}} \iff \pi^{-1}(Q) \in \mathcal{B},$$

and

$$\hat{\mu}(Q) = \mu(\pi^{-1}(Q)).$$

Definition 2.1. A *disintegration* of μ with respect to $\mathcal{P}$ is a family of probabilities $\{\mu_P; P \in \mathcal{P}\} \subset \mathcal{M}_1(M)$ such that for every $E \in \mathcal{B}$ one has

(1) $\mu_p(P) = 1$ for $\hat{\mu}$ almost every $P \in \mathcal{P}$.
(2) $P \in \mathcal{P} \mapsto \mu_P(E) \in \mathbb{R}$ is $\hat{\mathcal{B}}$-measurable.
(3) $\mu(E) = \int \mu_P(E) d\hat{\mu}(P)$.

Each measure μ_P is called a *conditional measure*.

Example 2.2. Let $\mathcal{P} = \{P_1, ..., P_n\}$ be a finite partition of M. Assume that no element of this partition has zero measure. In this case, the conditional measures are given by

$$\mu_i(E) = \frac{\mu(E \cap P_i)}{\mu(P_i)}, \text{ for every } E \in \mathcal{B}.$$

Indeed, we have $\hat{\mu}(\{P_i\}) = \mu(P_i)$ and

$$\mu(E) = \sum_{i=1}^{n} \mu(P_i) \frac{\mu(E \cap P_i)}{\mu(P_i)} = \sum_{i=1}^{n} \hat{\mu}(\{P_i\}) \mu_i(E).$$

In the same way, we can show that every countable partition admits a disintegration.

In the lemma below, we sate an important and useful property of a disintegration. The proof is left as an exercise.

Lemma 2.3. *Assume that $\mathcal{B}$ admits a countable generator. If $\{\mu_P; P \in \mathcal{P}\}$ and $\{\mu_P^*; P \in \mathcal{P}\}$ are disintegrations then $\mu_P = \mu_P^*$ for $\hat{\mu}$ almost every $P \in \mathcal{P}$.*

There are very natural examples of partitions for which no disintegration exist at all.

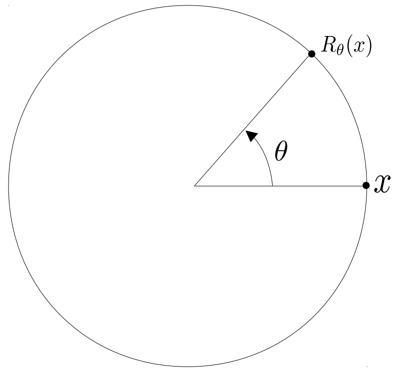

FIGURE 2. Circle rotations

Example 2.4. Let $\theta \in \mathbb{R}\backslash\mathbb{Q}$ be an irrational number, m be the normalised Lebesgue measure of the unit circle $\mathbb{S}^1$. We consider $R_\theta : \mathbb{S}^1 \to \mathbb{S}^1$ the circle rotation by an angle θ.

Let $\mathcal{P} = \big\{\{R_\theta^n(x)\}_{n\in\mathbb{Z}}; x \in \mathbb{S}^1\big\}$ be the partition into orbits of the irrational circle rotation. We claim that this partition admits no disintegration. Indeed, assume that there exists $\{\mu_P; P \in \mathcal{P}\}$ a disintegration of Lebesgue measure with respect to this partition.

We look at the push-forward measures $\{(R_\theta)_*\mu_P; P \in \mathcal{P}\}$. Then, as each set $P \in \mathcal{P}$ is invariant under the rotation R_θ we have:

(1) $(R_\theta)_*\mu_P(P) = \mu_P(R_\theta^{-1}(P)) = \mu_P(P) = 1$.
(2) For each Lebesgue measurable set E the map

$$P \in \mathcal{P} \mapsto (R_\theta)_*\mu_P(E) = \mu_P(R_\theta^{-1}(E))$$

is $\hat{m}$ measurable.
(3) $m(E) = m(R_\theta^{-1}(E)) = \int \mu_P(R_\theta^{-1}(E))d\hat{m}(P) = \int (R_\theta)_*\mu_P(E)d\hat{m}(P)$.

This shows that the family of push-forward measures $\{(R_\theta)_*\mu_P; P \in \mathcal{P}\}$ is a disintegration for m with respect to $\mathcal{P}$. By Lemma 2.3

$$(R_\theta)_*\mu_P = \mu_P, \text{ for } \hat{m} \text{ almost every } P \in \mathcal{P}.$$

Since irrational circle rotations are uniquely ergodic, i.e. the only measure left invariant under the map R_θ is Lebesgue measure, we conclude that $\mu_P = m$ for $\hat{m}$-almost every $P \in \mathcal{P}$.

However, this implies that $m(P) = m(\{R_\theta^n(x)\}_{n\in\mathbb{Z}}) = 1$, which is absurd.

3. Measurable Partitions

In this section we shall define a class of partitions for which we always can find a disintegration. Recall that a partition $\mathcal{P}$ is finer than a partition $\mathcal{Q}$, which we denote by $\mathcal{Q} \prec \mathcal{P}$, if every $P \in \mathcal{P}$ is contained in some $Q \in \mathcal{Q}$.

Definition 3.1. Let $(M, \mathcal{B}, \mu)$ be a probability space A partition $\mathcal{P}$ is measurable if there exists $M_0 \subset M$ with $\mu(M_0) = 1$ and a nested sequence of countable partitions $\mathcal{P}_1 \prec \mathcal{P}_2 \prec .... \prec \mathcal{P}_n \prec ...$ such that $\mathcal{P}|_{M_0} = \vee_{n=1}^{\infty}\mathcal{P}_n$. In other words, for every $P \in \mathcal{P}$ there exists a sequence P_n, with $P_n \in \mathcal{P}_n$ such that $P \cap M_0 = \cap_{n=1}^{\infty}(P_n \cap M_0)$.

Thus a measurable partition can be described as the joining of a nested sequence of countable partitions. Notice that, as in example 2.2, countable partitions always admit a disintegration.

From this fact and from a suitable martingale argument, one can prove the following fundamental theorem.

Theorem 3.2 (Rokhlin Disisntegration Theorem)**.** *If (M, d) is a complete and separable metric space and $\mathcal{P}$ is a measurable partition. Then, there exists $\{\mu_P; p \in \mathcal{P}\}$ a disintegration of μ.*

Let us see some examples of partitions which are, and which are not, measurable.

Example 3.3. In the two torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, consider for each pair i, n, with n a positive integer and $i \in \{1, 2, 3, ..., 2^n\}$ the interval $J(i, n) = [\frac{i-1}{2^n}, \frac{i}{2^n}]$. Then, the

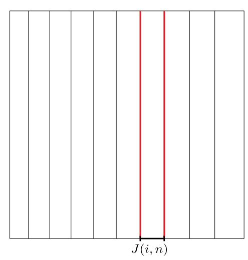

FIGURE 3. A measurable partition of $\mathbb{T}^2$.

partition $\mathcal{P}_n = \{\mathbb{S}^1 \times J(i, n)\}$ is a measurable partition.

Example 3.4. Let $f_A : \mathbb{S}^1 \to \mathbb{S}^1$ be the map induced by the integer matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then, f_A is an Anosov diffeomorphism. Let $\mathcal{P} = \{W^u(x); x \in \mathbb{T}^2\}$ be the partition into unstable manifolds. We claim that $\mathcal{P}$ is not measurable. Indeed, if $\mathcal{P}$ were measurable, as it is the partition into orbits of an irrational flow, $\mathcal{P} = \vee_{n=1}^{\infty}$ would imply that for each n there exists $P_n \in \mathcal{P}_n$, with $m(P_n) = 1$. Thus, $P = \cap_{n=1}^{\infty} P_n \in \mathcal{P}$, and $m(P) = 1$, which is absurd.

4. Ergodic Decomposition of Invariant Measures

We proceed to give an important application of the disintegration theorem, namely the decomposition of invariant measures into ergodic measures.

Let $(M, \mathcal{B}, \mu)$ be a probability space and $f : M \to M$ be a measurable map such that $f_*\mu = \mu$. We say that the measure preserving system $(f, M, \mathcal{B}, \mu)$ is ergodic if every measurable invariant set under f has either zero or full measure.

The goal of this section is to prove the following military principle: divide the space to conquer the ergodic decomposition.

Theorem 4.1 (Ergodic Decomposition)**.** *Let (M, d) be a complete and separable metric space and $(f, M, \mathcal{B}, \mu)$ a measure preserving system. Then there exists a measurable partition $\mathcal{P}$ whose disintegration $\{\mu_P; P \in \mathcal{P}\}$ satisfies $\hat{\mu}$ almost every μ_P is ergodic.*

The idea is that an ergodic system is dynamically indecomposable, since its orbits spread more or less uniformly over the configuration space, and thus it is possible split the M into the indecomposable components of the dynamics. Let us see this more closely by recalling a fundamental result in ergodic theory.

4.1. **The ergodic theorem.** Consider the following statistical question: given a point $p \in M$ and a certain positive measure set $A \subset M$, how often does the future orbit of p under f visit A?

From a more formal point of view this means to study the behaviour of the sequence

$$\frac{1}{n}\sum_{j=0}^{n-1} \chi_A(f^j(p)).$$

So, its natural to ask: does this sequence converges? If so, to what limit? From an heuristic point of view, it is reasonable to conjecture that a system if no positive measure invariant set (ergodic) is forced to visit every region of the configuration space uniformly, since otherwise some invariant with positive weight would be produced.

The ergodic theorem clarifies this clumsy reasoning.

Theorem 4.2. *Let $(f, M, \mathcal{B}, \mu)$ be a measure preserving system. Then for every measurable set $A \subset M$ the limit*

$$\frac{1}{n}\sum_{j=0}^{n-1} \chi_A(f^j(p))$$

exists for μ-almost every $p \in M$.

It is not hard to show (though we will not do this here) that the ergodic theorem implies the following.

Corollary 4.3. *A measure preserving system $(f, M, \mathcal{B}, \mu)$ is ergodic if and only if*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \chi_A(f^j(p)) = \mu(A),$$

for μ-almost every $p \in M$, and every measurable set A.

4.2. **Proof of Theorem 4.1.** As we said before, we need to dive the space to conquer the ergodic decomposition. So, our first task is to choose a suitable partition of M. Let $\mathcal{U}$ be a countable basis for the topology of M, and $\mathcal{A}$ the algebra generated by $\mathcal{U}$. Notice that $\mathcal{A}$ is countable and generates $\mathcal{B}$.

Then the ergodic theorem implies that for each $A \in \mathcal{A}$ there exists $M_A \subset M$ with $\mu(M_A) = 1$ and such that

$$\tau(A, x) = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \chi_A(f^j(x))$$

exists. Take $M_0 = \cap_{A\in\mathcal{A}} M_A$. Then $\mu(M_0) = 1$.

We insert the following equivalence relation in M_0: $x \sim y$ if, and only if, $\tau(A, x) = \tau(A, y)$, for every $A \in \mathcal{A}$.

Lemma 4.4. *The partition $\mathcal{P} = \{[x]; x \in M_0\}$ of M_0 into equivalence classes is measurable.*

We shall finish the proof assuming Lemma 4.4.

Proof of Theorem 4.1. Let $\{\mu_P; p \in \mathcal{P}\}$ be the associated disintegration. We only have to prove that each μ_P is ergodic. Fix $P \in \mathcal{P}$ and consider

$$\mathcal{C} = \{E \in \mathcal{B}; \tau(E, x) \text{ is defined and constant for every } x \in M_0 \cap P\}.$$

Notice that $\mathcal{A} \subset \mathcal{C}$, by definition of $\mathcal{P}$. Moreover, if $E_2 \subset E_1$ then

$$\tau(E_1 \setminus E_2, x) = \tau(E_1, x) - \tau(E_2, x)$$

exists and is constant over $M_0 \cap P$. If $\{E_i\}$ are two by two disjoint then

$$\tau(\cup_{i=1}^{\infty} E_i, x) = \sum_{i=1}^{\infty} \tau(E_i, x)$$

exists and is constant over $M_0 \cap P$. We conclude that $\mathcal{C}$ is a monotone class (it is stable under increasing unions and decreasing intersections). By the monotone class theorem we conclude that $\mathcal{C} = \mathcal{B}$. By Corollary 4.3 we deduce that μ_P is ergodic. □

Proof of Lemma 4.4. Let $\mathcal{A} = \{A_k\}$ be an enumeration of $\mathcal{A}$ and $\{q_k\} = \mathbb{Q}$ be an enumeration of the rational numbers. Fix $n \in \mathbb{N}$. We define a partition $\mathcal{P}_n$ in the following way: we mark in the line the points $q_1, ..., q_n$ and consider the partition of the line induced by these points.

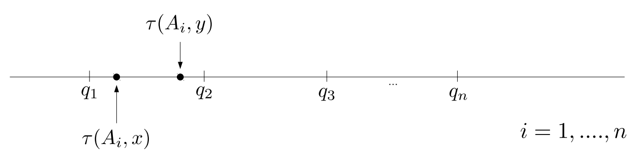

FIGURE 4. The partition $\mathcal{P}_n$.

We declare $x \sim_n y$ if and only if $\tau(A_i, x)$ and $\tau(A_i, y)$ belong to the same interval of this partition for every $i = 1, ..., n$ (see figure 4). Clearly, $\tau(A_i, x) = \tau(A_i, y)$ for every i if and only if $x, y \in \cap_{n=1}^{\infty} P_n$, with $P_n \in \mathcal{P}_n$, and thus $\mathcal{P} = \vee_{n=1}^{\infty} \mathcal{P}_n$. □

5. Measurable unstable partitions

We close this note by stating a deep generalization of Example 3.4, which is due to Leddrapier-Young.

Theorem 5.1. *Let $f : M \to M$ be a C^2 diffeomorphism, $\Lambda \subset M$ a hyperbolic set. Notice that the unstable W^u and the stable manifolds W^s form a partition of some invariant set which contains Λ. Then for every invariant and ergodic measure μ the following are equivalent:*

(1) $h_\mu(f) = 0$
(2) W^u *is measurable;*
(3) W^s *is measurable.*

Let us give a very rough idea of $2 \Rightarrow 1$. One first proves that if W^u comprises a measurable partition then the disintegration are Dirac. Indeed, take a compact set $K \subset W^u(f^n(x))$, with almost full measure (say $1-\varepsilon$). As f is C^2, using distortion arguments we show that almost all the mass of $\mu_{W(x)}$, say $1-\varepsilon$, is concentrated in a tiny neighbourhood of x inside $W^u(x)$. As ε is arbitrary, this shows that $\mu_{W^u(x)} = \delta_x$. From this, one deduces that the entropy vanishes.

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