RIGIDITY OF *U*-GIBBS MEASURES NEAR CONSERVATIVE ANOSOV DIFFEOMORPHISMS ON $\mathbb{T}^3$

SÉBASTIEN ALVAREZ, MARTIN LEGUIL, DAVI OBATA, BRUNO SANTIAGO

ABSTRACT. We show that within a C^1-neighbourhood $\mathcal{U}$ of the set of volume preserving Anosov diffeomorphisms on the three-torus $\mathbb{T}^3$ which are strongly partially hyperbolic with expanding center, any $f \in \mathcal{U} \cap \mathrm{Diff}^2(\mathbb{T}^3)$ satisfies the dichotomy: either the strong stable and unstable bundles E^s, E^u of f are jointly integrable, or any fully supported u-Gibbs measure of f is SRB.

CONTENTS

1. INTRODUCTION

1.1. **Context.** Invariant foliations play a key role in partially and uniformly hyperbolic dynamics. For example, they can be used to obtain ergodicity, topological transitivity and mixing for certain systems. In the path of trying to understand these foliations, one can investigate their topological and ergodic properties. For topological properties, one may ask about minimality, or robust minimality, of the invariant foliations. In this paper we are going to focus on understanding ergodic properties of the invariant foliations for a certain type of dynamical system.

We refer the reader to Section 2 for the definition of the dynamical objects that appear in this introduction. Let us denote by $\mathbb{T}^3 \stackrel{\text{def.}}{=} \mathbb{R}^3/\mathbb{Z}^3$ the three-dimensional torus. We let $\mathcal{A}^2(\mathbb{T}^3) \subset \mathrm{Diff}^2(\mathbb{T}^3)$ be the set of Anosov diffeomorphisms which are strongly partially hyperbolic with uniformly expanding center, that is, a diffeomorphism f belongs to $\mathcal{A}^2(\mathbb{T}^3)$ if f is Anosov and admits a splitting

$$T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u,$$

where E^c expands uniformly under the action of Df. A diffeomorphism in $\mathcal{A}^2(\mathbb{T}^3)$ can be seen as an Anosov and as a partially hyperbolic diffeomorphism. In this context the bundles $E^s, E^u, E^u \oplus E^c, E^s \oplus E^c$ and E^c integrate to invariant foliations respectively denoted by $\mathcal{W}^s, \mathcal{W}^u, \mathcal{W}^{cu}, \mathcal{W}^{cs}$ and $\mathcal{W}^c$, and called the stable, unstable, center-unstable, center-stable and center foliations (see [4, 38]). If we see f as an Anosov diffeomorphism, $\mathcal{W}^{cu}$ is a (two dimensional) unstable foliation. If we see it as a partially hyperbolic diffeomorphism, $\mathcal{W}^u$ is the (one dimensional) strong-unstable foliation.

There are two types of elements of $\mathcal{A}^2(\mathbb{T}^3)$: the *conservative elements*, which preserve some volume (that must be ergodic by Hopf's argument) and form a set denoted by $\mathcal{A}^2_m(\mathbb{T}^3)$, and the *dissipative ones*, which don't. In both cases there exists a unique invariant measure which is the "most compatible" with the volume and that is called the *SRB measure* (for Sinai-Ruelle-Bowen): *measures that are absolutely continuous with respect to the Lebesgue measure along center unstable leaves (see §2.5). In particular, they capture the "statistical" behavior of Lebesgue-almost every point (see* [49]*).* SRB measures are very important in the theory of smooth dynamics. Palis conjectured that for a typical dynamical system there are finitely many attractors, each attractor supporting a unique SRB measure and these measures capture the behavior of Lebesgue almost every point [36]. This conjecture remains open.

1.2. **Dynamics of (center)-unstable foliation.** Let $f \in \mathcal{A}^2(\mathbb{T}^3)$ with a splitting $T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u$. The dynamics of the center-unstable foliation $\mathcal{W}^{cu}$ is very well understood. It is minimal (i.e., every leaf is dense in $\mathbb{T}^3$) and there is a unique SRB measure.

On the other hand, recall that properties of the strong-unstable foliation $\mathcal{W}^u$ are especially interesting for dissipative dynamics: the study of topological and ergodic properties of attractors or quasi-attractors (which are $\mathcal{W}^u$-saturated, i.e., they contain its $\mathcal{W}^u$-leaves) is closely related to the problem of understanding properties of these foliations (or laminations). Apart from some finiteness results (see [10, 21, 23]), the dynamical properties of strong-unstable foliation are not well understood even in the uniformly hyperbolic setting.

For instance, it was only recently announced by Avila-Crovisier-Eskin-Potrie-Wilkinson-Zhang that $\mathcal{W}^u$ is minimal for any $C^{1+\alpha}$ Anosov diffeomorphism of $\mathbb{T}^3$. In higher dimensions, Avila-Crovisier-Wilkinson recently announced that C^1-openly and C^r-densely among the transitive Anosov diffeomorphisms admitting a decomposition $E^s \oplus E^c \oplus E^u$, where E^c is one dimensional and uniformly expanding, the strong unstable manifold is minimal.

A type of invariant measures that is associated with $\mathcal{W}^u$ are the so-called u-Gibbs measures. A measure is u-Gibbs if it admits conditional measures along $\mathcal{W}^u$ leaves that are absolutely continuous with respect to the Lebesgue measure of these leaves. In particular the support of such a measure is $\mathcal{W}^u$-saturated. Let us make a few remarks about u-Gibbs and SRB measures in our setting:

(1) SRB measures are absolutely continuous along two dimensional objects ($\mathcal{W}^{cu}$ leaves), while u-Gibbs are absolutely continuous along one dimensional objects ($\mathcal{W}^u$ leaves).
(2) SRB measures are also u-Gibbs measures.
(3) In general, we don't know when a u-Gibbs measure is an SRB measure.

1.3. **Main result.** The goal of this paper is related to item (3) above. We are interested in knowing when the u-Gibbs property implies SRB. In other words,

given a measure that is absolutely continuous along $\mathcal{W}^u$, when can we show that this measure is absolutely continuous along $\mathcal{W}^{cu}$?

For $\mathcal{A}^2(\mathbb{T}^3)$, we say that E^s and E^u are *jointly integrable* if there exists a two dimensional foliation $\mathcal{W}^{su}$ tangent to $E^s \oplus E^u$. It is known that C^1-openly and C^2-densely in $\mathcal{A}^2(\mathbb{T}^3)$ the directions E^s and E^u are not jointly integrable (see [45]).

Our main result is the following:

Theorem A. *There exists an open neighbourhood $\mathcal{U}$ of $\mathcal{A}_m^2(\mathbb{T}^3)$ within $\mathrm{Diff}^2(\mathbb{T}^3)$ so that for every $f \in \mathcal{U}$, either*

(1) E^s and E^u are jointly integrable, or
(2) any fully supported ergodic u-Gibbs measure μ is SRB.

Remark 1.1. *The hypothesis that the system is near a volume preserving one is used to have C^1-stable holonomies. The conclusion of Theorem A also holds for $f \in \mathcal{A}^2(\mathbb{T}^3)$ with C^1-stable holonomies, see Theorem 5.10.*

Remark 1.2. *As we mentioned before, Avila et al. announced that for any $C^{1+\alpha}$ Anosov diffeomorphism in $\mathbb{T}^3$ the strong unstable foliation is minimal. Since the support of any u-Gibbs measure is saturated by $\mathcal{W}^u$-leaves, their result would imply that the case (2) of Theorem A could be improved to every u-Gibbs measure is SRB. We could also apply the result announced by Avila-Crovisier-Wilkinson, that we mentioned, to obtain that open and densely in $\mathcal{U}$ every u-Gibbs is SRB, where $\mathcal{U}$ is the open set from Theorem A.*

Let us mention one application of our result. In [19], Gogolev, Kolmogorov and Maimon consider the linear Anosov diffeomorphism on $\mathbb{T}^3$ induced by the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The eigenvalues of A are real and approximately 0.2, 1.55 and 3.25. In particular, the diffeomorphism induced by this matrix belong to $\mathcal{A}_m^2(\mathbb{T}^3)$. In [19], the authors did a numerical study for two explicit families of perturbations of A, one conservative and one dissipative. Their numerical study indicates that for these families of perturbations of A, there is a unique u-Gibbs measure and this measure coincides with the SRB measure. They make the following conjecture.

Conjecture ([19], Conjecture 1.3). *For all analytic diffeomorphisms f in a sufficiently small neighbourhood of A there exists a unique u-Gibbs measure.*

Our result gives that for any C^2-diffeomorphism in a neighbourhood of A, either E^s and E^u are jointly integrable or any u-Gibbs fully supported is SRB. If we assume Avila et al.'s result, then we would obtain that either we have joint integrability or there is only one u-Gibbs measure. In Section 12 we introduce Gogolev-Kolmogorov-Maimon's conservative and dissipative families and prove that for both examples, E^s and E^u are not jointly integrable, applying a very convenient criterion of Gan-Shi [17]. That gives a theoretical explanation for their numerical study. Let us remark that in [19] the authors also make conjectures about transitivity and minimality of $\mathcal{W}^u$. In [46], Rodriguez Hertz-Ures gave a positive answer to their transitivity conjecture.

1.4. **Related works and further results.** One can think of a u-Gibbs measure as a measure that is "homogeneous" along strong unstable manifolds and an SRB measure as being "homogeneous" along entire unstable manifolds.

In a series of celebrated works, in the homogeneous setting, Ratner classified measures that are invariant by the action of a unipotent group [41, 40, 42, 43]. She proved that such measures are homogeneous, i.e., they are the Haar measure of some subgroup. Observe that unipotent flows parameterize unstable manifolds of the geodesic flow on surfaces with constant negative curvature (the horocycle flow). Hence, a consequence of Ratner's measure rigidity result is measure rigidity of the u-Gibbs measures of the geodesic flow on surfaces with constant negative curvature. A key idea in Ratner's approach is the so-called polynomial drift, which allowed her to obtain extra invariance of the measure from invariance along orbits of the unipotent flow.

In [2], Benoist-Quint introduced the idea of exponential drift to prove a measure rigidity result for stationary measures of a Zariski dense random walk on homogeneous spaces.

Outside the homogeneous setting, Eskin-Mirzakhani gave a non trivial modification of the exponential drift strategy, which is called the factorization method, to prove measure rigidity results for the action of $\mathrm{SL}(2, \mathbb{R})$ on moduli spaces [15]. Since then, these ideas were pushed to some different settings. In [7], Brown-Rodriguez Hertz classified the hyperbolic stationary measures of random products of surface diffeomorphisms. Cantat-Dujardin applied Brown-Rodriguez Hertz's result to classify random products of automorphisms of real and complex projective spaces [8].

In the partially hyperbolic setting, the third author adapted Brown-Rodriguez Hertz's result to obtain a rigidity result for u-Gibbs measures for partially hyperbolic skew products with two dimensional center [35].

In [30], Katz adapts the Eskin-Mirzakhani strategy for the smooth setting. He proved that for any C^∞ Anosov diffeomorphism f having a splitting $TM = E^s \oplus E^c \oplus E^u$, where E^c is one dimensional and expanding, any u-Gibbs measure that verifies a technical condition called *QNI* (quantified non-integrability) is SRB. After our work, Eskin-Potrie-Zhang [16] obtained equivalent notions to QNI that are easier to work with. In our setting, their result implies that, assuming C^∞ regularity, either any fully supported u-Gibbs measure is SRB, or E^s and E^u verify a condition that they called joint integrability up to order l, for every $l > 0$, which is slightly weaker than the usual joint integrability. Their result will be used by Avila et al., in another ongoing project, to prove that for any Anosov diffeomorphism in $\mathcal{A}^\infty(\mathbb{T}^3)$ either E^s and E^u are jointly integrable, or every u-Gibbs measure is SRB. All of these works are stated for C^∞ regularity, but they can be obtained for C^r regularity for $r \ggg 1$. Part of the goal of this paper is to obtain this type of measure rigidity result for u-Gibbs measures, but in lower regularity (in our case C^2).

1.5. **Ingredients of the proof.** The first ingredient concerns the transversality condition. We replace Katz's QNI condition by a zero-one law for angles inspired by Brown-Rodriguez Hertz [6, Lemma 7.1] (see also [2]). We use the fact that stable holonomies $(H^s_{x,y})_{x,y}$ are C^1 for any diffeomorphism $f \in \mathcal{A}^2(\mathbb{T}^3)$ close to a conservative one (see Lemma 2.6); thus, for x, y in the same stable manifold, we can define an (unoriented) angle (see Figure 1)

$$\alpha^s(x, y) \overset{\text{def.}}{=} \angle(DH^s_{x,y}E^u(x), E^u(y)).$$

Note that conditional measures μ^s_x on stable manifolds are not well defined. But full- and zero-measure sets for μ^s_x are well defined, see §2.4. We can now state our zero-one law (see Theorem 4.2 for a slightly more general statement).
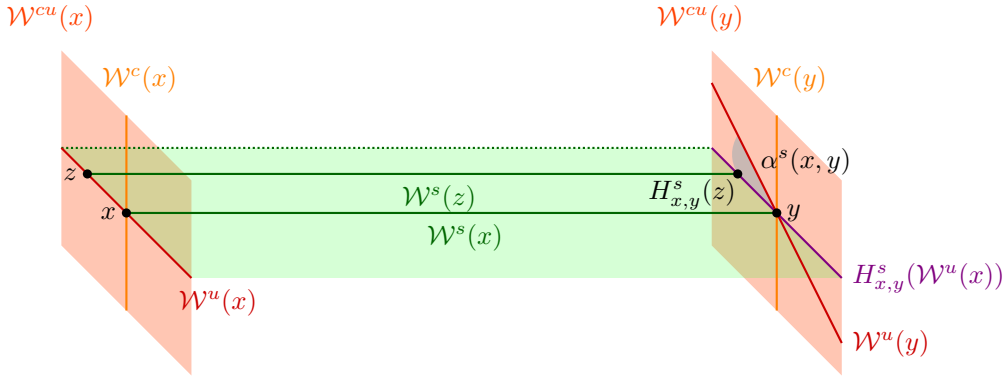
FIGURE 1. Stable holonomies and the angle function.

Theorem B (A zero-one law for angles)**.** *There exists an open neighbourhood $\mathcal{U}$ of $\mathcal{A}_m^2(\mathbb{T}^3)$ within $\mathrm{Diff}^2(\mathbb{T}^3)$ such that for any $f \in \mathcal{U}$, and for any ergodic f-invariant measure μ, the following dichotomy holds:*

(1) for μ-a.e. $x \in \mathbb{T}^3$ and μ_x^s-a.e. $y \in \mathcal{W}^s(x)$, $\alpha^s(x,y) = 0$;
(2) for μ-a.e. $x \in \mathbb{T}^3$ and μ_x^s-a.e. $y \in \mathcal{W}^s(x)$, $\alpha^s(x,y) > 0$.

This theorem is stated here in terms of the angle function α^s; yet, it seems possible to generalize it to a broader context (see Remark 4.4).

Another ingredient which plays an instrumental role in our proof is the existence of *normal forms* for the dynamics along two-dimensional unstable foliations. The dynamics along unstable manifolds is simplified when it is looked at in normal forms. The theory of non-stationary normal forms has been studied quite extensively since the pioneering work of Guysinsky-Katok [22], see for instance [26, 27, 18]. Katz uses the result from [26] where higher regularity is needed depending on the Lyapunov spectrum.

Yet, these results do not apply directly here, due namely to the fact that f is merely assumed to be C^2. We use instead an *ad hoc* construction, of different nature, based on one-dimensional normal forms along the center/unstable directions. Although the outcome is somewhat similar to the result in [26], we list a few differences with their work in Remark 1.3 below. We show:

Theorem C (Normal forms)**.** *Let $f \in \mathcal{A}^2(\mathbb{T}^3)$. Then, there exists a family $\{\Phi_x\}_{x \in \mathbb{T}^3}$ of C^1 diffeomorphisms $\Phi_x \colon \mathbb{R}^2 \to \mathcal{W}^{cu}(x)$ such that*

(1) $f \circ \Phi_x = \Phi_{f(x)} \circ N_x$, with $N_x \stackrel{\mathrm{def.}}{=} \begin{bmatrix} \lambda_x^u & 0 \\ 0 & \lambda_x^c \end{bmatrix}$, letting $\lambda_x^* \stackrel{\mathrm{def.}}{=} \|Df(x)|_{E^*}\|$;
(2) $\Phi_x(0) = x$ and $D\Phi_x(0)(1,0) = v^u(x)$, $D\Phi_x(0)(0,1) = v^c(x)$, $v^u(x)$, resp. $v^c(x)$ being a unit vector in $E^u(x)$, resp. $E^c(x)$;
(3) $\Phi_x(\cdot)$ depends continuously with the choice of x in the local C^1-topology[1];
(4) Φ_x is a foliated chart for $\mathcal{W}^u$, i.e., for all $s \in \mathbb{R}$, $\Phi_x(\mathbb{R} \times \{s\}) = \mathcal{W}^u(\Phi_x(0,s))$, and $\Phi_x(\{0\} \times \mathbb{R}) = \mathcal{W}^c(x)$.

Remark 1.3. *Compared with [26], our construction only requires f to be C^2, and does not need (pointwise) $1/2$ pinching. The price to pay is that the resulting change of coordinates is merely C^1, while it is C^∞ in [26]. Yet, this suffices for our purpose; indeed, u-Gibbs measures, when they are looked at in normal coordinates, still have the property of being absolutely continuous along the image of $\mathcal{W}^u$, namely, the horizontal foliation. Contrary to [26], the normal forms*

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[1]Uniform convergence of the function and its first derivative on compact sets.

$\{\Phi_x\}_{x \in \mathbb{T}^3}$ *we obtain do not in general define an affine structure on the foliation* $\mathcal{W}^{cu}$ *(see Remark 6.4); yet, we will investigate in Section 6 certain invariance properties of these normal forms under changes of charts.*
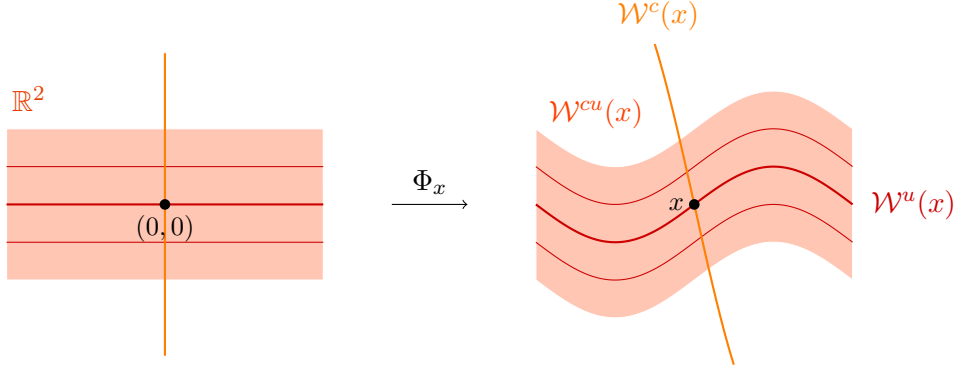

FIGURE 2.   Theorem C provides non-stationary C^1 linearisations of the dynamics along center unstable leaves. They send horizontal lines onto unstable manifolds.

Note that Katz also uses the C^∞ regularity many other times in his adaptation of Eskin-Mirzakhani's factorization method for Anosov systems. For example, at some moments, he has to approximate stable/unstable manifolds by Taylor polynomials with very high degree. In our setting, Theorems B and C are the two main reasons why we are able to adapt the Eskin-Mirzakhani's strategy in lower regularity.

We stress that, differently from previous works, we implement Y-configurations and a version of the factorization technique of [15] without using suspensions nor any reparametrization. We make all estimations directly with the diffeomorphism. This is possible because we can obtain uniform estimates for the drift of leaf-wise (quotient) measures along the center as well as synchronization estimates for stopping times (see §9.3), using only basic distortion estimates (see §2.3).

1.6. **Organization of the paper.** This paper is organized as follows: in Section 2 we introduce the basic definitions and results we need. In Section 3 we give an outline of the proof of Theorem A. In Section 4 we establish a zero-one law for transversality between the bundles E^s and E^u. In Section 5 we reduce the proof of Theorem A to a more technical result, see Theorem 5.10. In Section 6 we construct a non-stationary family of C^1 linearisations of the dynamics restricted to center unstable manifolds and use them to construct a family of measures $\{\hat{\nu}_x^c\}_{x \in \mathbb{T}^3}$ on the real line. We reduce the proof of our main technical result to proving that these measures are Lebesgue almost surely. In Section 8, we explain an argument from [7, 25] that shows that if the measures $\hat{\nu}_x^c$ are "invariant" by certain affine maps for many point x, then the measures $\hat{\nu}_x^c$ are actually the Lebesgue measure. We then explain how this is achieved by reducing the proof in proving Proposition 8.2. In Section 9 we introduce Y-configurations and other objects crucial for our argument. In Section 10 we introduce matched Y-configurations and in Section 11 we complete the proof.

2. Partially hyperbolic diffeomorphisms with expanding center

In this section we introduce the class of dynamical systems we work with as well as the main objects, taking the opportunity to fix notations and recall important basic facts.

2.1. **Definitions.** For any integer $r \geq 1$, we let $\mathcal{PH}^r(\mathbb{T}^3)$ be the set of all C^r *(strongly) partially hyperbolic diffeomorphisms* of $\mathbb{T}^3$ with one-dimensional stable/center/unstable bundles, i.e., the diffeomorphisms $f \colon \mathbb{T}^3 \to \mathbb{T}^3$ such that there exist a continuous splitting of $T\mathbb{T}^3$ into Df-invariant line bundles,

$$T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u,$$

as well as a Riemannian metric $\|.\|$ *adapted to this splitting* such that the functions

$$x \mapsto \lambda_x^* \stackrel{\text{def.}}{=} \|Df(x)|_{E^*}\|, \quad * \in \{s, c, u\},$$

are continuous and satisfy $\lambda_x^s < 1 < \lambda_x^u$ and $\lambda_x^s < \lambda_x^c < \lambda_x^u$, for all $x \in \mathbb{T}^3$. We will sometimes make the dependence on f explicit and write $\lambda_{f,x}^*$ instead of λ_x^*. We also let $E^{cs} \stackrel{\text{def.}}{=} E^c \oplus E^s$, resp. $E^{cu} \stackrel{\text{def.}}{=} E^c \oplus E^u$ be the center-stable, resp. center-unstable subbundle, and set $E^{su} \stackrel{\text{def.}}{=} E^s \oplus E^u$. We refer to Katok-Hasselblatt's book [28] for more details.

2.1.1. *Anosov diffeomorphisms with uniformly expanding center.* We denote by $\mathcal{A}^r(\mathbb{T}^3) \subset \mathcal{PH}^r(\mathbb{T}^3)$ the subset consisting of partially hyperbolic diffeomorphisms $f \in \mathcal{PH}^r(\mathbb{T}^3)$ with uniformly expanding center, i.e., such that $\lambda_x^c > 1$, for all $x \in \mathbb{T}^3$; in particular, any such diffeomorphism f is *Anosov*, for the hyperbolic splitting $E^s \oplus E^{cu}$. We also denote by $\mathcal{A}_m^r(\mathbb{T}^3) \subset \mathcal{A}^r(\mathbb{T}^3)$ the subset made of *conservative* Anosov diffeomorphisms (i.e., that preserve some volume).

To simplify the exposition, we assume that the bundles E^* are orientable and that f preserves their orientation (this can always be achieved by taking an orientable cover and considering powers of f). In particular, there are unitary vector fields $x \in \mathbb{T}^3 \mapsto v^*(x) \in E^*(x)$ such that

$$Df(x)v^*(x) \stackrel{\text{def.}}{=} \lambda_x^* v^*(f(x)).$$

2.1.2. *Notations for orbits and derivatives.* To give a more friendly aspect of some long calculations we shall make, we introduce the following notation. For a point $x \in \mathbb{T}^3$ we denote

$$x_n = f^n(x) \quad \text{for} \quad n \in \mathbb{Z}. \tag{1}$$

Also, for $n \in \mathbb{Z}$ and for $* = s, c, u$ we denote the derivative of f in restriction to the bundle E^* by

$$\lambda_x^*(n) \stackrel{\text{def.}}{=} \|Df^n(x)|_{E^*}\|. \tag{2}$$

The following *cocycle property* follows from the chain rule and the fact that the bundles are one-dimensional:

$$\lambda_x^*(n+m) = \lambda_{f^n(x)}^*(m)\lambda_x^*(n), \quad * \in \{s, c, u\}. \tag{3}$$

The following quantities associated to f will be useful for crude estimations

$$\|Df\| \stackrel{\text{def.}}{=} \max\{\|Df(x)v\| : x \in \mathbb{T}^3, v \in T_x\mathbb{T}^3, \|v\| = 1\}$$

and

$$m(Df) \stackrel{\text{def.}}{=} \min\{\|Df(x)v\| : x \in \mathbb{T}^3, v \in T_x\mathbb{T}^3, \|v\| = 1\}.$$

2.1.3. *Adapted metric and hyperbolic estimates.* The following quantity will play a key role later when we introduce stopping times and Y-configurations:

$$d_x^\ell \stackrel{\text{def.}}{=} \frac{\lambda_{x-\ell}^c(\ell)}{\lambda_{x-\ell}^u(\ell)}. \tag{4}$$

Notice that d_x^ℓ measures the amount of projective hyperbolicity we have for the dominated splitting $E^c \oplus E^u$.

We fix a Riemannian metric on $\mathbb{T}^3$ and constants $\chi_j^* \in \mathbb{R}$, for $* \in \{c, s, u, d\}$ and $j = 1, 2$ such that the following holds:

(1) $\chi_1^d < \chi_2^d < 0$ and $e^{\chi_1^d\ell} < d_x^\ell < e^{\chi_2^d\ell}$, for every $x \in \mathbb{T}^3$ and every $\ell \in \mathbb{Z}$.
(2) $\chi_1^s < \chi_2^s < 0$ and $e^{\chi_1^s\ell} < \lambda_x^s(\ell) < e^{\chi_2^s\ell}$, for every $x \in \mathbb{T}^3$ and every $\ell \in \mathbb{Z}$.
(3) $\chi_1^c > \chi_2^c > 0$ and $e^{\chi_1^c\ell} > \lambda_x^c(\ell) > e^{\chi_2^c\ell}$, for every $x \in \mathbb{T}^3$ and every $\ell \in \mathbb{Z}$.
(4) $\chi_1^u > \chi_2^u > 0$ and $e^{\chi_1^u\ell} > \lambda_x^u(\ell) > e^{\chi_2^u\ell}$, for every $x \in \mathbb{T}^3$ and every $\ell \in \mathbb{Z}$.

2.1.4. *Invariant manifolds.* Let $r \geq 1$, and let $f \in \mathcal{A}^r(\mathbb{T}^3)$. It is well-known (see [24]) that the strong bundles E^s and E^u are uniquely integrable to f-invariant continuous foliations with C^r-leaves $\mathcal{W}_f^s = \mathcal{W}^s$ and $\mathcal{W}_f^u = \mathcal{W}^u$ respectively, called the *strong stable* and *strong unstable* foliations. Since the splitting $E^s \oplus E^{cu}$ is Anosov, the center-unstable bundle E^{cu} also integrates uniquely to an f-invariant continuous foliation $\mathcal{W}_f^{cu} = \mathcal{W}^{cu}$, called the *center-unstable* foliation. For any $x \in \mathbb{T}^3$ and $* = s, u, cu$, we denote by $\mathcal{W}^*(x)$ the leaf of $\mathcal{W}^*$ through x; it is an immersed C^r manifold.

We now define the concept of *joint integrability* which appear in the statement of Theorem A

Definition 2.1 (Joint integrability)**.** *We say that $f \in \mathcal{A}^r(\mathbb{T}^3)$ is (or that the bundles E^s and E^u are) jointly integrable if the bundle $E^s \oplus E^u$ integrates to a continuous foliation with C^1 leaves.*

Let $r \geq 1$, and fix a C^r Anosov diffeomorphism $f \in \mathcal{A}^r(\mathbb{T}^3)$. By Corollary 1.3 in [34] the non-wandering set $\Omega(f)$ of f is equal to $\mathbb{T}^3$, and f is topologically conjugated to a *hyperbolic toral automorphism.* As a consequence, one has the following lemma (see Theorem 1.3 in [37] for a proof).

Lemma 2.2. *The stable foliation of f is minimal, i.e., each leaf of $\mathcal{W}^s$ is dense in $\mathbb{T}^3$.*

2.1.5. *Dynamical coherence.* As remarked before, any diffeomorphism $f \in \mathcal{A}^r(\mathbb{T}^3)$ can be seen either as a strongly partially hyperbolic diffeomorphism with respect to the splitting $E^s \oplus E^c \oplus E^u$ and an Anosov diffeomorphism with respect to the splitting $E^s \oplus E^{cu}$. By Lemma 2.2 we know that $\Omega(f) = \mathbb{T}^3$.

By the results of Potrie [38] and Brin-Burago-Ivanov [4] on partially hyperbolic diffeomorphisms of $\mathbb{T}^3$, we have that f is *dynamically coherent*. In particular, E^{cs} is also integrable to an f-invariant continuous foliation $\mathcal{W}_f^{cs} = \mathcal{W}^{cs}$, called the *center-stable foliation*. Moreover, $\mathcal{W}^s$ subfoliates $\mathcal{W}^{cs}$, while $\mathcal{W}^u$ subfoliates $\mathcal{W}^{cu}$, and the collection of all leaves $\mathcal{W}^c(x) \overset{\text{def.}}{=} \mathcal{W}^{cs}(x) \cap \mathcal{W}^{cu}(x)$, $x \in \mathbb{T}^3$, forms a foliation $\mathcal{W}_f^c = \mathcal{W}^c$, called the *center foliation*, which integrates E^c, and subfoliates both $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$. For $* \in \{u, c, s, cu, cs\}$, let d_* be the leaf-wise distance, and for $x \in \mathbb{T}^3$, $\sigma > 0$, set $\mathcal{W}_\sigma^*(x) \overset{\text{def.}}{=} \{y \in \mathcal{W}^*(x) \mid d_*(x, y) < \sigma\}$.

In our Anosov case we can show rather easily that the foliations $\mathcal{W}^c$ and $\mathcal{W}^u$ are globally transverse inside each $\mathcal{W}^{cu}$ leaf, as the two lemmas below demonstrate.

Lemma 2.3. *For any $x \in \mathbb{T}^3$, it holds $\mathcal{W}^{cu}(x) = \cup_{y \in \mathcal{W}^c(x)} \mathcal{W}^u(y)$.*

Proof. Let $z \in \mathcal{W}^{cu}(x)$ be arbitrary. We need to show that $\mathcal{W}^c(x) \cap \mathcal{W}^u(z) \neq \emptyset$. Since the bundles E^c and E^u are integrable and the local leaves have uniform size due to hyperbolicity, since the splitting $E^c \oplus E^u$ is dominated and backwards iteration under f contracts distances uniformly along $\mathcal{W}^{cu}$ we must have some $n \in \mathbb{N}$ such that

$$\mathcal{W}^c(f^{-n}(x)) \cap \mathcal{W}^u(f^{-n}(z)) \neq \emptyset,$$

with transverse intersection. By forward iteration and using that integral manifolds are invariant by the dynamics we obtain the conclusion of the lemma. $\square$

Lemma 2.4. *For every $y \in \mathcal{W}^c(x)$, it holds $\mathcal{W}^c(x) \cap \mathcal{W}^u(y) = \{y\}$.*
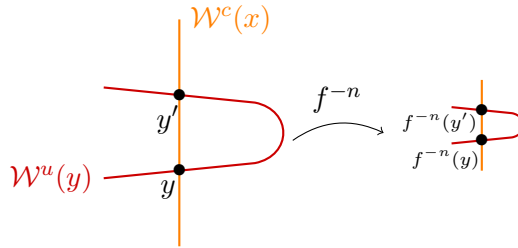
FIGURE 3. Proof of Lemma 2.4: iterating the picture of the left we arrive at a small scale where the picture violates the uniformly positive angle between E^c and E^u.

Proof. Assume by contradiction the existence of a point $y' \neq y$ in $\mathcal{W}^c(x) \cap \mathcal{W}^u(y)$. Consider the piece γ^u of unstable manifold joining y to y'. Since the length of $f^{-n}(\gamma^u)$ decreases exponentially there exists some n such that the curve $f^{-n}(\gamma^u)$ is entirely contained in a coordinate chart for which the line field E^c is almost vertical. Since this curve joins the points $f^{-n}(y)$ and $f^{-n}(y')$ which belong to the same local integral curve of E^c, this proves that the tangent space of $f^{-n}(\gamma^u)$ is almost vertical somewhere. This contradicts the dominated splitting $E^c \oplus E^u$ and completes the proof. $\square$

2.1.6. *Holonomies.* Let $x_1, x_2 \in \mathbb{T}^3$ be two *stably connected* points, i.e., such that $x_2 \in \mathcal{W}^s(x_1)$. Set $r \overset{\text{def.}}{=} d_s(x_1, x_2)$. By transversality, for any sufficiently small $\varepsilon > 0$, there is $\sigma_1 > 0$ such that for any point $y_1 \in \mathcal{W}^{cu}_{\sigma_1}(x_1)$, there exists a unique point $y_2 \in \mathcal{W}^s(y_1) \cap \mathcal{W}^{cu}_1(x_2)$ with $d_s(y_1, y_2) \in (r - \varepsilon, r + \varepsilon)$. We denote $H^s_{x_1, x_2}(y_1) \overset{\text{def.}}{=} y_2$. Thus, we get a well defined local homeomorphism

$$H^s_{x_1, x_2} : \mathcal{C}_1 \to \mathcal{C}_2,$$

from a neighbourhood $\mathcal{C}_1$ of x_1 within $\mathcal{W}^{cu}(x_1)$ to a neighbourhood $\mathcal{C}_2$ of x_2 within $\mathcal{W}^{cu}(x_2)$, called the (local) *stable holonomy map* between $\mathcal{C}_1$ and $\mathcal{C}_2$. Holonomies H^u, H^{cs}, H^{cu} along $\mathcal{W}^u, \mathcal{W}^{cs}, \mathcal{W}^{cu}$ are defined in a similar way.

2.2. **Regularity of extreme bundles and holonomy maps.** It is crucial to our proof that certain holonomy maps are of class C^1. This is the case when some *bunching* inequalities are satisfied between the rates of contraction/expansion of the system; they actually hold in a neighbourhood of volume preserving Anosov diffeomorphisms of $\mathbb{T}^3$, which is the main motivation behind this assumption in our result.

In our setting, we will say that f is *bunched* if there exists some $n \in \mathbb{N}$ such that for any $x \in \mathbb{T}^3$,

$$\lambda^s_{f,x}(n) < \frac{\lambda^c_{f,x}(n)}{\lambda^u_{f,x}(n)}, \tag{5}$$

where the notation $\lambda^*_{f,x}(n)$ was introduced in §2.1. In the equation above, we explicited the dependence of the contraction and expansion rates on f. We remark that this is clearly a C^1-open condition. This definition appears in [39] (see also [9, §4.7]).

2.2.1. *Regularity of the unstable bundle.* Let $f \in \mathcal{A}^2(\mathbb{T}^3)$. Then, f is a strongly partially hyperbolic diffeomorphism with one-dimensional center bundle, hence f is automatically center bunched. By [39, Theorem B], the unstable bundle E^u is C^1 when restricted to any center unstable leaf $\mathcal{W}^{cu}(x)$. In particular, the vector field $v^u|_{\mathcal{W}^{cu}}$ is a C^1 vector field over the immersed submanifold $\mathcal{W}^{cu}(x)$ and its C^1 norm depends continuously with respect to $x \in \mathbb{T}^3$.

As a corollary, we obtain that for any two small and nearby center curves $\gamma_1, \gamma_2 \subset \mathcal{W}^{cu}(x)$, in the same center unstable leaf, the unstable holonomy map $H^u \colon \gamma_1 \to \gamma_2$ is C^1. This regularity will play a role in our argument in Section 10.4. Moreover, for some estimations in our proof it is important to quantify the Lipschitz constant of these unstable holonomy maps, as in the following result, which is simply a more precise statement of [39, Theorem B] in our case.

Lemma 2.5. *Let $f \in \mathcal{A}^2(\mathbb{T}^3)$. There exists $\rho_0 > 0, C^u > 0$ such that for every $x, y \in \mathbb{T}^3$, if $x \in \mathcal{W}^u_2(y)$ then the unstable holonomy $H^u_{x,y}$ between local center manifolds is defined over $\mathcal{W}^c_{\rho_0}(x)$ and for every $z, z' \in \mathcal{W}^c_{\rho_0}(x)$ we have*

$$d(H^u_{x,y}(z), H^u_{x,y}(z')) \leq C^u d(z, z').$$

2.2.2. *Regularity of the stable bundle.* For the bundle E^s a stronger statement can be made when f is close to a volume preserving map. Indeed, fix an arbitrary $f_0 \in \mathcal{A}^1_m(\mathbb{T}^3)$.

Lemma 2.6. *There exists a neighbourhood $\mathcal{U}(f_0)$ of f_0 within $\mathrm{Diff}^2(\mathbb{T}^3)$ such that for any diffeomorphism $f \in \mathcal{U}(f_0)$, we have $f \in \mathcal{A}^2(\mathbb{T}^3)$ and the stable bundle E^s of f is of class C^1.*

Proof. Since f_0 is uniformly hyperbolic, there exists a continuous function $\mathbb{T}^3 \ni x \mapsto C(x) \in \mathrm{GL}(3, \mathbb{R})$ such that for every point $x \in \mathbb{T}^3$,

$$Df_0(x) = C(f_0(x))^{-1} \begin{bmatrix} \lambda^s_{f_0,x} & 0 & 0 \\ 0 & \lambda^c_{f_0,x} & 0 \\ 0 & 0 & \lambda^u_{f_0,x} \end{bmatrix} C(x).$$

The matrix $C(x)$ is a matrix with positive determinant that takes a basis formed by unit vectors in $E^*(x)$, and sends it to the orthogonal basis $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. In particular, there exists a uniform constant $C \geq 1$ such that

$$\max\{|\det C(x)|, |\det C(x)^{-1}|\} \leq C.$$

Consequently,

$$|\det Df_0(x)| = |\det C(f_0(x))^{-1}| \cdot |\det C(x)| \cdot \lambda^s_{f_0,x}\lambda^c_{f_0,x}\lambda^u_{f_0,x}.$$

Observing that for every $n \in \mathbb{N}$,

$$|\det Df_0^n(x)| = |\det C(f_0^n(x))^{-1}| \cdot |\det C(x)| \cdot \lambda^s_{f_0,x}(n)\lambda^c_{f_0,x}(n)\lambda^u_{f_0,x}(n).$$

The diffeomorphism f_0 is conservative so its Jacobian is a coboundary (by Livšic's Theorem [33] and [3, Theorem 4.14]). Thus, there exists a continuous function $\phi \colon \mathbb{T}^3 \to (0, \infty)$ bounded away from 0 and ∞ such that for any $x \in \mathbb{T}^3$, we have

$$|\det C(f_0(x))^{-1}| \cdot |\det C(x)| \cdot \lambda^s_{f_0,x}\lambda^c_{f_0,x}\lambda^u_{f_0,x} = \frac{\phi \circ f_0(x)}{\phi(x)}.$$

Hence,

$$
\begin{aligned}
\lambda^s_{f_0,x}(n) &= \frac{\lambda^s_{f_0,x}(n)\lambda^c_{f_0,x}(n)\lambda^u_{f_0,x}(n)}{\lambda^c_{f_0,x}(n)\lambda^u_{f_0,x}(n)} \\
&= |\det C(f_0^n(x))^{-1}|^{-1} \cdot |\det C(x)|^{-1} \cdot \frac{\phi \circ f^n(x)}{\phi(x)} \cdot \frac{1}{(\lambda^c_{f_0,x}(n))^2} \frac{\lambda^c_{f_0,x}(n)}{\lambda^u_{f_0,x}(n)} \\
&< \frac{\lambda^c_{f_0,x}(n)}{\lambda^u_{f_0,x}(n)},
\end{aligned}
$$

as long as n is large enough so that

$$|\det C(f_0(x))^{-1}| \cdot |\det C(x)| \frac{\phi \circ f^n(x)}{\phi(x)} < (\lambda^c_{f_0,x}(n))^2.$$

Such an integer n may be chosen independently of x because ϕ is uniformly bounded away from 0 and ∞, $\max\{|\det C(x)^{-1}|, |\det C(x)|\} \leq C$ and E^c is uniformly expanding. In particular, if $f \in \mathrm{Diff}^2(\mathbb{T}^3)$ is sufficiently C^1-close to f_0, then $f \in \mathcal{A}^2(\mathbb{T}^3)$, and for the same choice of $n \in \mathbb{N}$ and every $x \in \mathbb{T}^3$, it holds $\lambda^s_{f,x}(n) < \frac{\lambda^c_{f,x}(n)}{\lambda^u_{f,x}(n)}$ and f verifies (5). In other words, the hyperbolic splitting $E^s \oplus E^{cu}$ of f is bunched (see (5)). Then, according to the results of [39] (see also [9, Theorem 4.21]), the stable bundle E^s_f is C^1, as well as the stable holonomy maps. $\qquad\square$

Remark 2.7. *As noted in the above proof, the assumption that f is C^1-close to a conservative diffeomorphism ensures that the following* bunching condition *is automatically satisfied (hence that stable holonomies H^s are C^1, by [39]):*

$$\lambda^s_{f,x}(n) < \frac{\lambda^c_{f,x}(n)}{\lambda^u_{f,x}(n)}, \quad \forall\, x \in \mathbb{T}^3. \tag{6}$$

In other words, (6) means that the lack of conformality of Df along E^{cu} is dominated by the contraction along E^s. In particular, the conclusion of Theorem A holds true for any Anosov diffeomorphism $f \in \mathcal{A}^2(\mathbb{T}^3)$ satisfying (6).

2.2.3. *Hölder regularity of cs-holonomies.* Despite the above regularity results, a substantial source of technical difficulties for our strategy comes from the absence of smoothness for *center stable holonomies*. The quest for overcoming this is the main reason behind our *matching argument for Y-configurations* in Sections 10 and 11.3.2. In our setting, the best that can be said about center stable holonomies comes from Theorem A in [39], which we quote below in a convenient way for our purposes.

Lemma 2.8. *Let $f \in \mathcal{A}^2(\mathbb{T}^3)$. Then there exist $\rho_0 > 0$ (which we can assume is the same from Lemma 2.5), $C^{cs} > 0, \theta^{cs} > 0$ such that for every $x, y \in \mathbb{T}^3$ if $x \in \mathcal{W}^{cs}_{\rho_0}(y)$ then the center-stable holonomy $H^{cs}_{x,y}$ is defined on $\mathcal{W}^u_2(x)$ and for every $z, z' \in \mathcal{W}^u_2(x)$ we have*

$$d(H^{cs}_{x,y}(z), H^{cs}_{x,y}(z')) \leq C^{cs} d(z, z')^{\theta^{cs}}.$$

We remark that the roles of ρ_0 and 2 are "exchanged" when compared with the role of these constants in Lemma 2.5. Here, the transversal we are looking at for the center-stable holonomy has size 2, while in Lemma 2.5 the transversal has size ρ_0.

2.3. **Basic distortion estimates.** The goal of this subsection is to collect some classical distortion estimates and fix once and for all some constants which are going to be very important all along our constructions.

Lemma 2.9 (Basic distortion lemma). *Let $\varphi \colon \mathbb{T}^3 \to \mathbb{R}$ be a Hölder continuous function. Then, there exists a constant $C = C(\varphi) > 0$ such that*

    (a) If $y \in \mathcal{W}^s_1(x)$ and $n > 0$ or
    (b) if $f^n(y) \in \mathcal{W}^{cu}_1(f^n(x))$, for $n \in \mathbb{N}$, then

$$\left| \sum_{\ell=0}^{n-1} \varphi(f^\ell(x)) - \varphi(f^\ell(y)) \right| \leq C.$$

We omit the proof as it is quite classical. Applying the lemma to the functions $\varphi = \log \|Df(.)|_{E^*}\|$, $* = c, u$ we obtain a constant $C_0 = C_0(f) \geq 1$ such that if $y \in \mathcal{W}^s_1(x)$ and $n > 0$ or if $f^\ell(y) \in \mathcal{W}^{cu}_1(f^\ell(x))$ for every $\ell = 0, \dots, n$, then

$$C_0^{-1} \leq \frac{\|Df^n(x)|_{E^*}\|}{\|Df^n(y)|_{E^*}\|} \leq C_0. \tag{7}$$

Moreover, up to enlarging C_0, we also have that if $y \in \mathcal{W}^{cu}_1(x)$ then

$$C_0^{-1} \leq \frac{\|Df^{-n}(x)|_{E^*}\|}{\|Df^{-n}(y)|_{E^*}\|} \leq C_0. \tag{8}$$

for every $n \in \mathbb{N}$. Another important application is obtained by considering the function $\psi(x) = \log \frac{\|Df(x)|_{E^c}\|}{\|Df(x)|_{E^u}\|}$. We can assume that the constant C_0 also satisfies the following

Corollary 2.10. *Given $\ell \in \mathbb{N}$ recall from (4) that $d^\ell_x = \frac{\|Df^\ell(f^{-\ell}(x))|_{E^c}\|}{\|Df^\ell(f^{-\ell}(x))|_{E^u}\|}$. If $f^{-\ell}(y) \in \mathcal{W}^s_1(f^{-\ell}(x))$, then,*

$$C_0^{-1} \leq \frac{d^\ell_x}{d^\ell_y} \leq C_0.$$

2.3.1. *Distortion for quadrilaterals.* One of the key dynamical configuration for our strategy are the *quadrilaterals*. A *quadrilateral* is a quadruple $(x, x^u, y, y^u) \in (\mathbb{T}^3)^4$ such that $x^u \in \mathcal{W}_1^u(x)$, $y \in \mathcal{W}_1^s(x)$, and $y^u \in \mathcal{W}_1^u(y) \cap \mathcal{W}^{cs}(x^u)$. For such a quadrilateral, we define the point $z^u \stackrel{\text{def.}}{=} H_{x,y}^s(x^u)$, so that $z^u \in \mathcal{W}^s(x^u) \cap \mathcal{W}^c(y^u)$.
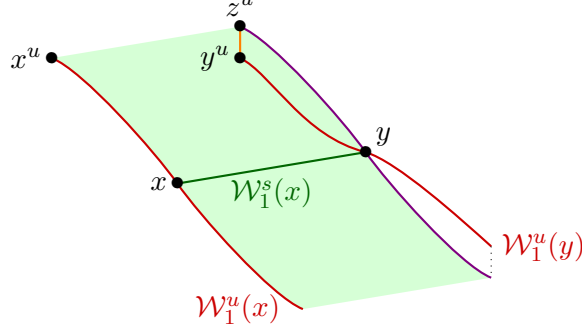

FIGURE 4. A quadrilateral.

From our previous distortion results, we can take a larger constant $C_0 = C_0(f)$ in order to have the following.

Corollary 2.11. *Now, assume that $y \in \mathcal{W}_1^s(x)$ and that $x^u \in \mathcal{W}_1^u(x)$ and $y^u \in \mathcal{W}_1^u(y)$ are such that $y^u = H_{x,y}^{cs}(x^u)$. Let $z^u = H_{x,y}^s(x^u)$. If $n \in \mathbb{N}$ satisfies $d(f^n(z^u), f^n(y^u)) \leq 1$ then*

$$C_0^{-1} \leq \frac{\|Df^\ell(x^u)|_{E^c}\|}{\|Df^\ell(y^u)|_{E^c}\|} \leq C_0,$$

for every $\ell = 0, \ldots, n$.

Proof. The result follows from (7) since

$$\frac{\|Df^\ell(x^u)|_{E^c}\|}{\|Df^\ell(y^u)|_{E^c}\|} = \frac{\|Df^\ell(x^u)|_{E^c}\|}{\|Df^\ell(z^u)|_{E^c}\|} \times \frac{\|Df^\ell(z^u)|_{E^c}\|}{\|Df^\ell(y^u)|_{E^c}\|}. \qquad \square$$

2.4. **Subordinate partitions and disintegrations.** We move now to the *ergodic theory* of Anosov diffeomorphisms of $\mathbb{T}^3$ with expanding center. Before giving the main definitions, we give some preliminaries on measurable partitions, disintegration and invariant measures.

2.4.1. *Measurable partitions.* Let μ be a probability measure of some standard Borel set X. Let ξ_1, ξ_2 be two partitions (mod 0) of X into measurable subsets. Say that ξ_1 *is finer than* (or *refines*) ξ_2 if for μ-a.e. $x \in X$ we have $\xi_1(x) \subset \xi_2(x)$ mod 0. This will be denoted by $\xi_2 \prec \xi_1$.

The *join* of ξ_1 and ξ_2 is the partition defined as $\xi_1 \vee \xi_2 = \{\xi_1(x) \cap \xi_2(x) : x \in X\}$.

A partition ξ of X is *measurable* whenever there exists a sequence $(\xi_n)_{n \in \mathbb{N}}$ of finite partitions of X by Borel subsets such that

$$\xi = \bigvee_{n=0}^{\infty} \xi_n.$$

Rokhlin proved the following fundamental result in [47]. The measure μ can be *disintegrated* into atoms of any measurable partition ξ. It means that there exists a family of probability measures $\{\mu_x^\xi\}_x$, called a *family of conditional measures of μ* relative to ξ defined for μ-a.e. $x \in X$ and satisfying for μ-a.e. $x \in X$:

(1) μ_x^ξ is a probability measure on X satisfying $\mu_x^\xi(\xi(x)) = 1$;

(2) if $y \in \xi(x)$ then $\mu_y^{\xi} = \mu_x^{\xi}$.

Moreover for every Borel subset $A \subset X$,

(3) $x \mapsto \mu_x^{\xi}(A)$ is measurable;

(4) $\mu(A) = \int \mu_x^{\xi}(A) \, d\mu(x)$.

Moreover such a family is unique modulo a null set of μ.

2.4.2. *Disintegration of invariant measures.* Assume now that μ is invariant by some invertible measurable transformation $f \colon X \to X$. Let ξ be a measurable partition and, for $n \in \mathbb{Z}$, let $\xi_n \overset{\text{def.}}{=} f^n \xi$. Let $\{\mu_x\}_x \overset{\text{def.}}{=} \{\mu_x^{\xi}\}_x$ and $\{\mu_{n,x}\}_x \overset{\text{def.}}{=} \{\mu_x^{\xi_n}\}_x$ be families of conditional measures of μ relative to ξ and ξ_n respectively. The following lemma will be useful for our purposes.

Lemma 2.12. *For every $n \in \mathbb{N}$ and μ-almost every $x \in X$, we have*

$$\mu_{n,x} = f_*^n \mu_{f^{-n}(x)}.$$

Proof. Let $\varphi \in L^1(X, \mu)$. We first disintegrate μ along ξ and use twice the f-invariance:

$$\int \varphi(x) \, d\mu(x) = \int \varphi \circ f^n(x) \, d\mu(x) = \int \left(\int_{\xi(x)} \varphi \circ f^n(y) \, d\mu_x(y) \right) d\mu(x)$$

$$= \int \left(\int_{\xi(f^{-n}(x))} \varphi \circ f^n(y) \, d\mu_{f^{-n}(x)}(y) \right) d\mu(x)$$

$$= \int \left(\int_{\xi_n(x)} \varphi(y) \, d\big(f_*^n \mu_{f^{-n}(x)}\big)(y) \right) d\mu(x).$$

We deduce that $\{f_*^n \mu_{f^{-n}(x)}\}_x$ is a system of conditional measures of μ with respect to ξ_n. The lemma follows by uniqueness μ-a.e. of conditional measures. $\square$

2.4.3. *Partitions subordinate to expanded and contracted foliations.* Let M be a closed manifold, $f \colon M \to M$ be a diffeomorphism of M, and μ be an ergodic invariant measure of f. Assume that $\mathcal{W}^+$ is a foliation invariant by f. Assume furthermore that it is uniformly expanded, that is $\|Df^{-1}|_{T\mathcal{W}^+}\| \le \lambda$ for some constant and $0 < \lambda < 1$.

A measurable partition ξ is *subordinate to* $\mathcal{W}^+$ if the following properties hold for μ-a.e. $x \in M$:

(1) $\xi(x)$ contains an open (in the internal topology) neighbourhood of x in $\mathcal{W}^+(x)$;

(2) $\xi \prec f^{-1}\xi$ (we say that ξ is *increasing*);

(3) $\bigvee_{n=0}^{\infty} f^{-n}\xi$ is the partition into points.

The existence of subordinate partitions was proven in [31] in a more general context (see also [48] and [5, Appendix D]).

Remark 2.13. *The subordinate partitions constructed in [48, 5] have atoms with uniformly bounded diameter. In particular, we can take subordinate partitions with diameter bounded by 1.*

Remark 2.14. *Assume that $\mathcal{W}' \subset \mathcal{W}^+$ is an f-invariant subfoliation with expansion constant $0 < \lambda' \le \lambda$ and that ξ is subordinate to $\mathcal{W}^+$. Then it follows from the proofs given in [5, 31, 48] that the partition $\xi' = \xi \vee \mathcal{W}'$ is subordinate to $\mathcal{W}'$.*

Similarly, assume $\mathcal{W}^-$ is invariant and *uniformly contracted* by f. This means that $\|Df|_{T\mathcal{W}^-}\| < \lambda$ for some $0 < \lambda < 1$. We say that ξ is *subordinate to $\mathcal{W}^-$* if properties (1)-(2)-(3) above hold replacing f by f^{-1}. We then say that ξ is *decreasing* for f.

Remark 2.15. *If $\mathcal{W}^+$ (resp. $\mathcal{W}^-$) has dimension 1 then atoms of the subordinate partition constructed in* [5, 31, 48] *are intervals.*

2.4.4. *Superposition property of subordinate partitions.* Let $M, f, \mathcal{W}^+$ be as in the above § 2.4.3. Let ξ be a partition subordinate to $\mathcal{W}^+$. Then, for every $n \geq 0$, $\xi_{-n} \overset{\text{def.}}{=} f^{-n}\xi$ is also a partition subordinate to $\mathcal{W}^+$. Since $\xi_{-n}(x)$ contains an open neighbourhood of x for μ-a.e. $x \in M$, an atom of ξ contains at most countably many atoms of ξ_{-n}. In the terminology of [12, Definition 5.15], we say that ξ and ξ_{-n} are *countably equivalent*.

From [12], Proposition 5.17 we obtain the following *superposition property*.

Lemma 2.16 (Superposition property)**.** *For μ a.e. $x \in M$ it holds that*

$$\mu_x(\xi_{-n}(x)) > 0,$$

for every $n \in \mathbb{N}$. Moreover, if $\mu_{-n,x}$ is the disintegration of μ with respect to the partition ξ_{-n}, then for any measurable set A,

$$\mu_{-n,x}(A) = \frac{\mu_x(A \cap \xi_{-n}(x))}{\mu_x(\xi_{-n}(x))}.$$

By the definition of a subordinate partition, for μ-a.e. $x \in M$, $\{\xi_{-n}(x)\}_{n \geq 0}$ contains a basis of open neighbourhoods of x.

Corollary 2.17. *For μ-a.e. $x \in M$, μ_x charges every open neighbourhood of x.*

Remark 2.18. *If $\mathcal{W}^-$ is contracting, and ξ is subordinate to $\mathcal{W}^-$, then the analogous statements to Lemma 2.16 and Corollary 2.17 hold for $\mathcal{W}^-$, replacing ξ_{-n} with ξ_n, for $n \geq 0$.*

2.4.5. *Uniform growth property.* Another useful property of the sequence of partitions ξ_n subordinate to a uniformly expanded foliation $\mathcal{W}^+$ is the following *uniform growth property*.

Lemma 2.19. *Given a real number $R > 0$, for μ almost every $x \in M$ there exists $n_0 = n_0(x, R) > 0$ such that if $n > n_0$ then*

$$\mathcal{W}_R^+(x) \subset \xi_n(x).$$

Proof. Define $\Lambda(\varepsilon) = \{x \in \mathbb{T}^3 : \mathcal{W}_\varepsilon^+(x) \subset \xi(x)\}$, and observe that $\mu(\Lambda(\varepsilon)) \to 1$ as $\varepsilon \to 0$, due to property (1) in the definition of subordinate partitions. The conclusion holds whenever $f^{-n_k}(x) \in \Lambda(1/k)$ where $n_k > \frac{\log(kR)}{-\log \lambda}$ (recall that $\|Df^{-1}|_{T\mathcal{W}^+}\| < \lambda < 1$). So the conclusion of the lemma holds in the set $\bigcup_k f^{n_k}(\Lambda(1/k))$ which has measure 1 by f-invariance of μ. $\qquad\square$

Remark 2.20. *The above lemma has an analogous version for measurable partitions subordinate to uniformly contracted foliations: the integer n_0 is then negative and the conclusion holds for all $n < n_0$.*

2.5. **SRB and u-Gibbs measures.** We return now to our partially hyperbolic setting where $f \in \mathcal{A}^2(\mathbb{T}^3)$. Let ξ^{cu} be a measurable partition of $\mathbb{T}^3$ subordinate to the center unstable foliation $\mathcal{W}^{cu}$. Let μ be an ergodic invariant measure for f. We say that μ is an *SRB measure* if its disintegration $\{\mu_x^{cu}\}_{x \in \mathbb{T}^3}$ with respect to the partition ξ^{cu} satisfies that μ_x^{cu} is absolutely continuous with respect to the inner Lebesgue measure Leb_x^{cu} of $\mathcal{W}^{cu}(x)$ for μ a.e. $x \in \mathbb{T}^3$.

Consider now ξ^u a measurable partition subordinate to the unstable foliation $\mathcal{W}^u$ and let $\{\mu^u_x\}_{x\in\mathbb{T}^3}$ denote the disintegration. Moreover, suppose that ξ^u has diameter bounded from above by 1 (see Remark 2.13). We say that μ is a *u-Gibbs measure* provided that μ^u_x is absolutely continuous with respect to the inner Lebesgue (length) measure Leb^u_x of $\mathcal{W}^u(x)$ for μ a.e. $x \in \mathbb{T}^3$.

Remark 2.21. *By Ledrappier-Young [32] none of the above definitions depend on the respective particular choice of subordinate partition.*

2.5.1. *Conditional measures along unstable leaves of an u-Gibbs measure.* We can (and we shall) consider the particular case in which the atoms of ξ^u are $\xi^u(x) = \mathcal{W}^u(x) \cap \xi^{cu}(x)$. By Remark 2.14 we know that this defines a partition subordinate to the unstable foliation $\mathcal{W}^u$. This is especially important in our setting, in which we want to prove that a given *u*-Gibbs measure is SRB. In a similar fashion, we consider the partition ξ^c whose atoms are $\xi^c(x) = \mathcal{W}^c(x) \cap \xi^{cu}(x)$.

Denote by $\xi^*_n \stackrel{\text{def.}}{=} f^n(\xi^*)$, for $* = u, cu, c$ and $n \geq 0$, with $\xi^*_0 = \xi^*$. Consider μ a *u*-Gibbs measure and let $\mu^{cu}_{n,x}$ denote the conditional measures along the partition ξ^{cu}_n. Notice that $\{\xi^u_n(y)\}_{y\in\xi^{cu}_n(x)}$ is a measurable partition of the probability space $(\xi^{cu}_n(x), \mu^{cu}_{n,x}, \mathcal{B}|_{\xi^{cu}_n(x)})$, where $\mathcal{B}|_{\xi^{cu}_n(x)}$ denotes the Borel sigma algebra of $\mathbb{T}^3$ restricted to the atom $\xi^{cu}_n(x)$. Rokhlin's disintegration theorem in this case give us a probability measure $\mu^c_{n,x}$ defined on $\xi^c_n(x)$ and a family of probability measures $\{\mu^u_{n,y}\}_{y\in\xi^c_n(x)}$ such that for every $A \subset \xi^{cu}_n(x)$ Borel measurable set we have

$$\mu^{cu}_{n,x}(A) = \int_{A\cap\mathcal{W}^c(x)} \mu^u_{n,y}(A \cap \mathcal{W}^u(y)) d\mu^c_{n,x}(y).$$

Remark 2.22. *For simplicity of the exposition in many situations, where no confusion may arise, given a set $A \subset \mathcal{W}^*(x)$, we shall denote $|A| \stackrel{\text{def.}}{=} \text{Leb}^*_x(A)$ for $* \in \{s, c, u\}$.*

An easy consequence of the *u*-Gibbs property is the following.

Lemma 2.23. *There exists $\beta > 1$ depending only on the diffeomorphism f such that for μ-almost every $x \in \mathbb{T}^3$,*

$$\frac{1}{\beta \, \text{Leb}^u_x[\xi^u(x)]} < \frac{d\mu^u_x}{d\text{Leb}^u_x} < \frac{\beta}{\text{Leb}^u_x[\xi^u(x)]}$$

inside $\xi^u(x)$.

Proof. This follows from the fact that μ^u_x is a probability measure that has a uniformly log-Hölder continuous density with respect to Leb^u_x and that the diameter of $\xi^u(x)$ is less than 1. $\square$

3. Heuristics of the proof

This section can be used to get an overview of the proof of Theorem A and also as a guide to read our paper.

From now on let $f \in \mathcal{A}^2(\mathbb{T}^3)$ be a diffeomorphism that is close to a conservative one. In this case, stable holonomies $H^s_{x,y}$ are C^1, and we can define angles $\alpha^s(x,y)$ for two points x, y in the same stable manifold (see Definition 4.1). The vanishing of the angle α^s can be seen as a kind of infinitesimal joint integrability.

Let μ be an ergodic *u*-Gibbs measure for f.

Starting point: a zero-one law. We first use Theorem B, which yields the following dichotomy.

Case 1: $\alpha^s(x, y) = 0$, for μ-a.e. every pair (x, y) in the same stable manifold;
Case 2: $\alpha^s(x, y) > 0$, for μ-a.e. every pair (x, y) in the same stable manifold.

The proof of this theorem will be given in Section 4 and relies on a martingale argument. In words, we either have positive angles almost everywhere along the stable manifold of almost every point, or we have zero angles almost everywhere along the stable manifold of almost every point.

   The rest of the proof is divided into two parts:

- [Case 1 + $\text{supp}(\mu) = \mathbb{T}^3$] $\implies$ joint integrability;
- Case 2 $\implies$ μ is SRB.

Part 1: joint integrability. Given a point $x \in \mathbb{T}^3$ we define the *Bad set* as

$$\mathbf{B} \overset{\text{def.}}{=} \{x \in \mathbb{T}^3 : \mu_x^s\{\alpha_x^s > 0\} = 0\},$$

where $\alpha_x^s \overset{\text{def.}}{=} [y \in \mathcal{W}^s(x) \mapsto \alpha^s(x, y)]$, and μ_x^s is the conditional measure of x along a stable manifold (see §5.1 for more details). The *Bad set* is the set of points such that there is an "infinitesimal" joint integrability with almost every other point in its stable manifold.

   The first part is an argument by contraposition that goes as follows (see Proposition 5.5):

- if there is no joint integrability, then for any $x \in \mathbb{T}^3$, the set $\{\alpha_x^s > 0\}$ is open and dense within the stable manifold $\mathcal{W}^s(x)$ (see Lemma 5.8);
- if, furthermore, the *Bad set* $\mathbf{B}$ has measure 1, the continuity of the angles implies that whenever $\alpha^s(x_0, y_0) > 0$, there exists a small foliated chart $\mathcal{V}$ around y_0 for $\mathcal{W}^s$ of measure 0 (see Lemma 5.9);
- hence, combining the two previous points, if there is no joint integrability and $\mathbf{B}$ has full measure, then the support of μ must have empty interior.

The last point is the only place in the paper where the support condition on μ is used.

Part 2: transversality implies SRB. Most of this paper is dedicated to proving that if μ is a u-Gibbs measure such that $\mu(\mathbf{B}) = 0$ then μ is SRB (Theorem 5.10).

   To check that a u-Gibbs measure μ is SRB, one can consider the disintegration of this measure along center-unstable manifolds $\{\mu_x^{cu}\}$ and then quotient it by the strong unstable manifolds $\{\hat{\mu}_x^c\}$. These are called the transverse measures. Then μ is SRB if and only if the transverse measures are equivalent to Lebesgue. In this approach, it is really convenient to consider certain parameterizations of center-unstable manifolds that simplify the dynamics, the so-called *normal forms* (see Section 6). These coordinates allow us to identify the quotient measures $\{\hat{\mu}_x^c\}$ with measures $\{\hat{\nu}_x^c\}$ in $\mathbb{R}$. To conclude that the measures $\{\hat{\mu}_x^c\}$ are equivalent to Lebesgue, we will show that the measures $\{\hat{\nu}_x^c\}$ are proportional to the Lebesgue measure on $\mathbb{R}$, where we say that two locally finite Borel measures ν, η are proportional, and we indicate it by $\nu \propto \eta$, if $\nu = c\eta$ for some $c > 0$.

   To do so, it is enough to prove that for μ-a.e. $x \in \mathbb{T}^3$, $\hat{\nu}_x^c$ is invariant by translation. Actually, thanks to an argument due originally to Katok-Spatzier [29] and Kalinin-Katok [25] (see also [7, Proposition 7.1]), which is a beautiful mixture of ergodic and Lie theoretic arguments, it is enough to prove something weaker: *for a set G of points $x \in \mathbb{T}^3$ with positive measure, there exist affine maps $\psi = as + b$, with $|a|$ uniformly bounded from above and $0 < |b|$ arbitrarily*

small, such that $\hat\nu_x^c \propto \psi_* \hat\nu_x^c$. This is Lemma 8.1 (largely inspired by [6, Lemma 6.1]).

The construction of these affine maps is where our key arguments are located. Our strategy is to use the *Y-configurations* introduced by Eskin-Mirzakhani [15] (see also Eskin-Lindenstrauss [14]). Below we outline this proof, splitting the explanation in two parts, for the sake of clarity. First, we describe Y-configurations. Then we explain how to use them in order to get invariance by affine maps in the way we described in last paragraph.

Y-configurations. See Figure 5. Fixing $\ell \in \mathbb{N}$ we find points x and y that are typical for μ such that:

(1) $y \in \mathcal{W}_{\mathrm{loc}}^s(x)$ and $d(x, y) \approx \|Df^\ell|_{E^s}\|$;
(2) if $x_{-\ell} = f^{-\ell}(x)$ and $y_{-\ell} = f^{-\ell}(y)$, then $d(x_{-\ell}, y_{-\ell}) \approx 1$ and the angle $\alpha^s(x_{-\ell}, y_{-\ell})$ is more than some constant $\frac{1}{C}$.

As a consequence of points (1) and (2), $\alpha^s(x, y) \geq \frac{1}{C} \frac{\|Df^\ell(x_{-\ell})|_{E^c}\|}{\|Df^\ell(x_{-\ell})|_{E^u}\|}$. Normal forms help a lot here. This computation is performed inside the proof of Lemma 9.2.

We can then find points $x^u \in \mathcal{W}_{\mathrm{loc}}^u(x)$ and $y^u \in \mathcal{W}_{\mathrm{loc}}^u(y)$ such that:

• there exists a point z^u with $\{z^u\} = \mathcal{W}_{\mathrm{loc}}^s(x^u) \cap \mathcal{W}_{\mathrm{loc}}^c(y^u)$;
• $d(z^u, y^u) \approx \frac{\|Df^\ell(x_{-\ell})|_{E^c}\|}{\|Df^\ell(x_{-\ell})|_{E^u}\|}$.

So far we obtained points x^u and y^u with some estimate of the displacement in the center direction that we get when projecting x^u to $\mathcal{W}_{\mathrm{loc}}^u(y)$ by center stable holonomy. Next, fixing $\varepsilon > 0$, we define the stopping time

$$\tau(\ell) = \tau(x, x^u, \varepsilon, \ell) \overset{\text{def.}}{=} \inf\left\{ n \in \mathbb{N} : \frac{\|Df^\ell(x_{-\ell})_{E^c}\|}{\|Df^\ell(x_{-\ell})|_{E^u}\|} \|Df^n(x^u)|_{E^c}\| \geq \varepsilon \right\}$$

and consider the points $f^{\tau(\ell)}(x^u)$ and $f^{\tau(\ell)}(y^u)$. The choice of the stopping time $\tau(\ell)$ is such that

$$d(f^{\tau(\ell)}(x^u), f^{\tau(\ell)}(y^u)) \approx \varepsilon, \tag{9}$$

see §9.3.2. Essentially, these points are related to the translational part of size ε of the affine maps that we desire to construct. To control the derivative of the affine maps and to actually obtain an "invariance" of the measures $\hat\nu_x^c$, as we mentioned before, we also define another stopping time

$$t(\ell) = t(x, x^u, \varepsilon, \ell) \overset{\text{def.}}{=} \inf\left\{ n \in \mathbb{N} : \frac{\|Df^n(x)|_{E^c}\|}{\|Df^{\tau(\ell)}(x^u)|_{E^c}\|} \geq 1 \right\}.$$

In particular, this condition implies that

$$\|Df^{t(\ell)}(x)|_{E^c}\| \approx \|Df^{\tau(\ell)}(x^u)|_{E^c}\|, \tag{10}$$

see §9.1.

How to use Y-configurations to get invariance by affine maps. By construction of measures $\{\hat\nu_p^c\}_{p \in \mathbb{T}^3}$ we understand how they change under three basic moves.

(1) *Applying the dynamics.* For μ-a.e. $p \in \mathbb{T}^3$, $\hat\nu_{f^n(p)}^c \propto \Lambda_* \hat\nu_p^c$, for every $n \in \mathbb{N}$, where $\Lambda(s) = \|Df^n(p)|_{E^c}\|s$ is a linear map of $\mathbb{R}$ (see Lemma 7.13).
(2) *Moving along unstable manifolds.* For $q \in \mathcal{W}^u(p)$ then $\hat\nu_q^c \propto L_* \hat\nu_p^c$ where $L(s) = \beta s$ is a linear map, for some $C_u^{-1} \leq \beta \leq C_u$, and $C_u = C_u(d_u(p, q)) > 1$ is a number which is bounded from above and it depends on the unstable distance $d_u(p, q)$ (see Lemma 7.14).
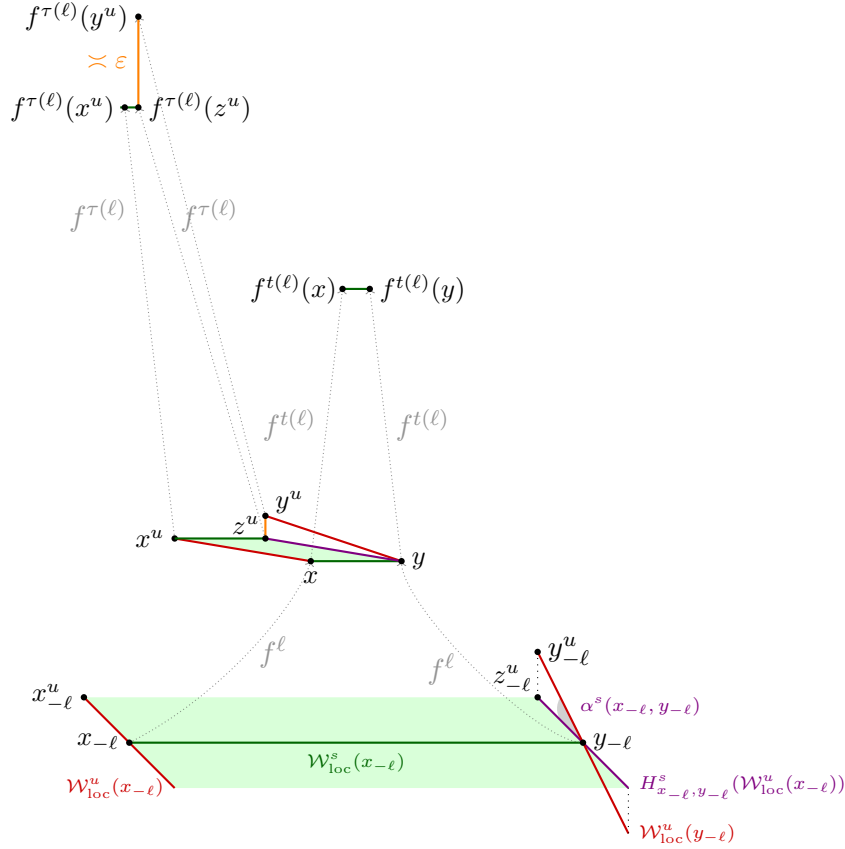
FIGURE 5. The points $x, x^u, x_{-\ell}, f^{\tau(l)}(x^u)$ and $f^{t(\ell)}(x)$ are called a Y-configuration (similarly the points involving y's).

(3) *Moving along center manifolds.* For $q \in \mathcal{W}^c(p)$ then $\hat{\nu}^c_q \propto \Psi_* \hat{\nu}^c_p$ where $\Psi(s) = as + b$, with $C_c^{-1} \leq a \leq C_c$ for some constant $C_c = C_c(d_c(p,q)) > 1$ bounded from above with $d_c(p,q)$, and with $b \approx d_c(p,q)$ (see Lemma 7.12).

Now we look at the points on the top part of Figure 5 and deduce two facts.

(a) $\hat{\nu}^c_{f^{\tau(\ell)}(x^u)} \propto (\Lambda_1)_* \hat{\nu}^c_{f^{t(\ell)}(x)}$ for some *linear* map $\Lambda_1 \colon \mathbb{R} \to \mathbb{R}$ with bounded norm independently of ℓ.

(b) $\hat{\nu}^c_{f^{\tau(\ell)}(y^u)} \propto (\Lambda_2)_* \hat{\nu}^c_{f^{t(\ell)}(y)}$ for some *linear* map $\Lambda_2 \colon \mathbb{R} \to \mathbb{R}$ with bounded norm ndependently of ℓ.

There is a subtlety here: stopping times $\tau(\ell)$ and $t(\ell)$ depend on x, x^u and not on y, y^u. This is treated by our *synchronization estimates* (Lemma 9.4). Recall that $z^u \in \mathcal{W}^s(x^u)$ and $y \in \mathcal{W}^s(x)$. Now note that as $\ell \to \infty$, $\tau(\ell), t(\ell) \to \infty$ so $d(f^{\tau(\ell)}(x^u), f^{\tau(\ell)}(z^u)) \to 0$ and $d(f^{t(\ell)}(x), f^{t(\ell)}(y)) \to 0$. If we knew that the family of measures $\hat{\nu}^c_z$ were continuous with z then we could hope to compare $\hat{\nu}^c_{f^{\tau(\ell)}(x^u)}$ with $\hat{\nu}^c_{f^{\tau(\ell)}(y^u)}$ and take accumulation points to construct the desired set G.

But the objects we are working with are only measurable, so we must first fix a large Lusin set for which the map $z \mapsto \hat{\nu}^c_z$ is continuous (among other dynamical objects that appear in the proof). We want to do the constructions of all of the points mentioned above, in a way that all of them belong to this Lusin set. For this, it is essential to obtain *quasi-isometric estimates* for the functions $\tau(\cdot)$ and

$t(\cdot)$ (see Lemma 9.1; see also Lemma 11.5 where the quasi-isometric estimates for stopping times are used in a crucial way).

By continuity, we have that $\hat{\nu}^c_{f^t(\ell)(x)} \approx \hat{\nu}^c_{f^t(\ell)(y)}$ and $\hat{\nu}^c_{f^{\tau(\ell)}(x^u)} \approx \hat{\nu}^c_{f^{\tau(\ell)}(z^u)}$. We conclude from this that $\hat{\nu}^c_{f^{\tau(\ell)}(x^u)}$ is *almost proportional to* $\hat{\nu}^c_{f^{\tau(\ell)}(y^u)}$, *up to some linear map with controlled derivative.*

After considering a subsequence ℓ_k we obtain points p and q (obtained as the limit of $f^{\tau(\ell_k)}(x^u)$ and $f^{\tau(\ell_k)}(y^u)$)) with the properties that:

- $q \in \mathcal{W}^c_{\mathrm{loc}}(p)$;
- $d(p, q) \approx \varepsilon$;
- p and q belong to the Lusin set;
- $\hat{\nu}^c_q \propto \hat{\Lambda}_* \hat{\nu}^c_p$, $\hat{\Lambda}$ being linear with uniformly bounded norm (see §10.4).

Finally we explained that $\hat{\nu}^c_p \propto \Psi_* \hat{\nu}^c_q$ for an affine map $\Psi = as + b$ with $|a|$ uniformly bounded and $|b| \approx \varepsilon$: we find $\hat{\nu}^c_p \propto (\Psi\hat{\Lambda})_* \hat{\nu}^c_p$: $\Psi\hat{\Lambda}$ is the desired affine map (with bounded slope and translational part $\approx \varepsilon$).

A technical difficulty. There is one delicate point in the above argument. Since the center stable foliation is not absolutely continuous, in general, we cannot choose the points x^u and y^u at the same time in the Lusin set and such that $y^u \in \mathcal{W}^{cs}_{\mathrm{loc}}(x^u)$. To overcome this difficulty, we introduce the notion of matched Y-configurations in Section 10 (the picture changes just a little bit from the one described above) and use it to prove our theorem (see Section 11).

Let us also emphasize that the control we obtain on the distance between the points z^u and y^u is possible because H^s is C^1, so essentially we can control this distance by looking at the angle between $DH^s_{x,y}E^u(x)$ and $E^u(y)$. This allows us to explicitly define the stopping time τ in our construction. This is not possible in Katz's proof since the holonomies are only Hölder. To get around this problem, he defines a much more complicated stopping time built from an operator using ideas from Eskin-Mirzakhani.

4. A ZERO-ONE LAW FOR ANGLES

The goal of this section is to prove Theorem B. In a slightly more general context, i.e., for some $f \in \mathcal{PH}^2(\mathbb{T}^3)$ with C^1 stable holonomies (for instance, if f is close to some $f_0 \in \mathcal{A}^2_m(\mathbb{T}^3)$, as we saw in Lemma 2.6), we introduce the angle function α^s, which measures the "twist" of unstable manifolds along stable manifolds.

Definition 4.1 (The angle function). *Let $x \in \mathbb{T}^3$ and $y \in \mathcal{W}^s(x)$. The* angle function *is the assignment* $(x, y) \mapsto \alpha^s(x, y) \stackrel{\text{def.}}{=} \angle(DH^s_{x,y}(x)E^u(x), E^u(y))$.

In this context, we have the following zero-one law, whose proof is inspired by the work of Brown-Rodriguez Hertz [6, Lemma 7.1].

Theorem 4.2 (A zero-one law for angles). *Let $f \in \mathcal{PH}^2(\mathbb{T}^3)$ be a partially hyperbolic diffeomorphism with a splitting $T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u$, whose stable holonomies H^s along the strong stable foliation $\mathcal{W}^s$ are C^1. Fix an ergodic f-invariant measure μ. Let ξ^s be a measurable partition subordinate to $\mathcal{W}^s$ and $\{\mu^s_x\}_x$ be a system of conditional measures relative to ξ^s. Then the following dichotomy holds:*

(1) either for μ-almost every $x \in \mathbb{T}^3$,

$$\mu^s_x \{y \in \xi^s(x) : \alpha^s(x, y) = 0\} = 1;$$

(2) or for μ-almost every $x \in \mathbb{T}^3$,

$$\mu_x^s \{y \in \xi^s(x) : \alpha^s(x,y) = 0\} = 0.$$

The second alternative will allow us to build "twisted" quadrilaterals (such as the one depicted in Figure 4) with points x, y in some good Lusin set while for x^u and y^u we will find points of the Lusin set arbitrarily close to them, so that the twisted quadrilateral will be part of a *matched Y-configurations* associated to x and y.

4.1. **Conditional expectation, σ-algebras and martingales.** The proof of Theorem 4.2 requires some preliminaires about martingales. If ξ is a measurable partition of a measurable space X then we let $\mathcal{F}_\xi$ denote the σ-algebra generated by unions of atoms of ξ. Let $\{\mu_x\}_x$ be a system of conditional measures of μ with respect to ξ. We define the *conditional expectation* of $\varphi \in L^1(X, \mu)$ as the following L^1-function

$$\mathbb{E}_\mu[\varphi \,|\, \mathcal{F}_\xi] \colon x \mapsto \int_{\xi(x)} \varphi \, d\mu_x^\xi.$$

Note that if $(\xi_n)_n$ is an increasing sequence of partitions (in the sense that $\xi_n \prec \xi_{n+1}$) then we have $\mathcal{F}_{\xi_n} \subset \mathcal{F}_{\xi_{n+1}}$.

The following result is a consequence of the increasing martingale theorem (for which we refer to [13, Theorem 5.5, p.126]).

Theorem 4.3 (Increasing martingale theorem)**.** *Let $(\xi_n)_{n \in \mathbb{N}}$ be an increasing sequence of measurable partitions of X such that $\bigvee_{n=0}^\infty \xi_n$ is the partition into points. Then, for every $\varphi \in L^1(X, \mu)$ and μ-almost every $x \in X$, we have*

$$\lim_{n \to \infty} \mathbb{E}_\mu[\varphi \,|\, \mathcal{F}_{\xi_n}](x) = \varphi(x).$$

4.2. **Proof of the zero-one law.** We are now ready to prove Theorem 4.2. We fix ξ^s, a measurable partition subordinate to $\mathcal{W}^s$. For $n \in \mathbb{N}$, set $\xi_n^s \overset{\text{def.}}{=} f^n \xi^s$. Systems of conditional measures relative to ξ^s and ξ_n^s are denoted respectively by $\{\mu_x^s\}_x$ and $\{\mu_{n,x}^s\}_x$. For μ-almost every $x \in \mathbb{T}^3$ and every $n \in \mathbb{N}$, set

- $\mathcal{P}^{\xi^s}(x) \overset{\text{def.}}{=} \{y \in \xi^s(x) : \alpha^s(x,y) = 0\}$,
- $\mathcal{P}_n^{\xi^s}(x) \overset{\text{def.}}{=} \{y \in \xi_n^s(x) : \alpha^s(x,y) = 0\}$.

By definition of ξ_n^s and invariance of the unstable bundle and of the stable foliation, we have the following commutation relation for μ-almost every $x \in \mathbb{T}^3$:

$$f^n \mathcal{P}^{\xi^s}\left(f^{-n}(x)\right) = \mathcal{P}_n^{\xi^s}(x). \tag{11}$$

Let $A \overset{\text{def.}}{=} \{x \in \mathbb{T}^3 : \mu_x^s[\mathcal{P}^{\xi^s}(x)] > 0\}$. This is a Borel set. We must prove that if $\mu(A) > 0$, then

$$\mu(A) = 1, \text{ and for } \mu\text{-almost every } x \in \mathbb{T}^3, \ \mu_x^s[\mathcal{P}^{\xi^s}(x)] = 1. \tag{12}$$

So let us suppose $\mu(A) > 0$ and prove (12). With that goal in mind, we claim that A is an f-invariant set (mod 0). Indeed, take a μ generic point $x \in A$. Notice that $\mathcal{P}_1^{\xi^s}(x) \subset \mathcal{P}^{\xi^s}(x)$, since $\xi_1^s(x) \subset \xi^s(x)$. Also, recall from Lemma 2.12 that

$$\mu_{1,f(x)}^s = f_* \mu_x^s.$$

Combining this with (11) one obtains that

$$\mu_{1,f(x)}^s(\mathcal{P}_1^{\xi^s}(f(x))) = \mu_x^s(f^{-1}(\mathcal{P}_1^{\xi^s}(f(x)))) = \mu_x^s(\mathcal{P}^{\xi^s}(x)) > 0.$$

Thus, from the superposition property (Lemma 2.16 and Remark 2.18) one gets

$$\mu_{f(x)}^s(\mathcal{P}^{\xi^s}(f(x))) \geq \mu_{f(x)}^s(\mathcal{P}_1^{\xi^s}(f(x))) = \mu_{1,f(x)}^s(\mathcal{P}_1^{\xi^s}(f(x)))\mu_{f(x)}^s(\xi_1^s(f(x))) > 0,$$

which implies that $f(x) \in A$, proving our claim.

From the ergodicity of μ, we have $\mu(A) = 1$. We define functions in $L^1(\mathbb{T}^3, \mu)$ by

- $\phi \colon x \mapsto \mu_x^s[\mathcal{P}^{\xi^s}(x)]$;
- $\phi_n \colon x \mapsto \mu_{n,x}^s[\mathcal{P}_n^{\xi^s}(x)] = \mu_{n,x}^s[\mathcal{P}^{\xi^s}(x)]$, for each $n \in \mathbb{N}$.

For μ-almost every $x \in \mathbb{T}^3$ and every $n \in \mathbb{N}$ we consider

- $\mathcal{F}_x^s = \mathcal{F}_{\xi^s(x)}^s = \{\emptyset, \xi^s(x)\}$, the trivial σ-algebra over $\xi^s(x)$;
- $\mathcal{F}_{n,x}^s = \mathcal{F}_{n,\xi^s(x)}^s$, the σ-algebra generated by unions of atoms $\xi_n^s(y)$, $y \in \xi^s(x)$ (note that $\mathcal{F}_{n,x}^s \subset \mathcal{F}_{n+1,x}^s$);
- $\mathcal{F}_{\infty,x}^s = \mathcal{F}_{\infty,\xi^s(x)}^s$, the smallest σ-algebra containing $\bigcup_{n=0}^\infty \mathcal{F}_{n,x}^s$. This is the Borel σ-algebra of $\xi^s(x)$.

For μ-almost every $x \in \mathbb{T}^3$ and $n \in \mathbb{N}$, we define the function $\psi_{n,x} \in L^1(\xi^s(x), \mu_x^s)$ by the following formula, for μ_x^s-almost every $y \in \xi^s(x)$:

$$\psi_{n,x}(y) = \mathbb{E}_{\mu_x^s}\big[\mathbf{1}_{\mathcal{P}^{\xi^s}(x)} \,\big|\, \mathcal{F}_{n,x}^s\big](y) = \mu_{n,y}^s[\mathcal{P}^{\xi^s}(x)].$$

Note that $\psi_{n,x}(y) = \phi_n(y)$ for all n and μ_x^s-almost every $y \in \mathcal{P}^{\xi^s}(x)$ (note that in that case $\mathcal{P}^{\xi^s}(x) = \mathcal{P}^{\xi^s}(y)$).

On the one hand, for μ-almost every $x \in \mathbb{T}^3$, by the increasing martingale theorem (we apply Theorem 4.3 to the probability space $(\xi^s(x), \mu_x^s)$), $\psi_{n,x}$ converges to $\mathbf{1}_{\mathcal{P}^{\xi^s}(x)}$ μ_x^s-almost surely as $n \to +\infty$. In particular,

$$\text{for } \mu_x^s\text{-a.e. } y \in \mathcal{P}^{\xi^s}(x), \quad \psi_{n,x}(y) = \phi_n(y) \to 1 \text{ as } n \to +\infty. \tag{13}$$

Let us define the set

$$\mathcal{S} \overset{\text{def.}}{=} \{x \in \mathbb{T}^3 : \phi_n(x) \to 1 \text{ as } n \to +\infty\}.$$

For each $n \in \mathbb{N}$, ϕ_n is measurable, hence $\mathcal{S}$ is a Borel set. Assume that $\mu[\mathcal{S}] = 0$. Since $\mu[\mathcal{S}] = \int \mu_x^s[\mathcal{S}]d\mu(x)$, we would then have $\mu_x^s[\mathcal{S}] = 0$, for μ-a.e. $x \in \mathbb{T}^3$. But (13) implies that for μ-a.e. $x \in A$, $\mu_x^s[\mathcal{S}] \geq \mu_x^s[\mathcal{S} \cap \mathcal{P}^{\xi^s}(x)] = \mu_x^s[\mathcal{P}^{\xi^s}(x)] > 0$, and by our assumption that $\mu[A] > 0$, we reach a contradiction. Therefore,

$$\mu[\mathcal{S}] = \mu\{x \in \mathbb{T}^3 : \phi_n(x) \to 1 \text{ as } n \to +\infty\} > 0.$$

On the other hand, we deduce from (11) and from Lemma 2.12 that

$$\phi_n(x) = \mu_{n,x}^s[\mathcal{P}_n^{\xi^s}(x)] = f_*^n \mu_{f^{-n}(x)}^s[f^n \mathcal{P}(f^{-n}(x))] = \phi(f^{-n}(x)).$$

The latter proves that $\phi \circ f^{-k}$ converges to 1 μ-almost surely on $\mathcal{S}$ as $k \to +\infty$. Therefore, by considering Cesàro averages $\frac{1}{n}\sum_{k=0}^{n-1} \phi \circ f^{-k}(x)$ for a μ-generic point $x \in \mathcal{S}$, and by Birkhoff's theorem, we conclude that $\int_{\mathbb{T}^3} \phi \, d\mu = 1$. As ϕ takes values in $[0,1]$, we must have $\phi(x) = 1$, for μ-almost every $x \in \mathbb{T}^3$. □

Remark 4.4. *It is clear that $\alpha^s(x,y) = 0$ is an equivalence relation on stable leaves, so Theorem 4.2 can be generalized as follows. Let f be as in Theorem 4.2. Let $\mathcal{R}$ be a measurable equivalence relation on stable leaves (i.e., such that $x\mathcal{R}y \Rightarrow y \in \mathcal{W}^s(x)$) such that $x\mathcal{R}y \Rightarrow f^n(x)\mathcal{R}f^n(y)$, for any $n \in \mathbb{N}$. Fix a measurable partition ξ^s subordinate to $\mathcal{W}^s$, a system $\{\mu_x^s\}_x$ of conditional measures relative to ξ^s, and for $x \in \mathbb{T}^3$, let $\mathcal{P}^{\xi^s}(x)$ be its $\mathcal{R}$-equivalence class. Then the following dichotomy holds:*

(1) either for μ-almost every $x \in M$, $\mu_x^s[\mathcal{P}^{\xi^s}(x)] = 1$;
(2) or for μ-almost every $x \in M$, $\mu_x^s[\mathcal{P}^{\xi^s}(x)] = 0$.

5. JOINT INTEGRABILITY AND THE *BAD SET*

We start in this section the formal proof of Theorem A. So we let $f \in \mathcal{A}^2(\mathbb{T}^3)$ be an Anosov diffeomorphism, strongly partially hyperbolic with expanding center and C^1 stable bundle. Recall from Lemma 2.6 that this is always satisfied when f is close to a conservative map f_0. Let μ denote an ergodic u-Gibbs measure, with full support.

We are going to reduce the proof of Theorem A to a more technical version of the result by studying the set of points x for which one sees almost no twist of the bundle E^u by the application of the stable holonomy. Our main technical result says that if the measure of this *Bad set* is zero then μ is SRB.

5.1. The *Bad set*. For any $x \in \mathbb{T}^3$, we denote

$$\mathcal{P}(x) \stackrel{\text{def.}}{=} \left\{ y \in \mathcal{W}^s(x) : \alpha^s(x, y) = 0 \right\}, \quad \mathcal{N}(x) \stackrel{\text{def.}}{=} \left\{ y \in \mathcal{W}^s(x) : \alpha^s(x, y) > 0 \right\}.$$

Observe that given any measurable partition ξ^s subordinate to $\mathcal{W}^s$ we have that for μ-a.e. $x \in \mathbb{T}^3$, $\mathcal{P}^{\xi^s}(x) = \mathcal{P}(x) \cap \xi^s(x)$, where $\mathcal{P}^{\xi^s}(x)$ was defined in the proof of Theorem 4.2, our zero-one law: see §4.2.

Remark 5.1. *Since the unstable bundle E^u is invariant under Df, and holonomy maps are equivariant with the dynamics, i.e.,*

$$f \circ H^s_{x,y} = H^s_{f(x),f(y)} \circ f, \tag{14}$$

we have $\forall x, y$,

$$\alpha^s(x, y) = 0 \quad \Longleftrightarrow \quad \forall n \in \mathbb{Z}, \ \alpha^s(f^n(x), f^n(y)) = 0. \tag{15}$$

For any $x \in \mathbb{T}^3$, the function $\alpha^s(x, \cdot)$ on $\mathcal{W}^s(x)$ is continuous. Therefore, the sets $\mathcal{P}(x)$ and $\mathcal{N}(x)$ are respectively closed and open. Moreover, by Remark 5.1, it holds

$$f^n(\mathcal{P}(x)) = \mathcal{P}(f^n(x)), \quad f^n(\mathcal{N}(x)) = \mathcal{N}(f^n(x)), \quad \forall n \in \mathbb{Z}. \tag{16}$$

Definition 5.2 (Bad set). *Let ξ^s be a measurable partition subordinate to the stable foliation $\mathcal{W}^s$, and let $\{\mu^s_x\}_{x \in \mathbb{T}^3}$ be a system of conditional measures relative to ξ^s. The* Bad set $\mathbf{B} = \mathbf{B}(\xi^s, \mu)$ *is defined as*

$$\mathbf{B} \stackrel{\text{def.}}{=} \{ x \in \mathbb{T}^3 : \mu^s_x(\mathcal{N}(x)) = 0 \}.$$

Recall that in Section 4.2, we introduced the set $A = \{ x \in \mathbb{T}^3 : \mu^s_x[\mathcal{P}^{\xi^s}(x)] > 0 \}$. Using our zero-one law, and the results established in §4.2, we have the following properties.

(1) $\mathbf{B}$ is equal μ-almost everywhere to the set A;
(2) $\mathbf{B}$ is measurable and f-invariant, and it satisfies that for all $\ell \in \mathbb{Z}$

$$\mathbf{B}(\xi^s, \mu) = \mathbf{B}(\xi^s_\ell, \mu), \tag{17}$$

where we recall $\xi^s_\ell = f^\ell(\xi^s)$ is still a measurable partition subordinate to $\mathcal{W}^s$;
(3) $\mathbf{B}$ has measure 0 or 1.

A priori the *Bad set* $\mathbf{B} = \mathbf{B}(\xi^s, \mu)$ depends on the particular choice of the measurable partition subordinate to $\mathcal{W}^s$. Nevertheless, the next lemma shows that the Bad set associated to another subordinate partition is equal to $\mathbf{B}$ modulo sets of measure 0.

Lemma 5.3. *For every measurable partition η^s subordinate to $\mathcal{W}^s$ we have*

$$\mu\left(\mathbf{B}(\xi^s, \mu)\right) = \mu\left(\mathbf{B}(\eta^s, \mu)\right) \in \{0, 1\}.$$

Proof. For $\ell \in \mathbb{Z}$, let us denote $\xi_\ell^s \overset{\text{def.}}{=} f^\ell(\xi^s)$. Then, all the partitions ξ_ℓ^s are measurable partitions subordinate to $\mathcal{W}^s$ and by (17) we have that for all $\ell \in \mathbb{Z}$, $\mathbf{B}(\xi^s, \mu) = \mathbf{B}(\xi_\ell^s, \mu)$. These sets have measure 0 or 1; assume they have measure 1. Consider a μ-generic point x so that

- $\eta^s(x)$ contains an open neighbourhood of x in $\mathcal{W}^s(x)$ so $\mu_x^{\xi_m^s}(\eta^s(x)) > 0$ for all $m \in \mathbb{Z}$ (see Corollary 2.17);
- $\eta^s(x)$ has diameter less than 1 and;
- $\mu_x^{\xi_m^s}(\mathcal{N}(x)) = 0$, for all $m \in \mathbb{Z}$.

Using Lemma 2.19 (specifically Remark 2.20) there exists $m < 0$ such that $\eta^s(x) \subset \xi_m^s(x)$ so we can apply the superposition property and

$$\mu_x^{\eta^s}(\mathcal{N}(x)) = \frac{\mu_x^{\xi_m^s}(\mathcal{N}(x) \cap \eta^s(x))}{\mu_x^{\xi_m^s}(\eta^s(x))} \leq \frac{\mu_x^{\xi_m^s}(\mathcal{N}(x))}{\mu_x^{\xi_m^s}(\eta^s(x))} = 0.$$

As a result, $x \in \mathbf{B}(\eta^s, \mu)$. This proves that $\mu(\mathbf{B}(\eta^s, \mu)) = 1$, confirming the second assertion and concluding. $\square$

Although we always need to fix a partition to speak about the bad set, Lemma 5.3 allows us to speak about *the measure of the Bad set* without fixing a particular choice of a partition.

Remark 5.4. *The conclusion of Lemma 5.3 remains valid even if the measurable partition η^s is only required to have the following properties for μ-a.e. $x \in \mathbb{T}^3$*

(1) $\eta^s(x) \subset \mathcal{W}^s(x)$;
(2) $\eta^s(x)$ contains an open neighbourhood of $\mathcal{W}^s(x)$ in the internal topology.
(3) $\eta^s(x)$ has uniformly bounded diameter;

Indeed the proof of Lemma 5.3 does not use that η^s is decreasing.

There are many (non decreasing) measurable partitions satisfying these three properties. For example, consider any finite foliated atlas $\mathcal{A}$ for $\mathcal{W}^s$ composed of open charts U_i such that $\mu(\partial U_i) = 0$. Define a finite partition mod 0 of $\mathbb{T}^3$ as

$$\mathcal{Q} = \bigvee_i \left\{ U_i, \mathbb{T}^3 \setminus \overline{U_i} \right\},$$

Each atom $\mathcal{Q}(x)$ is an open set included inside a foliated chart for $\mathcal{W}^s$. As a result it is trivially foliated by stable plaques: this defines a measurable partition of $\mathcal{Q}(x)$. We obtain a refinement of $\mathcal{Q}$, denoted by η^s, that satisfies the desired properties, and is referred to as the measurable partition associated to $\mathcal{A}$.

5.2. **Joint integrability.** When the Bad set has positive measure, by ergodicity we have that for almost every x, we see almost no twist of E^u along $\mathcal{W}^s(x)$. This can be read as an infinitesimal form of joint integrability between E^s and E^u. In this paragraph we shall improve this to actual joint integrability.

Proposition 5.5. *Let $f \colon \mathbb{T}^3 \to \mathbb{T}^3$ be a C^2 Anosov diffeomorphism, strongly partially hyperbolic with expanding center and C^1 stable holonomies. Let μ be a fully supported ergodic f-invariant measure. If the Bad set has full measure then E^s and E^u are jointly integrable.*

It is worth to point out that the proof of this proposition is the only place in our argument towards Theorem A where the full support assumption of the u-Gibbs measure is used. Moreover, in Proposition 5.5 the ergodic invariant measure μ does not need to be u-Gibbs.

5.2.1. *Local joint integrability.* In the proof of Proposition 5.5 we shall apply a criterion for joint integrability that comes from [11]. See also [44, §2.3].

Definition 5.6. *Given $x \in \mathbb{T}^3$ we say that the bundles E^s and E^u are jointly integrable at x if there exists $\delta, \varepsilon > 0$ such that for each $z \in \mathcal{W}^s_\delta(x)$ and $y \in \mathcal{W}^u_\delta(x)$ it holds*

$$\mathcal{W}^u_\varepsilon(z) \cap \mathcal{W}^s(y) \neq \emptyset.$$

The result below follows directly from [11, Lemma 5] (see also [44, Lemma 2.3.7]).

Lemma 5.7. *Let $f \in \mathcal{A}^2(\mathbb{T}^3)$. Assume that E^s and E^u are jointly integrable at each $x \in \mathbb{T}^3$, with uniform constants δ, ε. Then, f is jointly integrable (in the sense of Definition 2.1).*

5.2.2. *Proof of Proposition 5.5.* This Proposition follows directly from Lemma 5.8 and Lemma 5.9 below.

On the topological level, we have:

Lemma 5.8. *Let $f \colon \mathbb{T}^3 \to \mathbb{T}^3$ be a C^2 Anosov diffeomorphism, strongly partially hyperbolic with expanding center and C^1 stable holonomies. If for some $x \in \mathbb{T}^3$ the set $\mathcal{N}(x)$ is not open and dense in $\mathcal{W}^s(x)$, then E^s and E^u are jointly integrable.*

Proof. Let us assume that there exists $x_0 \in \mathbb{T}^3$ such that $\mathcal{P}(x_0)$ contains a non-trivial open interval, i.e., that for some $x \in \mathcal{W}^s(x_0)$, and $\varepsilon > 0$, it holds $\mathcal{W}^s_\varepsilon(x) \subset \mathcal{P}(x_0)$. Since $\mathcal{P}(x) = \mathcal{P}(x_0)$, we also have $\mathcal{P}(x) \supset \mathcal{W}^s_\varepsilon(x)$, hence by (16), for any integer $n \geq 1$,

$$\mathcal{P}\big(f^{-n}(x)\big) \supset f^{-n}\big(\mathcal{W}^s_\varepsilon(x)\big). \tag{18}$$

By compactness, we can take a subsequence $(f^{-n_k}(x))_{k \geq 0}$ such that $f^{-n_k}(x) \to y$ as $k \to +\infty$, for some point $y \in \mathbb{T}^3$. Since the restriction of f^{-1} to stable leaves is uniformly expanding, we deduce from (18) that

$$\mathcal{P}(y) = \mathcal{W}^s(y). \tag{19}$$

Let us now show that the same holds for any point, i.e., that $\mathcal{P}(z) = \mathcal{W}^s(z)$, for any $z \in \mathbb{T}^3$. Fix $z \in \mathbb{T}^3$ and $z' \in \mathcal{W}^s(z)$. By minimality of the stable foliation, the leaf $\mathcal{W}^s(y)$ is dense in $\mathbb{T}^3$, hence there exists a sequence $(y_n)_{n \geq 0} \in (\mathcal{W}^s(y))^{\mathbb{N}}$ such that $\lim_{n \to +\infty} y_n = z$. For each integer $n \geq 0$, we let $y'_n \stackrel{\text{def.}}{=} H^s_{z,z'}(y_n) \in \mathcal{W}^{cu}(z') \cap \mathcal{W}^s(y_n)$. In particular, we also have $\lim_{n \to +\infty} y'_n = H^s_{z,z'}(\lim_n y_n) = z'$. By continuity of the angle function α^s, we deduce that

$$\alpha^s(z, z') = \lim_{n \to +\infty} \alpha^s(y_n, y'_n) = 0,$$

i.e., $z' \in \mathcal{P}(z)$. Since $z' \in \mathcal{W}^s(z)$ was chosen arbitrarily within $\mathcal{W}^s(z)$, we deduce that $\mathcal{W}^s(z) = \mathcal{P}(z)$, for all $z \in \mathbb{T}^3$.

Now, fix an arbitrary $x \in \mathbb{T}^3$. Let $0 < \delta < \varepsilon$ be chosen so that the stable holonomy map $H^s_{x,z} \colon \mathcal{W}^{cu}_\delta(x) \to \mathcal{W}^{cu}_\varepsilon(z)$ is well defined for every $z \in \mathcal{W}^s_\delta(x)$. Notice that for any $y \in \mathcal{W}^u_\delta(x)$, for any $z \in \mathcal{W}^s_\delta(x)$, if we set $y' \stackrel{\text{def.}}{=} H^s_{x,z}(y) \in \mathcal{W}^s_\varepsilon(y)$, then we have

$$\angle\big(DH^s_{x,z} E^u(y), E^u(y')\big) = \alpha^s(y, y') = 0.$$

In particular, $H^s_{x,z}(\mathcal{W}^u_\delta(x))$ is a C^1 curve that is everywhere tangent to E^u and at y' it is tangent to $E^u(y')$. By the unique integrability of the E^u bundle, we conclude that

$$H^s_{x,z}(\mathcal{W}^u_{\mathrm{loc}}(x)) \subset \mathcal{W}^u(y').$$

Hence, we obtain that for any $x \in \mathbb{T}^3$, any $z \in \mathcal{W}^s_\delta(x)$ and $y \in \mathcal{W}^u_\delta(x)$ we have

$$\mathcal{W}^u_\varepsilon(z) \cap \mathcal{W}^s_\varepsilon(y) \neq \emptyset.$$

This proves that f fulfils the assumption of Lemma 5.7. Therefore, f is jointly integrable. $\qquad\square$

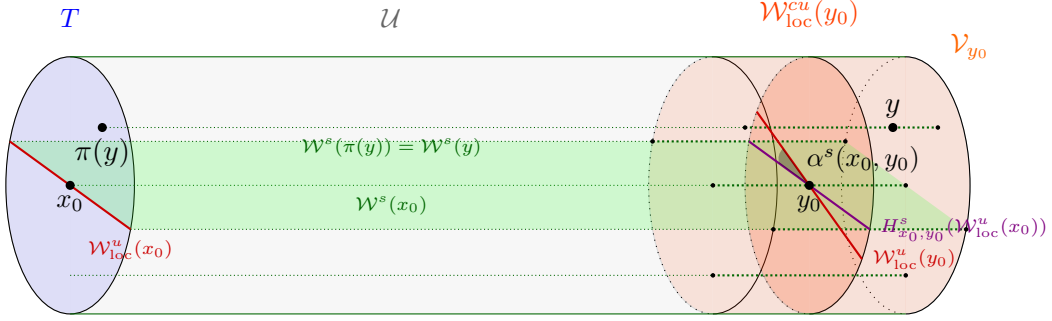The lemma below concludes the proof of Proposition 5.5.

FIGURE 6. Case where $E^s \oplus E^u$ is not integrable and $\nu(\mathbf{B}_\nu) = 1$.

Lemma 5.9. *Let $f \colon \mathbb{T}^3 \to \mathbb{T}^3$ be a C^2 Anosov diffeomorphism, strongly partially hyperbolic with expanding center and C^1 stable holonomies. Assume that E^s and E^u are not jointly integrable and let ν be an ergodic f-invariant Borel probability measure on $\mathbb{T}^3$. If $\mathrm{supp}(\mu)$ has non empty interior, then the* Bad set *associated to ν has zero measure.*

Proof. Fix ν as in the statement of the lemma, and let $x_0 \in \mathbb{T}^3$. Assume the first possibility does not occur. Then, the *Bad set* has full measure. As we assume that E^s and E^u are not jointly integrable, by Lemma 5.8, the set $\mathcal{N}(x_0)$ is open and dense in $\mathcal{W}^s(x_0)$. Fix $y_0 \in \mathcal{N}(x_0)$, i.e., such that $\alpha^s(x_0, y_0) \neq 0$. By the continuity of the angle function α^s, there exist:

- a foliated chart $\mathcal{U}$ for $\mathcal{W}^s$ containing x_0, y_0; stable plaques of $\mathcal{U}$ are denoted by $\{\mathcal{U}(x)\}_x$;
- a transversal T for $\mathcal{W}^s$ at x_0;
- a neighbourhood $\mathcal{V}_{y_0} \subset \mathcal{U}$ of y_0;
- a projection $\pi \colon \mathcal{V}_{y_0} \ni y \to \pi(y) \in T$ along stable plaques of $\mathcal{U}$;

such that

$$\forall\, y \in \mathcal{V}_{y_0}, \quad \alpha^s(\pi(y), y) > 0.$$

The open set $\mathcal{U}$ is obtained as a sufficiently small neighbourhood of a path inside $\mathcal{W}^s$ from x_0 to y_0 (which must be trivially foliated by $\mathcal{W}^s$ by Reeb's stability theorem). See Figure 6.

We claim that $\nu(\mathcal{V}_{y_0}) = 0$. To see this, we consider a finite foliated atlas $\mathcal{A}$ for $\mathcal{W}^s$ such that

(1) $\overline{\mathcal{U}}$ is included inside a foliated chart $\mathcal{U}'$ of $\mathcal{A}$ with $\mu(\partial \mathcal{U}') = 0$;
(2) every other foliated chart of $\mathcal{A}$ is disjoint from $\mathcal{U}$, and its boundary has measure 0.

To obtain such an atlas, first observe that from our construction, $\mathcal{U}$ can be obtained so that its closure lies inside an open set $\mathcal{U}'$ trivially foliated by $\mathcal{W}^s$. Next, we cover the compact set $\mathbb{T}^3 \setminus \mathcal{U}'$ by finitely many open balls B_i that

trivialize $\mathcal{W}^s$, have $\mu(\partial B_i) = 0$ and are disjoint from $\mathcal{U}$. Let η be the measurable partition associated to $\mathcal{A}$ (see Remark 5.4). By construction, stable plaques $\mathcal{U}(x)$ are contained inside atoms of η and we have that for almost every $x \in T$,

$$\eta(x) \cap \mathcal{V}_{y_0} = \mathcal{U}(x) \cap \mathcal{V}_{y_0} \subset \mathcal{N}(x),$$

hence $\nu_x^\eta(\eta(x) \cap \mathcal{V}_{y_0}) \leq \nu_x^\eta(\mathcal{N}(x)) = 0$ for μ-a.e. x. By the definition of conditional measures, this implies that $\nu(\mathcal{V}_{y_0}) = 0$. In other words, any point $y \in \mathcal{N}(x_0)$ has an open neighbourhood $\mathcal{V}_y \ni y$ such that $\mathbb{T}^3 \setminus \mathcal{V}_y$ has full measure therefore,

$$\operatorname{supp}(\nu) \subset \mathbb{T}^3 \setminus \bigcup_{y \in \mathcal{N}(x_0)} \mathcal{V}_y.$$

As $\mathcal{N}(x_0)$ is open and dense in $\mathcal{W}^s(x_0)$, and $\mathcal{W}^s(x_0)$ is dense in $\mathbb{T}^3$, the set $\bigcup_{y \in \mathcal{N}(x_0)} \mathcal{V}_y$ is open and dense in $\mathbb{T}^3$. Thus, $\operatorname{supp}(\nu)$ has empty interior. $\square$

5.3. **Main technical theorem.** We shall now show that our main result, Theorem A, can be reduced to a more technical statement involving the *Bad set*.

Theorem 5.10. *Let $f \colon \mathbb{T}^3 \to \mathbb{T}^3$ be a C^2 Anosov diffeomorphism, strongly partially hyperbolic with expanding center and C^1 stable holonomies. Let μ be an ergodic u-Gibbs measure. If the* Bad set *of μ has zero measure then μ is an SRB measure.*

5.3.1. *Proof of Theorem A assuming Theorem 5.10.* Recall that by Lemma 2.6 for every $f_0 \in \mathcal{A}_m^2(\mathbb{T}^3)$ there exists a small neighbourhood $\mathcal{U}(f_0)$ in $\mathrm{Diff}^2(\mathbb{T}^3)$ so that every $f \in \mathcal{U}(f_0)$ is an Anosov diffeomorphism, strongly partially hyperbolic with expanding center and C^1 stable holonomies. Let $\mathcal{U} \stackrel{\text{def.}}{=} \cup_{f_0 \in \mathcal{A}_m^2(\mathbb{T}^3)} \mathcal{U}(f_0)$. Take $f \in \mathcal{U}$. Then f fulfils the assumptions of Theorem 5.10. Assume that E^s and E^u are not jointly integrable and let μ be a fully supported ergodic *u*-Gibbs measure. Then, Lemma 5.9 implies that the *Bad set* has zero measure. By Theorem 5.10 it follows that μ is SRB, concluding. $\square$

It is important to remark that in this main technical result we *do not assume* that our *u*-Gibbs measure is fully supported. The result says that the existence of positive angles $\alpha^s(x, y)$ for almost every $y \in \mathcal{W}^s(x)$ and almost every $x \in \mathbb{T}^3$ suffices to convert μ into an SRB measure.

6. Normal forms

In this section, our goal is to prove Theorem C. This section is independent of the rest of the paper and we want to emphasize that we do not require any regularity of holonomies. We begin by revisiting the normal forms along one dimensional expanding foliations, as developed by Kalinin-Katok [25]. Then, using strongly that E^u is C^1 along the center-unstable manifold, we extend this construction to two dimensions. In the final part of the section, we study the change of coordinates for points on the same leaf of $\mathcal{W}^{cu}$.

Let $f \colon \mathbb{T}^3 \to \mathbb{T}^3$ be a C^2 Anosov diffeomorphism with a decomposition $T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u$, where E^c is uniformly expanded.

Let us recall some notations that will be frequently used in this section.

- Given a point $x \in \mathbb{T}^3$ we will write $\lambda_x^* \stackrel{\text{def.}}{=} \|Df(x)|_{E^*}\|$, for $* = c, u$. Observe that $\|Df^{-1}(x)|_{E^*}\| = (\lambda_{x_{-1}}^*)^{-1}$.
- Given $\ell \in \mathbb{Z}$ we write $x_\ell = f^\ell(x)$.
- Given $n \in \mathbb{Z}$, and $* = c, u$, we write

$$\lambda_x^*(n) \stackrel{\text{def.}}{=} \|Df^n(x)|_{E^*}\| = \begin{cases} \prod_{\ell=0}^{n-1} \lambda_{x_\ell}^*, & \text{when } n \geq 0; \\ \prod_{\ell=1}^{-n} \frac{1}{\lambda_{x_{-\ell}}^*} = \frac{1}{\lambda_{x_n}^*(-n)}, & \text{when } n < 0. \end{cases} \tag{20}$$

6.1. **One dimensional normal forms.** Given $x \in \mathbb{T}^3$ and $y \in \mathcal{W}^{cu}(x)$, we consider the functions

$$\rho_x^*(y) \stackrel{\text{def.}}{=} \prod_{\ell=1}^{+\infty} \frac{\lambda_{x-\ell}^*}{\lambda_{y-\ell}^*} = \lim_{n \to +\infty} \frac{\lambda_y^*(-n)}{\lambda_x^*(-n)}, \quad \text{for } * = c, u. \tag{21}$$

It follows readily from the definition that $\rho_y^*(x) = (\rho_x^*(y))^{-1}$. Furthermore, for any $z \in \mathcal{W}^{cu}(x) = \mathcal{W}^{cu}(y)$, we have

$$\rho_x^*(z) = \rho_y^*(z)\rho_x^*(y). \tag{22}$$

The basic distortion result (Lemma 2.9) implies that $\rho_x^* \colon \mathcal{W}^{cu}(x) \to (0, +\infty)$ is a continuous map that depends continuously on the base point (for further details we refer to [25]).

We then define

$$\mathcal{H}_x^*(y) \stackrel{\text{def.}}{=} \int_x^y \rho_x^*(\hat{y}) \, d\hat{y}, \tag{23}$$

which gives a C^1 diffeomorphism from $\mathcal{W}^*(x)$ to the real line. Now, consider the inverse map $\Phi_x^* = (\mathcal{H}_x^*)^{-1}$. Here's the proposition.

Proposition 6.1 (Kalinin-Katok – see [25], Section 3.1)**.** *For $* = c, u$, The family $\{\Phi_x^*\}_{x \in \mathbb{T}^3}$ of C^1 diffeomorphisms $\Phi_x^* \colon \mathbb{R} \to \mathcal{W}^*(x)$ satisfies*

   (1) $f \circ \Phi_x^*(s) = \Phi_{x_1}^*(\lambda_x^* s)$, for all $s \in \mathbb{R}$;
   (2) $\Phi_x^*(0) = x$;
   (3) $D\Phi_x^*(0)e_1 = v^*(x)$, where e_1 denotes the unitary tangent vector field of the real line $\mathbb{R}$.

For any $x \in \mathbb{T}^3$ and $y \in \mathcal{W}^*(x)$, we define the *change of charts* as

$$\mathcal{H}_{x,y}^* \stackrel{\text{def.}}{=} \mathcal{H}_y^* \circ (\mathcal{H}_x^*)^{-1} = (\Phi_y^*)^{-1} \circ \Phi_x^* \colon \mathbb{R} \to \mathbb{R}. \tag{24}$$

We remark that $\mathcal{H}_{x,y}^*(0) = \mathcal{H}_y^*(x)$.

Lemma 6.2. *For any $x \in \mathbb{T}^3$ and $y \in \mathcal{W}^*(x)$, and for any $s \in \mathbb{R}$, it holds*

$$\mathcal{H}_{x,y}^*(s) - \mathcal{H}_{x,y}^*(0) = \rho_y^*(x) \cdot s. \tag{25}$$

In other words, the change of one-dimensional normal form coordinates is affine with derivative $\rho_y^*(x)$.

Proof. Fix $s \in \mathbb{R}$, and let $z \stackrel{\text{def.}}{=} (\mathcal{H}_x^*)^{-1}(s) = (\mathcal{H}_y^*)^{-1}(\mathcal{H}_{x,y}^*(s)) \in \mathcal{W}^*(x) = \mathcal{W}^*(y)$. By the definition of $\mathcal{H}_x^*(x)$ (see (23)), after differentiating $\mathcal{H}_{x,y}^* = \mathcal{H}_y^* \circ (\mathcal{H}_x^*)^{-1}$ at s, we get

$$\frac{d\mathcal{H}_{x,y}^*}{ds}(s) = \frac{(\mathcal{H}_y^*)'(z)}{(\mathcal{H}_x^*)'(z)} = \frac{\rho_y^*(z)}{\rho_x^*(z)} = \rho_y^*(x),$$

thus $\mathcal{H}_{x,y}^*(s) - \mathcal{H}_{x,y}^*(0) = \int_0^s \rho_y^*(x) \, dt = \rho_y^*(x) \cdot s$. □

6.2. **Two dimensional normal forms.** Recall that there are unitary vector fields $\mathbb{T}^3 \ni x \mapsto v^*(x) \in E^*(x)$ such that

$$Df(x)v^*(x) = \lambda_x^* v^*(x_1).$$

Notice that since f is C^2, the bundle E^u is C^1 inside $\mathcal{W}^{cu}$ and since $\|v^u(x)\| > 0$ the map $x \mapsto \lambda_x^u \in \mathbb{R}$ is C^1 in restriction to center unstable manifolds. Moreover, for a fixed $R > 0$ the C^1 norm

$$\sup\{\|D(\lambda^u|_{\mathcal{W}^{cu}(x)})(y)\| : y \in \mathcal{W}_R^{cu}(x), \ x \in M\}$$

is bounded by some uniform constant $C = C(R, f) > 0$. These results follow from [39].

We restate here Theorem C, which gives a center-unstable version of Proposition 6.1.

Theorem 6.3. *There exists a family of C^1 diffeomorphisms $\Phi_x \colon \mathbb{R}^2 \to \mathcal{W}^{cu}(x)$, depending continuously on x such that:*

    (1) $f \circ \Phi_x(t,s) = \Phi_{f(x)}(\lambda_x^u t, \lambda_x^c s)$, for all $(s,t) \in \mathbb{R}^2$;

    (2) $\Phi_x(0,0) = x$;

    (3) $D\Phi_x(0,0)e_1 = v^u(x)$ and $D\Phi_x(0,0)e_2 = v^c(x)$;

    (4) $\Phi_x(\cdot,\cdot)$ depends continuously with the choice of x in the C^1-topology;

    (5) for all $s \in \mathbb{R}$, $\Phi_x(\mathbb{R} \times \{s\}) = \mathcal{W}^u(\Phi_x(0,s))$, and $\Phi_x(\{0\} \times \mathbb{R}) = \mathcal{W}^c(x)$ (see Figure 2).

The drawback of our construction is that the natural generalization of Lemma 6.2 is no longer true, as the remark below demonstrates.

Remark 6.4. *Note that in general, we cannot expect the change of normal charts $\mathcal{H}_{x,y} \stackrel{\text{def.}}{=} \Phi_y^{-1} \circ \Phi_x \colon \mathbb{R}^2 \to \mathbb{R}^2$ to be affine for any $x \in \mathbb{T}^3$ and $y \in \mathcal{W}^{cu}(x)$.*

Suppose it were the case. We claim that it implies that for any $x \in \mathbb{T}^3$, the center foliation $\mathcal{W}^c$ is C^1 within the center-unstable leaf $\mathcal{W}^{cu}(x)$. Indeed, for any $y \in \mathcal{W}^{cu}(x)$, by Theorem 6.3(5), we have

$$\Phi_x^{-1}(\mathcal{W}^c(y)) = \mathcal{H}_{y,x} \circ \Phi_y^{-1}(\mathcal{W}^c(y)) = \mathcal{H}_{y,x}(\{0\} \times \mathbb{R}),$$

which is a straight line since the map $\mathcal{H}_{y,x}$ is affine. Since two distinct center leaves cannot cross, and $\Phi_x^{-1}(\mathcal{W}^c(x)) = \{0\} \times \mathbb{R}$ is vertical, it follows that $\Phi_x^{-1}(\mathcal{W}^c(y))$ is vertical for any $y \in \mathcal{W}^{cu}(x)$. In particular, since the normal chart Φ_x is C^1, it does imply that $\mathcal{W}^c$ is C^1 within $\mathcal{W}^{cu}(x)$.

However, regularity of the center foliation $\mathcal{W}^c$ is a rare phenomenon. For instance, in [20, Lemma 3], it is shown that for $r \geq 2$, there exsits a C^1-open and C^r-dense set $\mathcal{V} \subset \mathcal{A}_m^r(\mathbb{T}^3)$ of the set of C^r conservative Anosov diffeomorphisms on $\mathbb{T}^3$ considered in the present work such that for $f \in \mathcal{A}_m^r(\mathbb{T}^3)$, the center foliation $\mathcal{W}^c$ is Lipschitz inside the center-unstable foliation $\mathcal{W}^{cu}$ if and only if $f \in \mathcal{A}_m^r(\mathbb{T}^3) \setminus \mathcal{V}$.

We now proceed to the proof of Theorem 6.3.

6.2.1. *A foliated chart.* A natural way of obtaining a parametrization (t,s) of $\mathcal{W}^{cu}(x)$ is to consider the arc-length parameter t along the curve $\mathcal{W}^u(\Phi_x^c(s))$. As we shall see, it is not hard to prove that this indeed gives a C^1 identification with nice properties (for instance horizontal lines are mapped onto unstable manifolds). The difficulty for proving Theorem 6.3 is that in these coordinates the map f *does not acts linearly*. For this reason in the forthcoming paragraphs we will perform suitable reparametrizations.

More formally, let $\phi \colon \mathbb{R} \times \mathcal{W}^{cu}(x) \to \mathcal{W}^{cu}(x)$ denote the flow of the vector field $x \mapsto v^u(x)$. Notice that $v^u|_{\mathcal{W}^{cu}(x)}$ is C^1, with its C^1 norm depending continuously on x on compact subsets of $\mathcal{W}^{cu}(x)$. With this flow at hand we define a map $\Gamma_x \colon \mathbb{R}^2 \to \mathcal{W}^{cu}(x)$ by

$$\Gamma_x(t,s) \stackrel{\text{def.}}{=} \phi^t(\Phi_x^c(s)), \quad \forall (t,s) \in \mathbb{R}^2,$$

where $\{\Phi_x^c\}_{x \in \mathbb{T}^3}$ are Kalinin-Katok's normal forms in $\mathcal{W}^c$ (see Theorem 6.3).

Lemma 6.5. *For each $x \in \mathbb{T}^3$, the map Γ_x is a C^1 diffeomorphism whose C^1 norm depends continuously with x.*

Proof. Let us show that this defines a C^1 diffeomorphism. Indeed, Γ_x is injective and surjective due to Lemmas 2.3 and 2.4 respectively. Therefore it suffices to prove that Γ_x is C^1 with invertible derivative at each point.

Choosing an arbitrary coordinate system locally in $\mathcal{W}^{cu}(x)$ we can think of Γ_x as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. Now, observe that the Jacobian matrix of the map $(t, s) \mapsto \phi^t(\Phi_x^c(s))$ is the 2×2 matrix whose columns are the vectors $v^u(\phi^t(\Phi_x^c(s)))$ and $D\phi^t(\Phi_x^c(s))\frac{d}{ds}\Phi_x^c(s)$. Since ϕ^t is a C^1 flow and Φ_x^c is a C^1 curve this proves that Γ_x has continuous derivative. Moreover, since $v^u(\Phi_x^c(s))$ and $\frac{d}{ds}\Phi_x^c(s)$ are transverse and since $D\phi^t(\Phi_x^c(s))$ is a linear isomorphism, we have that the vectors

$$v^u(\phi^t(\Phi_x^c(s))) = D\phi^t(\Phi_x^c(s))v^u(\Phi_x^c(s)) \quad \text{and} \quad D\phi^t(\Phi_x^c(s))\frac{d}{ds}\Phi_x^c(s)$$

are also transverse. Thus the Jacobian matrix of Γ_x is invertible, proving the assertion. The continuous dependence of the C^1 norm follows from that of v^u and Φ^c. $\qquad\square$

6.2.2. *Preliminary construction.* We shall start the construction of the map Φ with a slight modification of the construction of the foliated chart Γ_x. Instead of using the arc-length parameter along unstable manifolds we shall use normal forms. This choice will allow us to perform the required reparametrization for linearizing the action of $f|_{\mathcal{W}^{cu}(x)}$.

Lemma 6.6. *The map $\Psi_x \colon \mathbb{R}^2 \to \mathcal{W}^{cu}(x)$ given by $\Psi_x(t, s) \overset{\text{def.}}{=} \Phi_{\Phi_x^c(s)}^u(t)$ is a C^1 diffeomorphism, with C^1 norm depending continuously from x.*

Proof. Notice that Ψ_x is bijective because we can define directly an inverse map. Indeed, given $y \in \mathcal{W}^{cu}(x)$ we consider the point $y^c \in \mathcal{W}^c(x)$, whose existence is ensured by Lemmas 2.3 and 2.4, such that

$$\{y^c\} = \mathcal{W}^u(y) \cap \mathcal{W}^c(x).$$

Define then a map $\Xi_x \colon \mathcal{W}^{cu}(x) \to \mathbb{R}^2$ by

$$\Xi_x(y) \overset{\text{def.}}{=} (\mathcal{H}_{y^c}^u(y), \mathcal{H}_x^c(y^c)).$$

From the definitions we have

$$\Psi_x \circ \Xi_x(y) = \Phi_{\Phi_x^c(\mathcal{H}_x^c(y^c))}^u(\mathcal{H}_{y^c}^u(y)) = \Phi_{y^c}^u(\mathcal{H}_{y^c}^u(y)) = y.$$

In a similar way one sees that $\Xi_x \circ \Psi_x(t, s) = (t, s)$. It suffices then to check that Ξ_x is C^1 with invertible derivative at each point.

To prove that Ξ_x is a C^1 diffeomorphism then it suffices to establish that $\Xi_x \circ \Gamma_x \colon \mathbb{R}^2 \to \mathbb{R}^2$ is a C^1 diffeomorphism. To compute the derivative of this map fix a point $y = \Gamma_x(t, s) \in \mathcal{W}^{cu}(x)$. Then, since Γ_x is a foliated chart, we have $y^c = \Gamma_x(0, s)$. Thus,

$$\Xi_x \circ \Gamma_x(t, s) = (\mathcal{H}_{\Gamma_x(0,s)}^u(\Gamma_x(t, s)), \mathcal{H}_x^c(\Gamma_x(0, s))).$$

Note that $(t, s) \mapsto \mathcal{H}_x^c(\Gamma_x(0, s)) = s$ is a C^1 function. Moreover, as

$$\mathcal{H}_{\Gamma_x(0,s)}^u(\Gamma_x(t, s)) = \int_{\Gamma_x(0,s)}^{\Gamma_x(t,s)} \rho_{\Gamma_x(0,s)}(\Gamma_x(r, s))dr,$$

Leibniz rule will imply that $\Xi_x \circ \Gamma_x$ is a C^1 map as long as we prove that $(t, s) \mapsto \rho_{\Gamma_x(0,s)}(\Gamma_x(t, s)) \in (0, +\infty)$ is C^1. To verify this assertion, recall that

$$\rho_{\Gamma_x(0,s)}^u(\Gamma_x(t, s)) = \prod_{\ell=0}^{+\infty} \frac{\|Df^{-1}(f^{-\ell}(\Gamma_x(t, s)))|_{E^u}\|}{\|Df^{-1}(f^{-\ell}(\Gamma_x(0, s)))|_{E^u}\|}.$$

Consider the auxiliary function $g(t, s) = \log \rho_{\Gamma_x(0,s)}^u(\Gamma_x(t, s))$. Then we can write

$$g(t, s) = \sum_{\ell=0}^{+\infty} \left(\log \lambda_{f^{-\ell-1}(\Gamma_x(0,s))}^u - \log \lambda_{f^{-\ell-1}(\Gamma_x(t,s))}^u \right).$$

First notice that as $\lambda^u_{(.)} \colon \mathcal{W}^{cu} \to \mathbb{R}$ is C^1 and since $d(f^{-\ell}(\Gamma_x(0,s)), f^{-\ell}(\Gamma_x(t,s))) \to 0$ exponentially fast (with uniform rate, depending only on f), we deduce that g is the uniform limit on compact sets of the sequence of partial sums. Consider the function $g_\ell(t,s) = \log \lambda^u_{f^{-\ell-1}(\Gamma_x(t,s))}$. By the chain rule,

$$Dg_\ell(t,s) = \frac{D\lambda^u_{f^{-\ell-1}(\Gamma_x(t,s))} Df^{-\ell-1}(\Gamma_x(t,s)) D\Gamma_x(t,s)}{\lambda^u_{f^{-\ell-1}(\Gamma_x(t,s))}}.$$

Therefore, for $(t,s) \in B_R(0)$ there is a uniform constant $C = C(R,f) > 0$ so that

$$\|Dg_\ell(t,s)\| \leq C\|Df^{-\ell-1}(\Gamma_x(t,s))|_{E^{cu}}\|.$$

The right hand side above has a uniform bound decreasing exponentially as ℓ increases for $\|Df^{-\ell}(x)|_{E^{cu}}\| \leq e^{-\chi^c_2 \ell}$, for every $x \in \mathbb{T}^3$, with $\chi^c_2 > 0$ (recall our notations from §2.1.3). Thus,

$$\|Dg_\ell(t,s)\| \leq Ce^{-\chi^c_2 \ell}.$$

An analogous estimate holds for the function $\tilde{g}_\ell(t,s) = \log \lambda^u_{\Gamma_x(0,s)}$. This proves that the derivative of the truncated series defining g also converges uniformly on compact sets. By elementary calculus (see for instance Proposition 1.41 of [1]) this proves that g is C^1, and therefore we conclude that $(t,s) \mapsto \rho^u_{\Gamma_x(0,s)}(\Gamma_x(t,s)) \in (0, +\infty)$ is C^1, as desired. To complete the proof of the lemma, observe that the derivative of $\Xi_x \circ \Gamma_x$ is an upper triangular matrix with non-zero entries in the diagonal. By the inverse function theorem, as $\Xi_x \circ \Gamma_x$ is bijective, this proves that this map is in fact a C^1 diffeomorphism. This ends the proof of the lemma. $\qquad\square$

6.2.3. *A reparametrization function.* Here again, the diffeomorphism Ψ_x constructed in Lemma 6.6 does not satisfy all the requirements we need. Indeed, to obtain condition (1) from Theorem 6.3 it is necessary to perform a suitable reparametrization. The design of such a map is the content of next lemma.

Lemma 6.7. *For each $x \in \mathbb{T}^3$, there exists a C^1 function $\beta_x \colon \mathbb{R} \to \mathbb{R}^+$ such that*

$$Df(\Phi^c_x(s)) \cdot \beta_x(s)v^u(\Phi^c_x(s)) = \lambda^u_x \beta_{x_1}(\lambda^c_x s)v^u(\Phi^c_{x_1}(\lambda^c_x s)). \qquad (26)$$

Proof. Let us fix a point $x \in \mathbb{T}^3$. Consider the sequence of functions $\mathbb{R} \ni s \mapsto \psi^x_{-\ell}(s) \in \mathcal{W}^c(x_{-\ell})$ where

$$\psi^x_{-\ell}(s) \stackrel{\text{def.}}{=} \Phi^c_{x_{-\ell}}\left((\lambda^c_{x_{-\ell}} \cdots \lambda^c_{x_{-1}})^{-1}s\right).$$

For coherence of notation also make the convention that $\psi^x_0(s) = \Phi^c_x(s)$.

Remark 6.8. *Note that by property of Kalinin-Katok's normal forms, we have that if $y = \psi^x_0(s) = \Phi^c_x(s)$ for some $s \in \mathbb{R}$ and $\ell \in \mathbb{N}$, then*

$$f^{-\ell}(y) = \psi^x_{-\ell}(s).$$

Using this sequence we define $h_{n,x} \colon \mathbb{R} \to \mathbb{R}$,

$$h_{n,x}(s) \stackrel{\text{def.}}{=} \sum_{\ell=1}^n \left(\log \lambda^u_{\psi^x_{-\ell}(s)} - \log \lambda^u_{x_{-\ell}}\right).$$

Notice that $y \mapsto \lambda^u_y$ is uniformly bounded from above and from below, and the lower bound is larger than 1. Thus

$$|\log \lambda^u_{\psi^x_{-\ell}(s)} - \log \lambda^u_{x_{-\ell}}| \leq |\lambda^u_{\psi^x_{-\ell}(s)} - \lambda^u_{x_{-\ell}}|.$$

Now, as $y \mapsto \lambda_y^c$ is also uniformly bounded with a lower bound larger than 1, since the family $\{\Phi_x^c\}$ is continuous in the C^1 topology and $\Phi_{x_{-\ell}}^c(0) = x_{-\ell}$ it follows that $d(\psi_{-\ell}^x(s), x_{-\ell}) \to 0$ exponentially fast with uniform constants (keep in mind Remark 6.8). Moreover, as $y \mapsto \lambda_y^u$ is uniformly C^1 inside $\mathcal{W}^{cu}$ there exists some constant $C > 0$ such that

$$|\lambda_{\psi_{-\ell}^x(s)}^u - \lambda_{x_{-\ell}}^u| \leq Cd(\psi_{-\ell}^x(s), x_{-\ell}).$$

This proves that $h_{n,x}$ converges uniformly to a continuous function h_x. We define $\beta_x(s) \overset{\text{def.}}{=} e^{h_x(s)}$. Observe that with Notation (21), we can write

$$\beta_x(s) = \prod_{\ell=1}^{+\infty} \frac{\lambda_{\psi_{-\ell}^x(s)}^u}{\lambda_{x_{-\ell}}^u} = \rho_{\Phi_x^c(s)}^u(x) = \frac{1}{\rho_x^u(\Phi_x^c(s))}. \tag{27}$$

We claim that $\lambda_{\Phi_x^c(s)}^u \beta_x(s) = \lambda_x^u \beta_{x_1}(\lambda_x^c s)$. This formula immediately gives us (26). Indeed note that by property of Kalinin-Katok's normal forms

$$Df(\Phi_x^c(s)) \cdot v^u(\Phi_x^c(s)) = \lambda_{\Phi_x^c(s)}^u v^u(f(\Phi_x^c(s)))$$
$$= \lambda_{\Phi_x^c(s)}^u v^u(\Phi_{x_1}^c(\lambda_x^c s)),$$

and combining this equality with the claim provides (26).

To prove the claim we first remark that

$$\psi_{-\ell}^{x_1}(\lambda_x^c s) = \Phi_{x_{-\ell+1}}^c\left((\lambda_{x_{-\ell+1}}^c \cdots \lambda_{x_{-1}}^c \lambda_x^c)^{-1} \lambda_x^c s\right) = \psi_{-\ell+1}^x(s).$$

We deduce that

$$\beta_{x_1}(\lambda_x^c s) = \prod_{\ell=1}^{+\infty} \frac{\lambda_{\psi_{-\ell}^{x_1}(\lambda_x^c s)}^u}{\lambda_{x_{-\ell+1}}^u} = \prod_{\ell=1}^{\infty} \frac{\lambda_{\psi_{-\ell+1}^x(s)}^u}{\lambda_{x_{-\ell+1}}^u} = \frac{\lambda_{\Phi_x^c(s)}^u}{\lambda_x^u} \beta_x(s),$$

proving our claim.

To finish the lemma it remains to show that β_x is a C^1 function, depending continuously (with respect to the local C^1 topology) on x. With that goal in mind we compute

$$\frac{d}{ds} \psi_{-\ell}^x(s) = (\lambda_{x_{-\ell}}^c \cdots \lambda_{x_{-1}}^c)^{-1} D\Phi_{x_{-\ell}}^c\left((\lambda_{x_{-\ell}}^c \cdots \lambda_{x_{-1}}^c)^{-1} s\right) e_1.$$

Applying thus the chain rule we obtain

$$\frac{d}{ds} \lambda_{\psi_{-\ell}^x(s)}^u = D(\lambda_{\psi_{-\ell}^x(s)}^u) \frac{d}{ds} \psi_{-\ell}^x(s)$$
$$= (\lambda_{x_{-\ell}}^c \cdots \lambda_{x_{-1}}^c)^{-1} D(\lambda_{\psi_{-\ell}^x(s)}^u) D\Phi_{x_{-\ell}}^c\left((\lambda_{x_{-\ell}}^c \cdots \lambda_{x_{-1}}^c)^{-1} s\right) e_1,$$

where $D(\lambda_y^u)$ denotes the derivative of the function $y \mapsto \lambda_y^u$ at y. As $(\lambda_{x_{-\ell}}^c \cdots \lambda_{x_{-1}}^c)^{-1} \to 0$ exponentially fast with uniform constants, the left hand side above also vanishes exponentially fast because λ_y^u is C^1 and $x \mapsto \max_{|s| \leq 1} \|D\Phi_x^c(s)\|$ is uniformly bounded. As a consequence the sequence

$$h_{n,x}'(s) = \sum_{\ell=1}^{n} \frac{\frac{d}{ds} \lambda_{\psi_{-\ell}^x(s)}^u}{\lambda_{\psi_{-\ell}^x(s)}^u}$$

is uniformly bounded by a convergent geometric series, because λ_y^u is lower bounded by a constant larger than 1. Since $h_{n,x} \to h_x$ uniformly on compact sets, by elementary calculus (see for instance [1] Proposition 1.41) this proves that h_x is C^1 with $h_x' = \lim_{n \to +\infty} h_{n,x}'$. Moreover, the uniform estimates obtained show that the C^1 norm of f changes continuously with x. This completes the proof of the lemma. $\square$

6.2.4. *Defining the normal form.* With Lemmas 6.6 and 6.7 at hand we are in position to define our two dimensional parametrization of center-unstable manifolds $\mathcal{W}^{cu}(x)$. For each $x \in \mathbb{T}^3$ we consider

$$\Phi_x \colon \left\{ \begin{array}{ccc} \mathbb{R}^2 & \to & \mathcal{W}^{cu}(x), \\ (t,s) & \mapsto & \Psi_x(\beta_x(s)t, s) = \Phi^u_{\Phi^c_x(s)}(\beta_x(s)t). \end{array} \right.$$

Notice that, by Lemma 6.7, $s \mapsto \beta_x(s)$ is a C^1 positive function. It follows that $(t,s) \mapsto (\beta_x(s)t, s)$ is a C^1 diffeomorphism of $\mathbb{R}^2$. Therefore, applying Lemma 6.6 we obtain that Φ_x is a C^1 diffeomorphism. We denote $\mathcal{H}_x \overset{\text{def.}}{=} \Phi_x^{-1}$.

Proof of Theorem 6.3. Properties (2)-(4) are automatic from the construction of Φ_x. To prove (1) we apply Proposition 6.1 and obtain

$$f(\Phi_x(t,s)) = f\left(\Phi^u_{\Phi^c_x(s)}(\beta_x(s)t)\right) = \Phi^u_{f \circ \Phi^c_x(s)}\left(\lambda^u_{\Phi^c_x(s)}\beta_x(s)t\right).$$

Now, equality (26) implies that

$$\beta_x(s)\lambda^u_{\Phi^c_x(s)}v^u(f \circ \Phi^c_x(x)) = \lambda^u_x\beta_{x_1}(\lambda^c_x s)v^u\left(\Phi^c_{x_1}(\lambda^c_x s)\right).$$

Proposition 6.1 applied with Φ^c_x shows that $f \circ \Phi^c_x(s) = \Phi^c_{x_1}(\lambda^c_x s)$, where $x_1 = f(x)$. Combining we get $\beta_x(s)\lambda^u_{\Phi^c_x(s)} = \lambda^u_x\beta_{x_1}(\lambda^c_x s)$. One deduces then

$$f(\Phi_x(t,s)) = \Phi^u_{\Phi^c_{x_1}(\lambda^c_x s)}\left(\lambda^u_x\beta_{x_1}(\lambda^c_x s)t\right).$$

The very definition of Φ_x says that

$$\Phi^u_{\Phi^c_{x_1}(\lambda^c_x s)}\left(\lambda^u_x\beta_{x_1}(\lambda^c_x s)t\right) = \Phi_{x_1}(\lambda^u_x t, \lambda^c_x s). \qquad \square$$

Remark 6.9. *It follows readily from our construction that the inverse map $\mathcal{H}_x \colon \mathcal{W}^{cu}(x) \to \mathbb{R}^2$ sends each unstable manifold $\mathcal{W}^u(y)$, for $y \in \mathcal{W}^c(x)$ onto the horizontal line $\{s = \mathcal{H}^c_x(y)\}$.*

6.3. **Change of normal form coordinates.** As in (24) and Remark 6.4, for any $x \in \mathbb{T}^3$ and $y \in \mathcal{W}^{cu}(x)$, we define the change of charts

$$\mathcal{H}_{x,y} \overset{\text{def.}}{=} \mathcal{H}_y \circ \mathcal{H}_x^{-1} = \Phi_y^{-1} \circ \Phi_x \colon \left\{ \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2, \\ (t,s) & \mapsto & (h^1_{x,y}(t,s), h^2_{x,y}(t,s)). \end{array} \right. \qquad (28)$$

The function $h^2_{x,y}$ only depends on the second coordinate, as demonstrated by the following lemma.

Lemma 6.10. *For any $x \in \mathbb{T}^3$ and $y \in \mathcal{W}^{cu}(x)$, the diffeomorphism $\mathcal{H}_{x,y}$ preserves the foliation of $\mathbb{R}^2$ into horizontal lines $\{s = \text{cst}\}$.*

Proof. This is a straightforward consequence of the fact that normal forms sends horizontal lines to unstable manifolds (see Remark 6.9). More precisely, given $s \in \mathbb{R}$, using item (5) of Theorem 6.3 we have

$$\mathcal{W}^u(\Phi^c_x(s)) = \Phi_x(\mathbb{R} \times \{s\}).$$

Now define $\hat{y} = \mathcal{W}^c(y) \cap \mathcal{W}^u(\Phi^c_x(s))$ (recall Lemma 2.4). Applying item (5) of Theorem 6.3 once more we obtain

$$\mathcal{H}_y(\mathcal{W}^u(\hat{y})) = \mathbb{R} \times \{\mathcal{H}^c_y(\hat{y})\}$$

Since $\mathcal{W}^u(\hat{y}) = \mathcal{W}^u(\Phi^c_x(s))$, this shows that

$$\mathcal{H}_{x,y}(\mathbb{R} \times \{s\}) = \mathbb{R} \times \{\mathcal{H}^c_y(\hat{y})\},$$

which concludes the proof. $\qquad \square$

Lemma 6.11. *The diffeomorphism $\mathcal{H}_{x,y}(\cdot, \cdot)$ depends continuously in the C^1 topology with the pair (x,y), $x \in \mathbb{T}^3$ and $y \in \mathcal{W}^{cu}(x)$. Moreover, for any $\bar{x} \in \mathbb{T}^3$, and for $(\bar{t}, \bar{s}) \in \mathbb{R}^2$,*

(1) $\mathcal{H}_{x,y}(t,s) \to (\bar{t}, \bar{s})$ as $(x,y) \to (\bar{x}, \bar{x})$ and $(t,s) \to (\bar{t}, \bar{s})$;

(2) $D\mathcal{H}_{x,y}(t,s) \to \mathrm{Id}_{\mathbb{R}^2}$ as $(x,y) \to (\bar{x}, \bar{x})$ and $(t,s) \to (\bar{t}, \bar{s})$.

Proof. It follows immediately from Theorem 6.3(4), since $\mathcal{H}_{x,y} = \mathcal{H}_y \circ \mathcal{H}_x^{-1}$, and $D\mathcal{H}_{x,y}(t,s) = D\mathcal{H}_y(\mathcal{H}_x^{-1}(t,s)) \circ D\mathcal{H}_x^{-1}(t,s)$, with

$$D\mathcal{H}_x(t,s) \to D\mathcal{H}_{\bar{x}}(\bar{t}, \bar{s}),$$

as $x \to \bar{x}$ and $(t,s) \to (\bar{t}, \bar{s})$, and

$$D\mathcal{H}_y(\mathcal{H}_x^{-1}(t,s)) \to D\mathcal{H}_{\bar{x}}(\mathcal{H}_{\bar{x}}^{-1}(\bar{t}, \bar{s})) = (D\mathcal{H}_{\bar{x}}^{-1}(\bar{t}, \bar{s}))^{-1},$$

as $(x,y) \to (\bar{x}, \bar{x})$ and $(t,s) \to (\bar{t}, \bar{s})$.   □

6.3.1. *Change of normal charts for two points in the same center leaf.* We first study the change of normal charts between two points of the same center leaf.

Lemma 6.12. *Let $x \in \mathbb{T}^3$ and $y \in \mathcal{W}^c(x)$. For any $(t,s) \in \mathbb{R}^2$, it holds*

(1) $h_{x,y}^1(t,s) = \rho_y^u(x)t$;

(2) $h_{x,y}^2(s) = \mathcal{H}_{x,y}^c(s) = \rho_y^c(x)s + \mathcal{H}_y^c(x)$.

Proof. We start by proving (2). Consider the function $h_{x,y}^1$. As x and y belong to the same center leaf, for any $s \in \mathbb{R}$, it holds $h_{x,y}^1(0,s) = 0$, and

$$\Phi_x^c(s) = \Phi_x(0,s) = \Phi_y(h_{x,y}^1(0,s), h_{x,y}^2(s)) = \Phi_y(0, h_{x,y}^2(s)) = \Phi_y^c(h_{x,y}^2(s)),$$

hence by (25),

$$h_{x,y}^2(s) = (\Phi_y^c)^{-1} \circ \Phi_x^c(s) = \mathcal{H}_{x,y}^c(s).$$

The second claimed equality follows from Lemma 6.2.

Now, let us prove (1). Fix $(t,s) \in \mathbb{R}^2$, and denote $t' \overset{\text{def.}}{=} h_{x,y}^1(t,s)$ and $s' \overset{\text{def.}}{=} h_{x,y}^2(s)$ for simplicity. By definition, and as $\Phi_y^c(h_{x,y}^2(s)) = \Phi_x^c(s)$, we obtain

$$\Phi_{\Phi_x^c(s)}^u(\beta_x(s)t) = \Phi_x(t,s) = \Phi_y(t', s') = \Phi_{\Phi_y^c(s')}^u(\beta_y(s')t'),$$

hence

$$\beta_x(s)t = \beta_y(s')t'. \tag{29}$$

On the other hand, by (27) we have

$$\beta_y(s') = \rho_{\Phi_y^c(s')}^u(y) = \rho_{\Phi_x^c(s)}^u(y)$$
$$= \rho_{\Phi_x^c(s)}^u(x)\rho_x^u(y) = \beta_x(s)\rho_x^u(y), \tag{30}$$

where in the second line we applied (22). Combining this with (29) we deduce $t' = \rho_y^u(x)t$, concluding.   □

6.3.2. *Change of normal forms for two points in the same unstable leaf.* Let us now study the change of normal charts $\mathcal{H}_{x,x'} : (t,s) \mapsto (h_{x,x'}^1(t,s), h_{x,x'}^2(s))$ between two points x, x' of the same unstable leaf. Unlike the previous case, here we do not get an affine map.

Lemma 6.13. *Let $x \in \mathbb{T}^3$ and $x' \in \mathcal{W}^u(x)$. Then, there exists a C^1 function $a_{x,x'} : \mathbb{R} \to \mathbb{R}$ such that for any $(t,s) \in \mathbb{R}^2$, it holds*

$$h_{x,x'}^1(t,s) = \rho_{x'}^u(x)t + a_{x,x'}(s).$$

Proof. Consider $(t,s) \in \mathbb{R}^2$, and let us abbreviate $t' \overset{\text{def.}}{=} h_{x,x'}^1(t,s)$, $s' \overset{\text{def.}}{=} h_{x,x'}^2(s)$, $y = y(s) \overset{\text{def.}}{=} \Phi_x^c(s)$, $y' = y'(s) \overset{\text{def.}}{=} \Phi_{x'}^c(s')$, and $z = z(t,s) \overset{\text{def.}}{=} \Phi_x(t,s) = \Phi_{x'}(t', s')$. Recall that $z = \Phi_x(t,s) = \Phi_y^u(\beta_x(s)t)$. By (23), we thus obtain

$$\beta_x(s)t = \int_y^z \rho_y^u(\hat{y}) \, d\hat{y}. \tag{31}$$

Similarly, $z = \Phi_{x'}(t', s') = \Phi_{y'}^u(\beta_{x'}(s')t')$, and

$$\beta_x(s')t' = \int_{y'}^z \rho_{y'}^u(\hat{y}) \, d\hat{y}, \tag{32}$$

Observe that $y' \in \mathcal{W}^u(y)$. By (22), for any point $\hat{y} \in \mathcal{W}^u(y) = \mathcal{W}^u(y')$, we have $\rho_{y'}^u(\hat{y}) = \rho_y^u(\hat{y})\rho_{y'}^u(y)$. Hence,

$$\begin{aligned}
\beta_{x'}(s')t' &= \int_{y'}^z \rho_{y'}^u(\hat{y}) \, d\hat{y} = \int_{y'}^z \rho_{y'}^u(y)\rho_y^u(\hat{y}) \, d\hat{y} \\
&= \rho_{y'}^u(y) \left(\int_{y'}^z \rho_y^u(\hat{y}) \, d\hat{y} \right) = \rho_{y'}^u(y) \left(\int_{y'}^y \rho_y^u(\hat{y}) \, d\hat{y} + \int_y^z \rho_y^u(\hat{y}) \, d\hat{y} \right) \\
&= \rho_{y'}^u(y)\beta_x(s)t + \rho_{y'}^u(y) \int_{y'}^y \rho_y^u(\hat{y}) \, d\hat{y},
\end{aligned}$$

where in the last equality we used (31). We conclude that

$$t' = \rho_{y'}(y)\frac{\beta_x(s)}{\beta_{x'}(s')} \cdot t + \frac{\rho_{y'}^u(y)}{\beta_{x'}(s')} \int_{y'}^y \rho_y^u(\hat{y}) \, d\hat{y}.$$

Set $a_{x,x'}(s) \overset{\text{def.}}{=} \frac{\rho_{y'}^u(y)}{\beta_{x'}(s')} \int_{y'}^y \rho_y^u(\hat{y}) \, d\hat{y}$. This is the translation part of the change of coordinates. Recall that

$$\beta_x(s) = \rho_{\Phi_x^u(s)}^u(x) = \rho_y^u(x).$$

Similarly,

$$\beta_{x'}(s') = \rho_{\Phi_{x'}^u(s')}^u(x') = \rho_{y'}^u(x') = (\rho_{x'}^u(y'))^{-1}.$$

Therefore, by (22), it holds

$$\rho_{y'}^u(y)\frac{\beta_x(s)}{\beta_{x'}(s')} = \rho_y^u(x)\rho_{y'}^u(y)\rho_{x'}^u(y') = \rho_y^u(x)\rho_{x'}^u(y) = \rho_{x'}^u(x).$$

We conclude that $t' = \rho_{x'}^u(x) \cdot t + a_{x,x'}(s)$. Moreover,

$$\begin{aligned}
a_{x,x'}(s) &\overset{\text{def.}}{=} \frac{\rho_{y'}^u(y)}{\beta_{x'}(s')} \int_{y'}^y \rho_y^u(\hat{y}) \, d\hat{y} = \frac{\rho_{x'}^u(x)}{\beta_x(s)} \int_{y'}^y \rho_y^u(\hat{y}) \, d\hat{y} \\
&= \frac{\rho_{x'}^u(x)}{\rho_y^u(x)} \int_{y'}^y \rho_y^u(\hat{y}) \, d\hat{y} = \rho_{x'}^u(y) \int_{y'}^y \rho_y^u(\hat{y}) \, d\hat{y} = \int_{y'}^y \rho_y^u(\hat{y})\rho_{x'}^u(y) \, d\hat{y} \\
&= \int_{\mathcal{W}^u(y',y)} \rho_{x'}^u(\hat{y}) \, d\hat{y},
\end{aligned}$$

where in the last integral, $\hat{y}$ ranges over the segment $\mathcal{W}^u(y', y) \subset \mathcal{W}^u(y)$ of unstable manifold connecting the points y' and y. $\qquad\square$

Let us now study the map $h_{x,x'}^2 : \mathbb{R} \to \mathbb{R}$. The first step is to characterize it in terms of unstable holonomies.

Lemma 6.14. *Fix $x \in \mathbb{T}^3$ and $x' \in \mathcal{W}^u(x)$ and let $H_{x,x'}^u : \mathcal{W}^c(x) \to \mathcal{W}^c(x')$ denote the unstable holonomy map. Define the map $L_{x,x'}^u : \mathbb{R} \to \mathbb{R}$ by*

$$L_{x,x'}^u(s) \overset{\text{def.}}{=} \mathcal{H}_{x'}^c \circ H_{x,x'}^u \circ \Phi_x^c(s).$$

Then, we have $h_{x,x'}^2(s) = L_{x,x'}^u(s)$.

Proof. Consider an $s \in \mathbb{R}$ and define $y = \Phi_x^c(s)$ and $y' = H_{x,x'}^u(y)$. Set $s' \overset{\text{def.}}{=} \mathcal{H}_{x'}^c(y')$. Notice that, by definition, $s' = L_{x,x'}^u(s)$. By Remark 6.9 we know that

$\Phi_x(\mathbb{R} \times \{s\}) = \mathcal{W}^u(y)$, and similarly $\Phi_{x'}$ sends $\mathbb{R} \times \{s'\}$ onto $\mathcal{W}^u(y')$. Since $\mathcal{W}^u(y) = \mathcal{W}^u(y')$ this shows that

$$\mathcal{H}_{x'} \circ \Phi_x(\mathbb{R} \times \{s\}) = \mathbb{R} \times \{s'\},$$

and therefore $h^2_{x,x'}(s) = s'$. □

We now prove that unstable holonomies, when conjugated by normal forms (as in the previous lemma), become a linear maps of the real line.

Lemma 6.15. *Fix $x \in \mathbb{T}^3$ and $x' \in \mathcal{W}^u(x)$. Then, $L^u_{x,x'}(s) = \rho^c_{x'}(x)s$.*

Proof. For ease of notation in this proof, we will denote $L(s) = L^u_{x,x'}(s)$. By definition, we have that

$$\Phi^c_{x'} \circ L(s) = H^u_{x,x'} \circ \Phi^c_x(s).$$

Now, consider an integer $k \in \mathbb{Z}$ and define the functions $g_k \colon \mathbb{R} \to \mathbb{R}, s \mapsto \lambda^c_x(k) \cdot s$ (recall Notation (20)). Similarly let $g'_k \colon \mathbb{R} \to \mathbb{R}, s \mapsto \lambda^c_{x'}(k) \cdot s$. By applying f^k to both sides of the above equation and using the equivariance properties of normal forms and holonomy maps, we obtain

$$\Phi^c_{x'_k}(g'_k \circ L(s)) = H^u_{x_k,x'_k} \circ \Phi^c_{x_k}(g_k(s)).$$

Taking derivatives with respect to $s \in \mathbb{R}$ on both sides we get

$$D\Phi^c_{x'_k}(g'_k(s))\lambda^c_{x'}(k)\frac{dL}{ds}(s) = DH^u_{x_k,x'_k}(\Phi^c_{x_k}(g_k(s)))D\Phi^c_{x_k}(g_k(s))\lambda^c_x(k). \tag{33}$$

Recall that $\det D\Phi^c_{x_k}(g_k(s)) = \frac{1}{\rho^c_{x_k}(\Phi^c_{x_k}(g_k(s)))}$, and $\det D\Phi^c_{x'_k}(g'_k(s)) = \frac{1}{\rho^c_{x'_k}(\Phi^c_{x'_k}(g'_k(s)))}$. Passing to the determinant in (33), we deduce that

$$\frac{dL}{ds}(s) = \frac{\rho^c_{x'_k}(\Phi^c_{x'_k}(g'_k(s)))}{\rho^c_{x_k}(\Phi^c_{x_k}(s_k(s)))} \cdot \frac{\lambda^c_x(k)}{\lambda^c_{x'}(k)} \cdot \det DH^u_{x_k,x'_k}(\Phi^c_{x_k}(g_k(s))). \tag{34}$$

We now let $k \to -\infty$. By compactness, up to considering a subsequence, we can assume that $x_k, x'_k \to q \in \mathbb{T}^3$ as $k \to -\infty$. Moreover, $g_k(s), g'_k(s) \to 0$ as $k \to -\infty$, and by the properties of $\Phi^c_{(\cdot)}(\cdot)$ stated in Proposition 6.1, we have $\Phi^c_{x_k}(g_k(s)), \Phi^c_{x'_k}(g'_k(s)) \to \Phi^c_q(0) = q$ as $k \to -\infty$. By continuity of $\rho^c_{(\cdot)}(\cdot)$, we deduce that

$$\lim_{k \to -\infty} \frac{\rho^c_{x'_k}(\Phi^c_{x'_k}(g'_k(s)))}{\rho^c_{x_k}(\Phi^c_{x_k}(g_k(s)))} = \frac{\rho^c_q(q)}{\rho^c_q(q)} = 1.$$

Since the holonomy maps $H^u_{x_k,x'_k}$ converge uniformly to $H^u_{q,q} = \mathrm{Id}|_{\mathcal{W}^c(q)}$ in the C^1 topology as $k \to -\infty$, we also have $\lim_{k \to -\infty} \det DH^u_{x_k,x'_k}(\Phi^c_{x_k}(g_k(s))) = 1$.

By (34), (20) and (21), we conclude that

$$\frac{dL}{ds}(s) = \lim_{k \to -\infty} \frac{\lambda^c_x(k)}{\lambda^c_{x'}(k)} = \prod_{\ell=1}^{+\infty} \frac{\lambda^c_{x'_{-\ell}}}{\lambda^c_{x_{-\ell}}} = \rho^c_{x'}(x).$$

This completes the proof. □

7. LEAF-WISE QUOTIENT MEASURES

Let us denote by $\mathcal{M}(\mathbb{R})$ the set of locally finite Borel measures on the real line. Using the families of parametrizations $\{\Phi_x\}, \{\Phi^c_x\}_{x \in \mathbb{T}^3}$ we will construct a family of elements of $\mathcal{M}(\mathbb{R})$ which will be the main object of study in this paper. We summarize the result of the construction with the following statement.

Theorem 7.1. *Let $f \in \mathcal{A}^2(\mathbb{T}^3)$ be an Anosov diffeomorphism with expanding center. Let μ be an ergodic u-Gibbs measure of f. Then, there exists a family $\{\hat{\nu}_x^c\}_{x \in \mathbb{T}^3} \subset \mathcal{M}(\mathbb{R})$ of lacally finite Borel measures on the real line with the following property: if, for μ almost every x, the measure $\hat{\nu}_x^c$ is proportional to the Lebesgue measure of $\mathbb{R}$ then μ is an SRB measure.*

The family $\{\hat{\nu}_x^c\}_{x \in \mathbb{T}^3}$ is referred to as *leaf-wise quotient measures*. Once Theorem 7.1 is established, our main result is reduced to proving that, for μ almost every x, each $\hat{\nu}_x^c$ is a translation-invariant measure on the real line. The proposition below summarizes the properties of the family $\{\hat{\nu}_x^c\}_{x \in \mathbb{T}^3} \subset \mathcal{M}(\mathbb{R})$ that enable us to achieve this result.

Proposition 7.2 (Basic moves)**.** *The family $\{\hat{\nu}_x^c\}_{x \in \mathbb{T}^3} \subset \mathcal{M}(\mathbb{R})$ satisfies the following properties on a full measure subset Λ of $\mathbb{T}^3$, which is $\mathcal{W}^u$-saturated. For every $x \in \Lambda$*

(1) *if $y \in \mathcal{W}^c(x) \cap \Lambda$ then $\hat{\nu}_x^c \propto (\mathcal{H}_{x,y}^c)_* \hat{\nu}_y^c$, where $\mathcal{H}_{x,y}^c : \mathbb{R} \to \mathbb{R}$ is the change of normal form coordinates on the manifold $\mathcal{W}^c(x)$, which is an affine map.*

(2) *$\hat{\nu}_{f(x)}^c \propto (\Lambda_x^c)_* \hat{\nu}_x^c$, where $\Lambda_x^c : \mathbb{R} \to \mathbb{R}$ is the linear map $\Lambda_x^c(s) = \lambda_x^c s$.*

(3) *if $x' \in \mathcal{W}^u(x)$ then $\hat{\nu}_{x'}^c \propto (L_{x,x'})_* \hat{\nu}_x^c$, where $L_{x,x'} : \mathbb{R} \to \mathbb{R}$ is the linear map $L_{x,x'}(s) = \rho_{x'}^c(x)s$.*

The remainder of this section is dedicated to proving Theorem 7.1 and Proposition 7.2. Thus, we fix $f \in \mathcal{A}^2(\mathbb{T}^3)$ and μ an ergodic u-Gibbs measure.

7.1. **Conditional measures of μ in normal forms.** We will show that when we push the conditional measures of μ along center unstable manifolds using the normal forms and subsequently disintegrate this measure along the horizontal lines (corresponding to unstable manifolds), we obtain measures *proportional* to the Lebesgue measure. his fact plays a fundamental role in the proofs of Theorem 7.1 and Proposition 7.2.

7.1.1. *Some preliminary facts about the conditional measures $\mu_{n,x}^{cu}$.* Let us start by recalling the notation from §2.5.1 and stating some basic lemmas that will be useful throughout this section. Start with ξ_0^{cu}, an increasing measurable partition subordinate to the uniformly expanded foliation $\mathcal{W}^{cu}$. When we apply the dynamics, we obtain a sequence $\xi_n^{cu} \stackrel{\text{def.}}{=} f^n(\xi_0^{cu})$ of measurable partitions that remain subordinate to $\mathcal{W}^{cu}$. Moreover, for every $n \in \mathbb{N}$, we have $\xi_{n+1}^{cu} \prec \xi_n^{cu}$. Now consider $\{\mu_{n,x}^{cu}\}_{x \in \mathbb{T}^3}$ to be the disintegration of μ with respect to the partition ξ_n^{cu}. Similar to Lemma 2.16, we find that the following *superposition property* holds for μ-a.e. $x \in \mathbb{T}^3$:

$$\text{if } m > n, \text{ then } \mu_{n,x}^{cu}(A) = \frac{\mu_{m,x}^{cu}(A)}{\mu_{m,x}^{cu}(\xi_n^{cu}(x))}, \tag{35}$$

for every measurable set $A \subset \xi_n^{cu}(x)$. Furthermore, as μ is f-invariant, the distintegrations $\{\mu_{n,x}^{cu}\}_{n \in \mathbb{N}, x \in \mathbb{T}^3}$ satisfy:

Lemma 7.3. *For every $n \in \mathbb{N}$ and for μ-almost every $x \in \mathbb{T}^3$, we have*

$$\mu_{n+1,f(x)}^{cu} = f_* \mu_{n,x}^{cu}.$$

Proof. The proof follows a similar approach to that of Lemma 2.12. $\square$

By intersecting with the center foliation $\mathcal{W}^c$, or unstable foliation $\mathcal{W}^u$, we can refine each partition ξ_n^{cu}, $n \in \mathbb{N}$. More precisely, for $\star = c, u$, we consider a

partition $\xi_n^{cu} \prec \xi_n^\star = \{\xi_n^\star(y)\}_{y \in \mathbb{T}^3}$, with

$$\xi_n^\star(y) \stackrel{\text{def.}}{=} \mathcal{W}^\star(y) \cap \xi_n^{cu}(y), \quad \forall y \in \mathbb{T}^3.$$

By Remark 2.14, we obtain:

Lemma 7.4. *For $\star = c, u$, the family of sets $\xi_n^\star$ is a measurable partition subordinate to $\mathcal{W}^\star$.*

For every $n \in \mathbb{N}$ and for μ-almost every $x \in \mathbb{T}^3$, we define $\zeta_n^\star(x) \stackrel{\text{def.}}{=} \mathcal{H}_x(\xi_n^\star(x)) \subset \mathbb{R}^2$, and $\nu_{n,x}^{cu} \stackrel{\text{def.}}{=} (\mathcal{H}_x)_* \mu_{n,x}^{cu}$. Note that due to item (1) of Theorem 6.3, for each $x \in \mathbb{T}^3$, it holds

$$N_x \circ \mathcal{H}_x|_{\mathcal{W}^{cu}(x)} = \mathcal{H}_{f(x)} \circ f|_{\mathcal{W}^{cu}(x)}, \quad \text{where } N_x \colon (t, s) \mapsto (\lambda_x^u t, \lambda_x^c s). \qquad (36)$$

As a result of (36) and Lemma 7.3, we conclude:

Lemma 7.5. *For every $n \in \mathbb{N}$ and for μ-almost every $x \in \mathbb{T}^3$, it holds*

$$\nu_{n+1,f(x)}^{cu} = (N_x)_* \nu_{n,x}^{cu}.$$

7.1.2. *Disintegration of the conditional measures in normal forms.* As in §2.5.1 when we disintegrate $\mu_{n,x}^{cu}$ along the atoms $\{\xi_n^u(y)\}_{y \in \xi_n^{cu}(x)}$ we get a probability measure $\mu_{n,x}^c$ over $\xi_n^c(x)$ and a family $\{\mu_{n,y}^u\}_{y \in \xi_n^{cu}(x)}$, each of which is a probability measure on $\xi_n^u(y)$.

Now, let $\mathcal{H}_x : \mathcal{W}^{cu}(x) \to \mathbb{R}^2$ be the inverse map of the normal form coordinate chart. Define the measures

$$\nu_{n,x}^\star \stackrel{\text{def.}}{=} (\mathcal{H}_x)_* \mu_{n,x}^\star, \quad \text{for } \star = c, cu,$$

and

$$\nu_{n,s}^u \stackrel{\text{def.}}{=} (\mathcal{H}_x)_* \mu_{n,y}^u, \quad \text{for each } s = \mathcal{H}_x^c(y), \quad \text{with } y \in \xi_n^c(x).$$

A remarkable property of u-Gibbs measures is that their conditional measures along unstable manifolds in normal forms are *proportional* to Lebesgue.

Lemma 7.6. *For μ-a.e. $x \in \mathbb{T}^3$ and $\nu_{n,x}^{cu}$-a.e. $(t, s) \in \zeta_n^{cu}(x)$, there exists a constant $\gamma_{n,x}(s) > 0$ (independent of t) such that for every Borel set $B \subset \zeta_n^u(\Phi_x^c(s))$, the following holds*

$$\nu_{n,s}^u(B) = \gamma_{n,x}(s) \operatorname{Leb}(B).$$

Proof. Given that μ is a u-Gibbs measure, for μ-a.e. $x \in \mathbb{T}^3$, and $\mu_{n,x}^{cu}$-a.e. $y \in \xi_n^{cu}(x)$, the measure $\mu_{n,y}^u$ is absolutely continuous. Its density, up to a constant, is given by (referring to Notation (21))

$$\rho_y^u : \xi_n^u(y) \ni z \mapsto \rho_y^u(z) = \prod_{j=0}^{+\infty} \frac{\|Df^{-1}(f^{-j}(z))|_{E^u}\|}{\|Df^{-1}(f^{-j}(y))|_{E^u}\|}.$$

Let $(t, s) \stackrel{\text{def.}}{=} \mathcal{H}_x(y)$, so that $\nu_{n,s}^u = (\mathcal{H}_x)_* \mu_{n,y}^u$. Restricted to $\mathbb{R} \times \{s\}$, $\Phi_x = \mathcal{H}_x^{-1}$ is given by the map $\Phi_{\Phi_x^c(s)}^u(\beta_x(s)t)$, for some $\beta_x(s) \in \mathbb{R}$. Therefore, up to a constant that depends only on s, the density of $\nu_{n,s}^u = (\mathcal{H}_x)_* \mu_{n,y}^u$ is given by

$$\tilde{\rho}_s^u : t \mapsto \rho_y^u(\Phi_{\Phi_x^c(s)}^u(t)) D\Phi_{\Phi_x^c(s)}^u(t).$$

Since $D\Phi_{\Phi_x^c(s)}^u(t) = (D\mathcal{H}_y^u)^{-1}(\Phi_{\Phi_x^c(s)}^u(t)) = \rho_y^u(\Phi_{\Phi_x^c(s)}^u(t))^{-1}$, hence $\tilde{\rho}_s^u$ is constant. $\qquad \square$

7.2. **Construction of leaf-wise quotient measures.** The measures we define here are similar to the quotient measures $\nu_{n,x}^c$, but we will use unstable segments of a fixed length together with a normalization that allows for stabilization when n is sufficiently large large enough (see Lemma 7.8 below). Let $I \subset \mathbb{R}$ be any bounded interval centered at 0. It determines the length of unstable segments used in defining the leaf-wise quotient measures. Given a bounded Borel set $B \subset \mathbb{R}$ and a μ-typical point $x \in \mathbb{T}^3$ there exists $n_0 = n_0(x, I, B)$ such that for every $n \geq n_0$, $I \times B \subset \zeta_n^{cu}(x)$ and $I \times [-1,1] \subset \zeta_n^{cu}(x)$: see Lemma 2.19. We first show that the choice of length of unstable segments, and thus the interval $I \subset \mathbb{R}$ will not modify the definition.

Lemma 7.7. *For any other interval $J \subset \mathbb{R}$ centered at 0, there exists $n_0(J)$ such that*

$$\frac{\nu_{n,x}^{cu}(I \times B)}{\nu_{n,x}^{cu}(I \times [-1,1])} = \frac{\nu_{n,x}^{cu}(J \times B)}{\nu_{n,x}^{cu}(J \times [-1,1])},$$

for every $n \geq n_0(J)$.

Proof. Consider real numbers α and β such that the affine map $\psi(t) = \alpha t + \beta$ satisfies $\psi(I) = J$. Then, applying the disintegration in conditional measures and using Lemma 7.6 we obtain

$$\begin{aligned}
\frac{\nu_{n,x}^{cu}(J \times B)}{\nu_{n,x}^{cu}(J \times [-1,1])} &= \frac{\int_B \nu_{n,s}^u(J) d\nu_{n,x}^c(s)}{\int_{[-1,1]} \nu_{n,s}^u(J) d\nu_{n,x}^c(s)} \\
&= \frac{\int_B \gamma_{n,x}(s) \operatorname{Leb}(\alpha I + \beta) d\nu_{n,x}^c(s)}{\int_{[-1,1]} \gamma_{n,x}(s) \operatorname{Leb}(\alpha I + \beta) d\nu_{n,x}^c(s)} \\
&= \frac{\int_B \gamma_{n,x}(s) \operatorname{Leb}(I) d\nu_{n,x}^c(s)}{\int_{[-1,1]} \gamma_{n,x}(s) \operatorname{Leb}(I) d\nu_{n,x}^c(s)} \\
&= \frac{\int_B \nu_{n,s}^u(I) d\nu_{n,x}^c(s)}{\int_{[-1,1]} \nu_{n,s}^u(I) d\nu_{n,x}^c(s)} \\
&= \frac{\nu_{n,x}^{cu}(I \times B)}{\nu_{n,x}^{cu}(I \times [-1,1])}.
\end{aligned}$$

This equality holds for sufficiently large n, ensuring that $J \times B$ and $J \times [-1,1]$ are both subsets of $\zeta_n^{cu}(x)$. Notice that this can always be achieved due to Lemma 2.19. This completes the proof. $\qquad\square$

Given $B \subset \mathbb{R}$ a bounded Borel measurable subset, and given $n \in \mathbb{N}$ we define the following number

$$\hat{\nu}_{n,x}^c(B) \stackrel{\text{def.}}{=} \frac{\nu_{n,x}^{cu}(I \times B)}{\nu_{n,x}^{cu}(I \times [-1,1])}. \tag{37}$$

We now use the superposition property to show that this sequence stabilizes for large n.

Lemma 7.8. *For μ-almost every $x \in \mathbb{T}^3$, every bounded Borel set $B \subset \mathbb{R}$, and every interval I, there exists a number $n_0 = n_0(x, I, B) \in \mathbb{N}$ such that for all integers $m > n \geq n_0$ we have $\hat{\nu}_{m,x}^c(B) = \hat{\nu}_{n,x}^c(B)$.*

Proof. Firstly, observe that for each pair $m > n$, there exists a μ-full measure set of points $x \in \mathbb{T}^3$ for which $\mu_{m,x}^{cu}(\xi_{n,x}^{cu}(x)) > 0$. This follows from the superposition property (35). By taking the countable intersection of all these sets, and applying

(35) again, we deduce that for every bounded Borel set $B \subset \mathbb{R}$, and $m > n \geq n_0(x, I, B)$, so that $I \times B, I \times [-1, 1] \subset \zeta_n^{cu}(x) \subset \zeta_m^{cu}(x)$, the following holds

$$
\begin{aligned}
\hat{\nu}_{n,x}^c(B) &= \frac{\mu_{n,x}^{cu}(\Phi_x(I \times B))}{\mu_{n,x}^{cu}(\Phi_x(I \times [-1, 1]))} \\
&= \frac{\mu_{m,x}^{cu}(\Phi_x(I \times B))}{\mu_{m,x}^{cu}(\xi_{n,x}^{cu}(x))} \times \frac{\mu_{m,x}^{cu}(\xi_{n,x}^{cu}(x))}{\mu_{m,x}^{cu}(\Phi_x(I \times [-1, 1]))} = \hat{\nu}_{m,x}^c(B). \qquad \square
\end{aligned}
$$

Hence, the construction above defines a family of measures on $\mathbb{R}$.

Definition 7.9 (Leaf-wise quotient measure)**.** *For μ-a.e. $x \in \mathbb{T}^3$, let $\hat{\nu}_x^c$ be the unique locally finite Borel measure defined in $\mathbb{R}$ so that, for each bounded Borel $B \subset \mathbb{R}$, one has*

$$
\hat{\nu}_x^c(B) \overset{\text{def.}}{=} \hat{\nu}_{n_0(x,B),x}^c(B).
$$

By Lemma 7.8 the right hand side above is well defined.

Remark 7.10 (Choice of normalization)**.** *By construction, we have that $\hat{\nu}_x^c([-1, 1]) = 1$.*

Remark 7.11. *By construction, for μ-a.e. $x \in \mathbb{T}^3$, $\hat{\nu}_x^c$ gives positive measure to any open neighbourhood of 0. This follows from the definition and Corollary 2.17.*

7.3. **Obtaining the SRB property: Proof of Theorem 7.1.** To complete the proof of Theorem 7.1 we assume that for μ almost every $x \in \mathbb{T}^3$ the measure $\hat{\nu}_x^c$ is proportional to Lebesgue on $\mathbb{R}$. To conclude that μ is an SRB measure it suffices to show that the conditional measures $\mu_{0,x}^{cu}$ are absolutely continuous with respect to $\text{Leb}_{\mathcal{W}^{cu}(x)}$ for μ almost every x. To achieve this goal, consider $a < 0 < b$ with $b - a$ large enough so that, denoting $I = [a, b]$, we have

$$
\xi_0^{cu}(x) \subset \Phi_x(I \times I), \text{ for } \mu\text{-almost every } x \in \mathbb{T}^3.
$$

This can be done because the atoms of ξ_0^{cu} have uniformly bounded diameter and the C^1 norm of Φ_x depends continuously on x. Let $R = \max_{x \in \mathbb{T}^3} \text{diam}(\Phi_x(I \times I))$. Then, applying Lemma 2.19 with this number we get that for μ almost every x, there exists $n_0(x, I)$ such that if $n \geq n_0(x, I)$ then

(1) $I \subset \zeta_n^c(x)$;
(2) $I \times \{s\} \subset \zeta_n^u(\Phi_x^c(s))$, for every $s \in I$.

Now, given such x and $n \geq n_0(x, I)$ take $s \in I$. We observe

$$
\begin{aligned}
1 = \nu_{n,s}^u(\zeta_n^u(\Phi_x^c(s))) &\geq \nu_{n,s}^u(I) \\
&= \gamma_{n,x}(s) \text{Leb}(I) \\
&= \gamma_{n,x}(s)(b - a).
\end{aligned} \tag{38}
$$

Therefore,

$$
0 < \gamma_{n,x}(s) < \frac{1}{b - a}.
$$

We claim that $\nu_{n,x}^c|_I$ is absolutely continuous with respect to the leaf-wise quotient measure $\hat{\nu}_x^c|_I$. Indeed, let us denote for simplicity $\alpha = (\nu_{n,x}^{cu}(I \times [-1, 1]))^{-1}$. Then, if $B \subset I$ we have

$$
\begin{aligned}
\hat{\nu}_x^c(B) &= \alpha \nu_{n,x}^{cu}(I \times B) = \alpha \int_B \nu_{n,s}^u(I) \, d\nu_{n,x}^c(s) \\
&= \alpha \text{Leb}(I) \int_B \gamma_{n,x}(s) \, d\nu_{n,x}^c(s).
\end{aligned} \tag{39}
$$

Since $\gamma_{n,x} > 0$ if $\hat{\nu}_x^c(B) = 0$ then $\nu_{n,x}^c(B) = 0$, which proves our claim. Since, by assumption, $\hat{\nu}_x^c \propto \mathrm{Leb}$ we conclude that indeed $\nu_{n,x}^c|_I$ is absolutely continuous with $\mathrm{Leb}|_I$. As $d\nu_{n,x}^{cu}(t,s) = d\nu_{n,s}^u(t)d\nu_{n,x}^c(s)$, it follows immediately from Lemma 7.6 that $\nu_{n,x}^{cu}|_{I \times I}$ is absolutely continuous with respect to the two dimensional Lebesgue measure $\mathrm{Leb}|_{I \times I}$. Since Φ_x is C^1, we get that $\mu_{n,x}^{cu}|_{\Phi_x(I \times I)}$ is absolutely continuous with respect to $\mathrm{Leb}_{\mathcal{W}^{cu}(x)}$. By the superposition property we deduce that $\mu_{0,x}^{cu}$ is absolutely continuous, as desired. The theorem is proved. □

7.4. **Basic moves for leaf-wise measures.** We now proceed with the proof of Proposition 7.2. Our goal is to investigate how the measures $\hat{\nu}_x^c$ change as we move the base point x. In this analysis, the understanding of how the normal form coordinates change with the base point on the same center-unstable leaf plays a crucial role.
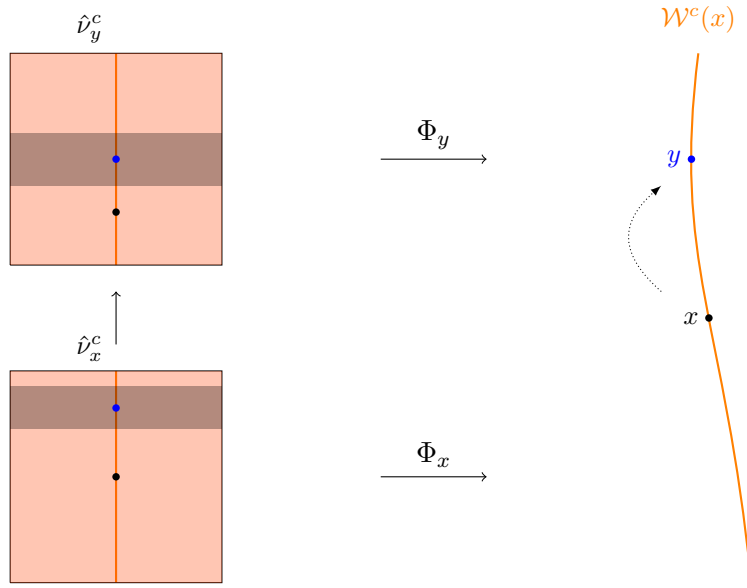

FIGURE 7. Moving the base point within a central leaf changes the leaf-wise measure by an affine map up to renormalization.

7.4.1. *First basic move: moving along center manifolds.* In the following lemma we show that for points on the same center leaf there exists an affine map that makes the corresponding leaf-wise quotient measure proportional to each other: see Figure 7. It is important to note that the derivative of this affine map is determined by normal form coordinates along the center manifold.

Lemma 7.12. *For μ-a.e. $x \in \mathbb{T}^3$, and $y \in \mathcal{W}^c(x)$, we have*

$$\hat{\nu}_y^c \propto (h_{x,y}^2)_* \hat{\nu}_x^c,$$

where $h_{x,y}^2 \colon \mathbb{R} \to \mathbb{R}$ is the affine map defined by $s \mapsto \rho_y^c(x) \cdot s + \mathcal{H}_y^c(x)$, corresponding to the action of the change of normal forms on the second coordinate given by Lemma 6.12.

Proof. Let I be a bounded interval centered at 0 and B be a bounded Borel subset of $\mathbb{R}$. Consider $n \in \mathbb{N}$, large enough so that $y \in \xi_n^{cu}(x)$ and

$$\Phi_y(I \times B) \cup \Phi_y(I \times [-1,1]) \subset \xi_n^{cu}(y) = \xi_n^{cu}(x).$$

This is possible after Lemma 2.19. Recall the normal form coordinate change $\mathcal{H}_{x,y}(t,s) = \mathcal{H}_y \circ \Phi_x = (h^1_{x,y}(t,s), h^2_{x,y}(s))$, which is given by (see Lemma 6.12)

$$h^1_{x,y}(t,s) = \rho^u_y(x)t \quad \text{and} \quad h^2_{x,y}(s) = \rho^c_y(x)s + \mathcal{H}^c_y(x).$$

Let $\tilde{B} = h^2_{x,y}(B)$, $I_1 = [-1,1]$ and $\tilde{I}_1 = h^2_{x,y}(I_1)$. Then, we can apply Lemma 7.6 as before to argue that

$$
\begin{aligned}
\hat{\nu}^c_x(B) &= \frac{\mu^{cu}_{n,x}(\Phi_x(I \times B))}{\mu^{cu}_{n,x}(\Phi_x(I \times I_1))} = \frac{\mu^{cu}_{n,y}(\Phi_y \circ \mathcal{H}_{x,y}(I \times B))}{\mu^{cu}_{n,y}(\Phi \circ \mathcal{H}_{x,y}(I \times I_1))} \\
&= \frac{\nu^{cu}_{n,y}(\rho^u_y(x)I \times \tilde{B})}{\nu^{cu}_{n,y}(\rho^u_y(x)I \times \tilde{I}_1)} = \frac{\int_{\tilde{B}} \nu^u_{n,s}(\rho^u_y(x)I)d\nu^c_{n,x}(s)}{\int_{\tilde{I}_1} \nu^u_{n,s}(\rho^u_y(x)I)d\nu^c_{n,x}(s)} \\
&= \frac{\int_{\tilde{B}} \nu^u_{n,s}(I)d\nu^c_{n,x}(s)}{\int_{\tilde{I}_1} \nu^u_{n,s}(I)d\nu^c_{n,x}(s)} = \frac{\nu^{cu}_{n,x}(I \times \tilde{B})}{\nu^{cu}_{n,x}(I \times \tilde{I}_1)} \\
&= \alpha(x,y)\hat{\nu}^c_y(\tilde{B}),
\end{aligned}
$$

where $\alpha(x,y) = \frac{\nu^{cu}_{n,x}(I \times I_1)}{\nu^{cu}_{n,x}(I \times \tilde{I}_1)}$. This is exactly the desired conclusion. $\qquad\square$
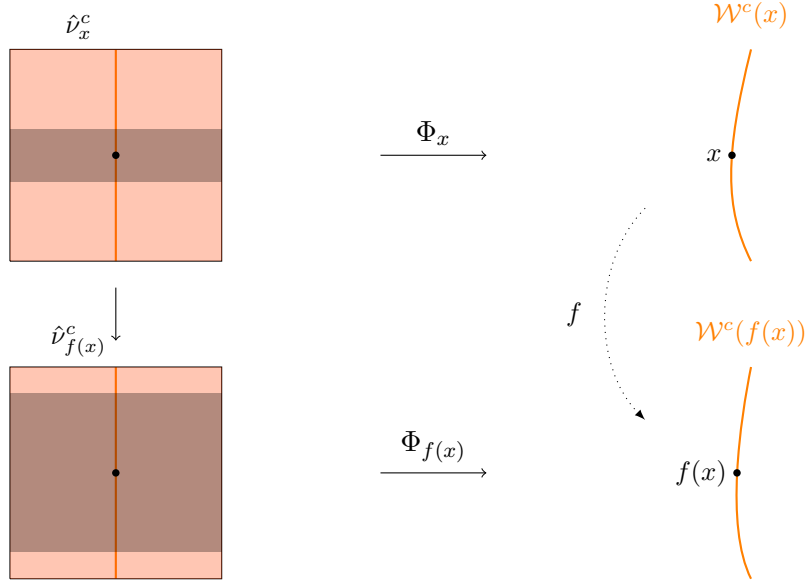

FIGURE 8. Applying the dynamics to the base point changes the leaf-wise measure by a linear map given by the differential of f along E^c up to renormalization.

7.4.2. *Second basic move: applying the dynamics.* When we push x to $f(x)$ the change on the measures is linear and the slope of the linear map is given by the derivative at x along the center direction. Recall that for $* = c, u$, we denote by Λ^*_x the linear map $t \mapsto \lambda^*_x \cdot t$ (where $\lambda^*_x \overset{\text{def.}}{=} \|Df(x)|_{E^*}\|$).

Lemma 7.13. *For μ-a.e. $x \in \mathbb{T}^3$, we have*

$$\hat{\nu}^c_{f(x)} \propto (\Lambda^c_x)_* \hat{\nu}^c_x.$$

Proof. Let I be a bounded interval centered at 0 and B be a bounded Borel subset of $\mathbb{R}$. Fix $n \in \mathbb{N}$ large so that $I \times B, I \times [-1, 1] \subset \zeta_{n+1}^{cu}(f(x))$ (see Lemma 2.19). By definition, we have

$$\hat{\nu}_{n+1,f(x)}^{c}(B) = \frac{\nu_{n+1,f(x)}^{cu}(I \times B)}{\nu_{n+1,f(x)}^{cu}(I \times [-1, 1])},$$

and by Lemmas 7.5 and 7.6

$$\frac{\nu_{n+1,f(x)}^{cu}(I \times B)}{\nu_{n+1,f(x)}^{cu}(I \times [-1, 1])} = \frac{\nu_{n,x}^{cu}(N_x^{-1}(I \times B))}{\nu_{n,x}^{cu}(N_x^{-1}(I \times [-1, 1]))}$$

$$= \frac{\int_{(\Lambda_x^c)^{-1}(B)} \nu_{n,s}^{u}((\lambda_x^u)^{-1}I) d\nu_{n,x}^{c}(s)}{\int_{(\Lambda_x^c)^{-1}([-1,1])} \nu_{n,s}^{u}((\lambda_x^u)^{-1}I) d\nu_{n,x}^{c}(s)}$$

$$= \frac{\nu_{n,x}^{cu}(I \times (\Lambda_x^c)^{-1}(B))}{\nu_{n,x}^{cu}(I \times (\Lambda_x^c)^{-1}([-1,1]))}$$

$$= \frac{\nu_{n,x}^{cu}(I \times (\Lambda_x^c)^{-1}(B))}{\nu_{n,x}^{cu}(I \times [-1,1])} \times \frac{\nu_{n,x}^{cu}(I \times [-1,1])}{\nu_{n,x}^{cu}(I \times (\Lambda_x^c)^{-1}([-1,1]))}$$

$$= \left(\frac{1}{\hat{\nu}_x^c((\Lambda_x^c)^{-1}([-1,1]))} \right) \hat{\nu}_x^c\left((\Lambda_x^c)^{-1}(B)\right)$$

$$= \alpha(x) \hat{\nu}_x^c((\Lambda_x^c)^{-1}(B)), \tag{40}$$

where $\alpha(x) = \hat{\nu}_x^c((\Lambda_x^c)^{-1}([-1, 1]))^{-1}$. We deduce that

$$\hat{\nu}_{f(x)}^{c}(B) \propto (\Lambda_x^c)_* \hat{\nu}_x^c(B). \quad \square$$
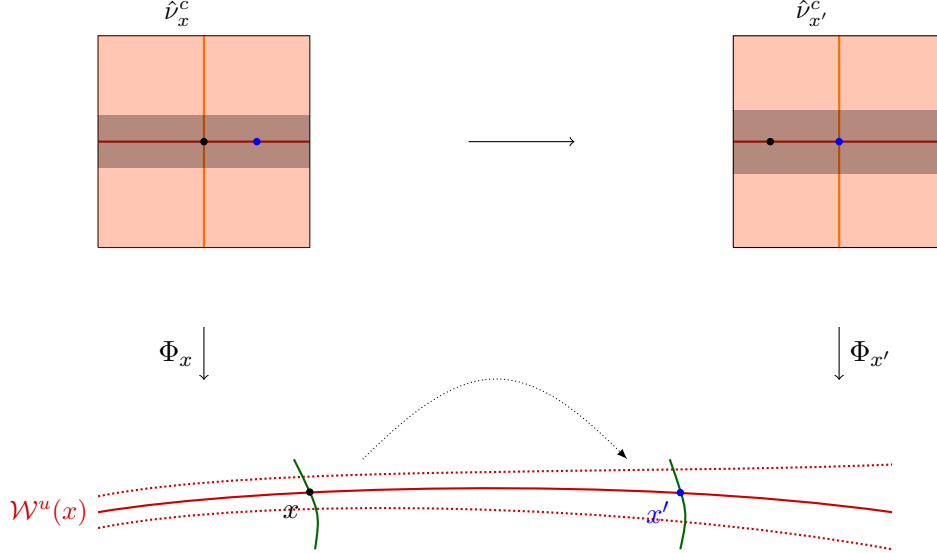
FIGURE 9.   Moving the base point within an unstable leaf changes the leaf-wise measure by a linear map up to renormalization.

7.4.3. *Third basic move: moving along unstable manifolds.* For points on the same strong unstable leaf the change is also linear, but the slope is determined by the derivative of the unstable holonomy map, which coincides with the derivative of the linear map giving the action of the change in normal form coordinates on the second variable. The lemma below completes the proof of Proposition 7.2.

Lemma 7.14. *For μ-a.e. $x \in \mathbb{T}^3$, and $x' \in \mathcal{W}^u(x)$, it holds*

$$\hat{\nu}_x^c \propto (L_{x,x'}^u)_* \hat{\nu}_{x'}^c,$$

for the linear map $L_{x,x'}^u \colon s \mapsto \rho_{x'}^c(x) \cdot s$ corresponding to the action of the change of normal coordinates on the second variable given by Lemmas 6.14 and 6.15.

Proof. For μ-a.e. $x \in \mathbb{T}^3$, and $x' \in \mathcal{W}^u(x)$, we proceed as follows. Let I be any bounded interval centered at 0 and B be any bounded Borel subset of $\mathbb{R}$. Then, there exists n large so that $x' \in \xi_n^{cu}(x)$ and $I \times B, I \times [-1,1] \times B \subset \zeta_n^{cu}(x)$. This is possible by Lemma 2.19. As before, let $\mathcal{H}_{x,x'}(t,s) = (h_{x,x'}^1(t,s), h_{x,x'}^2(s))$ be the change in normal form coordinate. By Lemmas 6.13, 6.14 and 6.15 we have

$$h_{x,x'}^1(t,s) = \rho_{x'}^u(x)t + a_{x,x'}(s), \quad \text{and} \quad h_{x,x'}^2(s) = \rho_{x'}^c(x)s.$$

Now, introduce $\tilde{B} = h_{x,x'}^2(B)$, $\tilde{I}_1 = h_{x,x'}^2(I_1)$, where $I_1 = [-1,1]$. By Lemma 7.6:

$$
\begin{aligned}
\hat{\nu}_{x'}^c(B) &= \frac{\mu_{n,x}^{cu}(\Phi_x \circ \mathcal{H}_{x,x'}(I \times B))}{\mu_{n,x}^{cu}(\Phi_x \circ \mathcal{H}_{x,x'}(I \times I_1))} \\
&= \frac{\nu_{n,x}^{cu}(h_{x,x'}^1(I \times B) \times \tilde{B})}{\nu_{n,x}^{cu}(h_{x,x'}^1(I \times I_1) \times \tilde{I}_1)} \\
&= \frac{\int_{\tilde{B}} \nu_{n,s}^u(\rho_{x'}^u(x)I + a_{x,x'}(\rho_x^c(x')s)) d\nu_{n,x}^c(s)}{\int_{\tilde{I}_1} \nu_{n,s}^u(\rho_{x'}^u(x)I + a_{x,x'}(\rho_x^c(x')s)) d\nu_{n,x}^c(s)} \\
&= \frac{\int_{\tilde{B}} \nu_{n,s}^u(I) d\nu_{n,x}^c(s)}{\int_{\tilde{I}_1} \nu_{n,s}^u(I) d\nu_{n,x}^c(s)} \\
&= \frac{\nu_{n,x}^{cu}(I \times \tilde{B})}{\nu_{n,x}^{cu}(I \times \tilde{I}_1)} \\
&= \alpha(x,x')\hat{\nu}_x^c(\tilde{B}), \quad\quad\quad\quad\quad\quad\quad\quad (41)
\end{aligned}
$$

where $\alpha(x,x') = \left(\hat{\nu}_x^c(\tilde{I}_1)\right)^{-1}$. This completes the proof of the lemma. $\square$

8. INVARIANCE BY AFFINE MAPS

Once we have constructed the *leaf-wise quotient measures* $\{\hat{\nu}_x^c\}_{x \in \mathbb{T}^3}$, Theorem 7.1 tells us that the proof of Theorem 5.10 reduces to show that $\hat{\nu}_x^c$ is a multiple of the Lebesgue measure of the real line. As in [7] this can be achieved by proving that $\hat{\nu}_x^c$ is, for many points x, invariant by affine maps with controlled slope and small translational part. More precisely, the lemma below is the analogue of [7, Proposition 7.1] in our context.

Lemma 8.1. *There exist constants $M_0 > 0$ and $\delta_0 \in (0,1)$ such that for every $\varepsilon > 0$ sufficiently small one can find a compact set $G(\varepsilon) \subset \mathbb{T}^3$ so that $\mu(G(\varepsilon)) \geq \delta_0$ and for every $p \in G(\varepsilon)$ there exists an affine map $\psi \colon \mathbb{R} \to \mathbb{R}$ satisfying*

(1) $\frac{1}{M_0} < |\psi'(0)| < M_0$;

(2) $\frac{\varepsilon}{M_0} < |\psi(0)| < \varepsilon M_0$;

(3) $\psi_* \hat{\nu}_p^c \propto \hat{\nu}_p^c$.

Furthermore, writing $G_0 \stackrel{\text{def.}}{=} \{p \in \mathbb{T}^3 : p \in G(\frac{1}{N})$ for infinitely many $N \in \mathbb{N}\}$, we have $\mu(G_0) \geq \delta_0$.

Exactly as in [7], the lemma above implies Theorem 5.10. The proof is a direct adaptation of [25, Lemma 3.10] and [7, Lemma 7.3]; for the convenience of the reader, we provide this beautiful argument below and we refer to [7, 25] for other applications of the same argument.

Proof of Theorem 5.10 assuming Lemma 8.1. Let $\mathrm{Aff}(\mathbb{R})$ denote the group of invertible affine maps of $\mathbb{R}$, and for $p \in \mathbb{T}^3$, let $\mathcal{A}(p) \subset \mathrm{Aff}(\mathbb{R})$ be the subgroup of affine maps $\psi \colon \mathbb{R} \to \mathbb{R}$ such that $\psi_* \hat{\nu}_p^c \propto \hat{\nu}_p^c$.

We claim that $\mathcal{A}(p)$ is a closed subgroup of $\mathrm{Aff}(\mathbb{R})$. Let $\psi_n \in \mathcal{A}(p)$ converge to $\psi \in \mathrm{Aff}(\mathbb{R})$. Since each element of $\mathrm{Aff}(\mathbb{R})$ is a homeomorphism of the real line and since the convergence in $\mathrm{Aff}(\mathbb{R})$ implies converge in the compact-open topology we have, for each continuous function with compact support $\phi \colon \mathbb{R} \to \mathbb{R}$ that
$$\int \phi \circ \psi_n \, d\hat{\nu}_p^c \to \int \phi \circ \psi \, d\hat{\nu}_p^c.$$
This implies that $(\psi_n)_* \hat{\nu}_p^c \to \psi_* \hat{\nu}_p^c$. On the other hand, for each n we have
$$(\psi_n)_* \hat{\nu}_p^c = c_n \hat{\nu}_p^c, \tag{42}$$
for some constant c_n, which can therefore be obtained by
$$c_n = \frac{\hat{\nu}_p^c(\psi_n^{-1}(K))}{\hat{\nu}_p^c(K)},$$
for any measurable set $K \subset \mathbb{R}$ with finite positive measure with respect to both $\hat{\nu}_p^c$ and $(\psi_n)_* \hat{\nu}_p^c$. Now, as all the measures are locally finite we can require further that such compact set K is also a continuity set (of finite measure) for $\psi_* \hat{\nu}_p^c$. In particular, since $(\psi_n)_* \hat{\nu}_p^c(K) \to \psi_* \hat{\nu}_p^c(K)$, we deduce that c_n converges to some positive real number c. We deduce from (42) and uniqueness of limits that $\psi_* \hat{\nu}_p^c = c \hat{\nu}_p^c$. This proves that $\psi \in \mathcal{A}(p)$ and establishes our claim. This implies in particular that $\mathcal{A}(p)$ is a Lie group.

By Lemma 8.1, for every $p \in G_0$, $\mathcal{A}(p)$ contains a sequence $(\psi_j)_{j \in \mathbb{N}}$ of elements $\psi_j \colon t \mapsto \lambda_j t + v_j$, with $v_j \neq 0$, $\lim_{j \to +\infty} v_j = 0$, and $\lim_{j \to +\infty} \lambda_j = \lambda$, for some $\lambda \neq 0$. In particular, $\mathcal{A}(p)$ contains the homothety $h_\lambda \colon t \mapsto \lambda t$, and $(h_\lambda^{-1} \circ \psi_j)_{j \in \mathbb{N}}$ converges to the identity within $\mathcal{A}(p)$. Therefore, $\mathcal{A}(p)$ is not discrete and must be of dimension 1 or 2. This implies that the identity component $\mathcal{A}^0(p) \subset \mathcal{A}(p)$ contains a one-parameter subgroup of $\mathrm{Aff}(\mathbb{R})$: such a group consists of translations, or is conjugate to homothety.

We now claim that the groups $\mathcal{A}^0(p)$ are isomorphic for a.e $p \in \mathbb{T}^3$. Indeed, recall the linear map $\Lambda_p^c \colon \mathbb{R} \to \mathbb{R}$ given by $\Lambda_p^c(t) = \lambda_p^c t$, where $\lambda_p^c = \|Df(p)|_{E^c}\|$. Then, it follows from Lemma 7.13 that $\psi \in \mathcal{A}^0(p)$ if, and only if, $\Lambda_p^c \circ \psi \circ (\Lambda_p^c)^{-1} \in \mathcal{A}^0(f(p))$, proving our claim.

Since isomorphisms classes of closed subgroups of $\mathrm{Aff}(\mathbb{R})$ form a separable space and since $\mu(G_0) > 0$, the claim then follows by ergodicity. In particular, for μ-a.e. $p \in \mathbb{T}^3$, $\mathcal{A}^0(p)$ contains a one-parameter subgroup of $\mathrm{Aff}(\mathbb{R})$.

Assume by contradiction that $\mathcal{A}^0(p)$ were conjugate to homothety for a positive measure set of $p \in \mathbb{T}^3$. Then, by ergodicity, for μ-a.e. $p \in \mathbb{T}^3$, the action of $\mathcal{A}^0(p)$ on $\mathbb{R}$ would have a unique fixed point $t(p) \in \mathbb{R}$. Since $\mathcal{A}^0(p)$ contains affine maps with arbitrarily small (and non-zero) translational part, we have $t(p) \neq 0$ for a set of positive measure.

Besides that, as we observed above $\mathcal{A}^0(f^n(p)) = \{\Lambda_{p,n}^c \circ \psi \circ (\Lambda_{p,n}^c)^{-1} : \psi \in \mathcal{A}^0(p)\}$, where $\Lambda_{p,n}^c(t) = \lambda_p^c(n) t$ (recall (2)). As a consequence we deduce
$$|t(f^n(p))| = \lambda_p^c(n) \cdot |t(p)|. \tag{43}$$
Consider a positive measure compact subset K of G_0 where the measurable function $p \mapsto t(p)$ is continuous and bounded. By Poincaré recurrence for some $p \in K$ we have $p_{n_k} = f^{n_k}(p) \in K$ for infinitely many iterates $n_k \in \mathbb{N}$. By compactness, we can assume $p_{n_k} \to q \in K$. However, (43) is incompatible with the boundedness of $t|_K$. We have thus reached a contradiction.

Therefore, $\mathcal{A}^0(p)$ contains the group of translations, for μ-a.e. $p \in \mathbb{T}^3$. For $t \in \mathbb{R}$, $p \in \mathbb{T}^3$, we let $g = g_t \colon s \mapsto s + t$ and $c(p,t) \overset{\text{def.}}{=} \hat{\nu}_p^c([-t-1, -t+1])$, so that

$$\frac{dg_* \hat{\nu}_p^c}{d\hat{\nu}_p^c} = c(p,t). \tag{44}$$

Let us see some properties of the function $(p,t) \mapsto c(p,t)$. Firstly, for μ-a.e. $p \in \mathbb{T}^3$, $\mathcal{A}^0(p)$ contains all translations. This implies that $\hat{\nu}_p^c$ has no atom, which implies that $c(p, \cdot)$ is continuous. We claim that

$$c(p,t) = c(p_n, \lambda_p^c(n)t), \tag{45}$$

where $p_n = f^n(p)$. To prove the claim, consider $\psi = \Lambda_{p,n}^c \circ g \circ (\Lambda_{p,n}^c)^{-1}$ and observe that $\psi(s) = s + \lambda_p^c(n)t$. From the definition it follows that

$$c(p_n, \lambda_p^c(n)) = \hat{\nu}_{p_n}^c(\psi^{-1}([-1,1])) = \frac{\hat{\nu}_{p_n}^c(\psi^{-1}([-1,1]))}{\hat{\nu}_{p_n}^c([-1,1])}.$$

In the last equality we have used our normalization choice $\hat{\nu}_{p_n}^c([-1,1]) = 1$ for leaf-wise quotient measures. Applying now Lemma 7.12 we deduce

$$\begin{aligned}
\frac{\hat{\nu}_{p_n}^c(\psi^{-1}([-1,1]))}{\hat{\nu}_{p_n}^c([-1,1])} &= \frac{\hat{\nu}_p^c((\lambda_p^c(n))^{-1} \times \psi^{-1}([-1,1]))}{\hat{\nu}_p^c((\lambda_p^c(n))^{-1} \times [-1,1])} \\
&= \frac{\hat{\nu}_p^c(g^{-1}(\lambda_p^c(n))^{-1} \times ([-1,1])))}{\hat{\nu}_p^c((\lambda_p^c(n))^{-1} \times [-1,1])} \\
&= \frac{g_* \hat{\nu}_p^c(\lambda_p^c(n))^{-1} \times ([-1,1]))}{\hat{\nu}_p^c((\lambda_p^c(n))^{-1} \times [-1,1])} \\
&= c(p,t), \tag{46}
\end{aligned}$$

where on the second equality we applied the definition of ψ and in the last equality we have used the fact that the measures $g_* \hat{\nu}_p^c$ and $\hat{\nu}_p^c$ are proportional by a factor precisely equal to $c(p,t)$. This establishes (45).

We now give the final argument for completing the proof. For each $\varepsilon > 0$, let $r > 0$ be chosen such that the set

$$B_{r,\varepsilon} \overset{\text{def.}}{=} \{p \in \mathbb{T}^3 : |c(q,t) - 1| < \varepsilon, \quad \forall |t| < r\}$$

satisfies $\mu(B_{r,\varepsilon}) > 0$. By ergodicity, μ-a.e. point $p \in \mathbb{T}^3$ visits $B_{r,\varepsilon}$ infinitely many times both in future and past; but (45) implies that $|c(p,t) - 1| < \varepsilon$ for all $t \in \mathbb{R}$ and μ-a.e. $p \in \mathbb{T}^3$. Letting $\varepsilon \to 0$, we conclude that $c(p,t) = 1$ for all $t \in \mathbb{R}$ and μ-a.e. $p \in \mathbb{T}^3$, hence, by (44), $\hat{\nu}_p^c$ is invariant under the group of translations, for μ-a.e. $p \in \mathbb{T}^3$, hence it is proportional to Lebesgue measure. $\square$

8.1. **Drift along the center.** Recall that the normal forms $\{\Phi_x^c \colon \mathbb{R} \to \mathcal{W}^c(x)\}_{x \in \mathbb{T}^3}$ give us parametrizations of the center manifolds whose change of coordinates are affine maps. Using these changes of coordinates one can build the maps $\psi \colon \mathbb{R} \to \mathbb{R}$ promised in Lemma 8.1. Indeed, we claim that it suffices to prove the result below.

Proposition 8.2. *There exist constants $M > 0$ and $\delta_0 \in (0,1)$ such that for every $\varepsilon > 0$ sufficiently small one can find a compact set $G = G(\varepsilon) \subset \mathbb{T}^3$ so that $\mu(G) \geq \delta_0$ and for every $p \in G$ there exists a point $q \in \mathcal{W}_1^c(p)$, so that*

$$M^{-1}\varepsilon \leq |\mathcal{H}_p^c(q)| \leq M\varepsilon \text{ and } \hat{\nu}_q^c \propto B_* \hat{\nu}_p^c, \tag{47}$$

for some linear map $B \colon \mathbb{R} \to \mathbb{R}$ of the form $s \mapsto \beta \cdot s$ so that $M^{-1} < |\beta| < M$.

8.1.1. *Proof of Lemma 8.1 assuming Proposition 8.2.* We only need to show that the set G of Proposition 8.2 satisfies the claims in Lemma 8.1 for a suitably chosen constant M (which will be perhaps a bit larger than the one already given by the proposition). To see this, take $q \in \mathcal{W}_1^c(p)$ for some $p \in \mathbb{T}^3$ and let us, for the sake of this proof, denote by $\xi = \mathcal{H}_{p,q}^c = \mathcal{H}_p^c \circ \Phi_q^c \colon \mathbb{R} \to \mathbb{R}$ the affine change of normal form coordinates along the center direction. Because normal forms depends continuously with respect to the base point, we may enlarge M if necessary so that $|\xi'(0)| \in (M^{-1}, M)$.

Now, take G the set given by Proposition 8.2 and let $p \in G$. We shall construct the affine map $\psi \colon \mathbb{R} \to \mathbb{R}$ claimed by the lemma. For this take $q \in \mathcal{W}_1^c(p)$ given by Proposition 8.2 and notice that, at one hand, Lemma 7.12 gives us that $\hat{\nu}_q^c \propto \xi_* \hat{\nu}_p^c$. On the other hand, Proposition 8.2 says that for some linear map $B \colon \mathbb{R} \to \mathbb{R}$ with derivative β bounded in between M^{-1} and M we have $\hat{\nu}_q^c \propto B_* \hat{\nu}_p^c$. These two properties give us that $\hat{\nu}_p^c \propto \psi_* \hat{\nu}_p^c$, where $\psi = B^{-1}\xi$ is an affine map. The bounds we have on β and on $|\xi'(0)|$ give

$$|\psi'(0)| \in (M^{-2}, M^2).$$

Moreover, by Proposition 8.2

$$|\psi(0)| = |B^{-1}\xi(0)| = |B^{-1}\mathcal{H}_p^c(q)| \in (M^{-2}\varepsilon, M^2\varepsilon).$$

This shows that ψ satisfies all the requirements in Lemma 8.1 with $M_0 = M^2$, thus completing the proof. □

8.1.2. *The Lusin set.* We now move on to the proof of Proposition 8.2, where our key arguments are concentrated. Our approach here draws inspiration from Eskin-Lindenstrauss' work [14]. To offer some insight of our reasoning, recall from Lemma 7.13 that the measures $\hat{\nu}_x^c$ change linearly when we shift the base point from x to $f^n(x)$, with the slope of this linear map being $\|Df^n(x)|_{E^c}\|$. Our strategy involves identifying specific dynamical configurations in which points, *almost* on the same center leaf, drift apart by a center distance proportional to ε. In these configurations, their corresponding leaf-wise quotient measures are *almost* proportional to each other, modulo a linear map whose slope can be controlled using the dynamics. To pinpoint the precise set where "almost" turns into "equality", we take limits. To achieve this we must consider points belonging to compact sets restricted to which the assignment $x \mapsto \hat{\nu}_x^c$ has good properties.

With that goal in mind, we denote by $C_c^0(\mathbb{R})$ the space of continuous functions with compact support of the real line.

Lemma 8.3. *For every $\delta > 0$ there exists a compact set $A \subset \mathbb{T}^3$ with $\mu(A) > 1-\delta$ such that for all $\{x_n\}_{n\in\mathbb{N}} \subset A$ converging to $x \in A$ the following holds*

$$\int \varphi d\hat{\nu}_{x_n}^c \to \int \varphi d\hat{\nu}_x^c, \quad \text{for every } \varphi \in C_c^0(\mathbb{R}).$$

Proof. Let us denote $K \stackrel{\text{def.}}{=} \text{supp}(\varphi)$. We first claim that, given $\varphi \in C_c^0(\mathbb{R})$, the function $x \in \mathbb{T}^3 \mapsto \int \varphi d\hat{\nu}_x^c \in \mathbb{R}$ is measurable. Indeed, consider the set

$$\Lambda(\varepsilon) = \{x \in \mathbb{T}^3; \mathcal{W}_\varepsilon^c(x) \subset \xi_0^c(x)\}.$$

Since $\mu(\Lambda(\varepsilon)) \to 1$ as $\varepsilon \to 0$ it suffices to check that, for a given $\varepsilon > 0$, the restriction of the function to the corresponding set $\Lambda(\varepsilon)$ is measurable (notice that the set $\Lambda(\varepsilon)$ itself is measurable). By compactness of K and uniform continuity

of the C^1-norm of our normal forms in compact sets, Lemma 2.19 ensures the existence of an $n \in \mathbb{N}$ such that

$$K \subset \zeta_n^c(x), \quad \text{for every} \quad x \in \Lambda(\varepsilon). \tag{48}$$

Define $\hat{\varphi} : \mathbb{R}^2 \to \mathbb{R}$ as $\hat{\varphi}(t,s) = \varphi(s)$. Then, denoting $\alpha(x) = 1/\nu_{n,x}^{cu}(I \times [-1,1])$, it follows from (48) that

$$\int \varphi d\hat{\nu}_x^c = \alpha(x) \int \hat{\varphi} \circ \mathcal{H}_x(z) d\mu_{n,x}^{cu}(z),$$

for every $x \in \Lambda(\varepsilon)$. Since the right-hand side above depends measurably on $x \in \Lambda(\varepsilon)$ our claim is proved.

As a second step in our proof, we claim the existence of a countable dense subset $\{\varphi_\ell\}_{\ell \in \mathbb{N}} \subset C_c^0(\mathbb{R})$ with the following property: for a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{T}^3$ converging to a point $x \in \mathbb{T}^3$, if

$$\int \varphi_\ell d\hat{\nu}_{x_n}^c \to \int \varphi_\ell d\hat{\nu}_x^c,$$

for all $\ell \in \mathbb{N}$, then we can conclude that

$$\int \varphi d\hat{\nu}_{x_n}^c \to \int \varphi d\hat{\nu}_x^c,$$

for every $\varphi \in C_c^0(\mathbb{R})$.

To construct such a dense subset, start with a countable dense subset $\mathcal{F}$ of $C_c^0(\mathbb{R})$ with the following property: if φ is in $C_c^0(\mathbb{R})$ and $K = \mathrm{supp}(\varphi)$, then define $\hat{K} = [\inf K - 1, \sup K + 1]$. For every $\varepsilon > 0$ small enough there exists $\psi \in \mathcal{F}$ such that $\|\varphi - \psi\|_\infty < \varepsilon$ and $\mathrm{supp}(\psi) \subset \hat{K}$. The existence of $\mathcal{F}$ follows from standard arguments.

Now, augment $\mathcal{F}$ by adding, for each positive integer n, a continuous bump function that equals 1 inside the interval $[-n, n]$ and 0 outside $(-n-1, n+1)$. The resulting set, still denoted as $\mathcal{F}$, remains countable and dense, but now has the property that for any compact set $K \subset \mathbb{R}$, there exists some function $\psi \in \mathcal{F}$ such that $\psi(x) \geq 1$ for every $x \in K$. To show that $\mathcal{F}$ meets the claim's requirements, consider φ in $C_c^0(\mathbb{R})$ and let K and $\hat{K}$ be as above. Choose $\psi \in \mathcal{F}$ such that $\psi|_{\hat{K}} \geq 1$. Then,

$$\hat{\nu}_{x_n}^c(\hat{K}) \leq \int \psi d\hat{\nu}_{x_n}^c.$$

By assumption, the right-hand side above converges to $\int \psi d\hat{\nu}_x^c$. Thus, $\hat{\nu}_{x_n}^c(\hat{K})$ is bounded by some number $\beta(K) > 0$, independent of n. Now, given $\varepsilon > 0$, choose $\hat{\psi}$ in the countable dense family such that

$$\|\hat{\psi} - \varphi\|_\infty < \frac{\varepsilon}{3\beta(K)},$$

and with $\mathrm{supp}(\hat{\psi}) \subset \hat{K}$. Since $\hat{\psi}$ is in $\mathcal{F}$ there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$

$$\left| \int \hat{\psi} d\hat{\nu}_{x_n}^c - \int \hat{\psi} d\hat{\nu}_x^c \right| < \frac{\varepsilon}{3}.$$

Thus, if $n \geq n_0$ we have

$$
\begin{aligned}
\left| \int \varphi d\hat{\nu}_{x_n}^c - \int \varphi d\hat{\nu}_x^c \right| &\leq \left| \int \varphi d\hat{\nu}_{x_n}^c - \int \hat{\psi} d\hat{\nu}_{x_n}^c \right| + \left| \int \hat{\psi} d\hat{\nu}_{x_n}^c - \int \hat{\psi} d\hat{\nu}_x^c \right| \\
&\quad + \left| \int \hat{\psi} d\hat{\nu}_x^c - \int \varphi d\hat{\nu}_x^c \right| \\
&< \frac{\varepsilon \hat{\nu}_{x_n}^c(\hat{K})}{3\beta(K)} + \frac{\varepsilon}{3} + \frac{\varepsilon \hat{\nu}_x^c(\hat{K})}{3\beta(K)} \\
&\leq \varepsilon.
\end{aligned}
$$

This proves our second claim. To conclude the proof of the lemma, take $\mathcal{F} = \{\varphi_\ell\}_{\ell \in \mathbb{N}}$ as the countable dense family of our previous claim. For each ℓ, according to our first claim, the function $g_\ell : x \in \mathbb{T}^3 \mapsto \int \varphi_\ell d\hat{\nu}_x^c$ is measurable. By Lusin's theorem there exists a compact set $A_\ell \subset \mathbb{T}^3$ with measure $\mu(A_\ell) > 1 - \delta/2^{\ell+1}$ and such that $g_\ell|_{A_\ell}$ is continuous. Define the compact set $A = \bigcap_{\ell=0}^{\infty} A_\ell$. Then, $\mu(A) > 1 - \delta$ and if a sequence $\{x_n\}_{n \in \mathbb{N}} \subset A$ converges to x, we have $g_\ell(x_n) \to g_\ell(x)$ for every ℓ, and the lemma follows from the above claims. $\qquad\square$

Let us take $\mathcal{L} \subset \mathbb{T}^3$ a compact set with large measure (we shall give precise estimates below) such that the conclusion of Lemma 8.3 holds for $A = \mathcal{L}$. We shall require further the following properties:

(1) For $\sigma = s, c, u, cu$ given ξ^σ, a measurable partition subordinate to $\mathcal{W}^\sigma$, there exists $r_0 > 0$ so that for every $x \in \mathcal{L}$,

$$
\mathcal{W}_{r_0}^\sigma(x) \subset \xi^\sigma(x).
$$

(2) For every $x \in \mathcal{L}$, $\xi^u(x)$ is an interval whose size $r^u(x)$ varies continuously with $x \in \mathcal{L}$ (see Remark 2.15).

The continuity of the function $\mathcal{L} \ni x \mapsto \hat{\nu}_x^c$ together with Remark 7.11 imply the following.

Corollary 8.4. *For each $M > 1$ there exists $c = c(M) > 0$ such that if $B : \mathbb{R} \to \mathbb{R}$ is a linear map with derivative in $[M^{-1}, M]$ then $\hat{\nu}_x^c(B^{-1}[-1,1]) \in [c^{-1}, c]$ for each $x \in \mathcal{L}$.*

Proof. Assume by contradiction that a lower bound does not hold. Denote $J_0 = [-M^{-1}, M^{-1}]$. Then, for each n there must exist $x_n \subset \mathcal{L}$ such that $\hat{\nu}_{x_n}^c(J_0) \leq 1/n$. We can assume, by compactness of $\mathcal{L}$, that $x_n \to x \in \mathcal{L}$. Since $y \in \mathcal{L} \mapsto \hat{\nu}_y^c$ is continuous, we have that

$$
\hat{\nu}_x^c(J_0) \leq \liminf_{n \to \infty} \hat{\nu}_{x_n}^c(J_0) = 0,
$$

which violates Remark 7.11. By a similar reasoning, considering $J_1 = [-M, M]$ and assuming that the upper bound does not hold we would obtain a point $x \in \mathcal{L}$ for which $\hat{\nu}_x^c(J_1) = \infty$, which is also impossible by local finiteness. $\quad\square$

8.1.3. *The main proposition.* Let us fix two numbers δ and δ_0 such that

$$
0 < \delta_0 < \frac{1}{10}, \qquad \text{and} \qquad \delta = 3\delta_0.
$$

We will require $\mu(\mathcal{L}) > 1 - \delta_0$. The constant δ_0 is the constant appearing in Lemma 8.1 and Proposition 8.2. Further assumption will be given on δ and δ_0 in Section 11, and more precisely in §11.3.6.

Given $M, \varepsilon > 0$ set

$$G(\varepsilon, M) \stackrel{\text{def.}}{=} \left\{ p \in \mathbb{T}^3 : \begin{array}{l} \exists\, q \in \mathcal{W}_1^c(p),\ \exists\, B(s) = \beta \cdot s \text{ linear map such that} \\ \hat{\nu}_q^c \propto B_* \hat{\nu}_p^c \\ M^{-1}\varepsilon \leq |\mathcal{H}_p^c(q)| \leq M\varepsilon \\ M^{-1} \leq |\beta| \leq M \end{array} \right\}.$$

(49)

Notice that $G(\varepsilon, M)$ is nothing but the set of points $p \in \mathbb{T}^3$ for which the conclusion of Proposition 8.2 holds with constants ε, M. Recall that in order to prove Proposition 8.2, and thus Lemma 8.1, we must show that $\mu(G(\varepsilon, M)) \geq \delta_0$.

Given this choice of parameters, Proposition 8.2 follows from the statement below.

Proposition 8.5. *There exists $M > 0$ such that for $\varepsilon > 0$ small enough and every compact set K_{00} with $\mu(K_{00}) > 1 - 2\delta_0$,*

$$K_{00} \cap \mathcal{L} \cap G(\varepsilon, M) \neq \emptyset.$$

Proof that Proposition 8.5 $\implies$ Proposition 8.2. Let $M > 0$ be the constant given in the statement above. Assume by contradiction that Proposition 8.2 does not hold with the constants M and δ_0. Then, for some small $\varepsilon > 0$ we must have $\mu(G(\varepsilon, M)) < \delta_0$. By regularity of μ there exists an open neighbourhood U of $G(\varepsilon, M)$ such that $\mu(U) < \delta_0$. Let $K_{00} \stackrel{\text{def.}}{=} \mathbb{T}^3 \setminus U$. This set is compact and satisfies $\mu(K_{00}) \geq 1 - \delta_0 > 1 - 2\delta_0$. Proposition 8.5 then implies that $K_{00} \cap G(\varepsilon, M) \neq \emptyset$, which is absurd. $\square$

In order to start the proof of Proposition 8.5, fix $\varepsilon > 0$ (independently of δ: notice that while δ is fixed we need to take $\varepsilon \to 0$). More assumptions on ε will be given later on. This is the expected size of the drift we want to see along the center direction. Given a compact set K_{00} whose measure is larger than $1 - 2\delta_0$, notice that the compact set

$$K_0 \stackrel{\text{def.}}{=} K_{00} \cap \mathcal{L}.$$

has measure $\mu(K_0) > 1 - 3\delta_0 = 1 - \delta$. We shall prove that $K_0 \cap G(\varepsilon, M) \neq \emptyset$. This will occupy the rest of the paper.

9. Stopping times, Y-configurations, quadrilaterals and synchronization

The goal of this section is to introduce the key dynamical ingredients involved in the proof of Proposition 8.5, which is the main part of our implementation of an exponential drift argument. We also use this section to derive some estimates relating these ingredients.

9.1. **Stopping times.** We introduce below the stopping time functions. They are devised to measure the appropriate time length of the top part of Figure 5 so that we get the precise drift we want along the center direction.

9.1.1. *Definition of stopping times.* Recall our concise notations for derivatives

$$\lambda_x^\star(n) = \|Df^n(x)|_{E^\star}\| \quad \star = c, s, u,$$

and

$$d_x^\ell = \frac{\lambda_{x-\ell}^c(\ell)}{\lambda_{x-\ell}^u(\ell)}$$

introduced in (2) of §2.1.2 and (4) of §2.1.3, respectively, where $x_n = f^n(x)$ is our concise notation for orbit points introduced in (1) of §2.1.2. Given $x \in \mathbb{T}^3$, $x^u \in \mathcal{W}^u(x)$, $\varepsilon > 0$, and $\ell \in \mathbb{N}$, we define

$$\tau(\ell) = \tau(x, x^u, \varepsilon, \ell) \stackrel{\text{def.}}{=} \inf \left\{ n \in \mathbb{N} : d_x^\ell \times \lambda_{x^u}^c(n) \geq \varepsilon \right\}.$$

We also define

$$t(\ell) = t(x, x^u, \varepsilon, \ell) \stackrel{\text{def.}}{=} \inf \left\{ n \in \mathbb{N} : \frac{\lambda_x^c(n)}{\lambda_{x^u}^c(\tau(\ell))} \geq 1 \right\}.$$

These functions are called the *stopping times.*

9.1.2. *Quasi-isometric estimates.* In the following, we fix $\varepsilon > 0$. Recall that for $x \in \mathbb{T}^3$, $x^u \in \mathcal{W}^u(x)$, we abbreviate $\tau(\ell) = \tau(x, x^u, \varepsilon, \ell)$ and $t(\ell) = t(x, x^u, \varepsilon, \ell)$.

Lemma 9.1 (Quasi-isometric estimates)**.** *There exists $\Theta = \Theta(f) > 1$ and $A = A(f) > 0$ so that given $\ell, m \in \mathbb{N}$, $x \in \mathbb{T}^3$ and $x^u \in \mathcal{W}^u(x)$ the following holds*

(1) $\Theta^{-1} m - A < \tau(\ell + m) - \tau(\ell) < \Theta m + A$ and
(2) $\Theta^{-1} m - A < t(\ell + m) - t(\ell) < \Theta m + A$.

Proof. In this proof we shall make use of the constants of hyperbolicity of f introduced in §2.1.3. With these constants at hand, we can now develop the argument for the *quasi-isometric estimates*. For this, fix $x \in \mathbb{T}^3$ and $x^u \in \mathcal{W}^u(x)$. Given some $\varepsilon > 0$, let us consider $\tau(\ell) = \tau(x, x^u, \varepsilon, \ell)$ and $t(\ell) = t(x, x^u, \varepsilon, \ell)$. From the definition of τ, we have that

$$d_x^\ell \lambda_{x^u}^c(\tau(\ell)) \geq \varepsilon \quad \text{and} \quad d_x^\ell \lambda_{x^u}^c(\tau(\ell) - 1) < \varepsilon, \quad \text{for each } \ell \in \mathbb{N}.$$

Thus,

$$\varepsilon \leq d_x^\ell \lambda_{x^u}^c(\tau(\ell)) < e^{\chi_1^c} \varepsilon, \quad \text{for each } \ell \in \mathbb{N}. \tag{50}$$

Fix $\ell, m \in \mathbb{N}$. On the one hand, by the cocycle property (3) we have that $d_x^{\ell+m} = d_x^\ell d_{x_\ell}^m$ and therefore we can use (50) to obtain

$$\varepsilon e^{m \chi_1^d} < d_x^{\ell+m} \lambda_{x^u}^c(\tau(\ell)) < \varepsilon e^{\chi_1^c + m \chi_2^d}.$$

On the other hand, we can use the cocycle property once more to write $\lambda_{f^{\tau(\ell)}(x^u)}^c(\tau(\ell+m) - \tau(\ell)) = \frac{\lambda_{x^u}^c(\tau(\ell+m))}{\lambda_{x^u}^c(\tau(\ell))}$ and thus obtain

$$e^{\chi_2^c(\tau(\ell+m) - \tau(\ell))} < \frac{\lambda_{x^u}^c(\tau(\ell+m))}{\lambda_{x^u}^c(\tau(\ell))} < e^{\chi_1^c(\tau(\ell+m) - \tau(\ell))}.$$

Therefore, since $d_x^{\ell+m} \lambda_{x^u}^c(\tau(\ell+m)) = d_x^{\ell+m} \lambda_{x^u}^c(\tau(\ell)) \frac{\lambda_{x^u}^c(\tau(\ell+m))}{\lambda_{x^u}^c(\tau(\ell))}$ we can combine the two above inequalities and obtain

$$\varepsilon e^{m \chi_1^d + \chi_2^c(\tau(\ell+m) - \tau(\ell))} < d_x^{\ell+m} \lambda_{x^u}^c(\tau(\ell+m)) < \varepsilon e^{\chi_1^c + m \chi_2^d + \chi_1^c(\tau(\ell+m) - \tau(\ell))}.$$

Putting $\ell + m$ instead of ℓ in (50) and combining with this we get the inequalities

$$\varepsilon e^{\chi_1^c + m \chi_2^d + \chi_1^c(\tau(\ell+m) - \tau(\ell))} > \varepsilon \quad \text{and} \quad \varepsilon e^{-\chi_1^c + m \chi_1^d + \chi_2^c(\tau(\ell+m) - \tau(\ell))} < \varepsilon.$$

Dividing by ε and taking logarithms we deduce that

$$m \left[\frac{-\chi_2^d}{\chi_1^c} \right] - 1 < \tau(m + \ell) - \tau(\ell) < m \left[\frac{-\chi_1^d}{\chi_2^c} \right] + \frac{\chi_1^c}{\chi_2^c}. \tag{51}$$

Let us deal with the function $\ell \mapsto t(\ell)$. From its very definition we have an inequality analogous to (50):

$$1 \leq \frac{\lambda_x^c(t(\ell))}{\lambda_{x^u}^c(\tau(\ell))} < e^{\chi_1^c}, \quad \text{for every } x \in \mathbb{T}^3, \ell \in \mathbb{N}. \tag{52}$$

To simplify the remainder of the exposition, we shall denote $\tau_{\ell,m} \overset{\text{def.}}{=} \tau(\ell + m) - \tau(\ell)$ and $t_{\ell,m} = t(\ell + m) - t(\ell)$. Notice that

$$e^{\chi_2^c t_{\ell,m} - \chi_1^c \tau_{\ell,m}} < \frac{\lambda_{f^{t(\ell)}(x)}^c \left(t(\ell + m) - t(\ell)\right)}{\lambda_{f^{\tau(\ell)}(x^u)}^c \left(\tau(\ell + m) - \tau(\ell)\right)} < e^{\chi_1^c t_{\ell,m} - \chi_2^c \tau_{\ell,m}}.$$

Combining this with (52), we can use the decomposition

$$\frac{\lambda_x^c \left(t(\ell + m)\right)}{\lambda_{x^u}^c \left(\tau(\ell + m)\right)} = \frac{\lambda_x^c \left(t(\ell)\right) \lambda_{f^{t(\ell)}(x)}^c \left(t(\ell + m) - t(\ell)\right)}{\lambda_{x^u}^c \left(\tau(\ell)\right) \lambda_{f^{\tau(\ell)}(x^u)}^c \left(\tau(\ell + m) - \tau(\ell)\right)}$$

and conclude that

$$e^{\chi_2^c t_{\ell,m} - \chi_1^c \tau_{\ell,m}} < \frac{\lambda_x^c \left(t(\ell + m)\right)}{\lambda_{x^u}^c \left(\tau(\ell + m)\right)} < e^{\chi_1^c t_{\ell,m} - \chi_2^c \tau_{\ell,m} + \chi_1^c}.$$

Now, using (52) with $\ell + m$ instead of ℓ, the above inequality implies that

$$e^{\chi_2^c t_{\ell,m} - \chi_1^c \tau_{\ell,m}} < e^{\chi_1^c} \quad \text{and} \quad 1 < e^{\chi_1^c t_{\ell,m} - \chi_2^c \tau_{\ell,m} + \chi_1^c},$$

and thus

$$e^{\chi_2^c t_{\ell,m} - \chi_1^c \tau_{\ell,m} - \chi_1^c} < 1 < e^{\chi_1^c t_{\ell,m} - \chi_2^c \tau_{\ell,m} + \chi_1^c}.$$

Taking logarithms leads us to

$$\left[\frac{\chi_2^c}{\chi_1^c}\right] \tau_{\ell,m} - 1 < t_{\ell,m} < \left[\frac{\chi_1^c}{\chi_2^c}\right] \tau_{\ell,m} + \frac{\chi_1^c}{\chi_2^c},$$

which combined with (51) ends the proof. □

9.2. Y-configurations.

We introduce below a dynamical ingredient inspired by [14, 15]. They allow us to "decompose" Figure 5 in the "x-side" and in the "y-side". Notice that each side has a kind of Y-shape. The idea from [14] for getting the points inside the Lusin set is to try to prove the existence of a large amount of these Y-shaped dynamical configurations and then try to find some of them which are linked through stable manifolds, as it appears in Figure 5. These measure theoretical arguments will be developed in Section 11. In this section, we introduce these configurations and establish *synchronization and (center) drift estimates for them.*

9.2.1. *Definition.* Given $\varepsilon > 0$ and $\ell \in \mathbb{N}$, a *Y-configuration* $Y = Y(x, x^u, \ell)$ is a quintuple of points $(x, x^u, x_{-\ell}, x_\tau^u, x_t)$, that depends on parameters $x, x^u \in \mathbb{T}^3$ and ℓ (the dependence on ε is implicit throughout the text), and such that

    (1) $x^u \in \mathcal{W}^u(x)$;
    (2) $x_{-\ell} = f^{-\ell}(x)$;
    (3) $x_\tau^u = f^\tau(x^u)$, where $\tau = \tau(x, x^u, \varepsilon, \ell)$;
    (4) $x_t = f^t(x^u)$, where $t = t(x, x^u, \varepsilon, \ell)$.

where $\tau(x, x^u, \varepsilon, \ell)$ and $t(x, x^u, \varepsilon, \ell)$ are the stopping times defined above.

We call ℓ the *length* of the Y-configuration. Moreover, given a set $\Lambda \subset \mathbb{T}^3$, we say that a Y-configuration $(x, x^u, x_{-\ell}, x_\tau^u, x_t)$ is Λ-*good* if $x, x^u, x_\tau^u, x_t \in \Lambda$.

In [14], the authors defined a notion of *pairs of coupled Y-configurations.* For technical reasons we shall replace this notion by that of pairs of *matched* Y-configurations that will appear in the next section.
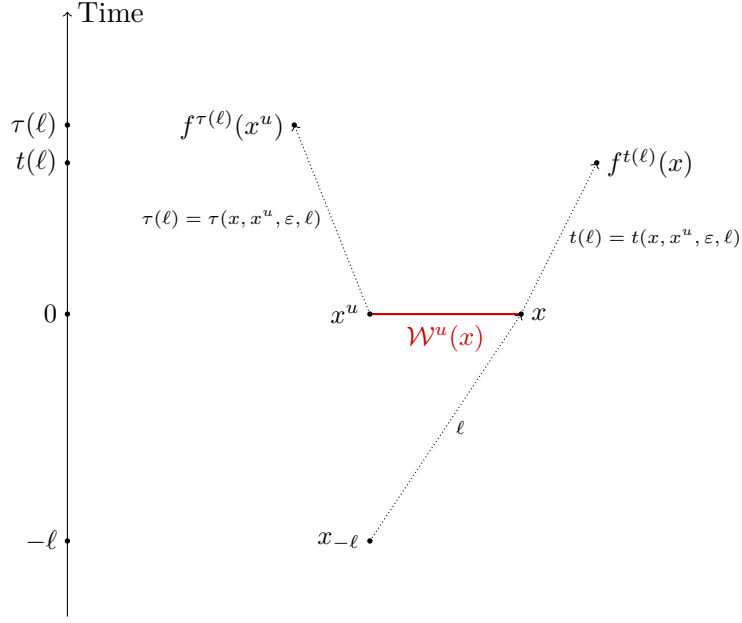
FIGURE 10.   A Y-configuration.

9.3. **Quadrilaterals and synchronization.** Another important aspect of Figure 5 is located at its middle, where we have a kind of a quadrilateral, twisted along the unstable direction. This, indeed, is the third dynamical ingredient in our implementation of an exponential drift argument. It will allow us to use the angle condition and to define the aforementioned notion of matching of Y-configurations.

9.3.1. *Quadrilaterals.* A *quadrilateral* is a quadruple $(x, x^u, y, y^u) \in (\mathbb{T}^3)^4$ such that

    (1) $y \in \mathcal{W}^s(x)$;
    (2) $x^u \in \mathcal{W}^u(x)$ belongs to the domain of a center-holonomy map $H^{cs}_{x,y}$;
    (3) $y^u = H^{cs}_{x,y}(x^u) \in \mathcal{W}^u(y) \cap \mathcal{W}^{cs}(x^u)$.

For such a quadrilateral, we define the point $z^u \overset{\text{def.}}{=} H^s_{x,y}(x^u)$, so that $z^u \in \mathcal{W}^s(x^u) \cap \mathcal{W}^c(y^u)$. Moreover, given $C > 1$, $\ell \in \mathbb{N}$, we say that (x, x^u, y, y^u) is a (C, ℓ)-*quadrilateral* if, besides,

    (4) $C^{-1} < d(x_{-\ell}, y_{-\ell}) < 1$,
    (5) $C^{-1} < \alpha^s(x_{-\ell}, y_{-\ell}) < C$ and
    (6) $C^{-1} < d(x, x^u) < C$,

where, as before, $x_{-\ell} = f^{-\ell}(x)$, $y_{-\ell} = f^{-\ell}(y)$.

9.3.2. *Drift estimates.* In Figure 5 we claim that the displacement along the center at the left top of the configuration is proportional to the parameter ε. We shall now make this assertion precise in terms of normal form coordinates. We first introduce a useful notation that will be used throughout the text, and in particular in the next proof.

    Given a set of real parameters $c_1, \ldots, c_n$ and two quantities a and b (that may or may not depend on other variables) we denote $a \asymp_{c_1, \ldots, c_n} b$ if there exists a real valued function $\rho = \rho(c_1, \ldots, c_n) \geq 1$ such that

$$\rho^{-1} a \leq b \leq \rho a.$$
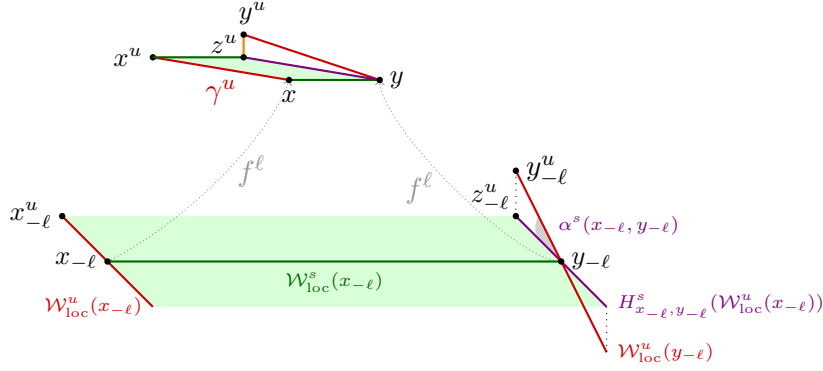
FIGURE 11.   A quadrilateral and its pre-image by f^ℓ.

Lemma 9.2. *Fix $\varepsilon > 0$. Given $C > 1$ and $T > 0$, there exists a constant $\kappa = \kappa(C, T) > 1$ such that for every $\ell \in \mathbb{N}$ large enough, every (C, ℓ)-quadrilateral (x, x^u, y, y^u), and every $\tau \geq 0$ such that $|\tau - \tau(y, y^u, \varepsilon, \ell)| \leq T$ we have*

$$\frac{\varepsilon}{\kappa} < \left| \mathcal{H}^c_{f^\tau(z^u)}(f^\tau(y^u)) \right| < \kappa \varepsilon. \tag{53}$$

Proof. Let $\gamma^u = [x, x^u] \subset \mathcal{W}^u(x)$ be the segment of strong unstable manifold connecting the points x and x^u. See Figure 11. Then, $f^{-\ell} \circ H^s_{x,y}(\gamma^u)$ is a C^1 curve joining $y_{-\ell}$ and $z^u_{-\ell}$.[2] Since, $f^{-\ell} \circ H^s_{x,y}(\gamma^u) = H^s_{x_{-\ell}, y_{-\ell}} \circ f^{-\ell}(\gamma^u)$ the assumption $C^{-1} < \alpha^s(x_{-\ell}, y_{-\ell}) < C$ implies that the vector tangent to the curve $f^{-\ell} \circ H^s_{x,y}(\gamma^u)$ at $y_{-\ell}$ and the strong unstable direction $E^u(y_{-\ell})$ are transverse with an angle between C^{-1} and C. Recall that $\mathcal{H}_{y_{-\ell}}: \mathcal{W}^{cu}(y_{-\ell}) \to \mathbb{R}^2$ sends $\mathcal{W}^u(y_{-\ell})$ onto the horizontal axis (see Theorem 6.3). As the length of $f^{-\ell} \circ H^s_{x,y}(\gamma^u)$ is exponentially small with ℓ and the bundle E^u is continuous, a standard compactness argument then ensures that for some constant $c_1 = c_1(C) > 0$ and for ℓ large enough, the curve $\mathcal{H}_{y_{-\ell}} \circ f^{-\ell} \circ H^s_{x,y}(\gamma^u)$ on $\mathbb{R}^2$ is contained in the cone

$$\mathcal{C}_1 = \left\{ (v_1, v_2) \in \mathbb{R}^2 : c_1^{-1}|v_1| \leq |v_2| \leq c_1|v_1| \right\}.$$

Now, observe that $\mathcal{H}_y \circ H^s_{x,y}(\gamma^u) = \mathcal{H}_y \circ f^\ell \circ \Phi_{y_{-\ell}} \circ \mathcal{H}_{y_{-\ell}} \circ f^{-\ell} \circ H^s_{x,y}(\gamma^u)$ and that $\mathcal{H}_y \circ f^\ell \circ \Phi_{y_{-\ell}}$ is the linear map $(t, s) \in \mathbb{R}^2 \mapsto (\lambda^u_{y_{-\ell}}(\ell)t, \lambda^c_{y_{-\ell}}(\ell)s)$. These two observations put together imply that the curve $\mathcal{H}_y \circ H^s_{x,y}(\gamma^u)$ is contained in the cone

$$\mathcal{C}_2 = \left\{ (v_1, v_2) \in \mathbb{R}^2 : c_1^{-1} \frac{\lambda^c_{y_{-\ell}}(\ell)}{\lambda^u_{y_{-\ell}}(\ell)} |v_1| \leq |v_2| \leq c_1 \frac{\lambda^c_{y_{-\ell}}(\ell)}{\lambda^u_{y_{-\ell}}(\ell)} |v_1| \right\}.$$

Now let γ^c denote the piece of center manifold connecting z^u to y^u. Then $\mathcal{H}_y \circ \gamma^c$ connects the point $\mathcal{H}_y(z^u)$, whose distance to the origin only depends on the constant C and is bounded from above and below, to the point $\mathcal{H}_y(y^u)$, which lies on the horizontal axis. Since $\mathcal{H}_y(z^u) \in \mathcal{C}_2$, we deduce that (see Figure 12)

$$\text{length}(\mathcal{H}_y \circ \gamma^c) \asymp_C \frac{\lambda^c_{y_{-\ell}}(\ell)}{\lambda^u_{y_{-\ell}}(\ell)}.$$

Since $C^{-1} < d^u(x, x^u) < C$ and we have C^1 holonomies, we also have that $C^{-1} < d(z^u, y) < C$ (upon enlarging the constant C if necessary in order to take into account the action of holonomy maps). Recall that normal forms change

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[2]Recall our notation convention for orbit points: $p_n = f^n(p)$ for $n \in \mathbb{Z}$.
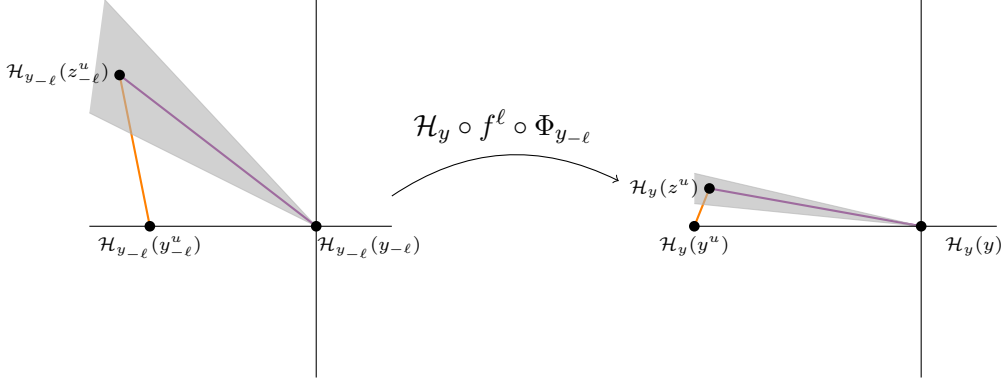
FIGURE 12. On normal form coordinates the dynamics on $\mathcal{W}^{cu}$ acts as a diagonal matrix, with stronger expansion on the horizontal.

continuously in the C^1 topology, so another compactness argument ensures that the C^1 norm of $\mathcal{H}_{z^u} \circ \Phi_y|_{W^{cu}_{2C}(y)}$ is bounded by some constant depending on C. Thus

$$\text{length}(\mathcal{H}_{z^u} \circ \gamma^c) \asymp_C \frac{\lambda^c_{y-\ell}(\ell)}{\lambda^u_{y-\ell}(\ell)}.$$

On the other hand, by the construction of normal forms, $\mathcal{H}_{z^u} \circ \gamma^c \subset \mathbb{R}^2$ is a vertical segment with length $|\mathcal{H}^c_{z^u}(y^u)|$. This proves that

$$|\mathcal{H}^c_{z^u}(y^u)| \asymp_C \frac{\lambda^c_{y-\ell}(\ell)}{\lambda^u_{y-\ell}(\ell)}.$$

As we did before it is possible to bound uniformly from above the C^1-norms of all $\mathcal{H}^c_*$ in restriction to center segments of uniform radius so (upon enlarging ℓ if necessary) we also have

$$d_c(y^u, z^u) \asymp_C \frac{\lambda^c_{y-\ell}(\ell)}{\lambda^u_{y-\ell}(\ell)}, \quad \text{and} \quad \left|\mathcal{H}^c_{y^u}(z^u)\right| \asymp_C \frac{\lambda^c_{y-\ell}(\ell)}{\lambda^u_{y-\ell}(\ell)}.$$

Since the dynamics on normal forms acts linearly, this implies

$$\left|\mathcal{H}^c_{f^{\tau(\ell)}(y^u)}(f^{\tau(\ell)}(z^u))\right| \asymp_C \frac{\lambda^c_{y-\ell}(\ell)}{\lambda^u_{y-\ell}(\ell)} \times \lambda^c_{y^u}(\tau(\ell))$$

where $\tau(\ell) = \tau(y, y^u, \varepsilon, \ell)$ is the stopping time. By definition this implies that

$$\left|\mathcal{H}^c_{f^{\tau(\ell)}(y^u)}(f^{\tau(\ell)}(z^u))\right| \asymp_C \varepsilon. \tag{54}$$

Since $|\tau - \tau(\ell)| \le T$ we obtain

$$\left|\mathcal{H}^c_{f^\tau(y^u)}(f^\tau(z^u))\right| \asymp_{C,T} \varepsilon. \quad \square$$

Corollary 9.3. *There exists a constant $\kappa = \kappa(C, T)$ such that for every τ satisfying $|\tau - \tau(y, y^u, \varepsilon, \ell)| \le T$ it holds*

$$d_c(f^\tau(y^u), f^\tau(z^u)) \le \kappa\varepsilon.$$

Proof. This follows from the proof of Lemma 9.2 and the fact that normal forms $\{\mathcal{H}^c_x\}_{x\in\mathbb{T}^3}$ are C^1 maps with continuously varying C^1 norm on compact sets. $\square$

9.3.3. *Synchronization estimates.* The last set of estimates we need concerns the oscillation of the stopping times from one side to the other when we have a (C, ℓ)-quadrilateral.

Lemma 9.4 (Synchronization for quadrilaterals)**.** *For any $C > 1$, there exists a constant $T_0 > 0$ depending only on f, C such that for any $\ell \in \mathbb{N}$, and any (C, ℓ)-quadrilateral (x, x^u, y, y^u), it holds*

$$|\tau(y, y^u, \varepsilon, \ell) - \tau(x, x^u, \varepsilon, \ell)| < T_0, \quad |t(y, y^u, \varepsilon, \ell) - t(x, x^u, \varepsilon, \ell)| < T_0.$$

Proof. Recall the constants $\chi_2^c, \chi_1^c > 0$ introduced in §2.1.3. They satisfy $e^{k\chi_2^c} \leq \lambda_x^c(k) \leq e^{k\chi_1^c}$ for every $k \in \mathbb{N}$ and every $x \in \mathbb{T}^3$ where $\lambda_x^c(k) = \|Df^k(x)|_{E^c}\|$ is our concise notation for derivatives from (2). For the sake of simplicity, denote $\tau(\ell) = \tau(x, x^u, \varepsilon, \ell)$ and $\tau'(\ell) = \tau(y, y^u, \ell, \varepsilon)$. Assume that $\tau(\ell) \geq \tau'(\ell)$. By definition

$$d_y^\ell \lambda_{y^u}^c(\tau'(\ell)) \geq \varepsilon. \tag{55}$$

Let us write for $k \in \mathbb{N}$

$$\frac{d_x^\ell \lambda_{x^u}^c(\tau'(\ell) + k)}{d_y^\ell \lambda_{y^u}^c(\tau'(\ell))} = \frac{d_x^\ell}{d_y^\ell} \times \frac{\lambda_{x^u}^c(\tau'(\ell))}{\lambda_{y^u}^c(\tau'(\ell))} \times \lambda_{x_{\tau'(\ell)}^u}^c(k). \tag{56}$$

We now bound from below each factor appearing in (56). We treat the third factor by observing that $\lambda_{x_{\tau'(\ell)}^u}^c(k) \geq e^{k\chi_2^c}$. To treat the first factor, we use that $y_{-\ell} \in \mathcal{W}_1^s(x_{-\ell})$ so the distortion control given by Corollary 2.10 gives $\frac{d_y^\ell}{d_x^\ell} \geq C_0^{-1}$ Finally, in order to treat the second factor, we notice that Corollary 9.3 provides that

$$d_c(f^j(z^u), f^j(y^u)) \leq C\varepsilon,$$

for some constant $C = C(f) > 0$ and for every $j = 0, \ldots, \tau'(\ell)$. With no loss of generality we can assume that $C\varepsilon < 1$. Applying the distortion control of Corollary 2.11 we obtain $\frac{\lambda_{x^u}^c(\tau'(\ell))}{\lambda_{y^u}^c(\tau'(\ell))} \geq C_0^{-1}$. These lower bounds together with (56) provide the following estimate

$$d_x^\ell \lambda_{x^u}^c(\tau'(\ell) + k) \geq \frac{e^{k\chi_2^c}}{C_0^2}\varepsilon \geq \varepsilon$$

as soon as we choose $k = k(f)$ such that

$$e^{k\chi_2^c} \geq C_0^2. \tag{57}$$

Hence by definition of the stopping time, we obtain $\tau'(\ell) + k \geq \tau(\ell)$ so $\tau(\ell) - \tau'(\ell) \leq k$. A symmetric argument shows that $\tau'(\ell) - \tau(\ell) \leq k$ if $\tau'(\ell) \geq \tau(\ell)$.

We consider now the stopping time t. As above, we denote $t(\ell) = t(x, x^u, \varepsilon, \ell)$ and $t'(\ell) = t'(y, y^u, \varepsilon, \ell)$. By definition

$$\frac{\lambda_y^c(t'(\ell))}{\lambda_{y^u}^c(\tau'(\ell))} \geq 1. \tag{58}$$

Let us write for $k' \in \mathbb{N}$:

$$\frac{\lambda_x^c(t'(\ell) + k')}{\lambda_{x^u}^c(\tau(\ell))} \frac{\lambda_{y^u}^c(\tau'(\ell))}{\lambda_y^c(t'(\ell)} = \frac{\lambda_{y^u}^c(\tau'(\ell))}{\lambda_{x^u}^c(\tau'(\ell))} \times \frac{\lambda_x^c(t'(\ell)}{\lambda_y^c(t'(\ell)}$$
$$\times \lambda_{x_{t'(\ell)}}^c(k') \times \frac{1}{\lambda_{x_{\tau'(\ell)}^u}^c(\tau(\ell) - \tau'(\ell))}.$$

We now bound from below each factor of the product above. We have already seen how to treat the first factor $\frac{\lambda_{x^u}^c(\tau'(\ell))}{\lambda_{y^u}^c(\tau'(\ell))} \geq C_0^{-1}$. To treat the second factor,

we use that $y \in \mathcal{W}_1^s(x)$ and apply Corollary 2.11 to get $\frac{\lambda_x^c(t'(\ell))}{\lambda_y^c(t'(\ell))} \geq C_0^{-1}$. We treat the third factor by observing that $\lambda_{x_{t'(\ell)}}^c(k') \geq e^{k'\chi_2^c}$. Note that $|\tau(\ell) - \tau'(\ell)| \leq k$ so the last bound follows from $\frac{1}{\lambda_{x_{\tau'(\ell)}^u}^c(\tau(\ell) - \tau'(\ell))} \geq e^{-k\chi_1^c}$. Finally we find

$$\frac{\lambda_x^c(t'(\ell) + k')}{\lambda_{x^u}^c(\tau(\ell))} \geq \frac{e^{k'\chi_2^c}}{C_0^2 e^{k\chi_1^c}} \times \frac{\lambda_y^c(t'(\ell))}{\lambda_{y^u}^c(\tau'(\ell))} \geq 1, \tag{59}$$

as soon as we choose $k' = k'(f)$ such that

$$e^{k'\chi_2^c} \geq C_0^2 e^{k\chi_1^c}. \tag{60}$$

Hence by definition we must have $t(\ell) \leq t'(\ell) + k'$ so $t(\ell) - t'(\ell) \leq k'$. Here again a symmetric argument yields $t'(\ell) - t(\ell) \leq k'$. Hence

$$T_0 = \max(k, k'), \tag{61}$$

is the desired constant. This ends the proof. $\qquad\square$

10. Matching of Y-configurations

In this section we develop a concept devised specifically to address the technical difficulty to implement the exponential drift idea we want to employ, a difficulty which was described at the end of Section 3. Namely, due to lack of absolute continuity of center stable holonomies we cannot ensure that the points x^u and y^u of Figure 5 belong both to the Lusin set. What we can actually prove is that small perturbations of these two points can indeed be put inside the Lusin set. This will lead us to the notion of matched configurations.

10.1. **Matching of dynamical balls.** We start the formal definition of matching in our scenario by introducing intervals along the unstable manifold which measure the amount of perturbation of the points $f^{\tau(\ell)}(x^u)$ and $f^{\tau(\ell)}(y^u)$ which are allowed, without breaking the estimates we performed for quadrilaterals in the previous section.

10.1.1. *Unstable dynamical balls.* Although the notion of dynamical balls is quite standard in ergodic theory, we use this name in this paper for a more specific object, adapted to our needs.

Definition 10.1. *Let $\varepsilon > 0$, $\ell \in \mathbb{N}$, $x \in \mathbb{T}^3$ and $x^u \in \mathcal{W}^u(x)$. Let $\tau(\ell) = \tau(x, x^u, \varepsilon, \ell)$. The (ε, ℓ)-unstable dynamical ball at x^u is defined as*

$$J(x^u) = J(x^u, \varepsilon, \ell) \stackrel{\text{def.}}{=} f^{-\tau(\ell)}\left(\mathcal{W}_1^u(f^{\tau(\ell)}(x^u))\right).$$

Remark 10.2. *By definition, for every $a \in J(x^u)$ and $j \in \{0, \ldots, \tau(x, x^u, \varepsilon, \ell)\}$*

$$d_u(f^j(x^u), f^j(a)) < 1.$$

10.1.2. *Synchronization inside a dynamical ball.* Before we define the notion of matching of Y-configuration it is useful to study the oscillation of stopping times inside an unstable dynamical ball.

Let $T_0 > 0$ be the constant obtained in Lemma 9.4.

Lemma 10.3. *For every $a \in J(x^u)$ it holds $|\tau(x, a, \varepsilon, \ell) - \tau(x, x^u, \varepsilon, \ell)| < T_0$ and $|t(x, a, \varepsilon, \ell) - t(x, x^u, \varepsilon, \ell)| < T_0$.*

Proof. The proof is almost identical to that of Lemma 9.4. In particular constants $0 < \chi_2^c < \chi_1^c$ are those defined in §2.1.3 and k and k', those defined by (57) and (60). Let us denote $\tau_a(l) \overset{\text{def.}}{=} \tau(x, a, \varepsilon, \ell)$. Suppose first that $\tau_a(\ell) \geq \tau(\ell)$.

$$\frac{d_x^\ell \lambda_a^c(\tau(\ell) + k)}{d_x^\ell \lambda_{x^u}^c(\tau(\ell))} = \frac{\lambda_a^c(\tau(\ell))}{\lambda_{x^u}^c(\tau(\ell))} \times \lambda_{a_{\tau(\ell)}}^c(k). \tag{62}$$

We have $\lambda_{a_{\tau(\ell)}}^c(k) \geq e^{k\chi_2^c}$. Finally we have by definition $f^{\tau(\ell)}(a) \in \mathcal{W}_1^u(f^{\tau(\ell)}(x^u))$ so by the distortion control (7) we have $\frac{\lambda_a^c(\tau(\ell))}{\lambda_{x^u}^c(\tau(\ell))} \geq C_0^{-1}$ and, by choice of k,

$$d_x^\ell \lambda_a^c(\tau(\ell) + k) \geq \frac{e^{k\chi_2^c}}{C_0} d_x^\ell \lambda_{x^u}^c(\tau(\ell)) \geq \varepsilon.$$

Hence $\tau_a(\ell) - \tau(\ell) \leq k$. A symmetric argument yields $\tau(\ell) - \tau_a(\ell) \leq k$.

We consider now the stopping time t. As above, we denote $t_a(\ell) \overset{\text{def.}}{=} \tau(x, a, \varepsilon, \ell)$ and $t(\ell) = t(x, a, \varepsilon, \ell)$. Let us first suppose $t_a(\ell) \geq t(\ell)$. We have

$$\frac{\lambda_x^c(t(\ell) + k')}{\lambda_a^c(\tau_a(\ell))} \times \frac{\lambda_{x^u}^c(\tau(\ell))}{\lambda_x^c(t(\ell))} = \frac{\lambda_{x^u}^c(\tau(\ell))}{\lambda_a^c(\tau(\ell))} \times \lambda_{x_{t'(\ell)}}^c(k')$$
$$\times \frac{1}{\lambda_{x_{\tau(\ell)}^u}^c(\tau(\ell) - \tau'(\ell))}.$$

The first factor is $\geq C_0^{-1}$. The second one is $\geq e^{k'\chi_2^c}$. The third one is $\geq e^{-k\chi_1^c}$. Hence by our choice of k',

$$\frac{\lambda_x^c(t(\ell) + k')}{\lambda_a^c(\tau_a(\ell))} \geq \frac{e^{k'\chi_2^c}}{C_0 e^{k\chi_1^c}} \times \frac{\lambda_{x^u}^c(\tau(\ell))}{\lambda_x^c(t(\ell))} \geq 1.$$

This proves that $t_a(\ell) - t(\ell) \leq k'$. Again, a symmetric argument gives $t_a(\ell) - t(\ell) \leq k'$ if $t(\ell) \geq t_a(\ell)$. This ends the proof of the lemma since $T_0 = \max(k, k')$. $\square$

10.1.3. *Matching of dynamical balls and distortion control.* We first define the notion of *matched unstable dynamical balls* and study their geometric properties.

Definition 10.4 (Matched unstable dynamical balls). *When (x, x^u, y, y^u) is a (C, ℓ)-quadrilateral, then we say that $J(x^u) = J(x^u, \varepsilon, \ell)$ and $J(y^u) = J(y^u, \varepsilon, \ell)$ are (C, ℓ)-matched.*

We will need the following distortion control for matched unstable dynamical balls.

Proposition 10.5. *For ε small enough, there exists $\kappa_1 = \kappa_1(f, \varepsilon)$ such that if ℓ is sufficiently large, for every (C, ℓ)-matched dynamical balls $J(x^u)$ and $J(y^u)$, and every $a \in J(x^u)$, $b \in J(y^u)$, we have*

$$\kappa_1^{-1} \leq \frac{\lambda_b^*(j)}{\lambda_a^*(j)} \leq \kappa_1,$$

for every integer $0 \leq j \leq \max(\tau(x, x^u, \varepsilon, \ell), \tau(y, y^u, \varepsilon, \ell))$ and $* = c, u$.

Proof. Let us assume that $\tau(\ell) \leq \tau'(\ell)$, where $\tau(\ell) = \tau(x, x^u, \varepsilon, \ell)$ and $\tau'(\ell) = \tau(y, y^u, \varepsilon, \ell)$. Let us first notice that by Lemma 9.4, we have $\tau' \leq \tau + T_0$.

Let $j \leq \tau(\ell)$ so in particular

$$|f^j(J(x^u))|, |f^j(J(y^u))| < 1.$$

Let $z^u = H_{x,y}^s(x^u)$ so $d(x^u, z^u) < 1$, for ℓ large enough, and, by Corollary 9.3, $d(f^j(y^u), f^j(z^u)) \leq d(f^{\tau'}(y^u), f^{\tau'}(z^u)) \asymp_{C, T_0} \varepsilon < 1$.

Let $\phi_* \stackrel{\text{def.}}{=} \log \|Df|_{E^*}\|$ and c_*, θ_* be the Hölder constant and exponent of ϕ_*, for $* = c, u$. We have

$$\left|\log \frac{\lambda_b^*(j)}{\lambda_a^*(j)}\right| \leq \sum_{i=0}^{j-1} |\phi_*(f^i(a)) - \phi_*(f^i(b))|.$$

Hence,

$$|\phi_*(f^i(a)) - \phi_*(f^i(b))| \leq |\phi_*(f^i(a)) - \phi_*(f^i(x^u))| + |\phi_*(f^i(x^u)) - \phi_*(f^i(z^u))|$$
$$+ |\phi_*(f^i(z^u)) - \phi_*(f^i(y^u))| + |\phi_*(f^i(y^u)) - \phi_*(f^i(b))|,$$

Note that for $i < j$, $d(f^i(a), f^i(x^u)), d(f^i(b), f^i(y^u)) \leq e^{\chi_1^u(i-j)}$. On the other hand, we have $d(f^i(x^u), f^i(z^u)) \leq e^{i\chi_2^s}$. Finally $d(f^i(z^u), f^i(y^u)) \leq e^{\chi_1^c(i-j)}$. It follows that

$$|\phi_*(f^i(a)) - \phi_*(f^i(b))| \leq c_* \left(2e^{\theta_* \chi_1^u(i-j)} + e^{i\theta_* \chi_2^s} + e^{\theta_* \chi_1^c(i-j)}\right).$$

By summing over i we deduced that $\left|\log \frac{\lambda_b^*(j)}{\lambda_a^*(j)}\right|$ is uniformly bounded from above by a constant depending only on f and ρ. Of course we can now bound these quotients for j up to $\tau'(\ell)$ by using that

$$d_0^{-T_0} \frac{\lambda_b^*(j)}{\lambda_a^*(j)} \leq \frac{\lambda_b^*(j+T_0)}{\lambda_a^*(j+T_0)} \leq d_0^{T_0} \frac{\lambda_b^*(j)}{\lambda_a^*(j)},$$

where $d_0 = \|Df\|/m(Df)$ (recall §2.1.2). $\qquad\square$

Corollary 10.6. *There exists $\kappa_2 = \kappa_2(f, \varepsilon)$ such that if ℓ is sufficiently large, for every (C, ℓ) matched dynamical balls $J(x^u)$ and $J(y^u)$,*

$$\kappa_2^{-1} \leq \frac{|f^j(J(y^u))|}{|f^j(J(x^u))|} \leq \kappa_2,$$

for every integer $0 \leq j \leq \max(\tau(x, x^u, \varepsilon, \ell), \tau(y, y^u, \varepsilon, \ell))$.

Proof. As before we denote $\tau(\ell) = \tau(x, x^u, \varepsilon, \ell)$ and $\tau'(\ell) = \tau(y, y^u, \varepsilon, \ell)$. Suppose $\tau(\ell) \leq \tau'(\ell) \leq \tau(\ell) + T_0$ (where T_0 is defined by Lemma 9.4). In particular $m(Df)^{T_0} \leq |f^{\tau(\ell)}(J(y^u))| \leq 1$ and $|f^{\tau(\ell)}(J(x^u))| = 1$.

For $j \in \{0, \ldots, \tau(\ell)\}$, we notice that

$$\frac{|f^j(J(x^u))|}{|f^j(J(y^u))|} = \frac{\int_{f^{\tau(\ell)}(J(x^u))} \lambda_q^u(k) dq}{\int_{f^{\tau(\ell)}(J(y^u))} \lambda_q^u(k) dq},$$

where $k = j - \tau(\ell) \leq 0$. By the bounded distortion along the strong unstable manifolds (7) we find

$$C_0^{-2} \frac{\lambda_{f^{\tau(\ell)}(x^u)}^u(k)}{\lambda_{f^{\tau(\ell)}(y^u)}^u(k)} \frac{|f^{\tau(\ell)}(J(x^u))|}{|f^{\tau(\ell)}(J(y^u))|} \leq \frac{|f^j(J(x^u))|}{|f^j(J(y^u))|}$$

$$\leq C_0^2 \frac{\lambda_{f^{\tau(\ell)}(x^u)}^u(k)}{\lambda_{f^{\tau(\ell)}(y^u)}^u(k)} \times \frac{|f^{\tau(\ell)}(J(x^u))|}{|f^{\tau(\ell)}(J(y^u))|}.$$

Notice that

$$\lambda_{f^{\tau(\ell)}(x^u)}^u(k) = \frac{\lambda_{x^u}^u(-k)}{\lambda_{x^u}^u(\tau(\ell))},$$

and a similar equation holds with y^u instead of x^u. Therefore, we can apply Proposition 10.5 to obtain

$$C_0^{-2} \kappa_1^{-2} \leq \frac{|f^j(J(x^u))|}{|f^j(J(y^u))|} \leq C_0^2 \|Df\|^{T_0} (\kappa_1)^2.$$

We conclude the proof of the corollary by noting that

$$m(Df)^{2T_0}\frac{|f^j(J(x^u))|}{|f^j(J(y^u))|} \leq \frac{|f^{j+T_0}(J(x^u))|}{|f^{j+T_0}(J(y^u))|} \leq \|Df\|^{2T_0}\frac{|f^j(J(x^u))|}{|f^j(J(y^u))|}. \quad \square$$

10.2. **Matching of Y-configurations.** We now define the notion of matched Y-configurations and show that our drift argument (Proposition 8.5) boils down to constructing arbitrarily long pairs of matched Y-configurations.

10.2.1. *Matching.* We first give the main definition.

Definition 10.7 (Matched Y-configurations). *Let $\mathcal{L} \subset \mathbb{T}^3$, $C > 0$, $\ell \in \mathbb{N}$. Let $X = X(x, x^u, \ell)$ and $Y = Y(y, y^u, \ell)$ be two Y configurations of length ℓ. We say that X and Y are $(\mathcal{L}, C, T)$-matched if there exist $a, b \in \mathbb{T}^3$, and $\tau, t \in \mathbb{N}$ such that*

(1) (x, x^u, y, y^u) is a (C, ℓ)-quadrilateral;
(2) $a \in J(x^u, \varepsilon, \ell)$ and $b \in J(y^u, \varepsilon, \ell)$;
(3) $|\tau - \tau(x, x^u, \varepsilon, \ell)| \leq T$ and $|t - t(x, x^u, \varepsilon, \ell)| \leq T$;
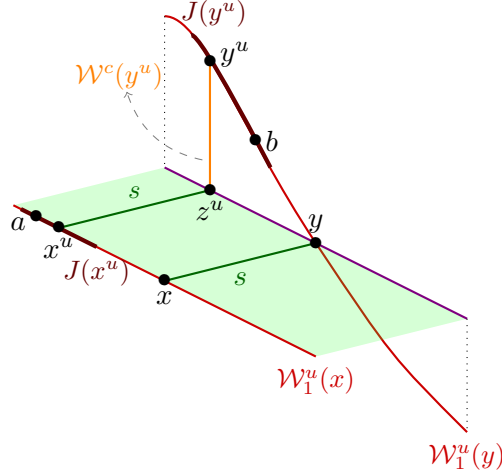*(4) $a, b, x, y, f^\tau(a), f^\tau(b), f^t(x), f^t(y) \in \mathcal{L}$.*

FIGURE 13. For matched Y-configurations we can only put inside the Lusin set "small perturbations" a and b of the endpoints x^y and y^u (respectively) of the quadrilateral.

Remark 10.8. *At first sight this definition is not symmetric (see item (3)). But the synchronization estimate (Lemma 9.4) implies that,*

$$|\tau - \tau(y, y^u, \varepsilon, \ell)| \leq T + T_0, \ and \ |t - t(y, y^u, \varepsilon, \ell)| \leq T + T_0.$$

10.2.2. *Finding pairs of long and matched Y-configurations.* We are now ready to state our main technical lemma which, as we shall see, implies Proposition 8.5.

Let $\mathcal{L}$ be the Lusin set defined in §8.1.2 and $\delta = 3\delta_0 > 0$ be the constant fixed in §8.1.3: we shall give further assumption on δ later on.

Lemma 10.9. *Let $K_0 \subset \mathcal{L}$ be a compact set of measure $\mu(K_0) > 1 - \delta$. There exist constants $C = C(\delta)$ and $T = T(\delta)$ and an infinite subset $\mathcal{D} \subset \mathbb{N}$ such that for every $\ell \in \mathcal{D}$ there exists a pair (X, Y) of (K_0, C, T)-matched Y-configurations of length ℓ.*

The end of the section is devoted to proving that Lemma 10.9 implies Proposition 8.5. The proof of Lemma 10.9 will be the object of Section 11.

10.3. **Asymptotic control of leaf-wise measures for matched configurations.** The goal of this paragraph is to study how leaf-wise quotient measures change along a pair of matched Y-configurations, and what happens when we have sequences of longer and longer pairs. To simplify the exposition, we shall break this explanation in two parts. First we deal with a single Y-configuration, and then we treat the full situation.

10.3.1. *Control along a Y-configuration.* The lemma below is essentially an easy corollary of Lemmas 7.13 and 7.14.

Lemma 10.10. *For $x \in \mathbb{T}^3$, $x^u \in \mathcal{W}_r^u(x)$, for some $r > 0$, $\varepsilon > 0$, and $\ell \in \mathbb{N}$ as above, it holds*

$$\hat{\nu}_{f^{\tau(\ell)}(x^u)}^c \propto (A_{x,x^u,\ell})_* \hat{\nu}_{f^{t(\ell)}(x)}^c,$$

for the linear map $A_{x,x^u,\ell} \colon s \mapsto a_{x,x^u,\ell} \cdot s$, with

$$a_{x,x^u,\ell} \overset{\text{def.}}{=} \rho_{x^u}^c(x) \frac{\lambda_x^c(t(\ell))}{\lambda_{x^u}^c(\tau(\ell))}.$$

Moreover, $|a_{x,x^u,\ell}| \in (a_0^{-1}, a_0)$ for some constant $a_0 = a_0(r) > 1$ depending only on the upper bound $r > 0$ on the distance along $\mathcal{W}^u$ between x and x^u; in particular, the linear map $A_{x,x^u,\ell}$ is uniformly bounded away from 0 and ∞, independently of ℓ.

Proof. Applying Lemma 7.13 and Lemma 7.14 we successively obtain:

$$\begin{aligned}
\hat{\nu}_{f^{\tau(\ell)}(x^u)}^c &\propto (\Lambda_{\tau(\ell),x^u}^c)_* \hat{\nu}_{x^u}^c \\
&\propto (L_{x,x^u} \circ \Lambda_{\tau(\ell),x^u}^c)_* \hat{\nu}_x^c \\
&\propto (L_{x,x^u} \circ \Lambda_{\tau(\ell),x^u}^c \circ (\Lambda_{t(\ell),x}^c)^{-1})_* \hat{\nu}_{f^{t(\ell)}(x)}^c,
\end{aligned}$$

where $\Lambda_{n,x}^c$ denotes the linear map $s \mapsto \lambda_x^c(n)s$, and L_{x,x^u} denotes the linear map $s \mapsto \rho_{x^u}^c(x) \cdot s$. Thus, $A_{x,x^u,\ell} \overset{\text{def.}}{=} \Lambda_{\tau(\ell),x^u}^c \circ L_{x,x^u} \circ (\Lambda_{t(\ell),x}^c)^{-1}$ is equal to

$$s \mapsto \rho_{x^u}^c(x) \frac{\lambda_x^c(t(\ell))}{\lambda_{x^u}^c(\tau(\ell))} \cdot s.$$

By the fact that $d_u(x, x^u) < r$, and by the definition of $t(\ell)$, $|A'_{x,x^u,\ell}(0)|$ is uniformly bounded, depending only on r and f, but not on ℓ, which concludes the proof. $\square$

10.3.2. *Control for matched configurations.* We now deal with the full picture of sequences of pairs of good and matched configurations. For the next result, we refer to §8.1.2 for the definition of the Lusin set $\mathcal{L}$, to §9.3 and to §10.2 for that of quadrilaterals and (C, ℓ)-matched Y-configurations respectively.

The result of this subsection is the core of the proof that Lemma 10.9 implies Proposition 8.5. *Until the end of this section, the notation x_n will NOT stand for $f^n(x)$ but for the usual notation of sequences.*

Lemma 10.11. *Let $\varepsilon > 0$ (ℓ_n) be an increasing sequence of integers. Suppose there exist constants $T, C > 0$ (independent of ε), and two sequences $X_n = (x_n, x_n^u, \ell_n)$ and $Y_n = (y_n, y_n^u, \ell_n)$ of Y-configurations of length ℓ_n that are $(\mathcal{L}, C, T)$-matched. Let τ_n, t_n, a_n, b_n be the objects given by the condition of matching and consider the sequences*

$$f^{\tau_n}(a_n), f^{\tau_n}(b_n) \in \mathcal{L}, \text{ and } f^{\tau_n}(x_n^u), f^{\tau_n}(y_n^u) \in \mathbb{T}^3.$$

Assume all these sequences converge to points a_∞, b_∞, p and q respectively. Then $q \in \mathcal{W}^c(p)$ and there exists $\gamma = \gamma(C, T) > 1$ and $M = M(C, T)$ such that

$$\frac{\varepsilon}{\gamma} \leq |\mathcal{H}_p^c(q)| \leq \gamma \varepsilon, \tag{63}$$

and

$$\hat{\nu}_{a_\infty}^c \propto B_* \hat{\nu}_{b_\infty}^c. \tag{64}$$

where $B(s) = \beta \cdot s$, is a linear map that satisfies $\frac{1}{M} < |\beta| < M$.

Proof. Note first that τ_n and t_n tend to infinity: indeed, this is a consequence of the matching condition and of the fact that $\ell_n \to \infty$ so the stopping times also tend to infinity (by Lemma 9.1).

Now suppose the sequences $f^{\tau_n}(a_n), f^{\tau_n}(b_n), f^{\tau_n}(x_n), f^{\tau_n}(y_n)$ converge to a_∞, b_∞, p, q respectively. Let $z_n^u = H_{x_n, y_n}^s(x_n^u) \in \mathcal{W}^s(x_n)$. The condition of matching implies that $d(z_n^u, x_n^u)$ is uniformly bounded so $d(f^{\tau_n}(z_n), f^{\tau_n}(x_n)) \to 0$ as $n \to +\infty$ and $f^{\tau_n}(z_n^u) \to p$. Since normal forms changes continuously, we have that

$$\mathcal{H}_{f^{\tau_n}(z_n^u)}^c(f^{\tau_n}(y_n^u)) \to \mathcal{H}_p^c(q), \quad \text{as} \quad n \to +\infty.$$

Thus (63) follows directly from Lemma 9.2 and Remark 10.8. Let us show (64). For this we use the matching condition: for all n, the points $f^{\tau_n}(a_n), f^{\tau_n}(b_n), f^{t_n}(x_n), f^{t_n}(y_n)$ all belong to the Lusin set $\mathcal{L}$.

Combining Lemma 10.10 with Lemma 7.14 and Remark 10.8 we deduce that there exist linear maps $B_n \colon \mathbb{R} \to \mathbb{R}$ and $\tilde{B}_n \colon \mathbb{R} \to \mathbb{R}$ and constants $C_n, \tilde{C}_n > 0$ such that the derivatives B_n' and $\tilde{B}_n'$ lie inside $[M^{-1}, M]$ for some constant $M = M(T, C)$, independent of n and

$$\hat{\nu}_{f^{\tau_n}(a_n)}^c = C_n(B_n)_* \hat{\nu}_{f^{t_n}(x_n)}^c \quad \text{and} \quad \hat{\nu}_{f^{\tau_n}(b_n)}^c = \tilde{C}_n(\tilde{B}_n)_* \hat{\nu}_{f^{t_n}(y_n)}^c.$$

We claim that C_n and $\tilde{C}_n$ are uniformly bounded away from 0 and ∞. Indeed, let us consider C_n. Let $c = c(M) > 0$ be the constant given by Corollary 8.4. Then, denoting $I = [-1, 1]$ we have that, by our choice of normalization (see Remark 7.10) $\hat{\nu}_{f^{\tau_n}(a_n)}^c(I) = 1$ and therefore

$$C_n = \frac{1}{\hat{\nu}_{f^{t_n}(x_n)}^c(B_n^{-1}(I))} \in [c^{-1}, c],$$

for $f^{t_n}(x_n) \in \mathcal{L}$ for each n. A similar argument treats the constants $\tilde{C}_n$. Therefore up to enlarging M, we can assume that $C_n, \tilde{C}_n \in [M^{-1}, M]$.

The condition of matching implies that $y_n \in \mathcal{W}^s(x_n)$ for every n and that $d(x_n, y_n)$ is uniformly bounded. Now recall that $f^{t_n}(x_n), f^{t_n}(y_n)$ belong to the Lusin set $\mathcal{L}$, which is compact. We can assume without loss of generality that there exists a point $x \in \mathcal{L}$ such that $f^{t_n}(x_n) \to x$ and $f^{t_n}(y_n) \to x$. For simplicity, let us denote $\hat{a}_n = f^{\tau_n}(a_n)$, $\hat{b}_n = f^{\tau_n}(b_n)$, $\hat{x}_n = f^{t_n}(x_n)$ and $\hat{y}_n = f^{t_n}(y_n)$. We can also assume that $C_n \to C$ and $\tilde{C}_n \to \tilde{C}$ and that there exist linear maps $B, \tilde{B} : \mathbb{R} \to \mathbb{R}$ such that $B_n \to B$ and $\tilde{B}_n \to \tilde{B}$ in the C^1 topology, i.e. the slopes of B_n converge to the slope of B and similarly for $\tilde{B}_n$ and $\tilde{B}$.

Now, we claim that $\hat{\nu}_{a_\infty}^c = CB_* \hat{\nu}_x^c$ and $\hat{\nu}_{b_\infty}^c = \tilde{C}\tilde{B}_* \hat{\nu}_x^c$. Once we prove this claim, the lemma will be established. To prove the claim, we first observe that

$$\int C_n(\varphi \circ B_n) d\hat{\nu}_{\hat{x}_n}^c \to \int C(\varphi \circ B) d\hat{\nu}_{\hat{x}}^c,$$

for every $\varphi \in C_c^0(\mathbb{R})$. Indeed, we can write

$$\left| \int C_n(\varphi \circ B_n)d\hat{\nu}_{\hat{x}_n}^c - \int C(\varphi \circ B)d\hat{\nu}_x^c \right| \leq \left| \int C_n(\varphi \circ B_n)d\hat{\nu}_{\hat{x}_n}^c - \int C(\varphi \circ B)d\hat{\nu}_{\hat{x}_n}^c \right|$$
$$+ \left| \int C(\varphi \circ B)d\hat{\nu}_{\hat{x}_n}^c - \int C(\varphi \circ B)d\hat{\nu}_x^c \right|.$$

The second term on the right-hand side above converges to zero as $n \to \infty$ because $\mathcal{L}$ satisfies the conclusion of Lemma 8.3 and $\hat{x}_n \in \mathcal{L}$ converges to x. The first term also converges to zero because $C_n(\varphi \circ B_n)$ converges uniformly to $C(\varphi \circ B)$ and $\hat{\nu}_{\hat{x}_n}^c(\text{supp}(\varphi))$ is bounded independently of n. We use this observation to conclude the proof of the claim as follows. Given any $\varphi \in C_c^0(\mathbb{R})$ we have that

$$\int \varphi d\hat{\nu}_{a_\infty}^c = \lim_{n \to \infty} \int \varphi d\hat{\nu}_{\hat{a}_n}^c = \lim_{n \to \infty} \int C_n(\varphi \circ B_n)d\hat{\nu}_{\hat{x}_n}^c$$
$$= \int C(\varphi \circ B)d\hat{\nu}_x^c,$$

proving that $\hat{\nu}_{a_\infty}^c = CB_*\hat{\nu}_x^c$. The proof that $\hat{\nu}_{b_\infty}^c = \tilde{C}\tilde{B}_*\hat{\nu}_x^c$ is similar so we omit it. The proof of the lemma is complete.   □

With Lemma 10.11 at hand we are now in position to reduce the proof of Theorem 5.10 to the proof of Lemma 10.9.

10.4. **Proof that Lemma 10.9 $\implies$ Proposition 8.5.** Let K_{00} be a compact set with $\mu(K_{00}) > 1 - 2\delta_0$. We apply Lemma 10.9 to the compact set $K_0 = K_{00} \cap \mathcal{L}$ which has measure $\mu(K_0) > 1 - 3\delta_0 = 1 - \delta$. So let $C = C(\delta) > 0$, $T_0 = T_0(\delta) > 0$, $K_0 = K_{00} \cap \mathcal{L}$, and $\mathcal{D}$ be the objects given by Lemma 10.9.

As the set $\mathcal{D}$ is infinite, there exists a sequence $\ell_n \to +\infty$ of integers belonging to $\mathcal{D}$. For each such integer, let $X_n = (x_n, x_n^u, \ell_n)$ and $Y_n = (y_n, y_n^u, \ell_n)$ be the pair of (K_0, C, T)-matched Y-configurations given by Lemma 10.9. We let a_n, b_n, τ_n, t_n be the points and times corresponding to the pair (X_n, Y_n) (see Definition 10.7). By definition, we have

$$f^{\tau_n}(a_n), f^{\tau_n}(b_n) \in K_0.$$

We also consider the sequences

$$f^{\tau_n}(x_n^u), f^{\tau_n}(y_n^u).$$

Upon extracting subsequences if necessary, we may assume that these four sequences converge respectively to points a_∞, b_∞, p and q. Observe that $a_\infty, b_\infty \in K_0$, but we do not know if the same holds for p and q. However, we have good estimates for the distance between these points (see Figure 14). Indeed, by Lemma 10.11 there exists $\gamma = \gamma(C, T) > 1$ and $M = M(C, T)$ such that

$$\frac{\varepsilon}{\gamma} \leq |\mathcal{H}_p^c(q)| \leq \gamma\varepsilon, \tag{65}$$

and

$$\hat{\nu}_{a_\infty}^c \propto B_*\hat{\nu}_{b_\infty}^c. \tag{66}$$

where $B(s) = \beta \cdot s$, satisfies $\frac{1}{M} < |\beta| < M$. We shall prove that these conditions ensure that $b_\infty \in G(\varepsilon, \tilde{M})$, for some constant $\tilde{M}$ to be defined later. This will show Proposition 8.5. Notice that by construction the four points a_∞, b_∞, p and q belong to the same center unstable leaf. Let us consider the local strong
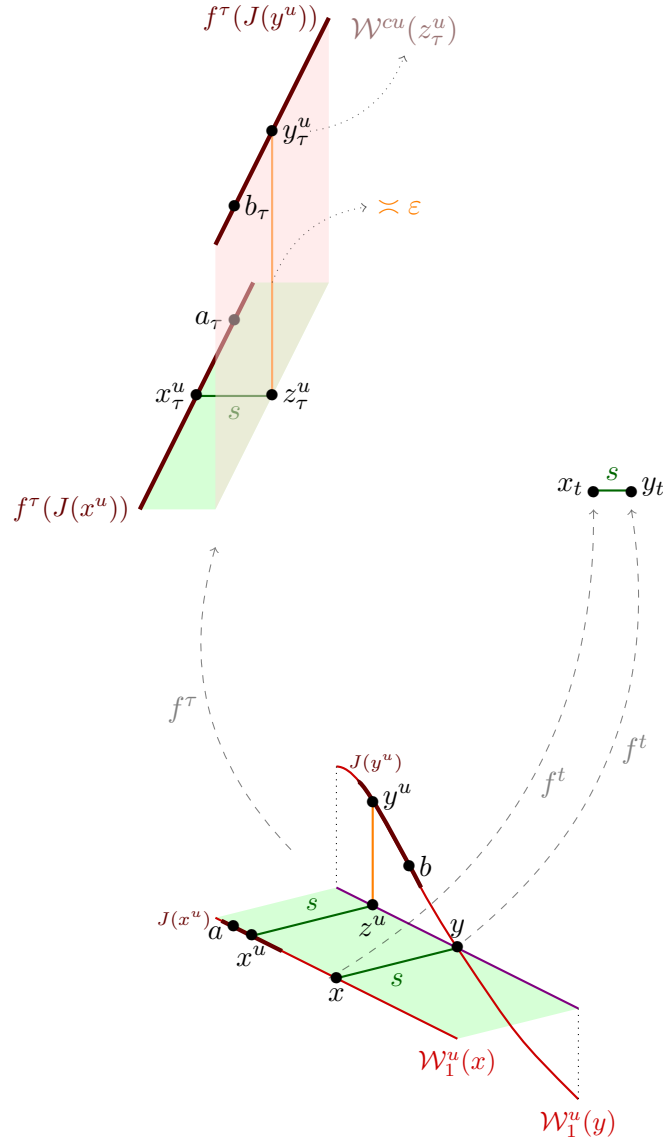
FIGURE 14.   For each ℓ_n we have a picture like this one (we have suppressed the dependence on n for simplicity). The top-left part will converge to Figure 15.

unstable manifolds $\mathcal{W}_3^u(p)$ and $\mathcal{W}_3^u(q)$. Notice that, also by construction we have $a_\infty \in \mathcal{W}_1^u(p)$ and $b_\infty \in \mathcal{W}_1^u(q)$. An application of Corollary 9.3 yields

$$d_c(p,q) \leq \kappa\varepsilon. \tag{67}$$

Hence we can choose ε small enough so that the intersection point $\hat{a} = \mathcal{W}^u(q) \cap \mathcal{W}^c(a_\infty)$ given by Lemmas 2.3 and 2.4 satisfies $d_u(\hat{a}, q) < 2$. Similarly we can choose $\hat{b} = W^u(p) \cap \mathcal{W}^c(b_\infty)$ so that $d_u(\hat{b}, p) < 2$ (see Figure 15). Denote by $\gamma_{a_\infty,\hat{a}}^c$ the segment of center manifold joining the points a_∞ and $\hat{a}$ and similarly consider the segments of center manifolds $\gamma_{p,q}^c$ and $\gamma_{b_\infty,\hat{b}}^c$. By (67) we have that

$$\text{length}(\gamma_{p,q}^c) = \text{length}(\Phi_p^c[0, \mathcal{H}_p^c(q)]) \asymp_{C,\gamma} \varepsilon.$$

Note that $d_u(a_\infty, p) < 2$, $d_u(b_\infty, q) < 2$ and $d_c(p,q) \leq \kappa\varepsilon < \rho_0$, where ρ_0 is the constant of Lemma 2.5 (provided ε is choosen small enough). Hence Lemma 2.5
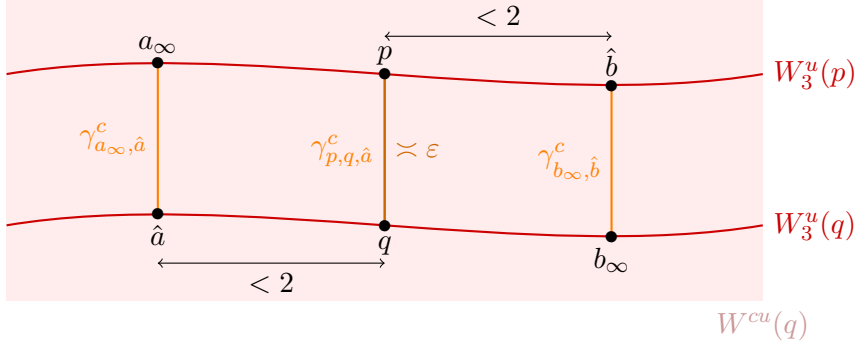
FIGURE 15. How to get invariance by an affine map using the exponential drift: we can move the leaf-wise measures from b_∞ to a_∞ with a linear map, and from a_∞ to $\hat{b}$ with a linear map and from $\hat{b}$ back to b_∞ with an affine map.

implies that the unstable holonomy maps H^u_{p,a_∞} and H^u_{q,b_∞} are bilipschitz with constants which depends only on f. Since $\gamma^c_{a_\infty,\hat{a}} = H^u_{p,a_\infty}(\gamma^c_{p,q})$ and similarly $\gamma^c_{b_\infty,\hat{b}} = H^u_{p,a_\infty}(\gamma^c_{p,q})$ we deduce that

$$\text{length}(\gamma^c_{a_\infty,\hat{a}}) \asymp_{C,\gamma} \varepsilon \quad \text{and} \quad \text{length}(\gamma^c_{b_\infty,\hat{b}}) \asymp_{C,\gamma} \varepsilon$$

Therefore, as ε is small we deduce that $\hat{b} \in \mathcal{W}^c_1(b_\infty)$. Using the uniform bound for the C^1 norm of the normal forms in segments of bounded length we have that

$$\mathcal{H}^c_{b_\infty}(\hat{b}) \asymp_{C,\gamma} \varepsilon. \tag{68}$$

We apply Lemma 7.14 to get a linear map $L \colon \mathbb{R} \to \mathbb{R}$ whose derivative satisfies $L'(0) \in [C^{-1}, C]$, for a constant $C = C(f)$ such that $\hat{\nu}_{\hat{b}} \propto L_* \hat{\nu}_{a_\infty}$. Therefore,

$$\hat{\nu}_{\hat{b}} \propto (LB)_* \hat{\nu}_{b_\infty}$$

and the linear map $L \circ B$ has a derivative bounded by $[C^{-1}M^{-1}, CM]$. This together with (68) implies that $b_\infty \in G(\varepsilon, \tilde{M})$ for some constant $\tilde{M} = \tilde{M}(\gamma, C, M)$ (see (49) for the definition of the set $G(\varepsilon, \tilde{M})$), concluding the proof of Proposition 8.5. $\qquad\square$

11. Construction of matched Y-configurations: end of the proof

This section is devoted to the proof of Lemma 10.9. Recall that we reduced our main Theorem A to Lemma 10.9. Let us recall what we want to do. We assume that $\mu(\mathbf{B}) = 0$, where $\mathbf{B} = \mathbf{B}$ is the Bad set introduced in Definition 5.2. In particular, it follows from the zero-one law (Theorem 4.2) that for μ-a.e. $x \in \mathbb{T}^3$,

$$\mu^s_x \{y \in \xi^s(x) : \alpha^s(x,y) = 0\} = 0. \tag{69}$$

where we recall that $\alpha^s(x,y) = \angle(DH^s_{x,y}(x)E^u(x), E^u(y))$ when $y \in \mathcal{W}^s(x)$. Given a large compact set K_0 we want to find arbitrarily long pairs of *matched* Y-configurations which are K_0-good (meaning that their points belong to K_0).

11.1. **Angle condition and absolute continuity.** We start this section by showing how to use the condition $\mu(\mathbf{B}) = 0$ to obtain abundance of pairs of points $y \in \mathcal{W}^s_1(x)$ so that $\alpha^s(x,y)$ is uniformly bounded from below.

Before carrying on the proof recall that for $* = s, u, c, cu$, we have fixed measurable partitions ξ^* subordinate to $\mathcal{W}^*$, whose atoms have diameter less than 1, as well as disintegrations $\{\mu^*_x\}_x$ relative to ξ^*. We fixed a Lusin set $\mathcal{L}$ of measure

$\mu(\mathcal{L}) > 1 - \delta$ (further assumptions on δ will be given later on) as given in §8.1.2. In particular there exists $r_0 > 0$ such that for every $x \in \mathcal{L}$,

$$\mathcal{W}_{r_0}^*(x) \subset \xi^*(x). \tag{70}$$

11.1.1. *A Markov type inequality.* Our first ingredient will be a simple inequality à la Markov that will be used several times throughout the section.

Lemma 11.1. *Let* $(X, \mathcal{B}, \mu)$ *be a probability space, and* $\eta \in (0, 1)$. *Let* $\psi \colon X \to [0, 1]$ *be a measurable function with* $\int \psi d\mu > 1 - \eta$. *Let* $B \stackrel{\text{def.}}{=} \{x \in X : \psi(x) > 1 - \sqrt{\eta}\}$. *Then,* $\mu(B) > 1 - \sqrt{\eta}$.

Proof. We have

$$1 - \eta < \int_B \psi d\mu + \int_{X \setminus B} \psi d\mu \le \mu(B) + (1 - \sqrt{\eta})(1 - \mu(B)) = 1 - \sqrt{\eta}(1 - \mu(B)),$$

which gives $\mu(B) > 1 - \sqrt{\eta}$. $\qquad\qquad\square$

11.1.2. *Bounding from below the angle function.* We fix a measurable partition ξ^s subordinate to $\mathcal{W}^s$ and a disintegration $\{\mu_x^s\}_x$ of μ relative to ξ^s. Below we use the condition $\mu(\mathbf{B}) = 0$: by (69) it means that for μ-a.e. $x \in \mathbb{T}^3$ and μ_x^s-a.e. $y \in \xi^s(x)$, $\alpha^s(x, y) > 0$.

Until the end of the section we fix $r_0 > 0$ such that (70) holds for all $x \in \mathcal{L}$. We consider a constant $\eta = \eta(\delta) > 0$ which goes to 0 with δ. This will be explicitly given later on.

Lemma 11.2. *Let* $B \subset \mathcal{L}$ *of measure* $\mu(B) > 1 - \eta$. *Then there exists a measurable set* $B' \subset B$ *as well as a number* $c = c(\eta)$ *such that* $0 < c \le r_0\eta$ *and such that the following properties hold:*

(1) $\mu(B') > 1 - 2\sqrt{\eta}$; *and for* μ-*a.e.* $x \in B'$ *there exists* $y \in B \cap \xi^s(x)$ *such that*
(2) $\alpha^s(x, y) > c$.

Proof. Let us consider the set $B_1 \stackrel{\text{def.}}{=} \{x \in \mathbb{T}^3 : \mu_x^s[B \cap \xi^s(x)] > 1 - \sqrt{\eta}\}$. It follows from Lemma 11.1 that $\mu(B_1) > 1 - \sqrt{\eta}$.

Now, given $\alpha \ge 0$, we define for μ-a.e. $x \in \mathbb{T}^3$:

$$\mathscr{A}_\alpha(x) \stackrel{\text{def.}}{=} \{y \in \xi^s(x) : \alpha^s(x, y) > \alpha\},$$

$$\mathscr{A}_{\alpha, \eta} \stackrel{\text{def.}}{=} \{x \in \mathbb{T}^3 : \mu_x^s[\mathscr{A}_\alpha(x)] > 1 - \eta\}.$$

Using that $\mu(\mathbf{B}) = 0$ we find $\mu_x^s[\mathscr{A}_0(x)] = 1$, for μ-almost every $x \in \mathbb{T}^3$. This implies that $\mu[\mathscr{A}_{0, \eta}] = 1$. Then, there exists $\alpha = \alpha(\eta) > 0$ such that the set $B_2 \stackrel{\text{def.}}{=} \mathscr{A}_{\alpha, \eta}$ has measure $\mu(B_2) > 1 - \eta$.

Let us define the set

$$B' \stackrel{\text{def.}}{=} B \cap B_1 \cap B_2,$$

so

$$\mu(B') > 1 - 2\eta - \sqrt{\eta} > 1 - 2\sqrt{\eta}$$

(provided $\eta < 1/4$, so $2\eta < \sqrt{\eta}$). Then for any $x \in B'$,

- $\mu_x^s(B \cap \xi^s(x)) > 1 - \sqrt{\eta}$;
- $\mu_x^s\{y \in \xi^s(x) : \alpha^s(x, y) > \alpha\} > 1 - \eta$;

Set $c = c(\eta) \stackrel{\text{def.}}{=} \min(\alpha(\eta), r_0\eta) \in (0, r_0\eta]$. Then, for every $x \in B'$, we have

$$\mu_x^s\{y \in B \cap \xi^s(x) : \alpha^s(x, y) > c\} \ge 1 - \sqrt{\eta} - \eta > 1 - 2\sqrt{\eta} > 0.$$

So we conclude that for every $x \in B'$, then there exists $y \in \xi^s(x)$ such that $\alpha^s(x, y) > c$. This concludes the proof of the lemma. $\qquad\square$

11.2. **Recurrence estimates: building long and good Y-configurations.**
In this subsection we fix a measurable set $K \subset \mathbb{T}^3$ of measure $\mu(K) > 1 - \delta$ for
some small $\delta > 0$. The main result of this paragraph is Proposition 11.6, that
builds K-good Y-configurations of length ℓ for every integer ℓ inside a subset
of $\mathbb{N}$ of positive density. We start by establishing some preliminary results from
ergodic theory.

11.2.1. *An elementary quantitative recurrence estimate.* For the sake of clarity
in the presentation, we introduce the following notion.

Definition 11.3. *Given $\gamma > 0$, $n > 0$ and B a measurable set, we say that a
point $x \in \mathbb{T}^3$ is (γ, n)-recurrent to B if $L > n$ implies that*

$$\# \left\{ \ell \in [0, L] : f^\ell(x) \in B \right\} > (1 - \gamma)L.$$

The following holds for any ergodic system $(f, \mathcal{B}, \mu)$.

Lemma 11.4. *For every measurable set B with $\mu(B) > 1 - \gamma$ there exist $T = T(\gamma)$ and a subset $B^\circ \subset B$ with $\mu(B^\circ) > 1 - \gamma$ so that any $x \in B^\circ$ is (γ, T)-recurrent to B.*

Proof. Let B be a measurable set with $\mu(B) > 1 - \gamma$. We consider the sequence
$(\varphi_n)_{n \in \mathbb{N}}$ of L^1 functions given by $\varphi_n(x) \stackrel{\text{def.}}{=} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_B(f^k(x))$. By Birkhoff's
Theorem, the sequence $(\varphi_n)_{n \in \mathbb{N}}$ converges almost surely to the constant function
$\mu(B)$. Moreover, by Egorov's Theorem, there exists a measurable subset $B^\circ \subset B$
of measure $\mu(B^\circ) > 1 - \gamma$ such that the sequence $(\varphi_n|_{B^\circ})_{n \in \mathbb{N}}$ converges uniformly
to $\mu(B)$. Then, there exists $T > 0$ such that for any $x \in B^\circ$ and $n > T$,
$\varphi_n(x) > 1 - \gamma$. Such an x is (γ, T)-recurrent to B by definition. $\qquad\square$

11.2.2. *Stopping times and return times to K for pairs (x, x^u).* Recall that we
fixed a measurable set $K \subset \mathbb{T}^3$ with measure $\mu(K) > 1 - \delta$. Applying Lemma 11.4
to K yields an integer $T_1 = T_1(\delta)$ and a measurable subset $K^\circ \subset K$ with measure
$\mu(K^\circ) > 1 - \delta$ consisting of (δ, T_1)-recurrent points to K. Up to enlarging T_1 we
can assume

$$T_1 > 4T_0 \tag{71}$$

where T_0 is the constant given by the synchronization estimates of Lemmas 9.4
and 10.3.

Given a pair $\omega = (x, x^u)$, with $x^u \in \xi^u(x)$ we set

$$E(x, x^u) = E(\omega) \stackrel{\text{def.}}{=} \{\ell \in \mathbb{N} : f^{\tau(\ell)}(x^u) \in K \text{ and } f^{t(\ell)}(x) \in K\}. \tag{72}$$

The objects T_1 and K° appearing in the next statement are the ones con-
structed in the previous paragraph.

Lemma 11.5. *There exists a constant $r > 0$ which only depends on the quasi
isometric estimates so that for every $L > T_1$, $x \in K^\circ$ and $x^u \in \xi^u(x) \cap K^\circ$ and
$\omega = (x, x^u)$,*

$$\# (E(\omega) \cap [0, L]) > (1 - r\delta)L.$$

Proof. Recall that the stopping times satisfy $t(\ell) \geq 0$ as well as the quasi-
isometric estimate

$$\Theta^{-1}|\ell - m| - A < |t(\ell) - t(m)| < \Theta|\ell - m| + A,$$

for some constants $\Theta > 1$ and $A > 0$ depending only on f: see Lemma 9.1.
In particular for every $L > 0$ we have $t(L) \in [0, L']$ where $L' = \Theta L + A$. By
hypothesis, $x \in K^\circ$ so whenever $L > T_1$ (and thus $L' > T_1$), we have

$$\#\{k \in [0, L'] : f^k(x) \notin K\} \leq \delta L'.$$

On the other hand, the quasi-isometric estimate also implies that an integer in $[0, L']$ has at most $2A\Theta$ preimages by t so we have

$$\#\{\ell \in [0, L] : f^{t(\ell)}(x) \notin K\} \leq 2A\Theta\#\{k \in [0, L'] : f^k(x) \notin K\} \leq 2A\Theta\delta(\Theta L + A).$$

This proves that for every $L > T_1$, the set $E_t(x) \stackrel{\text{def.}}{=} \{\ell \in \mathbb{N} : f^{t(\ell)}(x) \in K\}$ has density $> 1 - r_1\delta$ inside $[0, L]$ for some constant $r_1 > 0$ depending only on A and Θ. Now, since x^u also belongs to K°, the same property also holds for the set $E_\tau(x^u) \stackrel{\text{def.}}{=} \{\ell \in \mathbb{N} : f^{\tau(\ell)}(x^u) \in K\}$: it has density $> 1 - r_2\delta$ inside $[0, L]$ for $r_2 > 0$ depending only on the quasi-isometry constants of τ from Lemma 9.1.

Finally we can estimate from below the density of $E(\omega) = E(x, x^u) = E_\tau(x^u) \cap E_t(x)$ inside $[0, L]$ using the following inequality

$$\#(E(\omega) \cap [0, L]) \geq \#(E_\tau(x^u) \cap [0, L]) - \#([0, L] \setminus E_t(x)) > (1 - (r_1 + r_2)\delta)L.$$

This ends the proof of the lemma. $\qquad\square$

11.2.3. *Space of pairs* (x, x^u). It will be useful to consider a measurable structure on the space of pairs

$$\Omega \stackrel{\text{def.}}{=} \{\omega = (x, x^u) : x \in \mathbb{T}^3 \text{ and } x^u \in \xi^u(x)\}.$$

Note that Ω is contained inside the continuous submanifold of $\mathbb{T}^3 \times \mathbb{T}^3$ defined as $Y = \{(x, x^u) : x \in \mathbb{T}^3 \text{ and } x^u \in \mathcal{W}_1^u(x)\}$ (recall that 1 is a uniform upper bound of the diamaters of atoms of ξ^u). The topology on Y induced by the product topology of $\mathbb{T}^3 \times \mathbb{T}^3$ provides it with a Borel σ-algebra. Its restriction to Ω is denoted by $\mathcal{A}$. Let $\pi \colon \Omega \to \mathbb{T}^3$ be the projection on the first coordinate. So we have $B \in \mathcal{A}$ if and only if its projection $\pi(B)$ is a measurable subset of $\mathbb{T}^3$, and $B \cap (\{x\} \times \xi^u(x))$ is a measurable subset of $\{x\} \times \xi^u(x)$. Hence we can define a measure ν on Ω by

$$\nu(B) \stackrel{\text{def.}}{=} \int_{\pi(B)} \mu_x^u [B \cap (\{x\} \times \xi^u(x))] \, d\mu(x).$$

11.2.4. *Construction of many long and good Y-configurations.* Assume $\ell \in E(\omega)$ for $\omega = (x, x^u)$ with $x, x^u \in K^\circ$. Then $(x, x^u, x_{-\ell}, f^{\tau(\ell)}(x^u), f^{t(\ell)}(x))$ is a K-good Y-configuration of length ℓ. So constructing many long and good Y-configurations means constructing many integers $\ell \in \mathbb{N}$ for which the set of pairs $\omega = (x, x^u) \in K^\circ \times K^\circ$ with $\ell \in E(\omega)$ has large measure for ν.

More precisely, let us fix a continuous function $\eta \colon [0, 1] \to [0, \infty)$ vanishing at 0 and, abusing notation, write $\eta = \eta(\delta)$. An explicit construction will be given in the proof of the next proposition. We will assume that δ is small enough so that $\eta < 1$. For $\ell \in \mathbb{N}$ and $x \in K^\circ$ let us define

$$Q^u(x) \stackrel{\text{def.}}{=} K^\circ \cap \xi^u(x), \tag{73}$$

and

$$Q^u(x, \ell) \stackrel{\text{def.}}{=} \{x^u \in \xi^u(x) : x^u \in K^\circ, \text{ and } \ell \in E(x, x^u)\} \subset Q^u(x). \tag{74}$$

This yields a set of K-good Y-configurations with length ℓ. Next we define

$$K(\ell) \stackrel{\text{def.}}{=} \{x \in K^\circ : \mu_x^u[Q^u(x, \ell)] > 1 - \eta\}. \tag{75}$$

We want to construct a large set of integers ℓ with $\mu[K(\ell)]$ is large enough. This is provided by the following statement, inspired by Eskin-Lindestrauss' paper [14] and which is essentially a Fubini-like argument.

Proposition 11.6 (see Claim 6.4 in Eskin-Lindenstrauss [14])**.** *Set*

$$\mathcal{D} \stackrel{\text{def.}}{=} \{\ell \in \mathbb{N} : \mu[K(\ell)] > 1 - \eta\}. \tag{76}$$

Then for every $L > T_1$ we have

$$\#(\mathcal{D} \cap [0, L]) > (1 - \eta)L.$$

Proof. We start the proof by considering the set

$$K_1 \stackrel{\text{def.}}{=} \left\{ x \in K^\circ : \mu_x^u[Q^u(x)] > 1 - \sqrt{\delta} \right\},$$

so that after applying Lemma 11.1 with $\psi \colon x \mapsto \mu_x^u[Q^u(x)] = \mu_x^u[K^\circ \cap \xi^u(x)]$ we see that $\mu(K_1) > 1 - \sqrt{\delta}$. The set

$$B \stackrel{\text{def.}}{=} \{\omega = (x, x^u) \in \Omega : x \in K_1, \, x^u \in Q^u(x)\}$$

is a measurable subspace of Ω of measure

$$\nu(B) = \int_{K_1} \mu_x^u[Q^u(x)] d\mu(x) > (1 - \sqrt{\delta})^2 = 1 - \delta',$$

for some $\delta' = \delta'(\delta) > 0$ tending to zero with δ.

Now let $L > T_1$ and consider the space (I_L, m), for the set $I_L \stackrel{\text{def.}}{=} [0, L] \cap \mathbb{N}$ endowed with the counting measure that we denote by m. Let $F \subset \Omega \times I_L$ be the set of pairs (ω, ℓ) such that $\omega \in B$ and $\ell \in E(\omega)$. It follows from Lemma 11.5 that for all $\omega \in B$, $m[E(\omega)] > 1 - r\delta$. Hence

$$(\nu \times m)(F) = \int_B m[E(\omega)] \, d\nu(\omega) > (1 - r\delta)(1 - \delta') \stackrel{\text{def.}}{=} 1 - \delta'',$$

for some $\delta'' = \delta''(\delta)$ that tends to zero with δ.

On the other hand set $B(\ell) = \{\omega \in B : \ell \in E(\omega)\}$ so we have

$$F = \{(\omega, \ell) \in \Omega \times I_L : \omega \in B(\ell)\} = \bigcup_{\ell \in I_L} B(\ell) \times \{\ell\}.$$

Applying Fubini's theorem we get

$$\begin{aligned}
(\nu \times m)(F) &= \int_\Omega \int_{I_L} \mathbf{1}_F(\omega, \ell) \, dm(\ell) \, d\nu(\omega) = \int_{I_L} \int_\Omega \mathbf{1}_F(\omega, \ell) \, d\nu(\omega) \, dm(\ell) \\
&= \int_{I_L} \nu(B(\ell)) \, dm(\ell).
\end{aligned}$$

It follows from Lemma 11.1 that the set of integers $\mathcal{D}_L \stackrel{\text{def.}}{=} \{\ell \in I_L : \nu(B(\ell)) > 1 - \sqrt{\delta''}\}$ has measure $> 1 - \sqrt{\delta''}$ for m. Note that

$$\nu(B(\ell)) = \int_{K_1} \mu_x^u[Q^u(x, \ell)] \, d\mu(x),$$

hence, another application of Lemma 11.1 yields $\mu\{x \in K_1 : \mu_x^u[Q^u(x, \ell)] > 1 - \eta\} > 1 - \eta$ where $\eta \stackrel{\text{def.}}{=} \sqrt[4]{\delta''}$ is the function we were looking for. Now since by definition $\{x \in K_1 : \mu_x^u[Q^u(x, \ell)] > 1 - \eta\} \subset K(\ell)$ we conclude that $\mu[K(\ell)] > 1 - \eta$. This implies that $\mathcal{D}_L \subset \mathcal{D} \cap [0, L]$ where $\mathcal{D}$ is the set defined in (76). We deduce that $\#(\mathcal{D} \cap [0, L]) > (1 - \eta)L$ as claimed. $\square$

11.3. **Construction of matched Y-configurations.** We are now ready to finish the proof of Lemma 10.9, and thus that of our main Theorem. We will build arbitrarily long pairs of good and matched Y-configurations.

Let K_0 be a compact set included inside the Lusin set $\mathcal{L}$ defined in §8.1.2 of measure $\mu(K_0) > 1 - \delta$.

We consider the measurable set

$$K = (K_0)^\circ,$$

so points of K are (δ, T)-recurrent in K_0 for some $T = T(\delta)$, and $\mu(K) > 1 - \delta$. We apply the results of §11.2.1 to K. They yield a set $\mathcal{D} \subset \mathbb{N}$ as defined in (75) and Proposition 11.6. In particular, by definition, $\mathcal{D}$ has density $> 1 - \eta$ in $[0, L]$ for L large enough and for every $\ell \in \mathcal{D}$, $\mu[K(\ell)] > 1 - \eta$ (where $\eta = \eta(\delta) < 1$).

11.3.1. *Constructing the tails.* Recall that for $x, y \in \mathbb{T}^3$ and $\ell \in \mathbb{N}$ we let $x_{-\ell}$ and $y_{-\ell}$ denote $f^{-\ell}(x)$ and $f^{-\ell}(y)$ respectively.

Lemma 11.7. *For every $\ell \in \mathcal{D}$, there exists $K'(\ell) \subset K(\ell)$ such that $\mu[K'(\ell)] > 1 - 2\sqrt{\eta}$ and for every $x \in K'(\ell)$, there exists $y \in K(\ell) \cap \xi^s(x)$ such that*

$$\alpha^s(x_{-\ell}, y_{-\ell}) > c(\eta),$$

where $c(\eta)$ is the constant introduced in Lemma 11.2.

Proof. We apply Lemma 11.2 to $A = f^{-\ell}(K(\ell))$ which has measure $\mu(K(\ell)) > 1 - \eta$. Then we can define $K'(\ell) = f^\ell(A')$ where A' is the set provided by that lemma. It is clear that this set satisfies the desired properties (recall that ξ^s is decreasing). $\square$

11.3.2. *Constructing the quadrilaterals: the matching argument.* For $x \in K'(\ell)$, so $\xi^u(x)$ is an interval of size $r(x)$, we set

$$I^u(x) \stackrel{\text{def.}}{=} \{z \in \xi^u(x) : d^u(x, z) > c(\eta), \text{ and } d^u(z, \partial \xi^u(x)) > c(\eta)\}. \qquad (77)$$

Recall that $0 < c(\eta) \leq r_0 \eta \leq |\xi^u(x)|$ (see Lemma 11.2), hence $|I^u(x)| > (1 - 4\eta)|\xi^u(x)|$. Note furthermore that if $y \in \mathcal{L}$ is close enough to x then $H_{x,y}^{cs}(I^u(x)) \subset \xi^u(y)$ (we use here that $r(y) = |\xi^u(y)|$ is uniformly continuous in $\mathcal{L}$ and that center-stable holonomy maps converge uniformly to the identity as x tends to y).

Since μ_x^u is absolutely continuous continuous with respect to the inner Lebesgue length $|.|$ of $\mathcal{W}^u(x)$, with a uniform bound β on the densities (recall Lemma 2.23) it follows that

$$\frac{|Q^u(x, \ell)|}{|\xi^u(x)|} > 1 - \beta \eta,$$

Proposition 11.8 (Matching argument)**.** *There exists $L_0 \in \mathbb{N}$ such that for every $\ell \geq L_0$ and $x \in K'(\ell)$, if $y \in K(\ell) \cap \xi^s(x)$ is given by Lemma 11.7 then there exist $x^u \in \xi^u(x)$ and $y^u \in \xi^u(y)$ such that the following properties hold*

    (1) $d_u(x, x^u) > c(\eta)$;
    (2) $y^u = H_{x,y}^{cs}(x^u)$;
    (3) $J(x^u, \varepsilon, \ell) \cap Q^u(x, \ell) \neq \emptyset$ and $J(y^u, \varepsilon, \ell) \cap Q^u(y, \ell) \neq \emptyset$.

The next paragraphs are devoted to the proof of this proposition.

11.3.3. *Good matching.* Let $x, y \in \mathbb{T}^3$ be points verifying Lemma 11.7. Note that the lengths of dynamical balls $|J(z)| = |J(z, \varepsilon, \ell)|$ tend uniformly to 0 as $\ell \to \infty$. Hence we may suppose that ℓ is large enough so that for every $z \in I^u(x)$, $J(z) \subset \xi^u(x)$ and $J(H_{x,y}^{cs}(z)) \subset \xi^u(y)$.

Let $x_1, \ldots, x_k \in I^u(x) \subset \xi^u(x)$ and $J_1, \ldots, J_k \subset \mathcal{W}^u(x)$ be the dynamical balls defined by

$$J_i \stackrel{\text{def.}}{=} J(x_i, \varepsilon, \ell) \subset \xi^u(x).$$

We may assume that the following properties are satisfied

(1) for every $i \geq 1$, $d(x_i, x) > c(\eta)$;
(2) $I^u(x) \subset \bigcup_{i=1}^{k} J_i$;
(3) for every $z \in I^u(x)$, $\#\{i : z \in J_i\} \leq 2$.

Set

$$y_i \stackrel{\text{def.}}{=} H_{x,y}^{cs}(x_i) \in \xi^u(x), \quad \text{and} \quad J_i' \stackrel{\text{def.}}{=} J(y_i, \varepsilon, \ell) \subset \xi^u(x).$$

The dynamical balls J_i and J_i' are matched. In particular Corollary 10.6 implies that

$$\kappa_2^{-1} |J_i| \leq |J_i'| \leq \kappa_2 |J_i|.$$

Set

$$\tau_i \stackrel{\text{def.}}{=} \tau(x, x_i, \varepsilon, \ell), \quad \text{and} \quad \tau_i' \stackrel{\text{def.}}{=} \tau(y, y_i, \varepsilon, \ell),$$

so $|\tau_i - \tau_i'| < T_0$ for all i.

Lemma 11.9 (Control of overlap). *There exists $m = m(f) \in \mathbb{N}$ such that for every $z \in \xi^u(y)$*

$$\#\{i : z \in J_i'\} \leq m.$$

Proof. Let $J_{i_1}', \ldots, J_{i_m}'$ be the dynamical balls in $\mathcal{W}^u(y)$ that contain z. Note that for every j, $|\tau_{i_j}' - \tau_z'| \leq T_0 < T_1$ (see Lemma 10.3). Let

$$J \stackrel{\text{def.}}{=} \bigcup_{j=1}^{m} J_{i_j}, \qquad J' \stackrel{\text{def.}}{=} \bigcup_{j=1}^{m} J_{i_j}', \quad \text{and} \quad \tau^* \stackrel{\text{def.}}{=} \min_{j=1,\ldots m} \{\tau_{i_j}, \tau_{i_j}'\}.$$

We claim that the oscillations inside the set $\{\tau_{i_1}, \ldots, \tau_{i_m}, \tau_{i_1}', \ldots, \tau_{i_m}'\}$ are less than T_1. Indeed, by Lemma 9.4, for every j we have that $|\tau_{i_j} - \tau_{i_j}'| < T_0$. Also, by Lemma 10.3 we have $|\tau_{i_j}' - \tau(y, z, \varepsilon, \ell)| < T_0$ and thus $|\tau_{i_j}' - \tau_{i_k}'| < 2T_0$ and $|\tau_{i_j} - \tau_{i_k}| < 4T_0$. We conclude that for every $\alpha \in \{\tau_{i_1}, \ldots, \tau_{i_m}, \tau_{i_1}', \ldots, \tau_{i_m}'\}$ it holds

$$|\alpha - \tau^*| < 4T_0, \tag{78}$$

which proves our claim due to our choice of T_1 in (71)

On the one hand $f^{\tau_*}(z)$ belongs to the intersection of intervals $f^{\tau_*}(J_i')$, which have length ≤ 2, so $|f^{\tau_*}(J')| \leq 4$.

With this claim established we can bound from below the length $|f^{\tau^*}(J)|$. In fact, because at most two intervals J_i can overlap at the same time, we can estimate

$$|f^{\tau^*}(J)| = \left| \bigcup_{j=1}^{m} f^{\tau^*}(J_{i_j}) \right| \geq \frac{1}{2} \sum_{i=1}^{m} |f^{\tau^*}(J_{i_j})|.$$

Now, by definition of the unstable dynamical ball we have $|f^{\tau_{i_j}}(J_{i_j})| = 2$ and by (78) $|\tau^* - \tau_{i_j}| < T_1$ for every $j = 1, \ldots, m$. Therefore,

$$|f^{\tau^*}(J_{i_j})| \geq \frac{2}{\|Df^{-T_1}\|}.$$

Combining the last two inequalities one deduces that

$$|f^{\tau^*}(J)| \geq \frac{m}{\|Df^{-T_1}\|}.$$

This implies in particular that there exist two points of $f^{\tau*}(J)$ that are at distance of at least $\frac{m}{2\|Df^{-T_1}\|}$. Since such a point is at distance ≤ 1 of some $f^{\tau*}(x_i)$ we deduce that there exists j, l such that

$$d(f^{\tau^*}(x_{i_j}), f^{\tau^*}(x_{i_l})) \geq \frac{m}{2\|Df^{-T_1}\|} - 2.$$

On the one hand $d(f^{\tau^*}(y_{i_j}), f^{\tau^*}(y_{i_l})) \leq 2$ (this follows from the definition of τ^* and the fact that $f^{\tau^*}(z) \in \mathcal{W}_1^u(f^{\tau^*}(y_{i_j})) \cap \mathcal{W}_1^u(f^{\tau^*}(y_{i_l})))$. On the other hand if ℓ is large enough and ε small enough we may ask $d(f^{\tau^*}(y_{i_j}), f^{\tau^*}(x_{i_j})) \leq \rho_0$ (where ρ_0 is the constant of Lemma 2.8). Hence the Hölder regularity of center-stable holonomies provided by Lemma 2.8 yields

$$\frac{m}{2\|Df^{-T_1}\|} - 2 \leq d(f^{\tau^*}(x_{i_j}), f^{\tau^*}(x_{i_l}))$$
$$\leq C^{cs} d(f^{\tau^*}(y_{i_j}), f^{\tau^*}(y_{i_l}))^{\theta^{cs}}$$
$$\leq C^{cs} 2^{\theta^{cs}}.$$

We obtain

$$m \leq (C^{cs} 2^{1+\theta^{cs}} + 4)\|Df^{-T_1}\|.$$

This upper bound only depends on f, which concludes the proof. $\square$

11.3.4. *Proof of Proposition 11.8.* We are now ready to give a proof of Proposition 11.8 which is a modification of that of [15, Lemma 12.8]. Recall that x and y are given by Lemma 11.7 and their distance goes to zero as $\ell \to +\infty$. By continuity of the size of $\xi^u(x)$ inside the Lusin set K, we may suppose that for ℓ sufficiently large, $|\xi^u(x)| \leq 2|\xi^u(y)|$.

Set

$$\mathcal{I}^x \stackrel{\text{def.}}{=} \{i \in \{1, \dots, k\} : J_i \cap Q^u(x, \ell) \neq \emptyset\}, \qquad \mathcal{J}^x \stackrel{\text{def.}}{=} \{1, \dots, k\} \setminus \mathcal{I}^x,$$

and

$$\mathcal{I}^y \stackrel{\text{def.}}{=} \{i \in \{1, \dots, k\} : J_i' \cap Q^u(y, \ell) \neq \emptyset\}, \quad \text{and} \quad \mathcal{J}^y \stackrel{\text{def.}}{=} \{1, \dots, k\} \setminus \mathcal{I}^y.$$

Let

$$Q \stackrel{\text{def.}}{=} \{z \in Q^u(x, \ell) : \forall i, (z \in J_i \Rightarrow i \in \mathcal{I}^y)\}.$$

It suffices to show that $Q \cap I^u(x) \neq \emptyset$.

Note that

$$(Q^u(x, \ell) \cap I^u(x)) \setminus Q \subset \bigcup_{i \in \mathcal{I}^x \cap \mathcal{J}^y} Q^u(x, \ell) \cap J_i$$

so

$$|(Q^u(x, \ell) \cap I^u(x)) \setminus Q| \leq \sum_{i \in \mathcal{I}^x \cap \mathcal{J}^y} |Q^u(x, \ell) \cap J_i| \leq \sum_{i \in \mathcal{I}^x \cap \mathcal{J}^y} |J_i| \leq \kappa_2 \sum_{i \in \mathcal{J}^y} |J_i'|$$

$$\leq \kappa_2 m \left| \bigcup_{i \in \mathcal{J}^y} J_i' \right| \leq \kappa_2 m |\xi^u(y) \setminus Q^u(y, \ell)|$$

$$\leq \kappa_2 m \beta \eta |\xi^u(y)| \leq 2\kappa_2 m \beta \eta |\xi^u(x)|.$$

Hence there exists $\beta' = \beta'(f)$ such that

$$|Q \cap I^u(x)| \geq |Q^u(x, \ell) \cap I^u(x)| - |(Q^u(x, \ell) \cap I^u(x)) \setminus Q| \geq (1 - \beta'\eta)|\xi^u(x)| > 0,$$

as soon as $\eta < \frac{1}{\beta'}$.

Now let $a \in Q$. There exists i such that $d(x, x_i) > c$, $a \in J(x_i, \varepsilon, \ell)$ and $J(y_i, \varepsilon, \ell) \cap Q^u(y, \ell) \neq \emptyset$ where $y_i = H^{cs}_{x,y}(x_i)$: the proof of the proposition is over. $\qquad\square$

11.3.5. *The synchronization.* We are now ready to finish the proof of Lemma 10.9. Let $\ell \in \mathcal{D}$ such that $\ell \geq L_0$. Let $x \in K'(\ell)$ and $y \in K(\ell) \cap \xi^s(x)$ be given by Lemma 11.7. Let $x^u \in \xi^u(x)$ and $y^u \in \xi^u(y)$ be given by Proposition 11.8: there exist $a \in J(x^u, \varepsilon, \ell) \cap Q^u(x, \ell)$ and $b \in J(x^u, \varepsilon, \ell) \cap Q^u(y, \ell)$.

Lemma 11.10. *There exists $T > 0$ independent of ℓ and $\tau, t > 0$ such that*

(1) $|\tau - \tau(x, x^u, \varepsilon, \ell)| \leq T$ *and* $|t - t(x, x^u, \varepsilon, \ell)| \leq T$;
(2) $f^\tau(a), f^t(x), f^\tau(b), f^t(y) \in K_0$.

It follows from Lemma 11.10 that for $\ell \in \mathcal{D}$ larger than L_0, the Y-configurations $X = X(x, x^u, \ell)$ and $Y = Y(y, y^u, \ell)$ are (K_0, C, T)-matched. This proves Lemma 10.9.

Proof. Set

$$\tau(\ell) = \tau(x, a, \varepsilon, \ell), t(\ell) = t(x, a, \varepsilon, \ell), \tau'(\ell) = \tau(y, b, \varepsilon, \ell), t'(\ell) = t(y, b, \varepsilon, \ell).$$

By construction (x, x^u, y, y^u) is a (C, ℓ)-quadrilateral and $a \in J(x^u)$, $b \in J(y^u)$. Combining the results of Lemmas 9.4 and 10.3 we obtain

$$|\tau(\ell) - \tau(x, x^u, \varepsilon, \ell)| < T_0, \quad |t(\ell) - t(x, x^u, \varepsilon, \ell)| < T_0$$

as well as

$$|\tau'(\ell) - \tau(x, x^u, \varepsilon, \ell)| < 2T_0, \quad |t'(\ell) - t(x, x^u, \varepsilon, \ell)| < 2T_0,$$

and consequently

$$|\tau'(\ell) - \tau(\ell)| < 3T_0, \quad |t'(\ell) - t(\ell)| < 3T_0,$$

where $T_0 = T_0(\delta)$.

Proposition 11.8 implies that

$$f^{\tau(\ell)}(a), f^{t(\ell)}(x), f^{\tau'(\ell)}(b), f^{t'(\ell)}(y) \in K,$$

which means that these points are (δ, T)-recurrent inside the compact set K_0. Now we might have $\tau(\ell) \neq \tau'(\ell)$ or $t(\ell) \neq t'(\ell)$. Assume for example that $t(\ell) < t'(\ell)$. Let $T' = T'(\delta) > \max(T, T_0)$ such that

$$\frac{T_0}{T'} < \delta,$$

in such a way that $[t(\ell), t'(\ell)]$ has density $< 3\delta$ inside $[t(\ell), t'(\ell) + T']$. Since $T' > T$, this implies that

$$\left\{ k \in [t(\ell), t'(\ell) + T'] : f^k(x) \in K_0 \right\} \cap [t'(\ell), t'(\ell) + T']$$

has density $> 1 - 4\delta$ in $[t'(\ell), t'(\ell) + T']$. On the other hand the set

$$\left\{ k \in [t'(\ell), t'(\ell) + T'] : f^k(y) \in K_0 \right\}$$

has density $> 1 - \delta$ inside $[t'(\ell), t'(\ell) + T']$. Therefore these two sets must intersect (as soon as $\delta < 1/5$) so there exists $t \in [t'(\ell), t'(\ell) + T']$ such that $f^t(x), f^t(y) \in K_0$. This integer t satisfies

$$|t - t(x, x^u, \varepsilon, \ell)| \leq |t - t'(\ell)| + |t'(\ell) - t(x, x^u, \varepsilon, \ell)| < T' + 2T_0$$

so $T = T' + 2T_0$ is the desired constant. The same argument can be reproduced to treat a and b and find the number τ. $\qquad\square$

11.3.6. *Postliminary: choice of δ_0 and δ.* Let us list the requirements we made on the constant δ_0 and $\delta = 3\delta_0$. We needed to require

$$\delta < \min\left(\frac{1}{5}, \frac{1}{r}\right),$$

where r only depends on f (it is defined in Lemma 11.5).

We defined in Proposition 11.6 a function $\eta(\delta)$ (of the order of $\delta^{1/4}$) that depends only on a and required that

$$\eta < \min\left(\frac{1}{4}, \frac{1}{\beta}, \frac{1}{\beta'}\right),$$

where β is the constant of Lemma 2.23 and β' appears in §11.3.4.

12. REMARKS ON GOGOLEV-KOLMOGOROV-MAIMON'S PERTURBATIONS

In [19] the authors consider two families of perturbations of the linear Anosov diffeomorphism $f_0 \colon \mathbb{T}^3 \to \mathbb{T}^3$ induced by the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The *dissipative family* is given

$$f_{D,\varepsilon} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \frac{\varepsilon}{2\pi} \begin{bmatrix} \sin(2\pi x) \\ 0 \\ 0 \end{bmatrix} \quad \mathrm{mod}(1).$$

The Jacobian at the fixed point of $f_{D,\varepsilon}$ is given by

$$\mathrm{Jac} f_{D,\varepsilon} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{vmatrix} 2+\varepsilon & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 + \varepsilon,$$

so $f_{D,\varepsilon}$ is dissipative: it does not preserve any volume.

The *conservative family* is given by

$$f_{C,\varepsilon} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \frac{\varepsilon}{2\pi} \begin{bmatrix} \sin(2\pi x) \\ \sin(2\pi x) \\ 0 \end{bmatrix} \quad \mathrm{mod}(1).$$

The Jacobian at any point of $f_{C,\varepsilon}$ is given by

$$\mathrm{Jac} f_{C,\varepsilon} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{vmatrix} 2+\varepsilon\cos(2\pi x) & 1 & 0 \\ 1+\varepsilon\cos(2\pi x) & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1.$$

We apply Gan-Shi's criterion for joint integrability obtained in [17] in order to prove that these two families of perturbations are accessible and hence Theorem A applied to both of them gives the following theorem.

Theorem 12.1. *If $|\varepsilon| > 0$ is small enough $f_{D,\varepsilon}$ or $f_{C,\varepsilon}$ are accessible. Hence any fully supported ergodic u-Gibbs measure for $f_{D,\varepsilon}$ or $f_{C,\varepsilon}$ is SRB (this is Lebesgue in the conservative case).*

Proof. In our context, Gan-Shi's criterion for accessibility is the following. A $C^{1+\alpha}$ difeomorphism f C^1-close to f_0 is not accessible if and only if its central Lyapunov exponents at every perdiodic points coincide with those of f_0.

In order to prove the accessibility of $f_{D,\varepsilon}$ and $f_{C,\varepsilon}$ it is enough to prove that the central Lyapunov exponent at the fixed point differ from the one of A (which is

approximately 1.55). This can be done as follows. The characteristic polynomial of the differential at the fixed point for the dissipative family is

$$P_{D,\varepsilon}(X) = \begin{vmatrix} 2-X+\varepsilon & 1 & 0 \\ 1 & 2-X & 1 \\ 0 & 1 & 1-X \end{vmatrix} = -X^3 + (5+\varepsilon)X^2 - (6+3\varepsilon)X + 1 + \varepsilon.$$

The coefficients of $P_{D,\varepsilon}(X)$ vary smoothly with ε and when $\varepsilon = 0$, this polynomial has three disctinct roots. So by the implicit function theorem, for small $|\varepsilon|$, $P_{D,\varepsilon}$ has three distinct roots $\lambda_1(\varepsilon) < \lambda_2(\varepsilon) < \lambda_3(\varepsilon)$ that depend smoothly on ε and satisfy the relations

$$\begin{cases} \lambda_1\lambda_2\lambda_3 = 1 + \varepsilon \\ \lambda_1 + \lambda_2 + \lambda_3 = 5 + \varepsilon \\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 6 + 3\varepsilon \end{cases}.$$

We must have $\lambda_2'(0) \neq 0$, which yields the accessibility of $f_{D,\varepsilon}$. Indeed, if $\lambda_2'(0) = 0$, then derivating the relations above gives (the functions below are evaluated at 0)

$$\begin{cases} \lambda_2(\lambda_1'\lambda_3 + \lambda_1\lambda_3') = 1 \\ \lambda_1' + \lambda_3' = 1 \\ \lambda_2(\lambda_1' + \lambda_3') + \lambda_1'\lambda_3 + \lambda_1\lambda_3' = 3 \end{cases}.$$

Combining the above one finds $\lambda_2 + 1/\lambda_2 = 3$, which contradicts $\lambda_2 \simeq 1.55$ (so $\lambda_2 + 1/\lambda_2 \simeq 2.20$).

The same computation works for the conservative family. Indeed, the characteristic polynomial of the differential at the fixed point for this family is

$$P_{C,\varepsilon}(X) = \begin{vmatrix} 2-X+\varepsilon & 1 & 0 \\ 1+\varepsilon & 2-X & 1 \\ 0 & 1 & 1-X \end{vmatrix} = -X^3 + (5+\varepsilon)X^2 - (6+2\varepsilon)X + 1.$$

So the relations between roots and coefficients give

$$\begin{cases} \lambda_1\lambda_2\lambda_3 = 1 \\ \lambda_1 + \lambda_2 + \lambda_3 = 5 + \varepsilon \\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 6 + 2\varepsilon \end{cases}.$$

The same argument as before provides $\lambda_2'(0) \neq 0$ (the calculations are left to the reader), which yields the accessibility of $f_{C,\varepsilon}$. □

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SÉBASTIEN ALVAREZ
CMAT, Facultad de Ciencias, Universidad de la República
Igua 4225 esq. Mataojo. Montevideo, Uruguay.
email: `salvarez@cmat.edu.uy`

MARTIN LEGUIL
École polytechnique, CMLS
Route de Saclay, 91128 Palaiseau Cedex, France.
email: `martin.leguil@polytechnique.edu`

DAVI OBATA
Department of Mathematics, Brigham Young University
Provo, Utah 84602, USA.
email: `davi.obata@mathematics.byu.edu`

BRUNO SANTIAGO
Instituto de Matemática e Estatística, Universidade Federal Fluminense
Rua Prof. Marcos Waldemar de Freitas Reis, S/N – Bloco H, 4o Andar
Campus do Gragoatá, Niterói, Rio de Janeiro 24210-201, Brasil.
email: `brunosantiago@id.uff.br`