THE RIEMANN-HILBERT MAPPING FOR $\mathfrak{sl}_2$-SYSTEMS OVER GENUS TWO CURVES

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à Étienne Ghys

Abstract. We prove in two different ways that the monodromy map from the space of irreducible $\mathfrak{sl}_2$-differential-systems on genus two Riemann surfaces, towards the character variety of SL_2-representations of the fundamental group, is a local diffeomorphism. This is motivated by a question raised by Étienne Ghys about Margulis' problem: existence of curves of negative Euler characteristic in compact quotients of $\mathrm{SL}_2(\mathbb{C})$.

1. Introduction

Let S be a compact oriented topological surface of genus $g \geq 2$ and $X \in \mathrm{Teich}(S)$ a marked complex structure on S. Given a $\mathfrak{sl}_2$-matrix of holomorphic one forms on X:

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \in \mathfrak{sl}_2(\Omega^1(X))$$

we consider the system of differential equations for $Z \in \mathbb{C}^2$:

$$\mathrm{d}Z + AZ = 0. \tag{1}$$

A fundamental matrix $B(x)$ at a point $x_0 \in X$ is a two-by-two matrix $B \in \mathrm{SL}_2(\mathcal{O}_{x_0})$ whose columns form a base for the two-dimensional vector space of solutions, i.e. satisfying $\mathrm{d}B + AB = 0$, $\det(B) \equiv 1$. It can be analytically continued as a function $B : \widetilde{X} \to \mathrm{SL}_2(\mathbb{C})$ defined on the universal cover of X which satisfies an equivariance

$$\forall \gamma \in \pi_1(X), \quad B(\gamma \cdot x) = B(x) \cdot \rho_A(\gamma)^{-1}$$

for a certain representation $\rho_A : \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$. The conjugacy class of ρ_A in the $\mathrm{SL}_2(\mathbb{C})$-character variety

$$\Xi := \mathrm{Hom}(\pi_1(S), \mathrm{SL}_2(\mathbb{C}))//\mathrm{SL}_2(\mathbb{C})$$

does not depend on the initial solution and will be referred to as the **monodromy class** of the system. Also, for any $M \in \mathrm{SL}_2(\mathbb{C})$ the monodromy class of $MAM^{-1} \in$

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$\mathfrak{sl}_2(\Omega^1(X))$ coincides with that of A. It is therefore natural to consider the space of systems up to gauge equivalence:

$$\mathrm{Syst} := \{(X, A) : X \in \mathrm{Teich}(S), A \in \mathfrak{sl}_2(\Omega^1(X))\}//\mathrm{SL}_2(\mathbb{C}).$$

The Riemann-Hilbert mapping is the map

$$\mathrm{Mon} : \mathrm{Syst} \to \Xi$$

defined by $\mathrm{Mon}(X, [A]) := [\rho_A]$. Both Syst and Ξ are (singular) algebraic varieties of complex dimension $6g - 6$. The irreducible locus $\mathrm{Syst}^{\mathrm{irr}} \subset \mathrm{Syst}$ and $\Xi^{\mathrm{irr}} \subset \Xi$, characterized by those A and image(ρ) without non trivial invariant subspace, define smooth open subsets. Then, Mon induces a holomorphic mapping between these open sets. Our main aim is to prove

Theorem 1.1. *If S has genus two, the holomorphic map*

$$\mathrm{Mon} : \mathrm{Syst}^{irr} \to \Xi^{irr}$$

is a local diffeomorphism.

The equivalent statement is not true in general for higher genera. Easy counterexamples in genus at least 4 can be constructed by considering the pull back of a system on a genus two Riemann surface X by a parametrized family of ramified coverings over X. In genus $g = 3$ there are also counterexamples (see Section 6 for details). On the other hand, we note that irreducibility is a necessary assumption. Indeed, diagonal and nilpotent systems

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$$

admit non trivial isomonodromic deformations: this comes from isoperiodic deformations of pairs (X, α) that exist, as it can be seen just by counting dimensions (see also [9]).

Motivation. The question of determining the properties of the monodromy representations associated to holomorphic $\mathfrak{sl}_2$-systems on a Riemann surface X of genus $g > 1$ was raised by Ghys. The motivation comes from the study of quotients $M := \mathrm{SL}_2/\Gamma$ by cocompact lattices $\Gamma \subset \mathrm{SL}_2$. These compact complex manifolds are not Kähler. Huckleberry and Margulis proved in [8] that they admit no complex hypersurfaces (and therefore no non-constant meromorphic functions). Elliptic curves exist in such quotient, while the existence of compact curves of genus at least two remains open and is related to Ghys' question. Indeed, assuming that for a non trivial system on a curve X, its monodromy has image contained in Γ (up to conjugation), then the corresponding fundamental matrix induces a non trivial holomorphic map from X to M. Reciprocally, any curve X in M can be lifted to $\mathrm{SL}(2, \mathbb{C})$ and gives rise to the fundamental matrix of some system on X, whose monodromy is contained in Γ. In fact, it is not known whether holomorphic

$\mathfrak{sl}_2$-systems on Riemann surfaces of genus > 1 give rise to representations with discrete or real image. Although Ghys' question remains open, our result shows that we can locally realize arbitrary deformations of the monodromy representation by deforming the curve and the system.

Idea of the proofs. We propose two different proofs of our result, using isomonodromic deformations of two kinds of objects, namely vector bundles with connections used by the last two authors, and branched projective structures used by the first two authors. Both proofs were obtained independently. We decided to write them together in this paper.

We first develop the approach with flat vector bundles in Sections 2 and 3. It is based on the work [7] by the last two authors. One considers a system as a (holomorphic) $\mathfrak{sl}_2$-connection $\nabla = \mathrm{d} + A$ on the trivial bundle $E_0 \to X$, and look at the larger moduli space Con of all triples (X, E, ∇) where E is not necessarily trivial. The subspace Syst of those triples with $E = E_0$ has codimension 3. The monodromy map is defined on the larger space Con and defines a foliation $\mathcal{F}_{\mathrm{iso}}$ by 3-dimensional leaves: the isomonodromy leaves. We note that everything is smooth in restriction to the irreducible locus, and Mon is a submersion. Since $\mathcal{F}_{\mathrm{iso}}$ and Syst have complementary dimensions, the fact that the restriction of Mon to $\mathrm{Syst}^{\mathrm{irr}}$ is a local diffeomorphism is equivalent to the transversality of $\mathcal{F}_{\mathrm{iso}}$ to $\mathrm{Syst}^{\mathrm{irr}}$. To prove this, we strongly use the hyperellipticity of genus 2 curves to translate our problem to some moduli space of logarithmic connections on $\mathbb{P}^1$. There, isomonodromy equations are well-known, explicitely given by a Garnier system, and we can compute the transversality.

The approach using branched projective structures is developed in Sections 5, 6, 7, 8 and 9. We hope that it might be generalized to higher genus. It uses isomonodromic deformation spaces of branched complex projective structures over a surface S of genus $g \geq 2$, that were introduced in [2]; namely given a conjugacy class of irreducible representation $\rho : \pi_1(S) \to \mathrm{SL}(2, \mathbb{C})$, and an even integer k, the space of marked complex projective structures with k branch points (counted with multiplicity) and holonomy ρ has the structure of a smooth k-dimensional complex manifold denoted by $\mathcal{M}_{k,\rho}$. We establish a dictionary between

a) Systems on $X \in \mathrm{Teich}(S)$ with monodromy ρ
b) Regular holomorphic foliations on $X \times \mathbb{P}^1$ transverse to the $\mathbb{P}^1$-fibration and with monodromy $[\rho] \in \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{C}))$ (Riccati foliations)
c) Rational curves in the space $\mathcal{M}_{2g-2,\rho}$ of branched projective structures over S with branching divisor of degree $2g - 2$ and monodromy ρ.

The equivalence between (a) and (b) is an easy and well-known fact. The interesting aspect of the dictionary is between (a) and (c), it is discussed in Section 6. Injectivity of the differential of the Riemann-Hilbert mapping at a given $\mathfrak{sl}(2)$-system with monodromy ρ is then equivalent to first order rigidity of the corresponding rational curve in $\mathcal{M}_{2g-2,\rho}$. In the genus two case, the moduli space $\mathcal{M}_{2,\rho}$ is a complex surface, and the rigidity of the rational curve is established by showing its self-intersection is equal to -4. In higher genus, the infinitesimal rigidity of the rational curve does not hold, counter-examples are described in Section 10.

2. $\mathfrak{sl}_2$-systems on X

Let us consider the smooth projective curve of genus two defined in an affine chart by

$$X_{\boldsymbol{t}} := \{y^2 = x(x-1)(x-t_1)(x-t_2)(x-t_3)\} \tag{2}$$

for some parameter

$$\boldsymbol{t} = (t_1, t_2, t_3) \in T := \left(\mathbb{P}^1 \setminus \{0, 1, \infty\}\right)^3 \setminus \bigcup_{i \neq j} \{t_i = t_j\}. \tag{3}$$

A $\mathfrak{sl}_2$-system on $X_{\boldsymbol{t}}$ takes the form

$$\begin{cases} \mathrm{d}z_1 + \alpha z_1 + \beta z_2 = 0 \\ \mathrm{d}z_2 + \gamma z_1 - \alpha z_2 = 0 \end{cases}$$

where α, β, γ are holomorphic 1-forms on $X_{\boldsymbol{t}}$. In matrix form this system writes

$$dZ + AZ = 0 \quad \text{with} \quad A = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{4}$$

It can be seen as the equation $\nabla Z = 0$ for ∇-horizontal sections for the $\mathfrak{sl}_2$-connection $\nabla = \mathrm{d} + A$ on the trivial bundle over $X_{\boldsymbol{t}}$.

We denote by $\mathrm{Syst}(X_{\boldsymbol{t}})$ the moduli space of systems on $X_{\boldsymbol{t}}$ modulo SL_2-gauge action:

$$Z \mapsto MZ \quad \rightsquigarrow \quad A \mapsto MAM^{-1}, \quad \text{for } M \in \mathrm{SL}_2.$$

An invariant is evidently given by the determinant map

$$\mathrm{Syst}(X_{\boldsymbol{t}}) \to \mathrm{H}^0(X_{\boldsymbol{t}}, \Omega^1 \otimes \Omega^1) \ ; \ A \mapsto \det(A) = -(\alpha \otimes \alpha + \beta \otimes \gamma).$$

This map actually provides the categorical quotient for this action. More precisely, we have (see [7, section 3.3])

Proposition 2.1. *The system A in* (4) *is reducible if, and only if,* $\det(A)$ *is a square, i.e.* $\det(A) = \alpha' \otimes \alpha'$ *for a* 1*-form* α'*. Moreover, in the irreducible case, the conjugacy class of A is determined by* $\det(A)$*.*

More explicitely, in the irreducible case, we can conjugate A to a normal form

$$A = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} = \begin{pmatrix} 0 & (\beta_1 x + \beta_0)\frac{\mathrm{d}x}{y} \\ (\gamma_1 x + \gamma_0)\frac{\mathrm{d}x}{y} & 0 \end{pmatrix} \tag{5}$$

with $\beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{C}$. This normal form is unique up to conjugacy by (anti-) diagonal matrices

$$M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}$$

and the determinant is given by

$$\det(A) = \frac{(\beta_1 x + \beta_0)(\gamma_1 x + \gamma_0)}{x(x-1)(x-t_1)(x-t_2)(x-t_3)} \mathrm{d}x \otimes \mathrm{d}x.$$

The system of the form (5) is reducible if, and only if, $\beta_0\gamma_1 - \beta_1\gamma_0 = 0$. Denote by $\boldsymbol{\nu} = (\nu_0, \nu_1, \nu_2) \in \mathbb{C}^3$ the variable of the space of quadratic differentials

$$\mathrm{H}^0(X, \Omega^1 \otimes \Omega^1) \ni \frac{\nu_2 x^2 + \nu_1 x + \nu_0}{x(x-1)(x-t_1)(x-t_2)(x-t_3)} \mathrm{d}x \otimes \mathrm{d}x.$$

Then Proposition 2.1 can be reformulated as follows.

Corollary 2.2. *The moduli space* $\mathrm{Syst}^{irr}(X_{\boldsymbol{t}})$ *of irreducible* $\mathfrak{sl}_2$*-systems over the curve* $X_{\boldsymbol{t}}$ *identifies with*

$$\mathrm{Syst}^{irr}(X_{\boldsymbol{t}}) \xrightarrow{\sim} \mathcal{V} \ ; \ A \mapsto \boldsymbol{\nu} = \det(A),$$

where $\mathcal{V} = \mathbb{C}^3_{\boldsymbol{\nu}} \setminus \{\nu_1^2 - 4\nu_0\nu_2 = 0\}$.

Let now $\mathrm{Syst}^{\mathrm{irr}}$ denote the family of moduli spaces $\mathrm{Syst}^{\mathrm{irr}}(X_{\boldsymbol{t}})$, where the parameter $\boldsymbol{t}$ defining the curve varies in the space T defined in (3) :

$$\mathrm{Syst}^{\mathrm{irr}} := \left\{(X_{\boldsymbol{t}}, A) \mid \boldsymbol{t} \in T\, ,\ A \in \mathrm{Syst}^{\mathrm{irr}}(X_{\boldsymbol{t}})\right\}.$$

According to the above corollary, it identifies with

$$\mathrm{Syst}^{\mathrm{irr}} \xrightarrow{\sim} T \times \mathcal{V} \ ; \ (X_{\boldsymbol{t}}, A) \mapsto (\boldsymbol{t}, \boldsymbol{\nu} = \det(A)) \tag{6}$$

where $\mathcal{V} = \mathbb{C}^3_{\boldsymbol{\nu}} \setminus \{\nu_1^2 - 4\nu_0\nu_2 = 0\}$. Moreover, $\mathrm{Syst}^{\mathrm{irr}}$ can be seen as an open set of the moduli space of systems on curves defined by

$$\mathrm{Syst} := \bigcup_{\boldsymbol{t} \in T} \mathrm{Syst}(X_{\boldsymbol{t}}).$$

Remark 2.3. Note that here we have slightly modified the definition of Syst with respect to the introduction: the parameter $\boldsymbol{t}$ defining the curve varies not in the Teichmüller space, but in the affine variety T. The monodromy map can still be defined locally on the parameter space T and the fact that it is a local diffeomorphism obviously does not depend on the choice of generators for the fundamental group. The advantage in working with an algebraic family is that the isomonodromy foliation is defined by algebraic equations, which will allow us to compute transversality.

3. Fuchsian systems on $\mathbb{P}^1$

Following [7], we now describe generic $\mathfrak{sl}_2$-connections (E, ∇) on X as modifications of the pull-back via the hyperelliptic cover

$$h : X \to \mathbb{P}^1 \ ; \ (x, y) \mapsto x \tag{7}$$

of certain logarithmic connections $(\underline{E}, \underline{\nabla})$ on $\mathbb{P}^1_x$, actually systems.

We denote by $\underline{\mathrm{Con}}(\boldsymbol{t})$ the moduli space of logarithmic connections $(\underline{E},\underline{\nabla})$ on $\mathbb{P}^1$ with polar set $\{0,1,t_1,t_2,t_3,\infty\}$ and the following spectral data:

$$\left\{0\,,\,-\frac{1}{2}\right\}\quad\text{over}\quad x=0,1,\infty,\quad\text{and}\quad\left\{0\,,\,\frac{1}{2}\right\}\quad\text{over}\quad x=t_1,t_2,t_3. \tag{8}$$

We moreover define $\underline{\mathrm{Syst}}(\boldsymbol{t})$ to be the open subset of $\underline{\mathrm{Con}}(\boldsymbol{t})$ characterized by following the (generic) properties:

- $\underline{E}=\underline{E}_0$ is the trivial vector bundle;
- the $(-\frac{1}{2})$-eigendirections over $x=0,1$ and the 0-eigendirection over $x=\infty$ are pairwise distinct;
- the 0-eigendirection over $x=\infty$ is distinct from the $\frac{1}{2}$-eigendirection over any $x=t_i$ for $i\in\{1,2,3\}$.

Now the eigendirections of any fuchsian system in $\underline{\mathrm{Syst}}(\boldsymbol{t})$ can be normalized as follows (up to SL_2-gauge equivalence)

(9)

at the pole $x=$	0	1	t_1	t_2	t_3	∞
eigenvalue $\lambda=$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
eigendirection for λ	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}z_1\\1\end{pmatrix}$	$\begin{pmatrix}z_2\\1\end{pmatrix}$	$\begin{pmatrix}z_3\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\end{pmatrix}$

with $(z_1,z_2,z_3)\in\mathbb{C}^3$. On the other hand, any fuchsian system in $\underline{\mathrm{Syst}}(\boldsymbol{t})$ with parabolic data (9) writes (as a connection on the trivial bundle)

$$\underline{\nabla}=\nabla_0+c_1\Theta_1+c_2\Theta_2+c_3\Theta_3 \tag{10}$$

with

$$\nabla_0=\mathrm{d}+\begin{pmatrix}0&0\\0&-\frac{1}{2}\end{pmatrix}\frac{\mathrm{d}x}{x}+\begin{pmatrix}0&-\frac{1}{2}\\0&-\frac{1}{2}\end{pmatrix}\frac{\mathrm{d}x}{x-1}+\sum_{i=1}^{3}\begin{pmatrix}0&\frac{z_i}{2}\\0&\frac{1}{2}\end{pmatrix}\frac{\mathrm{d}x}{x-t_i}$$

and

$$\Theta_i=\begin{pmatrix}0&0\\1-z_i&0\end{pmatrix}\frac{\mathrm{d}x}{x}+\begin{pmatrix}z_i&-z_i\\z_i&-z_i\end{pmatrix}\frac{\mathrm{d}x}{x-1}+\begin{pmatrix}-z_i&z_i^2\\-1&z_i\end{pmatrix}\frac{\mathrm{d}x}{x-t_i}.$$

For given parameters $(\boldsymbol{z},\boldsymbol{c})=(z_1,z_2,z_3,c_1,c_2,c_3)\in\mathbb{C}^6$, we denote by $\underline{\nabla}_{\boldsymbol{z},\boldsymbol{c}}$ the corresponding connection (10). The moduli space $\underline{\mathrm{Syst}}(\boldsymbol{t})$ defined above is thus parameterized by $\mathbb{C}^6_{\boldsymbol{z},\boldsymbol{c}}$ as follows.

$$\left\{\begin{array}{ccccc}\mathbb{C}^6_{\boldsymbol{z},\boldsymbol{c}}&\stackrel{\sim}{\to}&\underline{\mathrm{Syst}}(\boldsymbol{t})&\subset&\underline{\mathrm{Con}}(\boldsymbol{t})\\(\boldsymbol{z},\boldsymbol{c})&\mapsto&(\underline{E}_0,\underline{\nabla}_{\boldsymbol{z},\boldsymbol{c}})&&\end{array}\right. \tag{11}$$

We note that the eigendirections of $\underline{\nabla}_{\boldsymbol{z},\boldsymbol{c}}$ with respect to the 0-eigenvalue satisfy

(12)

at the pole $x =$	0	1
eigenvalue $\lambda =$	0	0
eigendirection for λ	$\begin{pmatrix} \frac{1}{2\sum_{i=1}^{3} c_i(z_i-1)} \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 - \frac{1}{2\sum_{i=1}^{3} c_i z_i} \\ 1 \end{pmatrix}$

3.1. **Hyperelliptic cover.** Given a fuchsian system $(\underline{E}_0, \underline{\nabla}) \in \underline{\mathrm{Syst}}(\boldsymbol{t})$, we can pull it back to the curve $X = X_{\boldsymbol{t}}$ via the hyper-elliptic cover $h : X \to \mathbb{P}^1_x$ given in (7). We thus get a logarithmic connection $\widetilde{\nabla} := h^*\underline{\nabla}$ on the trivial bundle $E_0 \to X$, with poles at the 6 Weierstrass points with eigenvalues multiplied by 2:

$$\{0, -1\} \quad \text{over} \quad w_0, w_1, w_\infty, \quad \text{and} \quad \{0, 1\} \quad \text{over} \quad w_{t_1}, w_{t_2}, w_{t_3}$$

(w_i denotes the Weierstrass point over $x = i$). All these poles are "apparent singular points" in the sense that they disappear after a convenient birational bundle transformation. More precisely, $\widetilde{\nabla}$-horizontal sections have at most a single pole or zero at these points (depending on the sign of the non-zero-eigenvalue). If E denotes the rank 2 vector bundle locally generated by $\widetilde{\nabla}$-horizontal sections, and $\phi : E \dashrightarrow E_0$ the natural birational bundle isomorphism, then $\nabla := \phi^*\widetilde{\nabla}$ is a holomorphic connection, by construction. In terms of [7], section 1.5, this map ϕ is given by negative elementary transformations in the 0-eigendirections over $x = 0, 1, \infty$, and positive elementary transformations in the -1-eigendirections over $x = t_1, t_2, t_3$ of $\widetilde{\nabla}$. We denote by Φ the combined map $\Phi = \phi^* \circ h^* : (\underline{E}_0, \underline{\nabla}) \to (E, \nabla)$.

$$\begin{tikzcd}
(E,\nabla) & (E_0,\widetilde{\nabla}) \arrow[l, dotted, "\phi^*"'] \\
& (\underline{E}_0,\underline{\nabla}) \arrow[ul, "\Phi"] \arrow[u, "h^*"']
\end{tikzcd} \tag{13}$$

More generally, we can consider a logarithmic connection $(\underline{E}, \underline{\nabla})$ on $\mathbb{P}^1$ with polar set $\{0, 1, t_1, t_2, t_3, \infty\}$ and spectral data (8); by the same pull-back construction, we get a holomorphic connection (E, ∇) on X. Moreover, since the trace connection $\mathrm{tr}(\underline{\nabla}) = \mathrm{d} + \frac{1}{2}\mathrm{d}\log\left(\frac{(x-t_1)(x-t_2)(x-t_3)}{x(x-1)}\right)$ has trivial monodromy after pull-back on X, it follows that ∇ is a $\mathfrak{sl}_2$-connection on E. We have thus defined a map between the corresponding moduli spaces of connections

$$\Phi : \underline{\mathrm{Con}}(\boldsymbol{t}) \to \mathrm{Con}(X_{\boldsymbol{t}}) \ ; \ (\underline{E}, \underline{\nabla}) \mapsto (E, \nabla) \tag{14}$$

which has been studied by the last two authors in [7].

Theorem 3.1 ([7]). *The image of the map Φ defined in* (14) *is the moduli space* $\mathrm{Con}^{irr,ab}(X_{\boldsymbol{t}})$ *of* $\mathfrak{sl}_2$*-connections with irreducible or abelian monodromy. Moreover, the map Φ defines a 2-fold cover of* $\mathrm{Con}^{irr,ab}(X_{\boldsymbol{t}})$*, unramified over the open set* $\mathrm{Con}^{irr}(X_{\boldsymbol{t}}) \subset \mathrm{Con}^{irr,ab}(X_{\boldsymbol{t}})$ *of irreducible connections.*

Consider the set $\Sigma_{\boldsymbol{t}} \subset \underline{\mathrm{Syst}}(\boldsymbol{t})$ defined by

$$\Sigma_{\boldsymbol{t}} := \left\{(\boldsymbol{z},\boldsymbol{c}) \in \mathbb{C}^6 \mid z_1 - z_2 = z_2 - z_3 = c_1 + c_2 + c_3 = 0\right\} \subset \mathbb{C}^6_{\boldsymbol{z},\boldsymbol{c}} \simeq \underline{\mathrm{Syst}}(\boldsymbol{t}).$$

The rest of this section will be devoted to prove in a direct and explicit way that

- $\Phi(\Sigma_{\boldsymbol{t}}) \subset \mathrm{Syst}(X_{\boldsymbol{t}})$ and
- $\mathrm{Syst}(X_{\boldsymbol{t}})^{\mathrm{irr}} \subset \Phi(\Sigma_{\boldsymbol{t}})$.

More precisely, $\Sigma_{\boldsymbol{t}}$ consists in all those fuchsian systems in $\underline{\mathrm{Syst}}(\boldsymbol{t})$ whose image (E,∇) under Φ defines a holomorphic system on $X_{\boldsymbol{t}}$ (i.e. where E is the trivial bundle $E = E_0$). Moreover, every irreducible holomorphic system in $\mathrm{Syst}(X_{\boldsymbol{t}})$ can be obtained in that way. Note that $\Sigma_{\boldsymbol{t}} \subset \underline{\mathrm{Syst}}(\boldsymbol{t})$ is characterized by the fact that

- all three $\frac{1}{2}$-eigendirections coincide;
- all three 0-eigendirections over $x = 0, 1, \infty$ coincide.

Indeed, these conditions are equivalent to

$$z_1 = z_2 = z_3 =: z \quad \text{and} \quad c_1 + c_2 + c_3 = 0.$$

Consider now the connection $\underline{\nabla}_{\boldsymbol{z},\boldsymbol{c}}$ defined in (10) corresponding to a point in $\Sigma_{\boldsymbol{t}}$. After gauge transformation by the (constant) matrix

$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

the connection matrix becomes of the form

$$\begin{pmatrix} 0 & \frac{b(x)\mathrm{d}x}{x(x-1)} \\ \frac{c(x)\mathrm{d}x}{\prod_{i=1}^3 (x-t_i)} & \frac{1}{2}\mathrm{d}\log\left(\frac{\prod_{i=1}^3(x-t_i)}{x(x-1)}\right) \end{pmatrix} \tag{15}$$

$$\text{with} \quad \begin{cases} b(x) &= \frac{(2z-1)x - z}{2} \\ c(x) &= (t_1c_1 + t_2c_2 + t_3c_3)x + (t_1t_2c_3 + t_2t_3c_1 + t_3t_1c_2). \end{cases}$$

The connection matrix after lift to $X_{\boldsymbol{t}}$ via h (see (7)) can of course also be written as (15), we just have to keep in mind that x is not an appropriate local coordinate near a Weierstrass point (but y is, and hence the residues are double, see (2)). The birational bundle transformation ϕ defined in (13) consists in three negative elementary transformations (in the 0-eigendirections over $x = 0, 1, \infty$), and three positive elementary transformations (in the -1-eigendirections over $x = t_1, t_2, t_3$). In the particular case (15) we are considering here, ϕ^* is given explicitly by the meromorphic gauge transformation

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{x(x-1)}{y} \end{pmatrix},$$

yielding the following matrix connection on the (trivial) bundle $E := \phi^* E_0 \simeq E_0$

$$A = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} := \begin{pmatrix} 0 & b(x)\frac{dx}{y} \\ c(x)\frac{dx}{y} & 0 \end{pmatrix}. \tag{16}$$

Following Proposition 2.1, the system defined by A is reducible if, and only if, the holomorphic 1-forms β and γ are proportional, i.e. share the same zeros:

$$\frac{z}{1-2z} = \frac{t_1t_2c_3 + t_2t_3c_1 + t_3t_1c_2}{t_1c_1 + t_2c_2 + t_3c_3}. \tag{17}$$

The explicit restriction map $\Phi|_{\Sigma_{\boldsymbol{t}}} : \Sigma_{\boldsymbol{t}} \to \mathrm{Syst}^\times(X_{\boldsymbol{t}})$, where $\mathrm{Syst}^\times(X_{\boldsymbol{t}}) \subset \mathrm{Syst}(X_{\boldsymbol{t}})$ denotes the moduli space of all non trivial systems on $X_{\boldsymbol{t}}$ and Φ is defined by $\Phi = \phi^* \circ h^*$ as in (13), is surjective and étale over the open set $\mathrm{Syst}^{\mathrm{irr}}(X_{\boldsymbol{t}})$ of irreducible systems $(X_{\boldsymbol{t}}, A)$: given a decomposition $\nu_2x^2 + \nu_1x + \nu_0 = \nu(x - x_\beta)(x - x_\gamma)$, we get the explicit preimage

$$c_i = 2\nu \frac{t_i - x_\gamma}{(t_i - t_j)(t_i - t_k)}(2x_\beta - 1) \qquad \text{for } \{i,j,k\} = \{1,2,3\} \tag{18}$$

$$\text{and} \quad z = \frac{x_\beta}{2x_\beta - 1}.$$

In the irreducible case, we have $x_\beta \neq x_\gamma$ and up to permuting the roles of x_β and x_γ, we can assume $2x_\beta - 1 \neq 0$.

3.2. **Darboux coordinates.** Moduli spaces of connections on curves have a natural (holomorphic) symplectic structure. In the case of $\underline{\mathrm{Con}}(\boldsymbol{t})$, the symplectic two-form is given in the affine chart $\underline{\mathrm{Syst}}(\boldsymbol{t})$ by $\omega = dz_1 \wedge dc_1 + dz_2 \wedge dc_2 + dz_3 \wedge dc_3$. The classical Darboux coordinates, that will be needed to describe isomonodromy equations, are defined as follows.

The vector $e_1 = {}^t(1,0)$ becomes an eigenvector of the matrix connection $\nabla_{\boldsymbol{z},\boldsymbol{c}}$ for 3 values of $x = q_1, q_2, q_3$ (counted with multiplicity), namely at the zeros of the $(2,1)$-coefficient of the matrix connection:

$$\sum_{i=1}^{3} c_i \frac{(z_i - t_i)x - t_i(z_i - 1)}{x - t_i} = \left(\sum_{i=1}^{3} c_i(z_i - t_i)\right) \frac{\prod_{k=1}^{3}(x - q_k)}{\prod_{i=1}^{3}(x - t_i)} \tag{19}$$

At each of the three solutions $x = q_k$ of (19), the eigenvector $e_1 = {}^t(1,0)$ is associated to the eigenvalue

$$p_k := \sum_{i=1}^{3} c_i z_i \left(\frac{1}{q_k - 1} - \frac{1}{q_k - t_i} \right). \tag{20}$$

The equations (19) and (20) allow us to express our initial variables $(z_1, z_2, z_3, c_1, c_2, c_3)$ as rational functions of new variables $(q_1, q_2, q_3, p_1, p_2, p_3)$ as follows. Define

$$\Lambda := \sum_{\{i,j,k\}=\{1,2,3\}} \frac{p_i(q_i - t_1)(q_i - t_2)(q_i - t_3)}{(q_i - q_j)(q_i - q_k)}.$$

For $i = 1, 2, 3$, denote

$$\Lambda_i := \Lambda|_{t_i=1}$$

the rational function obtained by setting $t_i = 1$ in the expression of Λ. Then we have, for $\{i, j, k\} = \{1, 2, 3\}$

$$c_i = -\frac{(q_1 - t_i)(q_2 - t_i)(q_3 - t_i)}{t_i(t_i - 1)(t_i - t_j)(t_i - t_k)}\Lambda \quad \text{and} \quad z_i = t_i \frac{\Lambda_i}{\Lambda}. \tag{21}$$

The rational map

$$\Psi : \mathbb{C}^6_{\boldsymbol{q},\boldsymbol{p}} \dashrightarrow \mathbb{C}^6_{\boldsymbol{z},\boldsymbol{c}} \simeq \underline{\mathrm{Syst}}(\boldsymbol{t}) \tag{22}$$

has degree 6: the (birational) Galois group of this map is the permutation group on indices $k = 1, 2, 3$ for pairs (q_k, p_k). In these new coordinates, the symplectic form writes

$$\omega = \sum_{i=1}^{3} \mathrm{d}z_i \wedge \mathrm{d}c_i = \sum_{k=1}^{3} \mathrm{d}q_k \wedge \mathrm{d}p_k.$$

3.3. **Isomonodromy equations as a Hamiltonian system.** We now let the polar parameter $\boldsymbol{t} = (t_1, t_2, t_3)$ vary in the space T defined in (3). A local deformation $\boldsymbol{t} \mapsto (X_{\boldsymbol{t}}, E_{\boldsymbol{t}}, \nabla_{\boldsymbol{t}}) \in \mathrm{Con}(X_{\boldsymbol{t}})$ is said to be isomonodromic if the corresponding monodromy representation $\boldsymbol{t} \mapsto \rho_{\boldsymbol{t}} \in \Xi$ is constant. To define this latter arrow, we can choose and locally follow a system of generators for the fundamental group; the isomonodromy condition is clearly independent of the choice. In the moduli space of triples

$$\mathrm{Con} := \bigcup_{\boldsymbol{t} \in T} \mathrm{Con}(X_{\boldsymbol{t}}),$$

isomonodromic deformations parametrize the leaves of a smooth holomorphic foliation $\mathcal{F}$, namely the isomonodromic foliation. More precisely, $\mathcal{F}_{\mathrm{iso}}$ has dimension 3 and is transversal to the projection $X : \mathrm{Con} \to T$; moreover, holonomy induces symplectic analytic isomorphisms between fibers $\mathrm{Con}(X_{\boldsymbol{t}})$. The foliation $\mathcal{F}_{\mathrm{iso}}$ is also called the non-linear Gauss-Manin connection in [10, section 8], and it is proved there that Con is quasi-projective and $\mathcal{F}_{\mathrm{iso}}$ defined by polynomial equations. However, it is difficult to provide explicit equations in Con.

The construction of Φ in section 3.1 can be performed on the monodromy setting (see [7, section 2.1]) and therefore commutes with isomonodromic deformations. In Darboux coordinates, isomonodromic deformation equations are well-known as the Hamiltonian form of the Garnier system. Let us recall them explicitely. For $i = 1, 2, 3$ define H_i by

$$t_i(t_i - 1) \prod_{j \neq i} (t_j - t_i) \cdot H_i :=$$

$$\sum_{j=1}^{3} \frac{\prod_{k \neq j}(q_k - t_i)}{\prod_{k \neq j}(q_k - q_j)} F(q_j) \left(p_j^2 - G(q_j) p_j + \frac{p_j}{q_j - t_i} \right),$$

where $F(x) = x(x-1)(x-t_1)(x-t_2)(x-t_3)$ and $G(x) = \frac{F'(x)}{2F(x)}$ (where F' is the derivative with respect to x).

Any local analytic map $\boldsymbol{t} \mapsto (\boldsymbol{q}(\boldsymbol{t}), \boldsymbol{p}(\boldsymbol{t}))$ induces, via the 6-fold cover $\Psi : \mathbb{C}^6_{\boldsymbol{q},\boldsymbol{p}} \dashrightarrow \mathbb{C}^6_{\boldsymbol{z},\boldsymbol{c}} \simeq \underline{\mathrm{Syst}}(\boldsymbol{t})$, a deformation $\boldsymbol{t} \mapsto (\underline{E}_0, \underline{\nabla}_{\boldsymbol{z},\boldsymbol{c}}) \in \underline{\mathrm{Syst}}(\boldsymbol{t})$.

Theorem 3.2 (Malmquist)**.** *The local deformation induced by $\boldsymbol{t} \mapsto (\boldsymbol{q}(\boldsymbol{t}), \boldsymbol{p}(\boldsymbol{t}))$ is isomonodromic if, and only if,*

$$\frac{\partial q_k}{\partial t_i} = \frac{\partial H_i}{\partial p_k} \quad \textit{and} \quad \frac{\partial p_k}{\partial t_i} = -\frac{\partial H_i}{\partial q_k} \quad \forall i, k = 1, 2, 3. \tag{23}$$

In other words, the isomonodromic foliation $\underline{\mathcal{F}_{\mathrm{iso}}}$ on $\underline{\mathrm{Syst}} = \bigcup_{\boldsymbol{t}} \underline{\mathrm{Syst}}(\boldsymbol{t})$ is defined by the kernel of the 2-form

$$\Omega = \sum_{k=1}^{3} \mathrm{d}q_k \wedge \mathrm{d}p_k + \sum_{i=1}^{3} dH_i \wedge dt_i.$$

The tangent space to the foliation is also defined by the 3 vector fields

$$V_i := \frac{\partial}{\partial t_i} + \sum_{k=1}^{3} \left(\frac{\partial H_i}{\partial p_k}\right) \frac{\partial}{\partial q_k} - \sum_{k=1}^{3} \left(\frac{\partial H_i}{\partial q_k}\right) \frac{\partial}{\partial p_k} \tag{24}$$

fir $i = 1, 2, 3$. Note that the polar locus of these vector fields is given by

$$(q_1 - q_2)(q_2 - q_3)(q_1 - q_3) = 0,$$

namely the critical locus of the map (22).

3.4. **Transversality in Darboux coordinates.** Let $\Sigma := \bigcup_{\boldsymbol{t}} \Sigma_{\boldsymbol{t}}$ denote the locus of those systems in $\underline{\mathrm{Syst}}$ that lift under Φ as a connection (X, E, ∇) on the trivial bundle $E = E_0$. From the characterization described in section 3.1, we have

$$\Sigma := \{z_1 - z_2 = z_2 - z_3 = c_1 + c_2 + c_3 = 0\} \subset T \times \mathbb{C}^6_{\boldsymbol{z},\boldsymbol{c}}.$$

Condition $c_1 + c_2 + c_3 = 0$ implies that two of the q_i's are located on $x = 0, 1$; after fixing, say $q_1 = 0$ and $q_2 = 1$, we can determine q_3 from the matrix connection, as well as all p_i's. In particular, we find $p_3 = 0$. Denote

$$\Sigma^{\mathrm{Darb}} := \{q_1 = q_2 - 1 = p_3 = 0\} \subset T \times \mathbb{C}^6_{\boldsymbol{q},\boldsymbol{p}}.$$

If we assume $q_3 = q \neq 0, 1, \infty$ for the moment (which implies $q_i \neq q_j$ for all i, j), the locus of non-trivial connections on the trivial bundle E_0 in Con is parametrized by the image under Φ of

$$\Psi|_{\Sigma^{\mathrm{Darb}}} \ : \ \Sigma^{\mathrm{Darb}} \longrightarrow \Sigma \tag{25}$$

defined by

$$(p_1,p_2,q_3)\mapsto\begin{cases} 1-\frac{1}{z} &= \frac{(t_1-1)(t_2-1)(t_3-1)}{t_1t_2t_3}\frac{q_3}{q_3-1}\frac{p_2}{p_1}\\ c_1 &= \frac{t_1t_2t_3(q_3-1)p_1-(t_1-1)(t_2-1)(t_3-1)q_3p_2}{q_3(q_3-1)}\frac{q_3-t_1}{(t_1-t_2)(t_1-t_3)}\\ c_2 &= \frac{t_1t_2t_3(q_3-1)p_1-(t_1-1)(t_2-1)(t_3-1)q_3p_2}{q_3(q_3-1)}\frac{q_3-t_2}{(t_2-t_1)(t_2-t_3)}\\ c_3 &= \frac{t_1t_2t_3(q_3-1)p_1-(t_1-1)(t_2-1)(t_3-1)q_3p_2}{q_3(q_3-1)}\frac{q_3-t_3}{(t_3-t_1)(t_3-t_2)}\end{cases}$$

Note that $c_i=0$ if $q=t_i$. The locus of non-transversality of the isomonodromy foliation with Σ^{Darb} is contained in the zero locus of the determinant

$$\det(V_i\cdot F_j)\,|_{\Sigma^{\mathrm{Darb}}}=\frac{t_1t_2t_3(q_3-1)^2p_1+(t_1-1)(t_2-1)(t_3-1)q_3^2p_2}{8(t_1-t_2)(t_2-t_3)(t_1-t_3)q_3^2(q_3-1)^2}, \tag{26}$$

where $(F_1,F_2,F_3)=(q_1,q_2,p_3)$. Reversing the map (25) above, we get

$$\Psi^{-1}\ :\ \Sigma\ \longrightarrow\ \Sigma^{\mathrm{Darb}} \tag{27}$$

defined by

$$(z,c_2,c_3)\ \mapsto\ \begin{cases} p_1 &= \frac{(t_1-t_2)t_3c_2+(t_1-t_3)t_2c_3}{t_1t_2t_3}z\\ p_2 &= \frac{(t_1-t_2)(t_3-1)c_2+(t_1-t_3)(t_2-1)c_3}{(t_1-1)(t_2-1)(t_3-1)}(z-1)\\ q_3 &= \frac{(t_1-t_2)t_3c_2+(t_1-t_3)t_2c_3}{(t_1-t_2)c_2+(t_1-t_3)c_3}\end{cases}$$

After substitution, the numerator of the determinant (26) vanishes if, and only if,

$$\frac{z}{1-2z}=\frac{t_1t_2c_3+t_2t_3c_1+t_3t_1c_2}{t_1c_1+t_2c_2+t_3c_3}. \tag{28}$$

We here recognize the locus in equation (17) where the monodromy turns to be reducible. We have now established the main ingredients for the first proof of Theorem 1.1.

4. First proof of the main theorem

Consider the map

$$\underline{\mathrm{Syst}}\ \overset{\Phi}{\to}\ \mathrm{Con},$$

where we use the notations of the previous sections. In particular, Con denotes the moduli space of $\mathfrak{sl}_2$-connections over curves of the form $X_{\boldsymbol{t}}$ with $\boldsymbol{t}\in T$, $\underline{\mathrm{Syst}}$ denotes the moduli space of fuchsian rank 2 systems over $\mathbb{P}^1$ with polar set $\{0,1,t_1,t_2,t_3,\infty\}$ and spectral data (8) and Φ is the hyperelliptic lifting map (see section 3.1). We have seen that the locus $\mathrm{Syst}^{\mathrm{irr}}\subset\mathrm{Con}$ of irreducible connections defined on the trivial bundle over curves $X_{\boldsymbol{t}}$ (i.e. irreducible holomorphic systems over genus 2 curves) is contained in the image $\Phi(\underline{\mathrm{Syst}})$. Moreover, the preimage under Φ of the locus Syst $\subset$ Con of holomorphic systems over genus 2 curves is given by

$$\Sigma := \{z_1 - z_2 = z_2 - z_3 = c_1 + c_2 + c_3 = 0\} \subset T \times \mathbb{C}^6_{\boldsymbol{z},\boldsymbol{c}} \simeq \underline{\mathrm{Syst}}.$$

Since Φ is étale over $\mathrm{Syst}^{\mathrm{irr}}$, the isomonodromy foliation in Con is transversal to $\mathrm{Syst}^{\mathrm{irr}}$ if and only if the lift of the isomonodromy foliation is transversal to Σ^{irr} in $\underline{\mathrm{Syst}}$, where $\Sigma^{\mathrm{irr}} = \Sigma \setminus \Sigma^{\mathrm{red}}$ and Σ^{red} is the lift of the reducible locus in Syst calculated in equation (17)

$$\Sigma^{\mathrm{red}} := \left\{ (\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{c}) \in \Sigma \;\middle|\; \frac{z_1}{1-2z_1} = \frac{t_1t_2c_3 + t_2t_3c_1 + t_3t_1c_2}{t_1c_1 + t_2c_2 + t_3c_3} \right\}.$$

Now consider the rational map

$$T \times \mathbb{C}^6_{\boldsymbol{q},\boldsymbol{p}} \overset{\Psi}{\dashrightarrow} \underline{\mathrm{Syst}}$$

given by the Darboux-coordinates as in section 3.2 and the set

$$\Sigma^{\mathrm{Darb,irr}} := \Sigma^{\mathrm{Darb}} \setminus \Sigma^{\mathrm{Darb,red}}$$

where

$$\Sigma^{\mathrm{Darb,red}} := \left\{ (\boldsymbol{t}, \boldsymbol{q}, \boldsymbol{p}) \in \Sigma^{\mathrm{Darb}} \;\middle|\; t_1t_2t_3(q_3-1)^2p_1 + (t_1-1)(t_2-1)(t_3-1)q_3^2p_2 = 0 \right\}$$

and

$$\Sigma^{\mathrm{Darb}} := \{q_1 = q_2 - 1 = p_3 = 0\} \setminus \{Q^{\mathrm{Darb}} = 0\} \subset T \times \mathbb{C}^6_{\boldsymbol{q},\boldsymbol{p}}.$$

Here

$$Q^{\mathrm{Darb}} := q_3(q_3-1)(t_1t_2t_3(q_3-1)p_1 - (t_1-1)(t_2-1)(t_3-1)q_3p_2).$$

Lemma 4.1. *There is a neighborhood of $\Sigma^{Darb,irr}$ in $T \times \mathbb{C}^6_{\boldsymbol{q},\boldsymbol{p}}$ such that the restriction of Ψ to this neighborhood is a local diffeomorphism. Moreover, $\Sigma^{Darb,irr}$ is sent surjectively onto $\Sigma^{irr} \setminus \{Q_0Q_1Q_\infty = 0\}$, where*

$$\begin{array}{rcl} Q_0 &=& (t_1-t_2)t_3c_2 + (t_1-t_3)t_2c_3 \\ Q_1 &=& (t_1-t_2)c_2(t_3-1) + (t_1-t_3)(t_2-1)c_3 \\ Q_\infty &=& (t_1-t_2)c_2 + (t_1-t_3)c_3. \end{array}$$

Proof. We can check by direct computation from (19) and (20) that

- the c_i's have poles only at $q_k = q_l$, thus far from Σ^{Darb},
- the z_i's can have some indeterminacy point; however, a direct computation shows that the indeterminacy locus of Ψ intersects Σ^{Darb} at $p_1 = p_2 = 0$, which is contained in the reducible locus.

On the other hand, Ψ has a local analytic section (a right inverse) given by (19) and (20) and is rational. We have to check that Ψ^{-1} has no indeterminacy points on Σ^{irr}. Note that, formulae (20) involve indeterminacy summands like

$$\frac{\sum_{i=1}^3 c_iz_i}{q_2-1} \quad \text{and} \quad \frac{c_iz_i}{q_3-t_i};$$

however, we can modify these expressions by using the expression of $Q(z) = (z - q_1)(z - q_2)(z - q_3)$ defined by formula (19) and the fact that $q_2 - 1 = -\frac{Q(1)}{(q_1-1)(q_3-1)}$ for instance. By this way, we actually get an alternate formula for p_2 and p_3 which is now well defined near Σ^{irr}. The fact that the image of $\Sigma^{\mathrm{Darb,irr}}$ is precisely $\Sigma^{\mathrm{irr}} \setminus \{Q_0Q_1Q_\infty = 0\}$ follows directly from (25) and (27). □

Note that the set $Q_i = 0$ on Σ defined in the above lemma corresponds to $q_3 = i$ in Darboux coordinates. Now in the space $T \times \mathbb{C}^6_{\boldsymbol{q},\boldsymbol{p}}$ of Darboux coordinates we have explicit expressions of

- the isomonodromy foliation and
- the lift via $\Phi \circ \Psi$ of the locus Syst of the trivial bundle in Con: it contains $\Sigma^{\mathrm{Darb,irr}}$ as a large open subset (see Section 3.3).

We have seen by direct computation that the isomonodromy foliation is transversal to $\Sigma^{\mathrm{Darb,irr}}$. It follows that the isomonodromy foliation is transversal to $\Sigma^{\mathrm{irr}} \setminus \{Q_0Q_1Q_\infty = 0\}$. It remains to check transversality for systems corresponding to points in the special subset $\{Q_0Q_1Q_\infty = 0\}$ of Σ^{irr}. Yet in the definition of $\underline{\mathrm{Syst}}(\boldsymbol{t})$ and $\underline{\mathrm{Con}}(\boldsymbol{t})$ in Section 3, the polar set was split in half according to two types of parabolic data. The fact that we associated the $\left\{0, -\frac{1}{2}\right\}$-type to the poles at $I^- := \{x = 0, 1, \infty\}$ and the $\left\{0, \frac{1}{2}\right\}$-type to the poles at $I^+ := \{x = t_1, t_2, t_3\}$ was arbitrary. For any $i \in I^-$, we can perform the same construction setting $I^- := \{x = 0, 1, \infty, t_1\} \setminus \{x = i\}$ and $I^+ := \{x = i, t_2, t_3\}$ for instance. The correspondence between the former and the new construction of $\underline{\mathrm{Con}}(\boldsymbol{t})$ is given by a particular birational bundle isomorphism ϕ^{mod} on the corresponding logarithmic connections on $\mathbb{P}^1$, namely the combination of a negative elementary transformation in the 0-eigendirection over the pole $x = i$ and a positive elementary transformation in the $-\frac{1}{2}$-eigendirection over the pole $x = t_1$. Since $\phi^{\mathrm{mod}}(\underline{E}_0) \simeq \underline{E}_0$, where $\underline{E}_0$ denotes the trivial bundle on $\mathbb{P}^1$ as usual, ϕ^{mod} actually defines a reparametrization of $\underline{\mathrm{Syst}}(\boldsymbol{t})$ (and in particular, of Σ^{irr}), which can also be seen as a Moebius transformation in the base. One can easily check that a system in the former construction of Σ^{irr} corresponding to a point in $\{Q_i = 0\}$ is no longer contained in the special subset of the new construction of Σ^{irr}. In summary, transversality of the isomonodromy foliation to

$$\Phi(\Sigma^{\mathrm{irr}} \setminus \{Q_0Q_1Q_\infty = 0\}) \subset \mathrm{Syst}^{\mathrm{irr}}$$

implies transversality at any point of $\mathrm{Syst}^{\mathrm{irr}}$ by a reparametrization of the family of curves

$$\bigcup_{\boldsymbol{t} \in T} X_{\boldsymbol{t}}.$$

5. Branched projective structures

Given a compact oriented topological surface S, a branched projective structure σ over S is the data of a covering $S = \bigcup U_i$ by open sets and for each U_i a finite branched cover preserving the orientation $\varphi_i : U_i \to V_i \subset \mathbb{P}^1$ such that on any $U_i \cap U_j \neq \emptyset$ there exists a homography $A_{ij} : \mathbb{P}^1 \to \mathbb{P}^1$ satisfying

$$\varphi_i = A_{ij} \circ \varphi_j \quad \text{on} \quad U_i \cap U_j.$$

By abuse of language, we will call each φ_i a chart of the branched projective structure. The branching divisor of σ is the divisor $\operatorname{div}(\sigma) = \sum (n_p - 1)p$ where n_p is the order of branching of the map φ_i around $p \in U_i \subset S$. The complex structure of $\mathbb{P}^1$ can be pulled back to $S \setminus |\operatorname{div}(\sigma)|$ by imposing that each φ_i is holomorphic. The complex structure thus defined extends to a unique complex structure X on S. Note that any chart of σ can be analytically extended along any path in X.

Two branched projective structures σ_1 and σ_2 over S are said to be equivalent if there exists a homeomorphism $(S, \sigma_1) \to (S, \sigma_2)$ that is projective in the corresponding charts and that induces the identity on the fundamental group.

If we fix a universal covering map $\widetilde{S} \to S$ of S with $\pi_1(S) = \operatorname{Aut}(\widetilde{S}|S)$, we can associate a class of equivariant maps defined on $\widetilde{S}$ to any branched projective structure on S, called *developing map*. Indeed, take a chart φ of σ, we define a devoloping map $\mathcal{D} : \widetilde{S} \to \mathbb{P}^1$ by extending the chart φ along paths in S. The map $\mathcal{D}$ is equivariant with respect to a representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$, i.e. for each $\gamma \in \pi_1(S)$,

$$\mathcal{D}(\gamma \cdot z) = \rho(\gamma) \circ \mathcal{D}(z).$$

The map $\mathcal{D}$ is holomorphic with respect to the lift $\widetilde{X}$ of the complex structure X on S to $\widetilde{S}$. Reciprocally, given a complex structure X on S we can lift it to a complex structure $\widetilde{X}$ on $\widetilde{S}$. A holomorphic map $\mathcal{D} : \widetilde{X} \to \mathbb{P}^1$ that is equivariant with respect to some representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ induces a branched projective structure on S locally defined by $\mathcal{D}$.

From now on, we fix a universal covering map $\widetilde{S} \to S$. At the level of developing maps equivalence of branched projective structures can be read in the following terms: if $A \in \mathrm{PSL}_2(\mathbb{C})$ and $\mathcal{D}$ is a developing map of a branched projective structure with equivariance ρ, then

$$\mathcal{D}' = A \circ \mathcal{D} \tag{29}$$

defines an equivalent branched projective structure with equivariance

$$\rho' = A \circ \rho \circ A^{-1}. \tag{30}$$

Reciprocally, any pair of developing maps $(\mathcal{D}, \rho)$, $(\mathcal{D}', \rho')$ associated to equivalent branched projective structures satisfy equations (29) and (30) for some $A \in \mathrm{PSL}_2(\mathbb{C})$. In particular, two developing maps $\mathcal{D}$ and $\mathcal{D}'$ obtained from different charts of the same σ satisfy this last equivalence. Although developing maps are not in general unique, we have that if σ is a branched projective structure for which the image of the equivariance is a group with trivial centralizer, then for each representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ in the conjugacy class, there is a unique developing map $\mathcal{D}$ associated to σ having equivariance ρ.

Another natural way of constructing branched projective structures is to consider a complex manifold V endowed with a codimension one (singular) holomorphic foliation that is transversely projective[1] and a compact holomorphic curve $X \subset V$ that is generically transverse to the foliation and that does not pass through the singular set. The restriction to the curve X of the local submersions defining the foliation define the local charts of the branched projective structure on S. The points of tangency between X and the foliation produce the critical points of the charts. In fact, any projective structure on a *marked complex structure* $X \in \mathrm{Teich}(S)$ occurs in this way. Indeed, given a holomorphic developing map $\mathcal{D} : \widetilde{X} \to \mathbb{P}^1$ of a branched projective structure with equivariance ρ we can consider the section of the $\mathbb{P}^1$-bundle $X \times_\rho \mathbb{P}^1 \to X$ defined by $\mathcal{D}$. It defines a curve that is generically transverse to the transversely projective foliation induced on $X \times_\rho \mathbb{P}^1$ by the horizontal foliation on $\widetilde{X} \times \mathbb{P}^1$. Remark that, if $\rho(\pi_1(S))$ has trivial centralizer, the uniqueness of developing maps associated to ρ imply that two different sections of $X \times_\rho \mathbb{P}^1$ define different branched projective structures on the same marked complex structure X.

Given a natural number $k \in \mathbb{N}$ we define $\mathcal{M}_k$ to be the set of equivalence classes of branched projective structures σ on S having k critical points counted with multiplicity (i.e. $\deg(\mathrm{div}(\sigma)) = k$) whose equivariance has trivial centralizer. We have a natural 'monodromy' map

$$\mathcal{M}_k \to \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{C}))//\mathrm{PSL}_2(\mathbb{C}) \tag{31}$$

that associates, to each σ the point in the $\mathrm{PSL}_2(\mathbb{C})$-character variety associated to its equiviariance homomorphism. The fiber over a given a representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ is denoted by $\mathcal{M}_{k,\rho}$.

Theorem 5.1 ([2]). *If S has genus $g \geq 2$, ρ has trivial centralizer and $\mathcal{M}_{k,\rho} \neq \emptyset$[2] then it admits the structure of a complex manifold of dimension k.*

The complex structure on $\mathcal{M}_{k,\rho}$ comes from the deformation theory of projective structures. Namely, assume that Y is a complex analytic space and that we have the following data :

a) a holomorphic submersion $\Pi : Z \to Y$ having as fibers simply connected Riemann surfaces,
b) a free proper discontinuous action of $\Gamma = \pi_1(S)$ on Z preserving the fibration Π,

[1] Here, by transversely projective, we mean that there exists a system of local submersions with values in $\mathbb{P}^1$, which are first integrals of the foliation and which are well-defined up to post-composition by an automorphism of $\mathbb{P}^1$.

[2] If the representation is non elementary, then $\mathcal{M}_{k,\rho}$ is non empty when k is even and ρ lifts to $\mathrm{SL}(2,\mathbb{C})$ or when k is odd and ρ does not lift to $\mathrm{SL}(2,\mathbb{C})$, see [4]. The precise condition when ρ is not elementary but has trivial centralizer is not known, as far as we know.

c) a holomorphic map $\mathcal{D} : Z \to \mathbb{P}^1$ which is ρ-equivariant with respect to the Γ-action, and which is a local diffeomorphism on each fiber apart from k orbits counted with multiplicity.

Then, identifying the fibers with the universal covering of S in an equivariant way, we get a map from Y to $\mathcal{M}_{k,\rho}$ which is holomorphic. In fact, $\mathcal{M}_{k,\rho}$ is the universal complex space having properties (a)-(c). The fact that this latter is a smooth complex manifold comes from the fact that $\mathcal{M}_{k,\rho}$ is modelled on Hurwitz spaces, as is shown in the appendix of [2]. A particular system of charts showing this will be detailed in the proof of Lemma 7.1, Section 7.

Extending the ideas of the proof we can glue all the manifolds in Theorem 5.1 sharing the genus and the integer $k \geq 0$ to form a complex foliated manifold:

Theorem 5.2. *If S has genus $g \geq 2$ and $k \geq 0$, then the space $\mathcal{M}_k$ admits a complex structure compatible with those of Theorem 5.1 and for which the monodromy map (31) is a holomorphic submersion.*

The regular holomorphic foliation induced by this monodromy map will be referred to as the isomonodromy foliation on $\mathcal{M}_k$. Since the proof of Theorem 5.2 does not shed much light on that of Theorem 1.1 we leave it for the Appendix of this paper.

6. Dictionary between Systems and Rational curves in $\mathcal{M}_{2g-2,\rho}$

Let S be an orientable compact surface of genus $g \geq 2$ and $X \in \mathrm{Teich}(S)$. Take a system $(X, A) \in \mathrm{Syst}^{\mathrm{irr}}$ having monodromy $\rho_A : \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$. It induces on the trivial bundle the linear connection $\nabla = \nabla_0 + A$, whose flat sections satisfy (1). Its projectivization gives a flat connection ∇' on the trivial $\mathbb{P}^1$-bundle $X \times \mathbb{P}^1$. The foliation $\mathcal{F}$ induced by ∇' is transversely projective, hence for every $p \in \mathbb{P}^1$, it induces a branched projective structure $\sigma_A(p)$ on the horizontal $X \times p$ (notice that no horizontal section is flat, since otherwise the system would be reducible). The foliation $\mathcal{F}$ has been considered by Drach in another context (see [3]). If $B : \widetilde{X} \to \mathrm{SL}(2, \mathbb{C})$ denote the fundamental matrix of (1) whose value at a point x_0 is the identity – it satisfies that solutions to (1) takes values Z at x_0 and $B(x)Z$ at x – a developing map for the structure $\sigma_A(p)$ is defined as

$$\mathcal{D}_p(x) = B(x)^{-1}(p). \tag{32}$$

This latter is ρ-equivariant, where ρ is the composition of ρ_A with the natural projection $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{PSL}(2, \mathbb{C})$, and where we use the natural identification of $X \times p$ with X. In particular, the holonomy of $\sigma_A(p)$ is the conjugacy class of ρ. Moreover, the critical points of $\sigma_A(p)$ are the tangency points between $X_p = X \times p$ and $\mathcal{F}$, with the same multiplicities; by the formula [1, Proposition 2, p. 26]

$$|\mathrm{Tang}(\mathcal{F}, X_p)| = N_{\mathcal{F}} \cdot X_p - \chi(X_p) = X_p^2 - \chi(X_p) = 2g - 2, \tag{33}$$

where points in $\mathrm{Tang}(\mathcal{F}, X_p)$ are counted with multiplicities. Thus, we have built a well-defined map

$$\sigma_A : \mathbb{P}^1 \to \mathcal{M}_{2g-2,\rho}; \quad \sigma_A(p) := [\sigma(A,p)]. \tag{34}$$

It is holomorphic by (32). Notice that the monodromy ρ_A of ∇ is irreducible, so that $\mathcal{M}_{2g-2,\rho}$ is a smooth manifold of dimension $2g-2$. Indeed, assume the contrary. We then have a compact holomorphic curve s in $X \times \mathbb{P}^1$ which is a leaf of $\mathcal{F}$. Its self-intersection vanishes, so it is homologous to a horizontal $(X \times *)$ or to a vertical section $(* \times \mathbb{P}^1)$. By positivity of the intersection of complex submanifolds, this is only possible if s is a genuine horizontal or vertical. The first case is impossible since (X, A) is irreducible, and the second is impossible since $\mathcal{F}$ is transverse to the verticals.

Lemma 6.1. *The map $\sigma_A : \mathbb{P}^1 \to \mathcal{M}_{2g-2,\rho}$ defined in* (34) *is an embedding.*

Proof. We have shown that the representation ρ_A is irreducible, so the centralizer of the image of ρ_A in $\mathrm{PSL}(2,\mathbb{C})$ is trivial. In particular, for every $\sigma \in \mathcal{M}_{2g-2,\rho}$, there is a unique ρ-equivariant developing map $\mathcal{D}_\sigma$. In the case of $\sigma_A(p)$, this developing map is given by the formula (32), and we have $\mathcal{D}_{\sigma_A(p)}(x_0) = p$. This shows that σ_A is injective.

To see that it is an immersion, recall that we can glue all the developing maps $\mathcal{D}_\sigma$ for $\sigma \in \mathcal{M}_{2g-2,\rho}$ in the following data: a fibration $\pi : Z \to \mathcal{M}_{2g-2,\rho}$ with simply connected Riemann surfaces as fibers, a properly discontinuous action of $\pi_1(S)$ on Z preserving the fibration, and a ρ-equivariant map $\mathcal{D} : Z \to \mathbb{P}^1$ which induces on each fiber $\pi^{-1}(\sigma)$ the developing map $\mathcal{D}_\sigma$. Notice that the fibration π is the pull-back of the universal cover of the universal curve over Teichmüller space by the natural map $\mathcal{M}_{2g-2,\rho} \to \mathrm{Teich}(S)$. In particular, for any $p_0 \in \mathbb{P}^1$, we can choose a germ of section $\sigma \in \mathcal{M}_{2g-2,\rho} \to x(\sigma) \in Z$ of π defined at the neighborhood of $\sigma_A(p_0)$, taking every $\sigma_A(p)$ (for p close to p_0) to the point $x_0 \in \pi^{-1}(\sigma_A(p)) \simeq \widetilde{X}$. For σ in a neighborhood of $\sigma_A(p_0)$, we have a well-defined holomorphic function $f(\sigma) = \mathcal{D}(x(\sigma))$. By construction we have

$$f \circ \sigma_A(p) = \mathcal{D}(x(\sigma_A(p))) = \mathcal{D}_{\sigma_A(p)}(x_0) = p,$$

for p close to p_0, so σ_A is an immersion. $\square$

The following result is not necessary for the proof of Theorem 1.1. However it completes the proof of the dictionary between systems and rational curves in the isomonodromic moduli spaces $\mathcal{M}_{2g-2,\rho}$. An infinitesimal version of it will be used in Section 10 to provide counter-examples to Theorem 1.1 in genus ≥ 3.

Lemma 6.2. *Let $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ be an irreducible representation that lifts to* $\mathrm{SL}(2,\mathbb{C})$.[3] *Given a compact curve C and a non constant holomorphic map R :*

[3] Otherwise the space $\mathcal{M}_{2g-2,\rho}$ is empty, see [2].

$C \to \mathcal{M}_{2g-2,\rho}$, there exists $(X, A) \in \mathfrak{sl}_2(S)$ and a meromorphic map $R_0 : C \to \mathbb{P}^1$ such that

$$R = \sigma_A \circ R_0.$$

In particular the image of R is an embedded rational curve.

Proof. First remark that the composition of R with the natural (holomorphic) forgetful map $\mathcal{M}_{2,\rho} \to \mathrm{Teich}(S)$ must be some constant $X \in \mathrm{Teich}(S)$, since $\mathrm{Teich}(S)$ has the structure of a bounded domain of holomorphy. Denoting this constant by $X \in \mathrm{Teich}(S)$, let $S = X \ltimes_\rho \mathbb{P}^1$ be the flat $\mathbb{P}^1$-bundle having monodromy ρ, and $\mathcal{F}$ be the foliation whose leaves are the flat sections. In what follows we denote by ∇ the corresponding flat connection. For every $c \in C$, the unique ρ-equivariant developing map $\mathcal{D}_c : \tilde{X} \to \mathbb{P}^1$ defines a section $D_c : X \to S$. The formula (33) shows that $D_c^2 = D_c \cdot N_{\mathcal{F}} = \chi(D_c) + |\mathrm{Tang}(\mathcal{F}, D_c)| = 0$. Since the representation ρ lifts to $\mathrm{SL}(2, \mathbb{C})$, the fibration $S \to X$ is diffeomorphic to a product. See for instance [5]. Under this identification, the curves D_c are homologous to horizontal curves. In particular, they are all homologous, and we have $D_c \cdot D_{c'} = 0$ for any $c, c' \in C$. Since R is non constant, there exist three values c_i such that $R(c_i)$ are pairwise distinct. By the preceding discussion, the sections D_{c_i} of S are also disjoint. We infer that S is holomorphically equivalent as a $\mathbb{P}^1$-bundle to the trivial bundle $X \times \mathbb{P}^1$. Denote by ∇_0 the trivial connection on $X \times \mathbb{P}^1$. Under this identification, the original flat connection ∇ differs from ∇_0 by a holomorphic one form with values in the Lie algebra of $\mathrm{PSL}(2, \mathbb{C})$. Since this latter is naturally identified with $\mathrm{sl}(2, \mathbb{C})$, the flat connection ∇ is the one induced by a system $(X, A) \in \mathfrak{sl}_2(S)$. We have shown that there exists a map $r : C \to \mathbb{P}^1$ such that $R = \sigma_A \circ r$. This map is clearly holomorphic and the lemma follows. □

From Lemmas 6.1 and 6.2 we deduce that an isomonodromic deformation of systems of monodromy ρ corresponds to a non-trivial deformation of rational curves in $\mathcal{M}_{2g-2,\rho}$.

7. The tangent bundle to $\mathcal{M}_{k,\rho}$

In this section we fix $k > 0$ and a representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ with trivial centralizer such that $\mathcal{M}_{k,\rho}$ is non-empty. Whenever there is no risk of confusion, we note

$$\mathcal{M} = \mathcal{M}_{k,\rho}.$$

We will show that $T\mathcal{M}$ can be thought of as a push-forward of a line bundle.

Let $\Pi : \mathcal{C} \to \mathcal{M}$ denote the universal curve bundle over $\mathcal{M}$, that is, the fibre of Π over $\sigma \in \mathcal{M}$ is the Riemann surface X underlying σ. By the results of [2], there exists a transversely projective holomorphic foliation $\mathcal{G}$ of codimension one on $\mathcal{C}$ that induces on each fibre $\Pi^{-1}(\sigma)$ the projective structure σ.[4] The points of

[4]With the notation used just after Theorem 5.1, $\mathcal{G}$ is the quotient of the foliation whose leaves are the level sets of $\mathcal{D}$ on the fibration $Z \to \mathcal{M}_{k,\rho}$ by the action of $\pi_1(S)$.

tangency between $\mathcal{G}$ and a fiber of $\Pi^{-1}(\sigma)$ correspond to the points where σ has critical points. Let $B = \mathrm{Tang}(\mathcal{G}, \Pi)$ denote the tangency divisor between $\mathcal{G}$ and the fibration Π. By construction, $\Pi_{|B} : B \to \mathcal{M}$ is a branched cover of degree k and $\mathcal{G}$ is generically transverse to B.

Around a regular value $\sigma \in \mathcal{M}$ of $\Pi_{|B}$ we can define two regular transverse one dimensional holomorphic foliations, namely the push forward of the restriction of $\mathcal{G}$ to B around each of the k points in the fiber $\Pi_{|B}^{-1}(\sigma)$. With the help of these foliations we will analyze the sheaf of sections of $T\mathcal{M}$. Let us formalize this argument.

We know that a line bundle over the source space of a branched covering of degree k can be pushed forward to a rank k-vector bundle over the target space of the branched cover. Let $N_{\mathcal{G}|B}$ denote the restriction of the normal bundle of $\mathcal{F}$ to B. Its push forward

$$E := (\Pi_{|B})_*(N_{\mathcal{G}|B})$$

by $\Pi_{|B}$ is a rank k vector bundle over $\mathcal{M}$.

Lemma 7.1. *With the above notations, we have*

$$E \simeq T\mathcal{M}.$$

Proof. From the appendix in [2] we know how to describe local charts around a point in $\sigma \in \mathcal{M}_{k,\rho}$ for any ρ with trivial centralizer and k such that $\mathcal{M}_{k,\rho} \neq \emptyset$. Let us recall the construction. For each point q in the support of $\mathrm{div}(\sigma)$ consider a chart of the projective structure around it defined on a disc $D_q \subset S$, and fix a point $Q \in \partial D_q$ and its image in $\mathbb{P}^1$. Let $n_q + 1 = n + 1$ be the order of the projective structure around q and consider a conformal equivalence where the chart becomes the map $z \mapsto z^{n+1}$ defined on $\mathbb{D}$, $q = 0$ and $Q = 1$. Consider $a = (a_1, \ldots, a_n) \in (\mathbb{C}^n, 0)$ and the polynomial in the variable z

$$P_a(z) = z^{n+1} + a_n z^n + \ldots + a_1 z - (a_n + \ldots + a_1). \tag{35}$$

Then $P_a(1) = 1$ and its critical values lie on $\mathbb{D}$. P_0 corresponds to the normalized covering associated to σ at q. Let $\mathbb{D}_a := P_a^{-1}(\mathbb{D})$. We can topologically glue the covering $P_{a|\mathbb{D}_a} : \mathbb{D}_a \to \mathbb{D}$ to the projective structure induced by σ on the complement of D_q by gluing the boundaries with the help of the equivalence between the values of P_0 and P_a. By doing such a surgery independently at each critical point of σ we obtain a projective structure for each choice of n_q parameters at each q. Thus, the family of all such projective structures depends on $k = \sum n_q$ parameters. With the choices involved it is a chart in $\mathcal{M}_{k,\rho}$. In this chart, the tangent space to $\mathcal{M}_{k,\rho}$ is just the tangent space to the cartesian product of the parameters $(a_1, \ldots, a_{n_q})$ at each critical point q. With these coordinates we can define local coordinates of $\mathcal{C}$ around a point $q \in \Pi_B^{-1}(\sigma)$ where σ has a critical point of order $n = n_q$. Indeed, a neighbourhood corresponds to a neighbourhood of 0 in the coordinates $(a_1, a_2, \dots, a_n, z) = (a, z) \in \mathbb{C}^{n+1}$, the projection Π restricted to the neighbourhood is just the map $(a, z) \mapsto a$, the set B corresponds to the smooth analytic set

$$B \simeq \left\{ (a, z) \in \mathbb{C}^{n+1} \;\middle|\; \frac{\partial P_a}{\partial z}(z) = (n+1)z^n + n a_n z^{n-1} + \ldots + a_1 = 0 \right\}$$

and the foliation $\mathcal{G}$ has a first integral $v = P(a, z) := P_a(z)$ in the neighbourhood of 0. Let $\mathcal{G}_{|B}$ be the foliation restricted to B. Recall that local sections of the normal bundle to a codimension one foliation can be interpreted as tangent vectors to the values of a first integral of the foliation. The equivalence in the statement of the lemma will follow from analyzing the relation that $P_{|B}$ induces between the parameters of $\mathcal{M}_{k,\rho}$ and those of its values. Let $(a, c(a)) \in B$ be one of the critical points of P_a and $v = P_a(c(a))$ denote its critical value. Differentiation gives

$$dv = \sum_{i=1}^{n} (c(a)^i - 1) da_i$$

and hence for $(a, z) \in B$ and $i = 1, \dots, n$

$$\frac{\partial}{\partial a_i} = (z^i - 1)\frac{\partial}{\partial v}. \tag{36}$$

Recall that if $\sigma \in \mathcal{M}_{k,\rho}$ is a regular value of $\Pi_{|B}$, the local holomorphic sections of the bundle E around a point σ are given by direct sums of local sections of the normal bundle to the foliation at each of the preimages of σ by $\Pi_{|B}$. By the definition of the charts $(a_1, \dots a_k)$ of $\mathcal{M}_{k,\rho}$, at such a σ each coordinate function corresponds to a different point of the preimage and a coordinate of type $P_{a_i}(z) = z^2 + a_i z - a_i$ around it. Thus, each $\frac{\partial}{\partial a_i}$ corresponds to the section of the normal bundle to the foliation that is zero around all preimages, except for the chart corresponding to the a_i variable where the equivalence (36) takes place (with $n = 1$ since the branch point is simple).

To see that this equivalence defines an isomorphism of the sheaves of sections of $T\mathcal{M}$ and of E we first remark that any germ of holomorphic function f on B at the point $q = (0, 0) \in B$ of order n can be uniquely written as a sum

$$\sum_{i=1}^{n} f_i(a_1, \dots, a_n)(z^i - 1) \quad \text{for each } (a, z) \in B \tag{37}$$

where each f_i is holomorphic in $(a_1, \dots, a_n) \in (\mathbb{C}^n, 0)$. This follows from the Weierstrass Division Theorem applied to any holomorphic extension F of f to a neighbourhood of q in $\mathcal{C}$ and the Weierstrass polynomial $W := \frac{1}{n+1}\frac{\partial P_a}{\partial z}(z)$. Indeed, there exist holomorphic functions $\tilde{f}_i(a_1, \dots, a_n)$ such that

$$F(a, z) = \sum_{i=0}^{n} \tilde{f}_i(a_1, \dots, a_n) z^i + W(a, z) Q(a, z)$$

for some holomorphic $Q \in \mathcal{O}_{(\mathbb{C}^{n+1},0)}$. Defining $f_i = \tilde{f}_i$ for $i > 0$ and $f_0 = \tilde{f}_0 + \sum_{i=1}^n \tilde{f}_i$ and restricting to $B = \{W = 0\}$ we get the desired expression (37) for the germ $f : (B, q) \to \mathbb{C}$.

With this at hand we deduce that, outside the branching divisor of $\Pi_{|B}$, the correspondence $\frac{\partial}{\partial a_i} \mapsto (z^i - 1)\frac{\partial}{\partial v_i}$ defines the desired isomorphism.

Here we just treat the case where the critical fibers contain a single point, but the general case follows from the same type of argument. In the critical fibers we have a unique point $q \in B$ of order $n = k$ and we can consider a chart given by a polynomial as in (35). We need to extend the isomorphism to a neighbourhood of this point. The correspondence is already given by (36). To a holomorphic vector field $\sum f_i(a_1, \ldots, a_n)\frac{\partial}{\partial a_i}$ around $\sigma = (0, \ldots, 0)$ we associate

$$\sum_{i=1}^{n} f_i(a_1, \ldots, a_n)\left((z^i - 1)\frac{\partial}{\partial v}\right)$$

around the unique point $q = (0, 0)$ in the fiber of $\Pi_{|B}$ over σ. This last expression corresponds uniquely to the vector field $f(a, z)\frac{\partial}{\partial v}$ where f is defined by (37) in a neighbourhood of q in B. Since every holomorphic function on B can be uniquely written in this way, the equivalence is a sheaf isomorphism. □

8. Rigidity of rational curves in $\mathcal{M}_{2,\rho}$

Suppose $\mathcal{R}$ is a smooth embedded rational curve in $\mathcal{M} = \mathcal{M}_{k,\rho}$. Then the bundle $T\mathcal{M}_{|\mathcal{R}}$ splits as a direct sum of k line bundles over $\mathcal{R}$:

$$T\mathcal{M}_{|\mathcal{R}} = \bigoplus_{i=1}^{k} \mathcal{O}(d_i).$$

Among the factors, there is one coming from $\Pi_*(\Pi^*(T\mathcal{R}))$ that implies without loss of generality that d_1 can be assumed to be 2. The rigidity of the curve $\mathcal{R}$ in $\mathcal{M}_{k,\rho}$ is equivalent to proving that $d_i < 0$ for every $i \geq 2$.

The Chern class of the determinant bundle $\wedge^2 T\mathcal{M}_{|\mathcal{R}}$ is the sum $2 + (d_2 + \cdots + d_k)$. In particular, when $k = 2$, rigidity is equivalent to showing that this sum is at most 1. In fact the d_2 corresponds to the self-intersection of the curve $\mathcal{R}$ in this case. The problem is settled in the following statement.

Proposition 8.1. *Every smooth rational curve in $\mathcal{M}_{2,\rho}$ has self-intersection -4.*

Proof. By Lemma 7.1, we have $c_1(\wedge^2 T\mathcal{M}_{|\mathcal{R}}) = c_1(\wedge^2 E_{|\mathcal{R}})$. The latter can be calculated using the following relation between the Chern class of a line bundle and that of the determinant of the push forward by a branched covering.

Lemma 8.2. *If $L \to C$ is a line bundle over a smooth curve C and $f : C \to C'$ is a branched cover of degree 2, then*

$$c_1(\wedge^2 f_* L) = c_1(L) - \frac{1}{2}\sum(e_p - 1)$$

where e_p denotes the degree of f around $p \in C$.

Note that the number $\sum(e_p - 1)$ is always even.

In our case, we take $C = \Pi^{-1}(\mathcal{R}) \cap B$, $L = N_{\mathcal{G}|C}$, $C' = \mathcal{R}$ and $f = \Pi_{|C}$. The curve C is the projectivization of the set of eigenvectors of A in the product $X \times \mathbb{P}^1$, so we will call it the eigencurve in the sequel. By Lemma 6.2, the rational curve $\mathcal{R}$ can be parametrized by some embedding $\sigma_A : \mathbb{P}^1 \to \mathcal{M}$ which comes from a regular holomorphic foliation $\mathcal{F}$ on some $X \times \mathbb{P}^1$ that is transverse to the $\mathbb{P}^1$-fibers. In fact, by construction, the restriction of Π and $\mathcal{G}$ to $\Pi^{-1}(\mathcal{R})$ are naturally equivalent to the fibration $\Pi_2 : X \times \mathbb{P}^1 \to \mathbb{P}^1$ endowed with the foliation $\mathcal{F}$. Under this equivalence, $C = \mathrm{Tang}(\mathcal{F}, \Pi_2)$ and

$$L = N_{\mathcal{F}|C} \simeq N(\Pi_2)_{|C} \simeq T(\Pi_1)_{|C} \simeq \Pi^*_{1|C}(T(\mathbb{P}^1))$$

where $\Pi_1 : X \times \mathbb{P}^1 \to X$ is the canonical projection. Therefore, since $\Pi_{1|C}$ has degree two, we have $c_1(L) = 4$.

On the other hand, we need to calculate the degree of the branching divisor of $f = \Pi_{2|C} : C \to \mathbb{P}^1$. Since the branched covering has degree two, the only possibility is that the branching points are simple. Thus it suffices to count the number of fibers of type $X_p = X \times p$ such that $\mathcal{F}$ has a unique point of tangency with X_p of multiplicity two.

Lemma 8.3. *There are twelve points $p_1, \ldots, p_{12} \in \mathbb{P}^1$ counted with multiplicity such that X_{p_i} and $\mathcal{F}$ have a unique point of tangency of multiplicity two.*

Proof. The Riccati equation $\mathcal{F}$ on $X \times \mathbb{P}^1$ can be written as $dy = \alpha y^2 + \beta y + \gamma$ where $\alpha, \beta, \gamma \in \Omega^1(X)$ are holomorphic 1-forms. Since $\dim_{\mathbb{C}}(\Omega^1(X)) = 2$ we obtain a holomorphic map $\mathbb{P}^1 \to \mathbb{P}(\Omega^1(X)) \simeq \mathbb{P}^1$ defined by $y \mapsto [\alpha y^2 + \beta y + \gamma]$. It has at most degree two. If its degree is smaller than two, by using homogeneous coordinates in $\mathbb{P}^1$, we see that there is an invariant horizontal $X \times y$ for $\mathcal{F}$, which does not occur since our system (X, A) is irreducible (see also (5)). Hence the map is a degree two ramified covering. The fibers p_i of the statement correspond to forms in $\Omega^1(X)$ having a single zero. Recall that the hyperelliptic involution in X fixes precisely six points and that for each of them there corresponds a unique element in $\mathbb{P}(\Omega^1(X))$ having a single zero at the given fixed point of the involution. In fact these are the only one-forms in $\mathbb{P}(\Omega^1(X))$ with a single zero. Hence after the degree two branched covering there are 12 fibers with a single zero. □

By using Lemma 8.2 we get

$$c_1(\wedge^2 T\mathcal{M}_{|\mathcal{R}}) = 4 - \frac{12}{2} = -2.$$

By the argument at the beginning of the section, we have $2 + d_2 = -2$ and therefore $d_2 = -4$. □

9. Second proof of Theorem 1.1

Let $(X, A) \in \mathrm{Syst}^{irr}$. If Mon is locally injective at (X, A), then it is a non-constant holomorphic map between spaces of the same dimension, therefore open around (X, A). This implies that it is a local biholomorphism.

We will use the correspondence between systems in Syst^{irr} with monodromy ρ and embedded rational curves in $\mathcal{M}_{2,\rho}$ proved in Section 6 to see that the case where Mon is not locally injective at (X, A) cannot occur.

Suppose, to reach a contradiction, that for each neighbourhood U_ε of (X, A) there exist distinct (X_1, A_1^ε) and (X_2, A_2^ε) satisfying $\mathrm{Mon}(X_1, A_1^\varepsilon) = \mathrm{Mon}(X_2, A_2^\varepsilon) = \rho_\varepsilon$. Then, by Lemma 6.1 there exists, for each ε, a pair of disjoint embedded rational cuves $\sigma_{A_1^\varepsilon}, \sigma_{A_2^\varepsilon}$ in $\mathcal{M}_{2,\rho_\varepsilon} \subset \mathcal{M}_2$ that lie in the tubular ε-neighbourhood V_ε of the rational curve σ_A in the complex manifold $\mathcal{M}_2$ (we use Theorem 5.2 here). This implies that the distance between the rational curves $\sigma_{A_1^\varepsilon}, \sigma_{A_2^\varepsilon}$ tends to zero with ε.

On the other hand, by the tubular structure of V_ε and Proposition 8.1 we can find a uniform $\delta > 0$ so that in the δ-neighbourhood of a rational curve in $V_\varepsilon \cap \mathcal{M}_{k,\rho}$ there is no other rational curve. This contradiction shows that Mon is locally injective.

10. The higher genus case

Theorem 1.1 fails in higher genus. An instance of this phenomenon is given by the examples provided in the introduction, namely, a parametrized family of systems obtained by pulling back a system (X, A) on a genus two Riemann surface with irreducible monodromy by a parametrized family of ramified coverings $f_t : S \to X$ of fixed degree.

In general, infinitesimal rigidity of a system (X, A) can be detected on its eigencurve $C \subset X \times \mathbb{P}^1$, defined as the projectivization of the eigenvectors of A. More precisely, if we denote by $\Pi_2 : X \times \mathbb{P}^1 \to \mathbb{P}^1$ the horizontal fibration, the system (X, A) is rigid iff every section of $N_\mathcal{F} \simeq (\Pi_2)^*_{|C} T\mathbb{P}^1$ is the pull-back of a section of $T\mathbb{P}^1$. This is due to Lemma 7.1 together with an infinitesimal version of Lemma 6.2. This remark permits to construct infinitesimal deformations of other kind of systems.

The first ones are the systems (X, A) whose eigencurve $C \subset X \times \mathbb{P}^1$ has a vertical component and no horizontal one. They are generalizations of those coming from ramified covering techniques. In this case, the vertical component of C intersects the union of the other components in two points p and q (that might coincide): hence the section of $\Pi_2^*(T\mathbb{P}^1)_{|C}$ defined by a non zero holomorphic vector field on the vertical component that vanishes at p and q, and by zero on the other components, is not the pull-back of a section of $T\mathbb{P}^1$ by Π_2, hence the system is not infinitesimally rigid. As we proved, this is not possible in genus two to find such a system unless there is a component of the eigencurve that is horizontal (which is equivalent to the reducibility). However in higher genus it is always possible. In genus three examples can be provided by constructing explicit systems. Indeed, let X be any smooth curve of genus ≥ 3, and $x \in X$. The hyperplance H of $\Omega(X)$ consisting of forms vanishing at x has dimension ≥ 2. A generic degree two map $\mathbb{P}^1$ in $\mathbb{P}(H)$ comes, as in the proof of Lemma 8.3, from a system whose corresponding eigencurve curve contains the vertical $x \times \mathbb{P}^1$, but no horizontal.

Here is another family of examples where the Riemann-Hilbert mapping is not an immersion. Assume that the eigencurve C is symmetric with respect to the involution of $X \times \mathbb{P}^1$ given by $(x, y) \mapsto (x, -y)$, which happens if the coefficient α of the system vanishes identically. Notice that when β and γ are not $\mathbb{C}$-colinear, then the system is irreducible, and that if β and γ do not share a common zero then the curve C is smooth. We claim that in this case, we have a subspace of infinitesimal isomonodromic deformations for our system whose dimension is $\geq g - 3$. From Lemma 7.1, we know that this space has dimension

$$h^0(\Pi_2^* T\mathbb{P}^1, C) - h^0(T\mathbb{P}^1, \mathbb{P}^1) = h^0(\Pi_2^* O(2), C) - 3.$$

Denote $\pi_2 : X \to \mathbb{P}^1$ the meromorphic function $\pi_2 = \beta/\gamma$, and by $r : \mathbb{P}^1 \to \mathbb{P}^1$ the double covering $r(y) = y^2$. We have the relation

$$\pi_2 \circ \Pi_1 = r \circ \Pi_2.$$

Since the line bundle $O(2)$ over $\mathbb{P}^1$ is the preimage by r of $O(1)$, and that the preimage of $O(1)$ by π_2 is K_X, we get

$$\Pi_2^* O(2) = \Pi_1^*(\pi_2^* O(1)) = \Pi_1^*(K_X).$$

We thus obtain

$$h^0(\Pi_2^* O(2), C) \geq h^0(K_X, X) = g,$$

which shows our claim. The precise condition ensuring the Riemann-Hilbert mapping to be an immersion in terms of the eigencurve C seems hard to find in general.

11. Appendix: the complex structure on $\mathcal{M}_k$

In this appendix we prove Theorem 5.2.

Suppose S is an orientable compact connected topological surface of genus $g \geq 2$, $k \geq 0$ an integer and $\widetilde{S} \to S$ a fixed universal covering map.

As with $\mathcal{M}_{k,\rho}$, the complex structure on $\mathcal{M}_k$ is the complex structure defined by the deformation theory of projective structures. That is , assume that Y is a complex analytic space and that we have the following data :

a) a holomorphic submersion $\Pi : Z \to Y$ having as fibers simply connected Riemann surfaces,
b) a free proper discontinuous action of $\Gamma = \pi_1(S)$ on Z preserving the fibration Π,

c) a holomorphic map $\mathcal{D} : Z \to \mathbb{P}^1$ whose restriction to the fibre over each $y \in Y$ has a k critical orbits counted with multiplicity and is equivariant with respect to a representation $\rho(y) : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$.

Hence the restriction of $\mathcal{D}$ to the fiber over $y \in Y$ defines a branched projective structure over S that we call $\sigma(y)$. The complex structure that we are going to construct on $\mathcal{M}_k$ is such that the natural map $Y \to \mathcal{M}_k$ defined by $y \mapsto \sigma(y)$ is holomorphic.

Let $\sigma_0 \in \mathcal{M}_k$ be a point whose equivariance is $[\rho_0]$. Take a small neighbourhood $V \subset S$ formed by a union of disjoint round discs V_i (for σ_0), each containing one point of $\mathrm{div}(\sigma_0)$ and U neighbourhood of $[\rho_0]$ in the character variety. Let $\mathcal{H}_V = \Pi_i \mathcal{H}_{V_i}$ denote the product of Hurwitz spaces associated to each disc in V. We define a chart around σ_0 by using the following

Lemma 11.1. *There exists a holomorphic deformation $\Pi : Z \to U \times \mathcal{H}_V$ of branched projective structures such that the map $U \times \mathcal{H}_V \to \mathcal{M}_k$ is injective around the base point.*

In $\mathcal{M}_k$ we define the topology generated by images of open sets via the maps produced in Lemma 11.1. The induced topology on each $\mathcal{M}_{k,\rho} \subset \mathcal{M}_k$ coincides with the cut-and-paste topology defined in [2].

Lemma 11.2. *The topological space $\mathcal{M}_k$ is separated and the maps in Lemma 11.1 provide a topological atlas on $\mathcal{M}_k$.*

To check that the transition maps are holomorphic with respect to the natural product complex structures on the $U \times \mathcal{H}_V$'s it suffices to check that for every holomorphic deformation $Y \to \mathcal{M}_k$ whose image is contained in a chart, the natural projections onto U and $\mathcal{H}_{V_i}$ are holomorphic. The projection onto U is trivially holomorphic by definition of a holomorphic deformation. The holomorphicity of the other projections needs only be checked around the points in Y corresponding to the generic stratum of the deformation, since by continuity and Riemann Extension Theorem the map will be also holomorphic at the points on non-generic strata. We are therefore reduced to proving the following

Lemma 11.3. *Let $\Pi : Z \to (Y, y_0)$ be a germ of holomorphic deformation of branched projective structures all lying in a fixed stratum. Then, in any chart $U \times \mathcal{H}_V$ of $\mathcal{M}_k$ containing the point associated to y_0, the projection $(Y, y_0) \to \mathcal{H}_{V_i}$ is holomorphic.*

To finish the proof of Theorem 5.2 we just need to remark that the monodromy map read in any chart is given by a projection onto the U-factor, thus it is a local holomorphic submersion. Let us now prove the preceeding lemmas.

Proof of Lemma 11.1. Over each $\mathcal{H}_{V_i}$ there exists a holomorphic family $Z_i \to \mathcal{H}_i$ of branched projective structures on the disc V_i with prescribed boundary values that contains the restriction of σ_0 to V_i (see [2]).

Choose a lift of the chosen neighbourhood U of $[\rho_0]$ to $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{C}))$ around ρ_0 by imposing that no two points belong to the same $\mathrm{PSL}_2(\mathbb{C})$-orbit and, by abuse of language, call it U . We will first produce a holomorphic deformation of branched projective structures over S, $X_U \to U$ such that the monodromy of the branched projective structure over a point $\rho \in U$ is the homomorphism ρ. The construction is carried so as to be able to take out an open set of $W \subset X_U$ that is holomorphically equivalent to $U \times V$ in such a way that the holomorphic map $X_U \setminus W \to U$ is equipped with a deformation of branched projective structures on $S \setminus V$ with prescribed boundary values. The latter can be glued to the Z_i's to form a holomorphic deformation $Z \to U \times \mathcal{H}_V$ of σ_0, as desired.

As a smooth manifold, X_U is defined to be the product $U \times S$. Remark that σ_0 induces a natural complex structure τ_0 on S. The group $\pi_1(S)$ acts on $U \times \widetilde{S} \times \mathbb{P}^1$ by $\gamma \cdot (\rho, z, w) = (\rho, \gamma \cdot z, \rho(\gamma)(w))$. The action preserves the fibrations onto the first two factors, and the 'horizontal' smooth foliation by real surfaces. On the quotient space W_U, the $\mathbb{P}^1$-fibration $\pi_U : W_U \to U \times S$ is well defined and the induced foliation $\mathcal{F}$ is transversely holomorphic. The complex structure τ_0 on S defines a complex structure on W_U and on $U \times S$ and with respect to this complex structure, the fibration π_U together with the foliation $\mathcal{F}$ are locally holomorphically trivializable. Consider a covering $\{W_i\}$ of $\rho_0 \times S \subset U \times S$ where the trivialization is possible and such that some neighbourhood of each V_i is compactly contained in some W_i. Over each W_i the section σ_0 can be extended to a holomorphic section defined on W_i (with respect to the given holomorphic structure) by imposing $\sigma(\rho, z) = \sigma(\rho_0, z)$. At the intersections $W_i \cap W_j$ the chosen continuations do not coincide, but they are transverse to the foliation $\mathcal{F}$. By using a partition of unity associated to the W_i's, we can change the sections to a smooth section $\sigma : X_U \to W_U$ that is still transverse to the foliation on the $W_i \cap W_j$'s. On the neighbourhoods of the V_i's the section coincides with the first (holomorphic) extension. By construction, the tangencies between the foliation $\mathcal{F}$ and the section σ correspond by projection to the set $T = \sqcup U \times q_i$ where q_i is a point of $\mathrm{div}(\sigma_0)$. The transversely holomorphic structure of $\mathcal{F}$ induces a holomorphic structure on $X_U \setminus T$. Around T we also have a holomorphic structure defined by the initial complex structure. By construction these complex structures are compatible, so they produce a complex structure on X_U. This complex structure induces a complex structure on W_U such that the section σ, the foliation $\mathcal{F}$ and the projection $X_U \to U$ are holomorphic. The analytic continuation of any germ of σ at a point in $p_0 \times S$ produces a holomorphic map $\widetilde{X_U} \to \mathbb{P}^1$ defined on the universal cover of X_U with all the properties of a holomorphic deformation of σ_0. Over a point $\rho \in U$ the monodromy of the associated branched projective structure is the homomorphism ρ. The local holomorphic triviality of the fibration π_U equipped with the foliation $\mathcal{F}$ and the section σ around a $\rho_0 \times V_i$ allows to glue the given model to the holomorphic deformations $Z_i \to \mathcal{H}_{V_i}$ as a complex manifold and construct the wanted holomorphic deformation $Z \to U \times \mathcal{H}_V$ of σ_0.

To prove the injectivity of the associated map $c : U \times \mathcal{H}_V \to \mathcal{M}_k$ remark that for each point in $\mathcal{M}_k$ there is a well defined class of monodromy homomorphism. Since by construction no pair of points in U are conjugated by a non-trivial element in $\mathrm{PSL}_2(\mathbb{C})$ we have that $c(\rho_1, h_1) = c(\rho_2, h_2)$ implies $\rho_1 = \rho_2$. On the other hand, we know by [2] that the restriction $h \mapsto c(\rho_1, h)$ is injective; hence $h_1 = h_2$. □

Proof of Lemma 11.2. Suppose σ_0 and σ_1 are two points in $\mathcal{M}_k$ that are not separated. If their monodromies are not conjugated, we can find disjoint open neighbourhoods U_0 and U_1 around them and construct a chart using these open sets having disjoint images. Hence both can be thought as smooth sections of a flat bundle $S \times_{\rho_0} \mathbb{P}^1$. If the complex structures induced by σ_0 and σ_1 on S (and therefore on the flat bundle) are not equivalent then we can choose small open sets U_0, V_0 and U_1, V_1 that define charts and such that the projection of the image of the associated chart to Teichmüller space is contained in disjoint sets. Hence we can suppose σ_0 and σ_1 holomorphic with respect to a complex structure X on S. We know that there are three possibilities for the space of holomorphic sections of such a flat bundle: there are one, two or an infinite number of distinct sections. In either case, two distinct sections define non-equivalent branched projective structures. If the sections are distinct, we can choose some open set U around ρ_0 and V around the union of the divisors of σ_0 and σ_1 and carry the construction of Lemma 11.1 simultaneously for both sections, guaranteeing that the simultaneous smooth extension of the pair of sections produces holomorphic deformations that have no point in common in $\mathcal{M}_k$. The only left possibility is that both sections coincide, in which case $\sigma_0 = \sigma_1$. Therefore $\mathcal{M}_k$ is separated. By definition the maps defined in Lemma 11.1 are local homeomorphisms. □

Proof of Lemma 11.3. The key remark for the proof is that the complex structure on the stratum B of the base point of $\mathcal{H}_{V_i}$ has very special holomorphic charts. In fact, the map that sends, to every $b \in B$, the image of the branch point by the developing map of the associated holomorphic deformation, is a local holomorphic diffeomorphism (see [2]). Thus it suffices to check that for an arbitrary holomorphic deformation $Z \to (Y, y_0)$ lying in a fixed stratum the image of the critical point by the developing map defines a holomorphic map. This is obvious by the definition of a holomorphic deformation of branched projective structures. □

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