Partitioning a Graph into Convex Sets

D. Artigas\textsuperscript{a,*}, S. Dantas\textsuperscript{b}, M.C. Dourado\textsuperscript{d,e}, J.L. Szwarcfiter\textsuperscript{c,d,e}

\textsuperscript{a}Instituto de Ciência e Tecnologia, Universidade Federal Fluminense, Brazil
\textsuperscript{b}Instituto de Matemática, Universidade Federal Fluminense, Brazil
\textsuperscript{c}COPPE-Sistemas, Universidade Federal do Rio de Janeiro, Brazil
\textsuperscript{d}Instituto de Matemática, Universidade Federal do Rio de Janeiro, Brazil
\textsuperscript{e}NCE, Universidade Federal do Rio de Janeiro, Brazil

Abstract

Let $G$ be a finite simple graph. Let $S \subseteq V(G)$, its closed interval $I[S]$ is the set of all vertices lying on a shortest path between any pair of vertices of $S$. The set $S$ is convex if $I[S] = S$. In this work we define the concept of convex partition of graphs. If there exists a partition of $V(G)$ into $p$ convex sets we say that $G$ is $p$-convex. We prove that is $NP$-complete to decide whether a graph $G$ is $p$-convex for a fixed integer $p \geq 2$. We show that every connected chordal graph is $p$-convex, for $1 \leq p \leq n$. We also establish conditions on $n$ and $k$ to decide if a power of cycle is $p$-convex. Finally, we develop a linear-time algorithm to decide if a cograph is $p$-convex.

Key words: Chordal graphs, cographs, convex partition, convexity, powers of cycles.

1. Introduction

In recent years, many papers have appeared which, in some sense, extend concepts and methods from continuous mathematics to graph theory. The concept of convex sets is one of these topics of interest. The analogy between the concept of convex set in continuous and discrete mathematics can be

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\textsuperscript{*}Corresponding author

Email addresses: daniloartigas@puro.uff.br (D. Artigas), sdantas@im.uff.br (S. Dantas), mitre@nce.ufrj.br (M.C. Dourado), jayme@nce.ufrj.br (J.L. Szwarcfiter)
made by considering the vertex set of a connected graph and the distance between two vertices (number of edges in a shortest path between them) as a metric space. Thus, a vertex subset \( S \) of \( V(G) \) is said to be *convex* if it contains the vertices of all shortest paths connecting any pair of vertices in \( S \). Other definitions of convexity have been studied just by considering different path types such as chordless paths \([14, 18, 19]\) or triangle paths \([7]\).

Some of the early articles that generalized the Euclidean concepts of convex sets to graph theory are \([8, 9, 16, 17, 18, 21]\). But, convexity in graphs was also studied under different aspects like geodetic sets, geodetic, hull and convexity numbers \([4, 12, 13, 15]\).

The concept of convex \( p \)-partition in a graph was defined in \([1]\), as a partition of the vertex set of a graph into \( p \) convex sets. If \( G \) has a convex \( p \)-partition, then \( G \) is \( p \)-convex. In this paper we show that is \( NP \)-complete to decide if a graph is \( p \)-convex, for a fixed \( p \geq 2 \). So, a natural question is to study the complexity of determining if a graph is \( p \)-convex for different classes of graphs.

In the work \([3]\), the authors have studied the class of powers of chordal graphs. The class of powers of cycles has been studied on the domain of total coloring by \([6]\), and its coloring by \([5, 23]\). Characterizations and recognitions problems were developed in \([20, 22]\).

In this paper, we prove that all chordal graphs are \( p \)-convex for any value of \( p \). We show that it can be verified in linear-time if a cograph is \( p \)-convex for any value of \( p \). For the class of powers of cycles we determine the cases where the graph is biconvex. Also, we prove that any power of cycle is \( p \)-convex, for \( p \geq 3 \). Finally, we examine convex \( p \)-partitions of disconnected graphs.

### 2. Preliminaries

In this work, we denote by \( G \) a simple graph with vertex set \( V(G) \) and edge set \( E(G) \), where \( |V(G)| = n \) and \( |E(G)| = m \). Let \( S \subseteq V(G) \). We say that \( G[S] \) is the subgraph of \( G \) *induced* by \( S \). We denote by \( \overline{G} \) the *complement* of the graph \( G \).

A *geodesic* between \( v \) and \( w \) in \( G \) is a minimum path between \( v \) and \( w \) in the graph. The *closed interval* \( I[v, w] \) is the set of all vertices lying on a geodesic between \( v \) and \( w \). Given a set \( S \), \( I[S] = \bigcup_{u, v \in S} I[u, v] \). If \( I[S] = S \), then \( S \) is a *convex set*. The *convex hull* of \( S \), denoted \( I_h[S] \), is the smallest
convex set containing $S$. If $I_h[S] = V$, then $S$ is a hull set.

The length of a path $P$ between two vertices $v$ and $w$, denoted by $|P|$, is the number of edges in $P$. The distance in $G$ between $v$ and $w$, denoted by $d_G(v, w)$, is the length of a geodesic between $v$ and $w$ in $G$.

We define $N_G(v) = \{w \in V \mid d_G(v, w) = 1\}$ and $N_G[v] = \{w \in V \mid d_G(v, w) \leq 1\}$. Generalizing this concept, if $S \subseteq V$, then $N_G(S) = \{w \in V \setminus S \mid d_G(v, w) = 1, \forall v \in S\}$ and $N_G[S] = \{w \in V \mid d_G(v, w) \leq 1, \forall v \in S\}$.

A set $S \subseteq V(G)$ is an independent set if no two vertices of $S$ are adjacent in $G$. A set $K \subseteq V(G)$ is a clique if every two vertices of $S$ are adjacent in $G$.

We say that $v \in V(G)$ is a simplicial vertex of $G$ if $N_G(v)$ is a clique. We say that $v \in V(G)$ is a universal vertex of $G$ if $N_G[v] = V$.

A graph $C_n$ is a cycle, with length $n$, if it is a finite sequence $v_0, v_1, \ldots, v_n$ of vertices, $n \geq 3$, such that $(v_{i-1}, v_i) \in E(C_n)$, $1 \leq i \leq n$ and $v_0 = v_n$.

A graph $G$ is $p$-colorable if there exists an assignment of $p$ colors, to the vertices of $V(G)$, such that no two distinct adjacent vertices have the same color. The chromatic number of $G$, $\chi(G)$, is the minimum $p$ for which $G$ is $p$-colorable. See [2].

Let $V = (V_1, \ldots, V_p)$, $1 \leq p \leq n$, be a partition of $V(G)$. If $V$ contains only cliques we say that $V$ is a clique partition of $V(G)$. Denote by $\Theta(G)$ the minimum size of a clique partition of $V(G)$. If $V$ contains at most one non-clique, then $V$ is a quasi-clique partition of $V(G)$. If $V$ contains only convex sets, then $V$ is a convex partition of $V(G)$. Finally, if $V$ contains only convex sets and is a quasi-clique partition then we say that $V$ is a quasi-clique convex partition of $V(G)$. The latter concept appears naturally in the study of the convex partitions of cographs.

Given a graph $G$, a convex $p$-partition of $V(G)$ is a convex partition of $V(G)$ into $p$ sets. Clearly, every graph is 1-convex. So, we consider $p \geq 2$. We say that $G$ is $p$-convex if $V(G)$ has a convex $p$-partition. In particular, if $p = 2$, then $V(G)$ has a convex bipartition and $G$ is biconvex.

The convex partition number of a graph $G$, $\Theta_c(G)$, is the least integer $p \geq 2$ for which $G$ is $p$-convex. Denote by $\Theta_c'(G)$ the minimum integer $p \geq 2$ for which $G$ has a quasi-clique convex $p$-partition. A graph $G$ is strong $p$-convex if $G$ is $p$-convex and every convex $p$-partition of $G$ is a quasi-clique partition. Denote by $\Theta_c''(G)$ the minimum integer $p \geq 2$ for which $G$ is strong $p$-convex.

It is clear that, for any graph $G$, we have $\Theta_c(G) \leq \Theta_c'(G) \leq \Theta_c''(G) \leq |V(G)|$. An example where the equality holds is the complete bipartite graph,
that is, $\Theta_c(G) = \Theta_c'(G) = \Theta_c''(G) = q$, for $G = K_{q,q}$. We also have $\Theta_c(G) \leq \Theta_c'(G) \leq \Theta_c''(G) \leq |V(G)|$.

3. **NP-completeness**

In this section we discuss the complexity of the convex $p$-partition problem, i.e., the problem of deciding if a graph has a convex $p$-partition for a fixed $p$, $2 \leq p \leq n$.

**CONVEX $p$-PARTITION**

**Instance:** Graph $G$.

**Question:** Can $V(G)$ be partitioned into $p$ disjoint convex sets?

The clique $p$-partition problem is defined as follows:

**CLIQUE $p$-PARTITION**

**Instance:** Graph $G$.

**Question:** Can $V(G)$ be partitioned into $p$ disjoint cliques?

Note that, unlike the clique $p$-partition problem, the fact that $G$ is $p$-convex does not imply that $G$ is $(p+1)$-convex, for $p < |V(G)|$. For example, Figures 1(a) and 1(b) show a convex 2-partition and a convex 4-partition of a graph. However, this graph has not a convex 3-partition.

**Observation 1.** A clique $K$ of a graph $G$ is a convex set of $G$, consequently every clique partition of $V(G)$ is a convex partition of $G$.

**Observation 2.** If $\overline{G}$ is a $p$-colorable graph, for $p \geq 2$, then $G$ is $p$-convex. Furthermore, there is a convex partition of $V(G)$ formed by $p$ cliques.

Now we prove that deciding whether a graph is $p$-convex, for a fixed $p \geq 3$, is $NP$-complete.

**Theorem 3.** The convex $p$-partition problem is $NP$-complete, for a fixed $p \geq 3$. 
Proof. The problem is in \( \text{NP} \) because verifying if a subset of \( V(G) \) is convex can be done in polynomial-time [12]. The hardness proof is a reduction from the \text{clique} \( p \)-\text{partition} problem. Without loss of generality, let \( G \) be a graph with \( |V(G)| \geq 2 \), such that \( V(G) \) is not a clique. Let \( G' \) be the graph obtained from \( G \) by adding two non-adjacent vertices \( u \) and \( v \) with \( N(u) = N(v) = V(G) \).

First, we show that any proper convex set of \( G' \) is a clique. Suppose that \( C \) is a proper convex set of \( G' \) which is not a clique. In this case, \( u, v \in C \). But, since \( I[u, v] = V(G') \), we have that \( C = V(G') \), a contradiction.

If \( V(G) \) has a partition \( \mathcal{V} \) into \( p \) cliques, \( p \geq 3 \), then we can form a convex \( p \)-\text{partition} \( \mathcal{V}' \) of \( V(G') \) adding \( u, v \) in different sets of \( \mathcal{V} \).

Conversely, a convex \( p \)-\text{partition} \( \mathcal{V}' \) of \( V(G') \), \( p \geq 3 \), induces a partition of \( V(G) \) into \( \ell \) cliques, where \( p - 2 \leq \ell \leq p \). If \( \ell \neq p \), we divide a clique of \( \mathcal{V}' \) into two cliques in order to obtain a partition of \( V(G) \) into \( \ell + 1 \) cliques. If \( \ell + 1 \neq p \), then we repeat this argument until obtaining a clique \( p \)-\text{partition} of \( V(G) \).

Since \text{clique} 2-\text{partition} could be decided in polynomial-time, the above reduction is not valid when \( p = 2 \). The complexity of this case is proved by reducing the \( \text{NP} \)-complete 1-in-3 \text{3sat} problem to \text{convex} 2-\text{partition} problem.

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Figure 1: (a) Convex 2-partition and (b) Convex 4-partition of a graph that admits no convex 3-partition.
1-IN-3 3SAT

Instance: Set $X = \{x_1, \ldots, x_n\}$ of variables, collection $C = \{c_1, \ldots, c_m\}$ of clauses over $X$ such that each clause $c \in C$ has $|c| = 3$ and no negative literals.

Question: Is there a truth assignment for $X$ such that each clause in $C$ has exactly one true literal?

We say that $C$ is satisfiable if there exists a truth assignment for $X$ such that $C$ is satisfiable and each clause in $C$ has exactly one true variable.

Theorem 4. The convex 2-partition problem is NP-complete.

Proof. The problem is in $NP$, again, because verifying if a set is convex can be done in polynomial-time [12]. In order to reduce 1-IN-3 3SAT to CONVEX 2-PARTITION we construct a particular instance $G$ of CONVEX 2-PARTITION from a generic instance $(X, C)$ of 1-IN-3 3SAT, such that $C$ is satisfiable if and only if $G$ is biconvex. First we describe the construction of a particular instance $G$ of CONVEX 2-PARTITION; second we prove in Lemma 5 that a convex 2-partition of $V(G)$ defines a truth assignment that satisfies $(X, C)$; third we prove in Lemma 6 that a truth assignment that satisfies $(X, C)$ defines a graph $G$ which is biconvex. These steps are explained in detail below. □

The construction of a particular instance of CONVEX 2-PARTITION problem.

The vertex set $V(G)$ contains: for every variable $x_i \in X$, one vertex $x_i$ in $G$; for every clause $c_j$ in $C$ eleven vertices: $f_j, l_j^1, l_j^2, l_j^3, l_j^4, l_j^5, q_j^1, q_j^2, q_j^3, t_j$; and two auxiliary vertices: $f$ and $t$.

We denote by $F = \{f_j|1 \leq j \leq m\}$, $L = \{l_j^i|1 \leq j \leq m, 1 \leq i \leq 3\}$, $X = \{x_1, \ldots, x_n\}$, $Q = \{q_j^i|1 \leq j \leq m, 1 \leq i \leq 3\}$, $L = \{l_j^i|1 \leq j \leq m, 1 \leq i \leq 3\}$ and $T = \{t_j|1 \leq j \leq m\}$.

The edge set $E(G)$ is such that: $X \cup Q$ is a clique; $f$ is a universal vertex to $F \cup X \cup Q$, and $t$ is universal to $X \cup Q \cup T$; moreover, for every clause $c_j = \{x_b, x_c, x_d\}$, we add the edges $\{l_j^1, x_b\}$, $\{l_j^2, x_c\}$, $\{l_j^3, x_d\}$, $\{f_j, l_j^1\}$, $\{f_j, l_j^2\}$, $\{f_j, l_j^3\}$, $\{t_j, l_j^1\}$, $\{t_j, l_j^2\}$, $\{t_j, l_j^3\}$, $\{q_j^1, l_j^1\}$, $\{q_j^2, l_j^2\}$, $\{q_j^3, l_j^3\}$ and $\{t_j, q_j^1\}$, $\{t_j, q_j^2\}$, $\{t_j, q_j^3\}$, $\{l_j^1, q_j^1\}$, $\{l_j^2, q_j^2\}$, $\{l_j^3, q_j^3\}$. The construction of $G$ is finished.
Lemmas 5 and 6 prove the required equivalence for establishing Theorem 4. We exhibit in Figure 2 an example of a particular instance \((X, C) = (\{x_1, x_2, x_3, x_4, x_5\}, \{(x_1, x_2, x_3), (x_3, x_4, x_5)\})\).

**Figure 2:** The graph \(G\) for the instance \((X, C) = (\{x_1, x_2, x_3, x_4, x_5\}, \{(x_1, x_2, x_3), (x_3, x_4, x_5)\})\). We omit all edges between \(L\) and \(L\). The rectangle represents a clique, white vertices belong to \(V_t\) and black vertices belong to \(V_f\). White vertices of \(X\) represent the variables of \(X\) set to true.

**Lemma 5.** If \(G\) is biconvex, then \(C\) is satisfiable.

**Proof.** Let \(\mathcal{V} = (V_f, V_t)\) be a convex bipartition of \(V(G)\). First, we claim that \(f\) and \(t\) do not belong to the same set of \(\mathcal{V}\). Suppose that \(f, t \in V_f\), then \(X \cup Q \subseteq V_f\). Let \(v, w\) be two vertices of \(F \cup L \cup L \cup T\) generated by distinct clauses of \(C\). The vertices \(v\) and \(w\) do not belong to \(V_t\) since \(I[v, w] \cap V_f \neq \emptyset\). Hence, \(V_t\) is formed by at most eight vertices, the vertices of \(S = \{f_j, l_{1j}, l_{2j}, l_{3j}, \ell_{1j}, \ell_{2j}, \ell_{3j}, t_j\}\) generated by a unique clause \(c_j\) of \(C\). Observe that \(S\) is not a convex set, because there exists a geodesic between \(l_{1j}\) and \(\ell_{3j}\) that uses vertices of \(X \cup Q\). Hence \(V_t \subseteq S\). It is easy to see that, if one vertex of \(S' = \{f_j, l_{1j}, l_{2j}, l_{3j}\}\) belongs to \(V_f\), then all vertices of \(S'\) belong to \(V_f\). Therefore, we conclude that either \(V_t = S'\) or \(V_t = S \setminus S'\). Without loss of generality, suppose that \(V_t = S'\). Since \(\ell_{3j} \in I[l_{1j}, l_{2j}]\), \(V_t\) is not a convex set. Hence, \(\mathcal{V}\) is not a convex bipartition and we conclude that \(f\) and \(t\) belong to distinct sets of \(\mathcal{V}\). Let \(f \in V_f\) and \(t \in V_t\).

Since \(f \in I[f_j, t]\), then \(f_j \in V_f\) for all \(1 \leq j \leq m\). Analogously, \(t_j \in V_t\) for all \(1 \leq j \leq m\).
Now we prove that \( \mathcal{V} \) defines a satisfiable truth assignment for \((X, \mathcal{C})\). First, we observe that if vertex \( x_i \) belongs to \( V_f \), then \( N_{G[L \cup x_i]}(x_i) \subseteq V_f \). Let \( v \in N_{G[L \cup x_i]}(x_i) \), this property holds because there exists a geodesic between \( v \) and \( t \) using \( x_i \). Analogously, if vertex \( x_i \) belongs to \( V_t \), then \( N_{G[L \cup x_i]}(x_i) \subseteq V_t \). Consequently, we could associate the set \( \mathcal{X} \) with \( X \) and \( L \) with \( \mathcal{C} \) and \( \mathcal{V} \) would represent a truth assignment for the set of variables, where the variable \( x_i \) is true if and only if the vertex \( x_i \in V_t \). We refer to Figure 2, where white vertices belong to \( V_t \) and black vertices belong to \( V_f \). It remains to prove that for each set \( L_j = \{l^1_j, l^2_j, l^3_j\} \), 1 \( \leq \) \( j \) \( \leq \) \( m \), exactly one of the vertices belongs to \( V_t \). If at least two vertices \( v, w \) of \( L_j \) belong to \( V_t \), then \( f_j \in I[v, w] \), which is a contradiction. If \( L_j \subseteq V_f \), then \( \{l^1_j, l^2_j, l^3_j\} \subseteq V_f \), and consequently \( t_j \in V_f \), which is a contradiction. This concludes the proof. 

The converse of Lemma 5 is given next by Lemma 6.

**Lemma 6.** If \( \mathcal{C} \) is satisfiable, then \( G \) is biconvex.

**Proof.** Suppose that there exists a truth assignment which satisfies \((X, \mathcal{C})\). We construct a bipartition \((V_f, V_t)\) of \( V(G) \) as follows. First add to \( V_t \) the vertices \( t, t_1, \ldots, t_m \), the vertices \( x_i \) and \( l^i_j \in N_{G[L \cup x_i]}(x_i) \) such that the variable \( x_i \) is true; and the vertices \( q^i_j, \ell_i^j \) such that \( l^i_j \) has not been added to \( V_t \), for all \( 1 \leq j \leq m \) and \( 1 \leq i \leq 3 \). Define \( V_f = V(G) \setminus V_t \). We complete the proof showing that \( V_f \) and \( V_t \) are convex sets.

**Fact 1.** Vertex \( l^i_j \in V_f \) if and only if \( N_{G[X \cup L]}(l^i_j) \subseteq V_f \), for all \( 1 \leq j \leq m \) and \( 1 \leq i \leq 3 \).

**Fact 2.** Vertex \( l^i_j \in V_f \) if and only if \( l^i_j \in V_t \), for all \( 1 \leq j \leq m \) and \( 1 \leq i \leq 3 \).

**Fact 3.** Vertex \( l^i_j \in V_f \) if and only if \( q^i_j \in V_t \), for all \( 1 \leq j \leq m \) and \( 1 \leq i \leq 3 \).

**Fact 4.** For all \( l^i_j \in L \), if \( l^i_j \in V_f \), then \( N_{G[l^i_j]} \subseteq V_f \). Hence, if for some \( w \in V_f \), \( I[l^i_j, w] \not\subseteq V_f \), then there exists a vertex \( v \in N_{G[l^i_j]} \) such that \( I[v, w] \not\subseteq V_f \).

We prove that \( V_f \) is convex by showing that there does not exist a vertex in \( V_t \) lying in a geodesic between two non-adjacent vertices \( v, w \in V_f \). Consider the following cases:

Let \( v = f \). Case \( w \in L \): by Fact 4 we do not need to analyze this case. Case \( w \in L \): \( d(v, w) = 2 \) using a vertex \( z \) of \( Q \) and by Fact 3, \( z \in V_f \). Let \( v \in F \). Case \( w \in F \): trivial. Case \( w \in L \): fact 4. Case \( w \in X \cup Q \): trivial. Case \( w \in L \): let \( P \) be a geodesic between \( v \) and \( w \). If \(|P| = 2\), then
\( V(\mathcal{P}) \subseteq V_f \) by Fact 2; if \(|\mathcal{P}| = 3\), then \( V(\mathcal{P}) \subseteq V_f \) by Fact 3. Let \( v \in L \). By Fact 4 it is not necessary to analyze this case. Let \( v \in \mathcal{X} \cup Q \cup L \), trivial.

The argument to prove that \( V_t \) is convex is analogous. Hence, we conclude that if \( C \) is satisfiable, then \( V(G) \) has a convex bipartition. \( \square \)

4. Chordal graphs

In this section, we examine convex partitions of chordal graphs. A graph is \emph{chordal} if every cycle of length at least 4 has a chord.

**Theorem 7.** If \( G \) is a connected chordal graph, then \( G \) is \( p \)-convex for all \( 1 \leq p \leq n \).

**Proof.** Since \( G \) is chordal it admits a perfect elimination ordering \( L \) of its set of vertices \( V(G) \). We will prove that, given \( p \), if we divide \( V(G) \) into \( p \) sets, where \( p - 1 \) are unitary sets containing the first \( p - 1 \) vertices of \( L \), and the other set \( S \) is formed by the remaining vertices of \( V(G) \), this partition is a convex \( p \)-partition of \( V(G) \). Clearly, the unitary sets are convex, we just need to prove that \( S \) is convex.

Suppose that \( S \) is not a convex set. Therefore, there exists a geodesic \( \mathcal{P} \) between two vertices \( u, v \) of \( S \) using vertices outside \( S \). Let \( \mathcal{P} = w_0, w_1, \ldots, w_{d-1}, w_d \), where \( w_0 = u \) and \( w_d = v \). Let \( w_q \) be the first vertex of \( L \) which belongs to \( \mathcal{P} \), for some \( 1 \leq q \leq d - 1 \). Since \( G \) is chordal, we know that \( w_q \) is a simplicial vertex in the graph induced by \( w_q \) and all vertices greater than \( w_q \) in \( L \). Hence \( w_{q-1} \) and \( w_{q+1} \) are adjacent in \( G \). In this case, there exists a path \( \mathcal{P}' = u, \ldots, w_{q-1}, w_{q+1}, \ldots, v \) shorter than \( \mathcal{P} \), a contradiction. Then \( S \) is convex. \( \square \)

**Corollary 8.** If \( G \) is a connected chordal graph, then \( G \) has a convex quasi-clique \( p \)-partition, for all \( 1 \leq p \leq n \). \( \square \)

5. Powers of cycles

A \emph{power of cycle} \( C_n^k \), \( 1 \leq k \leq n \), is a graph such that \( V(C_n^k) = V(C_n) \) and \( E(C_n^k) = \{\{v_i, v_j\} | v_i, v_j \in V(C_n^k) \text{ and } d_{C_n}(v_i, v_j) \leq k\} \). The \emph{reach} of an edge \( \{v_i, v_j\} \) in \( C_n^k \) is the distance from \( v_i \) to \( v_j \) in \( C_n \). Let \( \{u, v\} \in E(C_n^k) \), we say that \( \{u, v\} \) is an edge of \emph{maximum reach} in \( C_n^k \) if \( d_{C_n}(u, v) = k \). We denote the vertices of \( C_n^k \) by \( v_0, \ldots, v_n \), where \( v_{i-1} \) and \( v_i \) are consecutive in \( C_n \) and \( v_n = v_0 \), for \( 1 \leq i \leq n \).
Next result states conditions to determine whether $C_n^k$ is $p$-convex, for $p \geq 2$.

**Theorem 9.** $C_n^k$ is $p$-convex if and only if $p \geq 3$ or $n \leq 2k + 2$ or $n \equiv 0, 1, 2 \pmod{2k}$.

**Proof.** It follows directly from Lemma 13 and Corollaries 11, 15 and 17. \qed

Lemma 10 establishes bounds for $p$ such that $C_n^k$ has a partition into $p$ cliques.

**Lemma 10.** $C_n^k$ is $p$-convex for $\left\lfloor \frac{n}{k+1} \right\rfloor \leq p \leq n$.

**Proof.** Let $\{v_0, v_k\}$ be an edge of maximum reach in $C_n^k$. The set $\{v_0, v_1, \ldots, v_k\}$ is a clique in $C_n^k$. By similarity, every edge of maximum reach in $C_n^k$ defines a clique of size $k + 1$. Hence $C_n^k$ has a partition into $\left\lfloor \frac{n}{k+1} \right\rfloor$ cliques. \qed

**Corollary 11.** If $n \leq 2k + 2$, then $C_n^k$ is $p$-convex, for all $1 \leq p \leq n$. \qed

Let $v, w$ be a pair of vertices of $C_n^k$ and $V_1, V_2$ be the sets of vertices of the two different paths from $v$ to $w$ in $C_n^k$. In the following observation we prove that the geodesics between $v$ and $w$ in $C_n^k$ are the geodesics between $v$ and $w$ either in $C_n^k[V_1]$ or $C_n^k[V_2]$.

**Observation 12.** Let $S = \{v_1, v_2, \ldots, v_{|S|}\}$ be a subset of $V(C_n^k)$. Then for every geodesic between $v_1$ and $v_{|S|}$, $\mathcal{P}(v_1, v_{|S|}) = \{u_1, u_2, \ldots, u_{|\mathcal{P}|}\}$, where $u_1 = v_1$ and $u_{|\mathcal{P}|} = v_{|S|}$ either $U = \{u_2, \ldots, u_{|\mathcal{P}|-1}\} \subseteq S$ or $U \subseteq (V(C_n^k) \setminus S)$.

**Proof.** Suppose that there exists a geodesic $\mathcal{P}(v_1, v_{|S|}) = u_1, u_2, \ldots, u_{|\mathcal{P}|}$ such that $U \cap S \neq \emptyset$ and $U \cap (V(C_n^k) \setminus S) \neq \emptyset$. Then there exists a vertex $u_i$, $2 \leq i \leq |\mathcal{P}| - 1$, such that either $\{u_2, \ldots, u_i\} \subseteq S$ and $u_{i+1} \in V(C_n^k) \setminus S$, or $\{u_2, \ldots, u_i\} \subseteq V(C_n^k) \setminus S$ and $u_{i+1} \in S$.

Let $\{u_2, \ldots, u_i\} \subseteq S$ and $u_{i+1} \in V(C_n^k) \setminus S$. Since $u_i \in S$, $u_{i+1} \in V(C_n^k) \setminus S$ and $d_{C_n^k}(u_i, u_{i+1}) \leq k$, either $\{v_1, u_i\} \in E(C_n^k)$ or $\{v_i, v_{|S|}\} \in E(C_n^k)$. Then $\mathcal{P}$ is not a geodesic. The case $\{u_2, \ldots, u_i\} \subseteq V(C_n^k) \setminus S$ and $u_{i+1} \in S$ is analogous. \qed

Now we show that all powers of cycles are $p$-convex for $3 \leq p < \left\lceil \frac{n}{k+1} \right\rceil$. The idea is to divide $V(G)$ into $p$ sets of consecutive vertices of $C_n$ such that each set is formed by at most $\left\lceil \frac{n}{p} \right\rceil$ vertices, and then we prove that these sets are convex.

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Lemma 13. $C^k_n$ is $p$-convex for $p \geq 3$.

Proof. First let $p = 3$ and $\mathcal{V} = (V_1, V_2, V_3)$ be a partition of $V(C^k_n)$, such that, $|V_1| = \lceil \frac{n}{3} \rceil$, $|V_2| = \lceil \frac{n}{3} \rceil$, $|V_3| = n - 2 \lceil \frac{n}{3} \rceil$, and each $V_i$ contains consecutive vertices of $C_n$. We assume that $k \leq \lceil \frac{n}{3} \rceil$, otherwise $\mathcal{V}$ is a clique partition. Let $V_1 = \{v_1, \ldots, v_{|V_1|}\}$ and $v_r, v_s \in V_1$ two vertices such that $1 < r < s \leq |V_1|$. We want to show that if a pair of vertices of $V_1$ has a geodesic using vertices outside $V_1$, then $v_1$ and $v_1$ contains one. Define $U = \{v_r, v_{r+1}, \ldots, v_s\}$, $U' = \{v_s, v_{s+1}, \ldots, v_r\}$ and $V'_1 = \{v_1, v_2, \ldots, v_{|V_1|}\}$. Suppose that there exists a geodesic $\mathcal{P}(r, s)$ between $r$ and $s$ such that $V(\mathcal{P}) \subseteq U'$. Since $|V'_1| < |U'|$, $|U| < |V'_1|$ and by Observation 12, we conclude that there exists a geodesic $\mathcal{P}'(v_1, v_{|V_1|})$ such that $V(\mathcal{P}') \subseteq V'_1$. Then it is sufficient to show that there exists no geodesic between $v_1$ and $v_{|V_1|}$ containing vertices outside $V_1$.

Suppose that there exists geodesics between $u$ and $v$, $\mathcal{P}(u, v) \subseteq C^k_n[V_1]$ and $\mathcal{P}'(u, v) \subseteq C^k_n[(V \setminus V_1) \cup \{v_1, v_{|V_1|}\}]$, in $C^k_n$. Then $|\mathcal{P}| = \lceil \frac{n}{k} - 1 \rceil$ and $|\mathcal{P}'| = \lceil \frac{n - \lceil \frac{2}{k} \rceil ^{-1}}{k} \rceil$.

Since $k \leq \lceil \frac{n}{3} \rceil$ and $n - \lceil \frac{2}{k} \rceil \geq 2 \lceil \frac{n}{3} \rceil$, we have that $|\mathcal{P}| < |\mathcal{P}'|$. Hence $\mathcal{P}'(u, v)$ is not a geodesic, a contradiction. It is clear that a similar argument holds for $p > 3$.

For case $p = 2$ there exist values of $n$ and $k$ such that $C^k_n$ is not biconvex.

Lemma 14. If $n \equiv 0, 1, 2 \pmod{2k}$, then a subset $S \subseteq V(C^k_n)$ formed by $\lceil \frac{n}{2} \rceil$ consecutive vertices of $C^k_n$ is convex.

Proof. We prove that there does not exist a geodesic between each pair of vertices of $S$ using vertices outside $S$.

Without loss of generality, let $S = \{v_1, \ldots, v_{\lceil n/2 \rceil}\}$. Similarly to the proof of Lemma 13, we restrict our attention to vertices $v_1$ and $v_{\lceil n/2 \rceil}$. Since $n = 2kq + r$, where $q$ and $r$ are positive integers and $0 \leq r \leq 2$, then $|S|$ is at most $qk + 1$. Hence, $|\mathcal{P}_{C^k_n[S]}(v_1, v_{\lceil n/2 \rceil})| = \left\lfloor \frac{|S| - 1}{2} \right\rfloor = q$, for some geodesic $\mathcal{P}$ between $v_1$ and $v_{\lceil n/2 \rceil}$ in $C^k_n[S]$.

Analogously, let $S' = (V(C^k_n) \setminus S) \cup \{v_1, v_{\lceil n/2 \rceil}\}$. Clearly, $|S'|$ is at least $qk + 2$, consequently $|\mathcal{P}_{C^k_n[S']} (v_1, v_{\lceil n/2 \rceil})| = q + 1$, for some geodesic $\mathcal{P}$ between $v_1$ and $v_{\lceil n/2 \rceil}$ in $C^k_n[S']$. Therefore, by Observation 12, $S$ is convex.

Corollary 15. $C^k_n$ is biconvex for $n \equiv 0, 1, 2 \pmod{2k}$.
Lemma 16. Let \( S \subset V(C_n^k) \) be a non-clique convex set of \( C_n^k, n > 2k + 2 \) and \( n \not\equiv 0, 1, 2 \pmod{2k} \). Then \( |S| < \left\lceil \frac{n}{2} \right\rceil \).

Proof. Suppose that there exists a non-clique convex set \( S \subset V(C_n^k) \) such that \( |S| \geq \left\lceil \frac{n}{2} \right\rceil \). We show that \( |S| \geq \left\lceil \frac{n}{2} \right\rceil \) implies that \( S \) contains a pair of vertices \( u, w \) such that \( I_h[u, w] = V(C_n^k) \).

First, we claim that \( S \) has a pair of vertices \( u \) and \( w \) such that \( \left\lceil \frac{n}{2} \right\rceil - 1 \leq d_{C_n^k}(u, w) \leq \left\lceil \frac{n}{2} \right\rceil \). We denote \( a + b \pmod{n} \) by \( a + b \). We denote by \( B(v_i) \) the vertex \( v_{i+D} \), such that either \( D = \left\lceil \frac{n}{2} \right\rceil - 1 \) or \( D = \left\lfloor \frac{n}{2} \right\rfloor \), and \( B(S) = \{ B(v) \in V(C_n^k) \mid v \in S \} \). Clearly, \( |B(S)| = |S| \). We analyze two cases: \( n \) odd and \( n \) even. If \( n \) is odd, let \( D = \left\lceil \frac{n}{2} \right\rceil - 1 \). Suppose that the claim is false, then \( S \cap B(S) = \emptyset \). Since \( n \) is odd, \( |S| + |B(S)| > n \), which is a contradiction. If \( n \) is even, let \( D = \frac{n}{2} \). We define \( S' = \{ v_1, \ldots, v_q \} \) as a maximal subset of consecutive vertices of \( S \) in \( C_n, 1 \leq q \leq |S| \). Since \( S' \) is maximal, \( v_0, v_{q+1} \notin S \), which implies that \( v_D, v_{q+1+D} \notin B(S) \). But \( v_D \) and \( v_{q+1+D} \) have distance \( \frac{n}{2} - 1 \) from \( v_1 \) and \( v_q \), respectively. Suppose that the claim is false. Analogously to the odd case, \( |S| + |B(S) \cup \{ v_D, v_{q+1+D} \}| > n \), a contradiction.

Let \( u, w \in S \) and \( \left\lceil \frac{n}{2} \right\rceil - 1 \leq d_{C_n^k}(u, w) \leq \left\lceil \frac{n}{2} \right\rceil \). Now we prove that \( I_h[u, w] = V(C_n^k) \). Let \( d_{C_n^k}(u, w) = \left\lceil \frac{n}{2} \right\rceil - 1 \), and without loss of generality, \( u = v_0 \) and \( w = v_{\left\lceil \frac{n}{2} \right\rceil - 1} \). We denote by \( R = \{ v_0, v_1, \ldots, v_{\left\lceil \frac{n}{2} \right\rceil - 1} \} \) and \( R' = \{ v_{\left\lceil \frac{n}{2} \right\rceil - 1}, v_{\left\lceil \frac{n}{2} \right\rceil}, \ldots, v_0 \} \). Analogously to the proof of Lemma 14, since \( n = 2kq + r, 3 \leq r < 2k, d_{C_n^k}[R](v_0, v_{\left\lceil \frac{n}{2} \right\rceil - 1}) = d_{C_n^k}[R'](v_0, v_{\left\lceil \frac{n}{2} \right\rceil - 1}) = q + 1 \). We remark that, since \( n > 2k + 2 \), \( d_{C_n^k}(v_0, v_{\left\lceil \frac{n}{2} \right\rceil - 1}) \geq 2 \). Moreover, a geodesic between \( v_0 \) and \( v_{\left\lceil \frac{n}{2} \right\rceil - 1} \) in \( C_n^k[R] \) is not only formed by edges of maximum reach, which implies that there exist at least two geodesics between \( v_0 \) and \( v_{\left\lceil \frac{n}{2} \right\rceil - 1} \) in \( C_n^k[R], \mathcal{P} \) and \( \mathcal{P}' \).

Let \( \mathcal{P}(v_0, v_{\left\lceil \frac{n}{2} \right\rceil - 1}) \) be a geodesic constructed using edges of maximum reach until it is possible, then \( V(\mathcal{P}) = \{ v_0, v_k, v_{2k}, \ldots, v_{qk}, v_{\left\lceil \frac{n}{2} \right\rceil - 1} \} \). Clearly, if \( V(\mathcal{P}') = \{ v_0, v_{k-1}, v_{2k-1}, \ldots, v_{(q-1)k}, v_{\left\lceil \frac{n}{2} \right\rceil - 1} \} \), then \( \mathcal{P}'(v_0, v_{\left\lceil \frac{n}{2} \right\rceil - 1}) \) is also a geodesic.

Since \( v_{ik-1} \) and \( v_{i(k+1)} \) belong to \( I[v_0, v_{\left\lceil \frac{n}{2} \right\rceil - 1}], \) for \( 1 \leq i \leq q - 1 \), we have that \( X = \bigcup_{1 \leq i \leq q - 1} I[v_{ik-1}, v_{i(k+1)}] = \bigcup_{1 \leq i \leq q - 1} \{ v_{ik-1}, v_k, \ldots, v_{i(k)+1} \} \subseteq I_h[v_0, v_{\left\lceil \frac{n}{2} \right\rceil - 1}] \). There also exist geodesics between \( v_0 \) and \( v_{\left\lceil \frac{n}{2} \right\rceil - 1} \) using vertices of \( R' \). Therefore, \( X' = \{ v_{\left\lceil \frac{n}{2} \right\rceil - 1+k}, v_{\left\lceil \frac{n}{2} \right\rceil - 1+2k}, \ldots, v_{\left\lceil \frac{n}{2} \right\rceil - 1+(q-1)k} \} \subseteq \)
I[v_0, v_{\lfloor \frac{n}{2}\rfloor}]. Consequently, \{v_{gk}, v_{gk+1}, \ldots, v_{\lfloor \frac{n}{2}\rfloor+1+k}\} \subseteq I_h[X \cup \{v_{\lfloor \frac{n}{2}\rfloor} \}]. Similarly, we conclude that \{v_{0}, v_{\lfloor \frac{n}{2}\rfloor} \} \subseteq I_h[X \cup \{v_{\lfloor \frac{n}{2}\rfloor} \}].

Corollary 17. \(C_n^k\) is not biconvex, for \(n > 2k + 2\) and \(n \not\equiv 0, 1, 2 \pmod{2k}\).

Proof. Follows from Corollary 11 and Lemma 16.

6. Disconnected graphs

In this section, we describe a method for reducing the problem of deciding whether a disconnected graph admits a convex \(p\)-partition into a similar problem for a connected graph.

Note that if a disconnected graph contains \(\omega\) connected components then it is trivially \(p\)-convex, for any \(p \leq \omega\).

Theorem 18. Let \(G\) be a graph with connected components \(G_1, \ldots, G_\omega\). Graph \(G\) is \(p\)-convex if and only if for each \(G_i\) there exists \(p_i, 1 \leq i \leq \omega\), such that:

(i) \(G_i\) is \(p_i\)-convex;
(ii) \(\sum_{1 \leq i \leq \omega} p_i \geq p\), and each \(p_i \leq p\).

Proof. Let \(V = (V_1, \ldots, V_p)\) be a convex \(p\)-partition of \(V(G)\). We define \(V_i = (V_1 \cap G_i, \ldots, V_p \cap G_i)\) by only considering cases \(V_j \cap G_i \neq \emptyset\), \(1 \leq j \leq p\) and \(1 \leq i \leq \omega\). Note that \(V_i\) is a convex \(p_i\)-partition of \(V(G_i)\), where \(p_i \leq p\).

Furthermore, since each set \(V_j\) has vertices of one or more partitions \(V_i\), we have \(\sum_{1 \leq i \leq \omega} p_i \geq p\).

Conversely, let \(G_i\) be \(p_i\)-convex, \(1 \leq i \leq \omega\), and \(\sum_{1 \leq i \leq \omega} p_i \geq p\). The convex sets which form the convex \(p_i\)-partitions of graphs \(G_i\) is a convex \(\ell\)-partition of \(G\), where \(\ell = \sum_{1 \leq i \leq \omega} p_i \geq p\). If \(\ell > p\), we construct a convex \((\ell - 1)\)-partition of \(G\) performing the union between a convex set of a connected component \(G_i\) and one convex set of \(G_j\), where \(i \neq j\). We note that the union of convex sets of distinct connected components is also convex, and the union of convex sets of the same connected component could not be convex. So, we repeat this process until obtaining partitions with less than \(\ell - 1\) convex sets. Then,
by the pigeonhole principle, $\max \{ p_i \mid 1 \leq i \leq \omega \}$ is the lower bound for the minimum number of sets in a convex partition obtained in this way. Since each $p_i \leq p$, with this procedure we have a convex $p$-partition for $G$. \hfill \Box

Theorem 18 reduces the problem of deciding whether a disconnected graph $G$, with connected components $G_1, \ldots, G_\omega$, is $p$-convex, to the problem of deciding whether its connected components $G_i$ are $p_i$-convex, for $1 \leq p \leq n$. This theorem leads to Algorithm 1.

\textbf{Algorithm 1} Algorithm for convex $p$-partition of a disconnected graph.

(i) For each $i$, $1 \leq i \leq \omega$, determine the largest $p_i \leq p$ such that $G_i$ is $p_i$-convex;

(ii) If $\sum_{1 \leq i \leq \omega} p_i \geq p$, then $G$ is $p$-convex; otherwise $G$ is not $p$-convex.

We remark that using Algorithm 1, we can determine in polynomial-time if a disconnected graph is $p$-convex, for graph classes for which there exist a polynomial-time algorithm to determine if a connected graph is $p$-convex. The complexity of a brute force algorithm based on Algorithm 1 is $O(p^\omega X)$, where $O(X)$ is the complexity to test if the connected graph, $G_i$, is $p_i$-convex.

7. Cographs

Finally we examine convex partitions of cographs. A graph is a cograph if it does not contain $P_4$ as an induced subgraph. We note that $G$ is a non-trivial connected cograph if and only if $\overline{G}$ is a disconnected cograph.

\textbf{Theorem 19}. Let $p \geq 2$, the following sentences are equivalent for a connected cograph $G$:

(i) $G$ is $p$-convex;

(ii) $G$ is strong $p$-convex;

(iii) Either $\overline{G}$ is $p$-colorable or $\overline{G}$ contains exactly one non-trivial connected component $\overline{H}$, such that $H = G[V(\overline{H})]$ has a quasi-clique convex $p$-partition.

\textbf{Proof}. (i) $\Rightarrow$ (ii) Let $G$ be a convex graph and consider any convex $p$-partition $\mathcal{V} = \{V_1, \ldots, V_p\}$ of $G$. Suppose that $\mathcal{V}$ contains two sets that are not cliques, for instance $V_1$ and $V_2$. This implies that two non-adjacent vertices $v, v' \in V_1$
belong to a same connected component of $\overline{G}$. Similarly for two non-adjacent vertices $u, u' \in V_2$. Suppose that these four vertices are in distinct connected components of $\overline{G}$ then $v \in I[u, u'] \subseteq V_2$, which is a contradiction. Hence, these four vertices belong to the same connected component of $\overline{G}$. But this implies that a vertex which is not in this connected component belongs to both $V_1$ and $V_2$, another contradiction.

(ii) $\Rightarrow$ (iii) Let $G$ be a strong $p$-convex graph. If $\overline{G}$ has only trivial connected components, then $V(G)$ is a clique and $\overline{G}$ is $p$-colorable.

Suppose that $\overline{G}$ has exactly one non-trivial connected component $\overline{H}$. Clearly, if $V(H) \leq p$, then $\overline{G}$ is $p$-colorable. From now on we consider $|V(H)| > p$. Let $\mathcal{V} = (V_1, \ldots, V_p)$ be a quasi-clique convex $p$-partition of $G$. If $\mathcal{V}$ only contains cliques, then $\overline{G}$ is $p$-colorable. If $\mathcal{V}$ contains exactly one non-clique, then let $v, v'$ be two non-adjacent vertices of $V_1$. All trivial connected components of $\overline{G}$ belong to $I[v, v'] \subseteq V_2$. Hence the sets $V_2, \ldots, V_p$ are formed by vertices of $H$. Consequently, $\mathcal{V}' = (V_1 \cap H, V_2, \ldots, V_p)$ is a quasi-clique convex $p$-partition of $H$.

Now consider that $\overline{G}$ has at least two non-trivial connected components and suppose by contradiction that $\overline{G}$ is not $p$-colorable. Let $\mathcal{V} = (V_1, \ldots, V_p)$ be a quasi-clique convex $p$-partition of $G$. Then there exists a set of $\mathcal{V}$, for instance $V_1$, with non-adjacent vertices $u, u'$, otherwise $\overline{G}$ would be $p$-colorable. Hence, $u$ and $u'$ belong to the same connected component of $\overline{G}$, say $H_1$. This implies that any vertex of any other connected component of $\overline{G}$ must belong to $V_1$. But, since $\overline{G}$ has at least two non-trivial connected components, there exists a connected component $H_2$ with two non-adjacent vertices $v, v' \in V_1$. Since $H_1 \subseteq I[v, v']$, we conclude that $V_1 = V(G)$, a contradiction.

(iii) $\Rightarrow$ (i) If $\overline{G}$ is $p$-colorable then $G$ is $p$-convex. If $\overline{G}$ is not $p$-colorable and has exactly one non-trivial connected component $\overline{H}$, such that $H$ contains a quasi-clique convex $p$-partition $\mathcal{V} = (V_1, \ldots, V_p)$. Then we can obtain a convex $p$-partition for $G$ by adding the vertices $V(G) \setminus V(H)$ to the set of $\mathcal{V}$ that is not a clique.

The previous theorem gives conditions to develop an algorithm to decide if a connected cograph $G$ is $p$-convex. This algorithm uses the cotree of the graph $G$ [11]. The cotree $T_G$ of $G$ is a tree rooted at $G$ such that the children of each node of $T_G$ are the connected components of its complement. The leaves of $T_G$ are the vertices of $G$.

In Figure 3, we schematically exhibit the first levels of the cotree of $G$. 


Figure 3: Scheme of the cotree of cograph $G$. White vertices are non-trivial connected components and the black vertices are trivial connected components.

The black vertices represent trivial connected components and the white ones are non-trivial. By Theorem 19, to decide if $G$ is $p$-convex we need to check the number of non-trivial connected components of $\overline{G}$. Since $\overline{G}$ has just one non-trivial connected component $G'$, we need to verify if $G[V(G')]$ is $p$-convex. Since $G[V(G')]$ is disconnected we can not use an algorithm based on Theorem 19. By Theorem 18, it is important to determine the largest $p'_i$, less than or equal to $p$, such that $G'_i$ is $p'_i$-convex for all connected components of the graph $G[V(G')]$. Therefore, we use Theorem 19 to determine $p'_i$, for all $G'_i$. First we note that $G'_3$ is trivial, since $|V(G'_3)| \leq p$, then $p'_3 = |V(G'_3)| = 1$; suppose that $|V(G'_1)| > p$ and $|V(G'_2)| > p$, to $G'_1$ and $G'_2$ we need to apply Theorem 19. Since $G'_1$ has two non-trivial connected components, then we need to examine if $G'_1$ is $p$-colorable. Since $G'_2$ does not have a non-trivial connected component, then $G'_2$ is $p$-convex. Although the cotree $T_G$ has more vertices, we do not need to analyze all the vertices of $T_G$ to answer whether $G$ is $p$-convex.

We describe Algorithm 2 based on Theorems 18 and 19. Let $G$ be a connected cograph. The algorithm decides the largest $p_G \leq p$, such that $G$ is $p_G$-convex by analyzing the children of $G$ in the cotree $T_G$. If it is not possible to determine $p_G$, we recursively repeat the process to the children of $G$ in $T_G$ (possibly, not all of them). We modify Algorithm 1 for disconnected cographs. We also use the linear-time algorithm to determine the cotree [11] of a cograph.

Before presenting the algorithm, we need some definitions. Let $H$ be
a connected cograph, \(\omega(H)\) is the number of connected components of \(H\), while \(\omega'(H)\) denotes the number of non-trivial connected components of \(H\). If \(H\) has just one non-trivial connected component we denote this component by \(H'\); the connected components of a cograph \(H\) are called \(H_1, \ldots, H_{\omega(H)}\);

\[
f(H, p) \leq p\]
is the largest integer such that \(H\) is \(f(H, p)\)-convex.

**Algorithm 2** Algorithm for computing \(f(H, p)\).

**Input:** Connected cograph \(H\).

**function** \(f(H, p)\)

If \(|V(H)| \leq p\), then return \(|V(H)|\); otherwise

- If \(H\) is in an odd level of \(T_G\):
  - If \(\omega'(H) = 0\), then return \(p\);
  - If \(\omega'(H) = 1\), then return \(f(H', p)\);
  - If \(\omega'(H) \geq 2\), then determine \(\chi(H)\). If \(\chi(H) \leq p\), then return \(p\), otherwise \(G\) is not \(p\)-convex;

- Otherwise return \(\min\{p, \sum_{1 \leq i \leq \omega(H)} f(H_i, p)\}\).

The Algorithm 2 determines \(f(H, p)\) for a cograph \(H\) in \(T_G\). Hence, to determine if a connected cograph \(G\) is \(p\)-convex we determine the cotree \(T_G\) and check if \(f(G, p) = p\).

**Theorem 20.** If \(G\) is a cograph, then we can decide in time \(O(n + m)\) if \(G\) is \(p\)-convex.

**Proof.** The complexity of determining the cotree is \(O(n + m)\) and the cotree has \(O(n)\) nodes [11]. At each visited node \(H\) the algorithm can: (i) determine in \(O(1)\) time the value of \(f(H, p)\); (ii) visit the children of \(H\); (iii) decide if \(\chi(H) \leq p\). In steps (i) and (iii), Algorithm 2 does not make recursive calls to the children of \(H\). In (iii), if we determine \(\chi(H_{i_1})\) and \(\chi(H_{i_2})\), for two different nodes \(H_1\) and \(H_2\) of \(T_G\), then \(V(H_1)\) and \(V(H_2)\) are disjoints. We know, by [10, 11], that \(\chi(H)\) can be calculated in \(O(V(H) + E(H))\) time, for any node \(H\) of \(T_G\). Hence the complexity of the Algorithm 2 is
Figure 4: Graph which has a cover into 2 convex sets and it is not 2-convex.

$O(n + m)$. By Theorem 18, for a disconnected cograph $G$, we need to verify if $\sum_{1 \leq i \leq \omega(G)} f(G_i, p) \geq p$. Hence, it is easy to see that the Algorithm 2 could be extended to disconnected cographs.

8. Conclusion

We have considered the problem of the partition of $V(G)$ into $p$ convex sets. We have proved that the problem is \textit{NP}-complete for fixed values of $p \geq 2$.

We also have shown that chordal graphs are $p$-convex, for $1 \leq p \leq n$, and described a linear-time algorithm to decide whether a cograph is $p$-convex. We have shown that powers of cycles $C_n^k$ are $p$-convex, for $p \geq 3$. We also have determined conditions on $n$ and $k$, which determine whether a power of cycle is biconvex.

Finally, we mention that we have also considered the problem of deciding whether a graph has a cover into $p$ convex sets. We define the convex cover of a graph $G$ as a family of convex subsets of $V(G)$, such that the union of these sets is equal to $V(G)$ and none of this sets is contained in the union of other sets of the family. The concepts of convex partitions and convex covers are distinct. Figure 4 shows an example of a graph that has a cover into 2 convex sets but no partition into 2 convex sets. The results presented in this article are directly extensible for the convex cover problem. However, we do not know if there exists a linear-time algorithm to decide if a cograph has a cover into $p$ convex sets.
References


