Partitioning a Graph into Convex Sets[☆]

D. Artigas[a,*], S. Dantas[b], M.C. Dourado[d,e], J.L. Szwarcfiter[c,d,e]

[a] *Instituto de Ciência e Tecnologia, Universidade Federal Fluminense, Brazil*
[b] *Instituto de Matemática, Universidade Federal Fluminense, Brazil*
[c] *COPPE-Sistemas, Universidade Federal do Rio de Janeiro, Brazil*
[d] *Instituto de Matemática, Universidade Federal do Rio de Janeiro, Brazil*
[e] *NCE, Universidade Federal do Rio de Janeiro, Brazil*

Abstract

Let G be a finite simple graph. Let $S \subseteq V(G)$, its closed interval $I[S]$ is the set of all vertices lying on a shortest path between any pair of vertices of S. The set S is convex if $I[S] = S$. In this work we define the concept of convex partition of graphs. If there exists a partition of $V(G)$ into p convex sets we say that G is p-convex. We prove that is NP-complete to decide whether a graph G is p-convex for a fixed integer $p \geq 2$. We show that every connected chordal graph is p-convex, for $1 \leq p \leq n$. We also establish conditions on n and k to decide if a power of cycle is p-convex. Finally, we develop a linear-time algorithm to decide if a cograph is p-convex.

Key words: Chordal graphs, cographs, convex partition, convexity, powers of cycles.

1. Introduction

In recent years, many papers have appeared which, in some sense, extend concepts and methods from continuous mathematics to graph theory. The concept of convex sets is one of these topics of interest. The analogy between the concept of convex set in continuous and discrete mathematics can be

[☆]This research was partially supported by CAPES, CNPq and FAPERJ, Brazil.
[*]Corresponding author

Email addresses: daniloartigas@puro.uff.br (D. Artigas), sdantas@im.uff.br (S. Dantas), mitre@nce.ufrj.br (M.C. Dourado), jayme@nce.ufrj.br (J.L. Szwarcfiter)

made by considering the vertex set of a connected graph and the distance between two vertices (number of edges in a shortest path between them) as a metric space. Thus, a vertex subset S of $V(G)$ is said to be *convex* if it contains the vertices of all shortest paths connecting any pair of vertices in S. Other definitions of convexity have been studied just by considering different path types such as chordless paths [14, 18, 19] or triangle paths [7].

Some of the early articles that generalized the Euclidean concepts of convex sets to graph theory are [8, 9, 16, 17, 18, 21]. But, convexity in graphs was also studied under different aspects like geodetic sets, geodetic, hull and convexity numbers [4, 12, 13, 15].

The concept of convex p-partition in a graph was defined in [1], as a partition of the vertex set of a graph into p convex sets. If G has a convex p-partition, then G is p-convex. In this paper we show that is NP-complete to decide if a graph is p-convex, for a fixed $p \geq 2$. So, a natural question is to study the complexity of determining if a graph is p-convex for different classes of graphs.

In the work [3], the authors have studied the class of powers of chordal graphs. The class of powers of cycles has been studied on the domain of total coloring by [6], and its coloring by [5, 23]. Characterizations and recognitions problems were developed in [20, 22].

In this paper, we prove that all chordal graphs are p-convex for any value of p. We show that it can be verified in linear-time if a cograph is p-convex for any value of p. For the class of powers of cycles we determine the cases where the graph is biconvex. Also, we prove that any power of cycle is p-convex, for $p \geq 3$. Finally, we examine convex p-partitions of disconnected graphs.

2. Preliminaries

In this work, we denote by G a simple graph with vertex set $V(G)$ and edge set $E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. Let $S \subseteq V(G)$. We say that $G[S]$ is the subgraph of G *induced* by S. We denote by $\overline{G}$ the *complement* of the graph G.

A *geodesic* between v and w in G is a minimum path between v and w in the graph. The *closed interval* $I[v, w]$ is the set of all vertices lying on a geodesic between v and w. Given a set S, $I[S] = \bigcup_{u,v \in S} I[u, v]$. If $I[S] = S$, then S is a *convex set*. The *convex hull* of S, denoted $I_h[S]$, is the smallest convex set containing S. If $I_h[S] = V$, then S is a *hull set*.

The *length* of a path $\mathcal{P}$ between two vertices v and w, denoted by $|\mathcal{P}|$, is the number of edges in $\mathcal{P}$. The *distance* in G between v and w, denoted by $d_G(v, w)$, is the length of a geodesic between v and w in G.

We define $N_G(v) = \{w \in V \mid d_G(v, w) = 1\}$ and $N_G[v] = \{w \in V| \; d_G(v, w) \leq 1\}$. Generalizing this concept, if $S \subseteq V$, then $N_G(S) = \{w \in V \setminus S| \; d_G(v, w) = 1, \forall v \in S\}$ and $N_G[S] = \{w \in V| \; d_G(v, w) \leq 1, \forall v \in S\}$.

A set $S \subseteq V(G)$ is an *independent set* if no two vertices of S are adjacent in G. A set $K \subseteq V(G)$ is a *clique* if every two vertices of S are adjacent in G.

We say that $v \in V(G)$ is a *simplicial* vertex of G if $N_G(v)$ is a clique. We say that $v \in V(G)$ is a *universal* vertex of G if $N_G[v] = V$.

A graph C_n is a *cycle*, with length n, if it is a finite sequence $v_0, v_1, \ldots, v_n$ of vertices, $n \geq 3$, such that $\{v_{i-1}, v_i\} \in E(C_n)$, $1 \leq i \leq n$ and $v_0 = v_n$.

A graph G is *p-colorable* if there exists an assignment of p colors, to the vertices of $V(G)$, such that no two distinct adjacent vertices have the same color. The *chromatic number* of G, $\chi(G)$, is the minimum p for which G is p-colorable. See [2].

Let $\mathcal{V} = (V_1, \ldots, V_p)$, $1 \leq p \leq n$, be a partition of $V(G)$. If $\mathcal{V}$ contains only cliques we say that $\mathcal{V}$ is a *clique partition* of $V(G)$. Denote by $\Theta(G)$ the minimum size of a clique partition of $V(G)$. If $\mathcal{V}$ contains at most one non-clique, then $\mathcal{V}$ is a *quasi-clique partition* of $V(G)$. If $\mathcal{V}$ contains only convex sets, then $\mathcal{V}$ is a *convex partition* of $V(G)$. Finally, if $\mathcal{V}$ contains only convex sets and is a quasi-clique partition then we say that $\mathcal{V}$ is a *quasi-clique convex partition* of $V(G)$. The latter concept appears naturally in the study of the convex partitions of cographs.

Given a graph G, a *convex p-partition* of $V(G)$ is a convex partition of $V(G)$ into p sets. Clearly, every graph is 1-convex. So, we consider $p \geq 2$. We say that G is *p-convex* if $V(G)$ has a convex p-partition. In particular, if $p = 2$, then $V(G)$ has a convex bipartition and G is *biconvex*.

The *convex partition number* of a graph G, $\Theta_c(G)$, is the least integer $p \geq 2$ for which G is p-convex. Denote by $\Theta'_c(G)$ the minimum integer $p \geq 2$ for which G has a quasi-clique convex p-partition. A graph G is *strong p-convex* if G is p-convex and every convex p-partition of G is a quasi-clique partition. Denote by $\Theta''_c(G)$ the minimum integer $p \geq 2$ for which G is strong p-convex.

It is clear that, for any graph G, we have $\Theta_c(G) \leq \Theta'_c(G) \leq \Theta''_c(G) \leq |V(G)|$. An example where the equality holds is the complete bipartite graph, that is, $\Theta_c(G) = \Theta'_c(G) = \Theta''_c(G) = q$, for $G = K_{q,q}$. We also have $\Theta_c(G) \leq \Theta'_c(G) \leq \Theta(G) \leq |V(G)|$.

3. *NP*-completeness

In this section we discuss the complexity of the CONVEX p-PARTITION problem, i.e., the problem of deciding if a graph has a convex p-partition for a fixed p, $2 \leq p \leq n$.

CONVEX p-PARTITION

Instance: Graph G.

Question: Can $V(G)$ be partitioned into p disjoint convex sets?

The CLIQUE p-PARTITION problem is defined as follows:

CLIQUE p-PARTITION

Instance: Graph G.

Question: Can $V(G)$ be partitioned into p disjoint cliques?

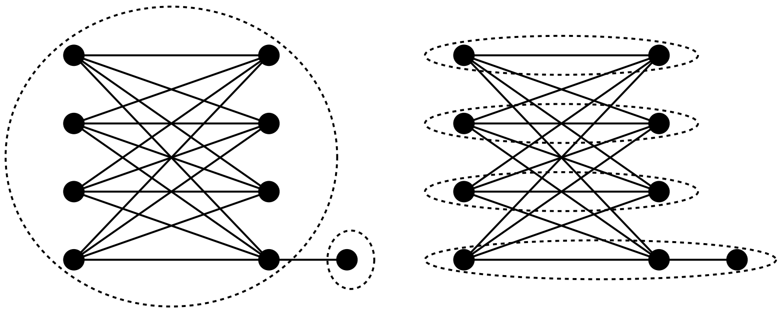
Note that, unlike the CLIQUE p-PARTITION problem, the fact that G is p-convex does not imply that G is $(p+1)$-convex, for $p < |V(G)|$. For example, Figures 1(a) and 1(b) show a convex 2-partition and a convex 4-partition of a graph. However, this graph has not a convex 3-partition.

Observation 1. A clique K of a graph G is a convex set of G, consequently every clique partition of $V(G)$ is a convex partition of G.

Observation 2. If $\overline{G}$ is a p-colorable graph, for $p \geq 2$, then G is p-convex. Furthermore, there is a convex partition of $V(G)$ formed by p cliques.

Now we prove that deciding whether a graph is p-convex, for a fixed $p \geq 3$, is NP-complete.

Theorem 3. *The* CONVEX p-PARTITION *problem is NP-complete, for a fixed $p \geq 3$.*



(a) (b)

Figure 1: (a) Convex 2-partition and (b) Convex 4-partition of a graph that admits no convex 3-partition.

Proof. The problem is in NP because verifying if a subset of $V(G)$ is convex can be done in polynomial-time [12]. The hardness proof is a reduction from the CLIQUE p-PARTITION problem. Without loss of generality, let G be a graph with $|V(G)| \geq 2$, such that $V(G)$ is not a clique. Let G' be the graph obtained from G by adding two non-adjacent vertices u and v with $N(u) = N(v) = V(G)$.

First, we show that any proper convex set of G' is a clique. Suppose that C is a proper convex set of G' which is not a clique. In this case, $u, v \in C$. But, since $I[u, v] = V(G')$, we have that $C = V(G')$, a contradiction.

If $V(G)$ has a partition $\mathcal{V}$ into p cliques, $p \geq 3$, then we can form a convex p-partition $\mathcal{V}'$ of $V(G')$ adding u, v in different sets of $\mathcal{V}$.

Conversely, a convex p-partition $\mathcal{V}'$ of $V(G')$, $p \geq 3$, induces a partition of $V(G)$ into ℓ cliques, where $p - 2 \leq \ell \leq p$. If $\ell \neq p$, we divide a clique of $\mathcal{V}'$ into two cliques in order to obtain a partition of $V(G)$ into $\ell + 1$ cliques. If $\ell + 1 \neq p$, then we repeat this argument until obtaining a clique p-partition of $V(G)$. □

Since CLIQUE 2-PARTITION could be decided in polynomial-time, the above reduction is not valid when $p = 2$. The complexity of this case is proved by reducing the NP-complete 1-IN-3 3SAT problem to CONVEX 2-PARTITION problem.

1-IN-3 3SAT

Instance: Set $X = \{x_1, \ldots, x_n\}$ of variables, collection $\mathcal{C} = \{c_1, \ldots, c_m\}$ of clauses over X such that each clause $c \in C$ has $|c| = 3$ and no negative literals.

Question: Is there a truth assignment for X such that each clause in $\mathcal{C}$ has exactly one true literal?

We say that $\mathcal{C}$ is *satisfiable* if there exists a truth assignment for X such that $\mathcal{C}$ is satisfiable and each clause in $\mathcal{C}$ has exactly one true variable.

Theorem 4. *The* CONVEX 2-PARTITION *problem is NP-complete.*

Proof. The problem is in NP, again, because verifying if a set is convex can be done in polynomial-time [12]. In order to reduce 1-IN-3 3SAT to CONVEX 2-PARTITION we construct a particular instance G of CONVEX 2-PARTITION from a generic instance $(X, \mathcal{C})$ of 1-IN-3 3SAT, such that $\mathcal{C}$ is satisfiable if and only if G is biconvex. First we describe the construction of a particular instance G of CONVEX 2-PARTITION; second we prove in Lemma 5 that a convex 2-partition of $V(G)$ defines a truth assignment that satisfies $(X, \mathcal{C})$; third we prove in Lemma 6 that a truth assignment that satisfies $(X, \mathcal{C})$ defines a graph G which is biconvex. These steps are explained in detail below. ☐

The construction of a particular instance of CONVEX 2-PARTITION *problem.*

The vertex set $V(G)$ contains: for every variable $x_i \in X$, one vertex x_i in G; for every clause c_j in $\mathcal{C}$ eleven vertices: $f_j, l_j^1, l_j^2, l_j^3, \ell_j^1, \ell_j^2$, ℓ_j^3, q_j^1, q_j^2, q_j^3, t_j; and two auxiliary vertices: f and t.

We denote by $F = \{f_j | 1 \leq j \leq m\}$, $L = \{l_j^i | 1 \leq j \leq m, 1 \leq i \leq 3\}$, $\mathcal{X} = \{x_1, \ldots, x_n\}$, $Q = \{q_j^i | 1 \leq j \leq m, 1 \leq i \leq 3\}$, $\mathcal{L} = \{\ell_j^i | 1 \leq j \leq m, 1 \leq i \leq 3\}$ and $T = \{t_j | 1 \leq j \leq m\}$.

The edge set $E(G)$ is such that: $\mathcal{X} \cup Q$ is a clique; f is a universal vertex to $F \cup \mathcal{X} \cup Q$, and t is universal to $\mathcal{X} \cup Q \cup T$; moreover, for every clause $c_j = \{x_b, x_c, x_d\}$, we add the edges $\{l_j^1, x_b\}$, $\{l_j^2, x_c\}$, $\{l_j^3, x_d\}$, $\{f_j, l_j^1\}, \{f_j, l_j^2\}, \{f_j, l_j^3\}$, $\{t_j, \ell_j^1\}$, $\{t_j, \ell_j^2\}, \{t_j, \ell_j^3\}$, $\{q_j^1, \ell_j^1\}$, $\{q_j^2, \ell_j^2\}$, $\{q_j^3, \ell_j^3\}$ and $\{l_j^1, \ell_j^2\}$, $\{l_j^1, \ell_j^3\}$, $\{l_j^2, \ell_j^1\}$, $\{l_j^2, \ell_j^3\}$, $\{l_j^3, \ell_j^1\}$, $\{l_j^3, \ell_j^2\}$. The construction of G is finished.

Lemmas 5 and 6 prove the required equivalence for establishing Theorem 4. We exhibit in Figure 2 an example of a particular instance $(X, \mathcal{C}) = (\{x_1, x_2, x_3, x_4, x_5\}, \{(x_1, x_2, x_3), (x_3, x_4, x_5)\})$.

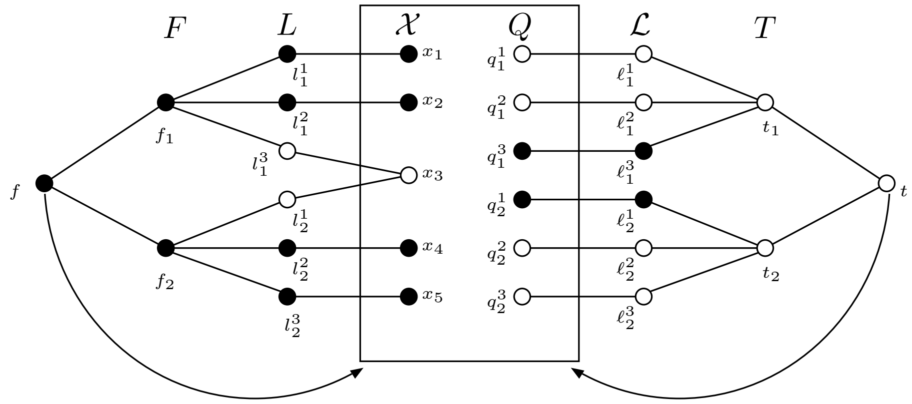


Figure 2: The graph G for the instance $(X, \mathcal{C}) = (\{x_1, x_2, x_3, x_4, x_5\}, \{(x_1, x_2, x_3), (x_3, x_4, x_5)\})$. We omit all edges between L and $\mathcal{L}$. The rectangle represents a clique, white vertices belong to V_t and black vertices belong to V_f. White vertices of $\mathcal{X}$ represent the variables of X set to true.

Lemma 5. *If G is biconvex, then $\mathcal{C}$ is satisfiable.*

Proof. Let $\mathcal{V} = (V_f, V_t)$ be a convex bipartition of $V(G)$. First, we claim that f and t do not belong to the same set of $\mathcal{V}$. Suppose that $f, t \in V_f$, then $\mathcal{X} \cup Q \subseteq V_f$. Let v, w be two vertices of $F \cup L \cup \mathcal{L} \cup T$ generated by distinct clauses of $\mathcal{C}$. The vertices v and w do not belong to V_t since $I[v, w] \cap V_f \neq \emptyset$. Hence, V_t is formed by at most eight vertices, the vertices of $S = \{f_j, l_j^1, l_j^2, l_j^3, \ell_j^1, \ell_j^2, \ell_j^3, t_j\}$ generated by a unique clause c_j of $\mathcal{C}$. Observe that S is not a convex set, because there exists a geodesic between l_j^1 and ℓ_j^1 that uses vertices of $\mathcal{X} \cup Q$. Hence $V_t \subset S$. It is easy to see that, if one vertex of $S' = \{f_j, l_j^1, l_j^2, l_j^3\}$ belongs to V_f, then all vertices of S' belong to V_f. Therefore, we conclude that either $V_t = S'$ or $V_t = S \backslash S'$. Without loss of generality, suppose that $V_t = S'$. Since $\ell_j^3 \in I[l_j^1, l_j^2]$, V_t is not a convex set. Hence, $\mathcal{V}$ is not a convex bipartition and we conclude that f and t belong to distinct sets of $\mathcal{V}$. Let $f \in V_f$ and $t \in V_t$.

Since $f \in I[f_j, t]$, then $f_j \in V_f$ for all $1 \leq j \leq m$. Analogously, $t_j \in V_t$ for all $1 \leq j \leq m$.

Now we prove that $\mathcal{V}$ defines a satisfiable truth assignment for $(X, \mathcal{C})$. First, we observe that if vertex x_i belongs to V_f, then $N_{G[L\cup x_i]}(x_i) \subseteq V_f$. Let $v \in N_{G[L\cup x_i]}(x_i)$, this property holds because there exists a geodesic between v and t using x_i. Analogously, if vertex x_i belongs to V_t, then $N_{G[L\cup x_i]}(x_i) \subseteq V_t$. Consequently, we could associate the set $\mathcal{X}$ with X and L with $\mathcal{C}$ and $\mathcal{V}$ would represent a truth assignment for the set of variables, where the variable x_i is true if and only if the vertex $x_i \in V_t$. We refer to Figure 2, where white vertices belong to V_t and black vertices belong to V_f. It remains to prove that for each set $L_j = \{l_j^1, l_j^2, l_j^3\}$, $1 \le j \le m$, exactly one of the vertices belongs to V_t. If at least two vertices v, w of L_j belong to V_t, then $f_j \in I[v, w]$, which is a contradiction. If $L_j \subseteq V_f$, then $\{\ell_j^1, \ell_j^2, \ell_j^3\} \subseteq V_f$, and consequently $t_j \in V_f$, which is a contradiction. This concludes the proof. □

The converse of Lemma 5 is given next by Lemma 6.

Lemma 6. *If $\mathcal{C}$ is satisfiable, then G is biconvex.*

Proof. Suppose that there exists a truth assignment which satisfies $(X, \mathcal{C})$. We construct a bipartition (V_f, V_t) of $V(G)$ as follows. First add to V_t the vertices $t, t_1, \ldots, t_m$, the vertices x_i and $l_j^i \in N_{G[L\cup x_i]}(x_i)$ such that the variable x_i is true; and the vertices q_j^i, ℓ_j^i such that l_j^i has not been added to V_t, for all $1 \le j \le m$ and $1 \le i \le 3$. Define $V_f = V(G) \backslash V_t$. We complete the proof showing that V_f and V_t are convex sets.

Fact 1. Vertex $l_j^i \in V_f$ if and only if $N_{G[\mathcal{X}\cup l_j^i]}(l_j^i) \in V_f$, for all $1 \le j \le m$ and $1 \le i \le 3$.

Fact 2. Vertex $l_j^i \in V_f$ if and only if $\ell_j^i \in V_t$, for all $1 \le j \le m$ and $1 \le i \le 3$.

Fact 3. Vertex $l_j^i \in V_f$ if and only if $q_j^i \in V_t$, for all $1 \le j \le m$ and $1 \le i \le 3$.

Fact 4. For all $l_j^i \in L$, if $l_j^i \in V_f$, then $N_G(l_j^i) \subseteq V_f$. Hence, if for some $w \in V_f$ $I[l_j^i, w] \not\subseteq V_f$, then there exists a vertex $v \in N_G(l_j^i)$ such that $I[v, w] \not\subseteq V_f$.

We prove that V_f is convex by showing that there does not exist a vertex in V_t lying in a geodesic between two non-adjacent vertices $v, w \in V_f$. Consider the following cases:

Let $v = f$. Case $w \in L$: by Fact 4 we do not need to analyze this case. Case $w \in \mathcal{L}$: $d(v, w) = 2$ using a vertex z of Q and by Fact 3, $z \in V_f$. Let $v \in F$. Case $w \in F$: trivial. Case $w \in L$: fact 4. Case $w \in \mathcal{X} \cup Q$: trivial. Case $w \in \mathcal{L}$: let $\mathcal{P}$ be a geodesic between v and w. If $|\mathcal{P}| = 2$, then

$V(\mathcal{P}) \subseteq V_f$ by Fact 2; if $|\mathcal{P}| = 3$, then $V(\mathcal{P}) \subseteq V_f$ by Fact 3. Let $v \in L$. By Fact 4 it is not necessary to analyze this case. Let $v \in \mathcal{X} \cup Q \cup \mathcal{L}$, trivial.

The argument to prove that V_t is convex is analogous. Hence, we conclude that if $\mathcal{C}$ is satisfiable, then $V(G)$ has a convex bipartition. □

4. Chordal graphs

In this section, we examine convex partitions of chordal graphs. A graph is *chordal* if every cycle of length at least 4 has a chord.

Theorem 7. *If G is a connected chordal graph, then G is p-convex for all $1 \leq p \leq n$.*

Proof. Since G is chordal it admits a perfect elimination ordering L of its set of vertices $V(G)$. We will prove that, given p, if we divide $V(G)$ into p sets, where $p-1$ are unitary sets containing the first $p-1$ vertices of L, and the other set S is formed by the remaining vertices of $V(G)$, this partition is a convex p-partition of $V(G)$. Clearly, the unitary sets are convex, we just need to prove that S is convex.

Suppose that S is not a convex set. Therefore, there exists a geodesic $\mathcal{P}$ between two vertices u, v of S using vertices outside S. Let $\mathcal{P} = w_0, w_1, ..., w_{d-1}, w_d$, where $w_0 = u$ and $w_d = v$. Let w_q be the first vertex of L which belongs to $\mathcal{P}$, for some $1 \leq q \leq d-1$. Since G is chordal, we know that w_q is a simplicial vertex in the graph induced by w_q and all vertices greater than w_q in L. Hence w_{q-1} and w_{q+1} are adjacent in G. In this case, there exists a path $\mathcal{P}' = u, ..., w_{q-1}, w_{q+1}, ..., v$ shorter than $\mathcal{P}$, a contradiction. Then S is convex. □

Corollary 8. *If G is a connected chordal graph, then G has a convex quasi-clique p-partition, for all $1 \leq p \leq n$.* □

5. Powers of cycles

A *power of cycle* C_n^k, $1 \leq k \leq n$, is a graph such that $V(C_n^k) = V(C_n)$ and $E(C_n^k) = \{\{v_i, v_j\}|\ v_i, v_j \in V(C_n^k) \text{ and } d_{C_n}(v_i, v_j) \leq k\}$. The *reach* of an edge $\{v_i, v_j\}$ in C_n^k is the distance from v_i to v_j in C_n. Let $\{u, v\} \in E(C_n^k)$, we say that $\{u, v\}$ is an edge of *maximum reach* in C_n^k if $d_{C_n}(u, v) = k$. We denote the vertices of C_n^k by $v_0, \ldots, v_n$, where v_{i-1} and v_i are consecutive in C_n and $v_n = v_0$, for $1 \leq i \leq n$.

Next result states conditions to determine whether C_n^k is p-convex, for $p \geq 2$.

Theorem 9. *C_n^k is p-convex if and only if $p \geq 3$ or $n \leq 2k+2$ or $n \equiv 0, 1, 2$ (mod $2k$).*

Proof. It follows directly from Lemma 13 and Corollaries 11, 15 and 17. □

Lemma 10 establishes bounds for p such that C_n^k has a partition into p cliques.

Lemma 10. *C_n^k is p-convex for $\lceil \frac{n}{k+1} \rceil \leq p \leq n$.*

Proof. Let $\{v_0, v_k\}$ be an edge of maximum reach in C_n^k. The set $\{v_0, v_1, \ldots, v_k\}$ is a clique in C_n^k. By similarity, every edge of maximum reach in C_n^k defines a clique of size $k+1$. Hence C_n^k has a partition into $\lceil \frac{n}{k+1} \rceil$ cliques. □

Corollary 11. *If $n \leq 2k+2$, then C_n^k is p-convex, for all $1 \leq p \leq n$.* □

Let v, w be a pair of vertices of C_n^k and V_1, V_2 be the sets of vertices of the two different paths from v to w in C_n. In the following observation we prove that the geodesics between v and w in C_n^k are the geodesics between v and w either in $C_n^k[V_1]$ or $C_n^k[V_2]$.

Observation 12. Let $S = \{v_1, v_2, \ldots, v_{|S|}\}$ be a subset of $V(C_n^k)$. Then for every geodesic between v_1 and $v_{|S|}$, $\mathcal{P}(v_1, v_{|S|}) = u_1, u_2, \ldots, u_{|\mathcal{P}|}$, where $u_1 = v_1$ and $u_{|\mathcal{P}|} = v_{|S|}$ either $U = \{u_2, \ldots, u_{|\mathcal{P}|-1}\} \subseteq S$ or $U \subseteq (V(C_n^k) \backslash S)$.

Proof. Suppose that there exists a geodesic $\mathcal{P}(v_1, v_{|S|}) = u_1, u_2, \ldots, u_{|\mathcal{P}|}$ such that $U \cap S \neq \emptyset$ and $U \cap V(C_n^k) \backslash S \neq \emptyset$. Then there exists a vertex u_i, $2 \leq i \leq |\mathcal{P}| - 1$, such that either $\{u_2, \ldots, u_i\} \subseteq S$ and $u_{i+1} \in V(C_n^k) \backslash S$, or $\{u_2, \ldots, u_i\} \subseteq V(C_n^k) \backslash S$ and $u_{i+1} \in S$.

Let $\{u_2, \ldots, u_i\} \subseteq S$ and $u_{i+1} \in V(C_n^k) \backslash S$. Since $u_i \in S$, $u_{i+1} \in V(C_n^k) \backslash S$ and $d_{C_n}(u_i, u_{i+1}) \leq k$, either $\{v_1, u_i\} \in E(C_n^k)$ or $\{u_i, v_{|S|}\} \in E(C_n^k)$. Then $\mathcal{P}$ is not a geodesic. The case $\{u_2, \ldots, u_i\} \subseteq V(C_n^k) \backslash S$ and $u_{i+1} \in S$ is analogous. □

Now we show that all powers of cycles are p-convex for $3 \leq p < \lceil \frac{n}{k+1} \rceil$. The idea is to divide $V(G)$ into p sets of consecutive vertices of C_n such that each set is formed by at most $\left\lceil \frac{n}{p} \right\rceil$ vertices, and then we prove that these sets are convex.

Lemma 13. *C_n^k is p-convex for $p \geq 3$.*

Proof. First let $p = 3$ and $\mathcal{V} = (V_1, V_2, V_3)$ be a partition of $V(C_n^k)$, such that, $|V_1| = \lceil \frac{n}{3} \rceil$, $|V_2| = \lceil \frac{n}{3} \rceil$, $|V_3| = n - 2\lceil \frac{n}{3} \rceil$, and each V_i contains consecutive vertices of C_n. We assume that $k \leq \lceil \frac{n}{3} \rceil$, otherwise $\mathcal{V}$ is a clique partition. Let $V_1 = \{v_1, \ldots, v_{|V_1|}\}$ and $v_r, v_s \in V_1$ two vertices such that $1 < r < s \leq |V_1|$. We want to show that if a pair of vertices of V_1 has a geodesic using vertices outside V_1, then v_1 and $v_{|V_1|}$ has also one. Define $U = \{v_r, v_{r+1}, \ldots, v_s\}$, $U' = \{v_s, v_{s+1}, \ldots, v_r\}$ and $V_1' = \{v_{|V_1|}, v_{|V_1|+1}, \ldots, v_1\}$. Suppose that there exists a geodesic $\mathcal{P}(r, s)$ between r and s such that $V(\mathcal{P}) \subseteq U'$. Since $|V_1'| < |U'|$, $|U| < |V_1|$ and by Observation 12, we conclude that there exists a geodesic $P'(v_1, v_{|V_1|})$ such that $V(P') \subseteq V_1'$. Then it is sufficient to show that there does not exist a geodesic between v_1 and $v_{|V_1|}$ containing vertices outside V_1.

Suppose that there exist geodesics between u and v, $\mathcal{P}(u, v) \subseteq C_n^k[V_1]$ and $\mathcal{P}'(u, v) \subseteq C_n^k[(V \backslash V_1) \cup \{v_1, v_{|V_1|}\}]$, in C_n^k. Then $|\mathcal{P}| = \lceil \frac{\lceil \frac{n}{3} \rceil - 1}{k} \rceil$ and $|\mathcal{P}'| = \lceil \frac{n - \lceil \frac{n}{3} \rceil + 1}{k} \rceil$.

Since $k \leq \lfloor \frac{n}{3} \rfloor$ and $n - \lceil \frac{n}{3} \rceil \geq 2\lfloor \frac{n}{3} \rfloor$, we have that $|\mathcal{P}| < |\mathcal{P}'|$. Hence $\mathcal{P}'(u, v)$ is not a geodesic, a contradiction. It is clear that a similar argument holds for $p > 3$. □

For case $p = 2$ there exist values of n and k such that C_n^k is not biconvex.

Lemma 14. *If $n \equiv 0, 1, 2$ (mod $2k$), then a subset $S \subseteq V(C_n^k)$ formed by $\lceil \frac{n}{2} \rceil$ consecutive vertices of C_n^k is convex.*

Proof. We prove that there does not exist a geodesic between each pair of vertices of S using vertices outside S.

Without loss of generality, let $S = \{v_1, \ldots, v_{\lceil \frac{n}{2} \rceil}\}$. Similarly to the proof of Lemma 13, we restrict our attention to vertices v_1 and $v_{\lceil \frac{n}{2} \rceil}$. Since $n = 2kq + r$, where q and r are positive integers and $0 \leq r \leq 2$, then $|S|$ is at most $qk + 1$. Hence, $|\mathcal{P}_{C_n^k[S]}(v_1, v_{\lceil \frac{n}{2} \rceil})| = \left\lceil \frac{|S|-1}{2} \right\rceil = q$, for some geodesic $\mathcal{P}$ between v_1 and $v_{\lceil \frac{n}{2} \rceil}$ in $C_n^k[S]$.

Analogously, let $S' = (V(C_n^k) \backslash S) \cup \{v_1, v_{\lceil \frac{n}{2} \rceil}\}$. Clearly, $|S'|$ is at least $qk + 2$, consequently $|\mathcal{P}_{C_n^k[S']}(v_1, v_{\lceil \frac{n}{2} \rceil})| = q + 1$, for some geodesic $\mathcal{P}$ between v_1 and $v_{\lceil \frac{n}{2} \rceil}$ in $C_n^k[S']$. Therefore, by Observation 12, S is convex. □

Corollary 15. *C_n^k is biconvex for $n \equiv 0, 1, 2$ (mod $2k$).* □

Lemma 16. *Let $S \subset V(C_n^k)$ be a non-clique convex set of C_n^k, $n > 2k+2$ and $n \not\equiv 0, 1, 2 \pmod{2k}$. Then $|S| < \lceil \frac{n}{2} \rceil$.*

Proof. Suppose that there exists a non-clique convex set $S \subset V(C_n^k)$ such that $|S| \geq \lceil \frac{n}{2} \rceil$. We show that $|S| \geq \lceil \frac{n}{2} \rceil$ implies that S contains a pair of vertices u, w such that $I_h[u, w] = V(C_n^k)$.

First, we claim that S has a pair of vertices u and w such that $\lceil \frac{n}{2} \rceil - 1 \leq d_{C_n}(u, w) \leq \lceil \frac{n}{2} \rceil$. We denote $a + b \pmod{n}$ by $a + b$. We denote by $B(v_i)$ the vertex v_{i+D}, such that either $D = \lceil \frac{n}{2} \rceil - 1$ or $D = \lceil \frac{n}{2} \rceil$, and $B(S) = \{B(v) \in V(C_n) | v \in S\}$. Clearly, $|B(S)| = |S|$. We analyze two cases: n odd and n even. If n is odd, let $D = \lceil \frac{n}{2} \rceil - 1$. Suppose that the claim is false, then $S \cap B(S) = \emptyset$. Since n is odd, $|S| + |B(S)| > n$, which is a contradiction. If n is even, let $D = \frac{n}{2}$. We define $S' = \{v_1, \ldots, v_q\}$ as a maximal subset of consecutive vertices of S in C_n, $1 \leq q \leq |S|$. Since S' is maximal, $v_0, v_{q+1} \notin S$, which implies that $v_D, v_{q+1+D} \notin B(S)$. But v_D and v_{q+1+D} have distance $\frac{n}{2} - 1$ from v_1 and v_q, respectively. Suppose that the claim is false. Analogously to the odd case, $|S| + |B(S) \cup \{v_D, v_{q+1+D}\}| > n$, a contradiction.

Let $u, w \in S$ and $\lceil \frac{n}{2} \rceil - 1 \leq d_{C_n}(u, w) \leq \lceil \frac{n}{2} \rceil$. Now we prove that $I_h[u, w] = V(C_n^k)$. Let $d_{C_n}(u, w) = \lceil \frac{n}{2} \rceil - 1$, and without loss of generality, $u = v_0$ and $w = v_{\lceil \frac{n}{2} \rceil - 1}$. We denote by $R = \{v_0, v_1, \ldots, v_{\lceil \frac{n}{2} \rceil - 1}\}$ and $R' = \{v_{\lceil \frac{n}{2} \rceil - 1}, v_{\lceil \frac{n}{2} \rceil}, \ldots, v_0\}$. Analogously to the proof of Lemma 14, since $n = 2kq + r$, $3 \leq r < 2k$, $d_{C_n^k[R]}(v_0, v_{\lceil \frac{n}{2} \rceil - 1}) = d_{C_n^k[R']}(v_0, v_{\lceil \frac{n}{2} \rceil - 1}) = q + 1$. We remark that, since $n > 2k + 2$, $d_{C_n^k}(v_0, v_{\lceil \frac{n}{2} \rceil - 1}) \geq 2$. Moreover, a geodesic between v_0 and $v_{\lceil \frac{n}{2} \rceil - 1}$ in $C_n^k[R]$ is not only formed by edges of maximum reach, which implies that there exist at least two geodesics between v_0 and $v_{\lceil \frac{n}{2} \rceil - 1}$ in $C_n^k[R]$, $\mathcal{P}$ and $\mathcal{P}'$.

Let $\mathcal{P}(v_0, v_{\lceil \frac{n}{2} \rceil - 1})$ be a geodesic constructed using edges of maximum reach until it is possible, then $V(\mathcal{P}) = \{v_0, v_k, v_{2k}, \ldots, v_{qk}, v_{\lceil \frac{n}{2} \rceil - 1}\}$. Clearly, if $V(\mathcal{P}') = \{v_0, v_{k-1}, v_{2k-1}, \ldots, v_{qk-1}, v_{\lceil \frac{n}{2} \rceil - 1}\}$, then $\mathcal{P}'(v_0, v_{\lceil \frac{n}{2} \rceil - 1})$ is also a geodesic.

Since v_{ik-1} and $v_{(i+1)k}$ belong to $I[v_0, v_{\lceil \frac{n}{2} \rceil - 1}]$, for $1 \leq i \leq q - 1$, we have that $X = \bigcup_{1 \leq i \leq q-1} I[v_{ik-1}, v_{(i+1)k}] = \bigcup_{1 \leq i \leq q-1} \{v_{ik-1}, v_{ik}, \ldots, v_{(i+1)k}\} \subseteq I_h[v_0, v_{\lceil \frac{n}{2} \rceil - 1}]$. There also exist geodesics between v_0 and $v_{\lceil \frac{n}{2} \rceil - 1}$ using vertices of R'. Therefore, $X' = \{v_{\lceil \frac{n}{2} \rceil - 1 + k}, v_{\lceil \frac{n}{2} \rceil - 1 + 2k}, \ldots, v_{\lceil \frac{n}{2} \rceil - 1 + (q-1)k}\} \subseteq$

$I[v_0, v_{\lceil \frac{n}{2} \rceil -1}]$. Consequently, $\{v_{qk}, v_{qk+1}, \ldots, v_{\lceil \frac{n}{2} \rceil -1+k}\} \subseteq I_h[X \cup \{v_{\lceil \frac{n}{2} \rceil -1}, v_{\lceil \frac{n}{2} \rceil -1+k}\}] \subseteq I_h[v_0, v_{\lceil \frac{n}{2} \rceil -1}]$. Similarly, we conclude that $I_h[v_0, v_{\lceil \frac{n}{2} \rceil -1}] = I_h[X \cup X' \cup \{v_0, v_{\lceil \frac{n}{2} \rceil -1}\}] = V(C_n^k)$, which is a contradiction. The case where the distance between u and w is $\lceil \frac{n}{2} \rceil$ is analogous to this one. □

Corollary 17. *C_n^k is not biconvex, for $n > 2k+2$ and $n \not\equiv 0, 1, 2$ (mod $2k$).*

Proof. Follows from Corollary 11 and Lemma 16. □

6. Disconnected graphs

In this section, we describe a method for reducing the problem of deciding whether a disconnected graph admits a convex p-partition into a similar problem for a connected graph.

Note that if a disconnected graph contains ω connected components then it is trivially p-convex, for any $p \leq \omega$.

Theorem 18. *Let G be a graph with connected components $G_1, \ldots, G_\omega$. Graph G is p-convex if and only if for each G_i there exists p_i, $1 \leq i \leq \omega$, such that:*

(i) *G_i is p_i-convex;*
(ii) *$\sum_{1 \leq i \leq \omega} p_i \geq p$, and each $p_i \leq p$.*

Proof. Let $\mathcal{V} = (V_1, \ldots, V_p)$ be a convex p-partition of $V(G)$. We define $\mathcal{V}_i = (V_1 \cap G_i, \ldots, V_p \cap G_i)$ by only considering cases $V_j \cap G_i \neq \emptyset$, $1 \leq j \leq p$ and $1 \leq i \leq \omega$. Note that $\mathcal{V}_i$ is a convex p_i-partition of $V(G_i)$, where $p_i \leq p$. Furthermore, since each set V_j has vertices of one or more partitions $\mathcal{V}_i$, we have $\sum_{1 \leq i \leq \omega} p_i \geq p$.

Conversely, let G_i be p_i-convex, $1 \leq i \leq \omega$, and $\sum_{1 \leq i \leq \omega} p_i \geq p$. The convex sets which form the convex p_i-partitions of graphs G_i is a convex ℓ-partition of G, where $\ell = \sum_{1 \leq i \leq \omega} p_i \geq p$. If $\ell > p$, we construct a convex $(\ell - 1)$-partition of G performing the union between a convex set of a connected component G_i and one convex set of G_j, where $i \neq j$. We note that the union of convex sets of distinct connected components is also convex, and the union of convex sets of the same connected component could not be convex. So, we repeat this process until obtaining partitions with less than $\ell - 1$ convex sets. Then, by the pigeonhole principle, $max\{p_i|\ 1 \leq i \leq \omega\}$ is the lower bound for the minimum number of sets in a convex partition obtained in this way. Since each $p_i \leq p$, with this procedure we have a convex p-partition for G. □

Theorem 18 reduces the problem of deciding whether a disconnected graph G, with connected components $G_1, \ldots, G_\omega$, is p-convex, to the problem of deciding whether its connected components G_i are p_i-convex, for $1 \leq p \leq n$. This theorem leads to Algorithm 1.

Algorithm 1 Algorithm for convex p-partition of a disconnected graph.

(i) For each i, $1 \leq i \leq \omega$, determine the largest $p_i \leq p$ such that G_i is p_i-convex;

(ii) If $\sum_{1 \leq i \leq \omega} p_i \geq p$, then G is p-convex; otherwise G is not p-convex.

We remark that using Algorithm 1, we can determine in polynomial-time if a disconnected graph is p-convex, for graph classes for which there exist a polynomial-time algorithm to determine if a connected graph is p-convex. The complexity of a brute force algorithm based on Algorithm 1 is $O(p\omega X)$, where $O(X)$ is the complexity to test if the connected graph, G_i, is p_i-convex.

7. Cographs

Finally we examine convex partitions of cographs. A graph is a *cograph* if it does not contain P_4 as an induced subgraph. We note that G is a non-trivial connected cograph if and only if $\overline{G}$ is a disconnected cograph.

Theorem 19. *Let $p \geq 2$, the following sentences are equivalent for a connected cograph G:*

(i) *G is p-convex;*

(ii) *G is strong p-convex;*

(iii) *Either $\overline{G}$ is p-colorable or $\overline{G}$ contains exactly one non-trivial connected component $\overline{H}$, such that $H = G[V(\overline{H})]$ has a quasi-clique convex p-partition.*

Proof. (i) ⇒ (ii) Let G be a convex graph and consider any convex p-partition $\mathcal{V} = (V_1, \ldots, V_p)$ of G. Suppose that $\mathcal{V}$ contains two sets that are not cliques, for instance V_1 and V_2. This implies that two non-adjacent vertices $v, v' \in V_1$

belong to a same connected component of $\overline{G}$. Similarly for two non-adjacent vertices $u, u' \in V_2$. Suppose that these four vertices are in distinct connected components of $\overline{G}$ then $v \in I[u, u'] \subseteq V_2$, which is a contradiction. Hence, these four vertices belong to the same connected component of $\overline{G}$. But this implies that a vertex which is not in this connected component belongs to both V_1 and V_2, another contradiction.

(ii) $\Rightarrow$ (iii) Let G be a strong p-convex graph. If $\overline{G}$ has only trivial connected components, then $V(G)$ is a clique and $\overline{G}$ is p-colorable.

Suppose that $\overline{G}$ has exactly one non-trivial connected component $\overline{H}$. Clearly, if $V(H) \leq p$, then $\overline{G}$ is p-colorable. From now on we consider $|V(H)| > p$. Let $\mathcal{V} = (V_1, \ldots, V_p)$ be a quasi-clique convex p-partition of G. If $\mathcal{V}$ only contains cliques, then $\overline{G}$ is p-colorable. If $\mathcal{V}$ contains exactly one non-clique, then let v, v' be two non-adjacent vertices of V_1. All trivial connected components of $\overline{G}$ belong to $I[v, v'] \subseteq V_1$. Hence the sets $V_2, \ldots, V_p$ are formed by vertices of H. Consequently, $\mathcal{V}' = (V_1 \cap H, V_2, \ldots, V_p)$ is a quasi-clique convex p-partition of H.

Now consider that $\overline{G}$ has at least two non-trivial connected components and suppose by contradiction that $\overline{G}$ is not p-colorable. Let $\mathcal{V} = (V_1, \ldots, V_p)$ be a quasi-clique convex p-partition of G. Then there exists a set of $\mathcal{V}$, for instance V_1, with non-adjacent vertices u, u', otherwise $\overline{G}$ would be p-colorable. Hence, u and u' belong to the same connected component of $\overline{G}$, say H_1. This implies that any vertex of any other connected component of $\overline{G}$ must belong to V_1. But, since $\overline{G}$ has at least two non-trivial connected components, there exists a connected component H_2 with two non-adjacent vertices $v, v' \in V_1$. Since $H_1 \subseteq I[v, v']$, we conclude that $V_1 = V(G)$, a contradiction.

(iii) $\Rightarrow$ (i) If $\overline{G}$ is p-colorable then G is p-convex. If $\overline{G}$ is not p-colorable and has exactly one non-trivial connected component $\overline{H}$, such that H contains a quasi-clique convex p-partition $\mathcal{V} = (V_1, \ldots, V_p)$. Then we can obtain a convex p-partition for G by adding the vertices $V(G) \backslash V(H)$ to the set of $\mathcal{V}$ that is not a clique. $\square$

The previous theorem gives conditions to develop an algorithm to decide if a connected cograph G is p-convex. This algorithm uses the cotree of the graph G [11]. The *cotree* T_G of G is a tree rooted at G such that the children of each node of T_G are the connected components of its complement. The leaves of T_G are the vertices of G.

In Figure 3, we schematically exhibit the first levels of the cotree of G.

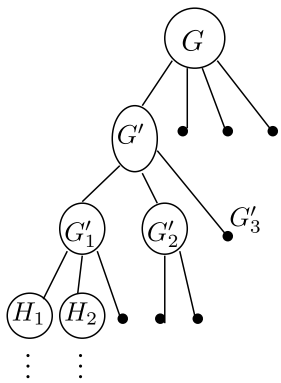


Figure 3: Scheme of the cotree of cograph G. White vertices are non-trivial connected components and the black vertices are trivial connected components.

The black vertices represent trivial connected components and the white ones are non-trivial. By Theorem 19, to decide if G is p-convex we need to check the number of non-trivial connected components of $\overline{G}$. Since $\overline{G}$ has just one non-trivial connected component G', we need to verify if $G[V(G')]$ is p-convex. Since $G[V(G')]$ is disconnected we can not use an algorithm based on Theorem 19. By Theorem 18, it is important to determine the largest p'_i, less than or equal to p, such that G'_i is p'_i-convex for all connected components of the graph $G[V(G')]$. Therefore, we use Theorem 19 to determine p'_i, for all G'_i. First we note that G'_3 is trivial, since $|V(G'_3)| \leq p$, then $p'_3 = |V(G'_3)| = 1$; suppose that $|V(G'_1)| > p$ and $|V(G'_2)| > p$, to G'_1 and G'_2 we need to apply Theorem 19. Since $\overline{G'_1}$ has two non-trivial connected components, then we need to examine if $\overline{G'_1}$ is p-colorable. Since G'_2 does not have a non-trivial connected component, then G'_2 is p-convex. Although the cotree T_G has more vertices, we do not need to analyze all the vertices of T_G to answer whether G is p-convex.

We describe Algorithm 2 based on Theorems 18 and 19. Let G be a connected cograph. The algorithm decides the largest $p_G \leq p$, such that G is p_G-convex by analyzing the children of G in the cotree T_G. If it is not possible to determine p_G, we recursively repeat the process to the children of G in T_G (possibly, not all of them). We modify Algorithm 1 for disconnected cographs. We also use the linear-time algorithm to determine the cotree [11] of a cograph.

Before presenting the algorithm, we need some definitions. Let H be a connected cograph, $\omega(\overline{H})$ is the number of connected components of $\overline{H}$, while $\omega'(\overline{H})$ denotes the number of non-trivial connected components of $\overline{H}$. If $\overline{H}$ has just one non-trivial connected component we denote this component by $\overline{H'}$; the connected components of a cograph $\overline{H}$ are called $\overline{H_1}, \ldots, \overline{H_{\omega(\overline{H})}}$; $f(H,p) \leq p$ is the largest integer such that H is $f(H,p)$-convex.

Algorithm 2 Algorithm for computing $f(H,p)$.

Input: Connected cograph H.

function $f(H,p)$

If $|V(H)| \leq p$, **then return** $|V(H)|$;
otherwise
If H is in an odd level of T_G:

- **If** $\omega'(\overline{H}) = 0$, **then return** p;
- **If** $\omega'(\overline{H}) = 1$, **then return** $f(\overline{H'}, p)$;
- **If** $\omega'(\overline{H}) \geq 2$, **then** determine $\chi(\overline{H})$. **If** $\chi(\overline{H}) \leq p$, **then return** p, **otherwise** G is not p-convex;

otherwise
return $min\{p, \sum_{1 \leq i \leq \omega(\overline{H})} f(\overline{H_i}, p)\}$.

The Algorithm 2 determines $f(H,p)$ for a cograph H in T_G. Hence, to determine if a connected cograph G is p-convex we determine the cotree T_G and check if $f(G,p) = p$.

Theorem 20. *If G is a cograph, then we can decide in time $O(n+m)$ if G is p-convex.*

Proof. The complexity of determining the cotree is $O(n+m)$ and the cotree has $O(n)$ nodes [11]. At each visited node H the algorithm can: (i) determine in $O(1)$ time the value of $f(H,p)$; (ii) visit the children of H; (iii) decide if $\chi(\overline{H}) \leq p$. In steps (i) and (iii), Algorithm 2 does not make recursive calls to the children of H. In (iii), if we determine $\chi(\overline{H_1})$ and $\chi(\overline{H_2})$, for two different nodes H_1 and H_2 of T_G, then $V(H_1)$ and $V(H_2)$ are disjoints. We know, by [10, 11], that $\chi(\overline{H})$ can be calculated in $O(V(H) + E(H))$ time, for any node H of T_G. Hence the complexity of the Algorithm 2 is

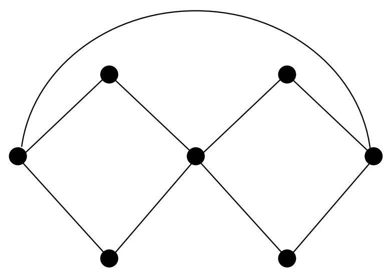


Figure 4: Graph which has a cover into 2 convex sets and it is not 2-convex.

$O(n+m)$. By Theorem 18, for a disconnected cograph G, we need to verify if $\sum_{1\leq i\leq \omega(\overline{G})} f(\overline{G_i}, p) \geq p$. Hence, it is easy to see that the Algorithm 2 could be extended to disconnected cographs. □

8. Conclusion

We have considered the problem of the partition of $V(G)$ into p convex sets. We have proved that the problem is NP-complete for fixed values of $p \geq 2$.

We also have shown that chordal graphs are p-convex, for $1 \leq p \leq n$, and described a linear-time algorithm to decide whether a cograph is p-convex. We have shown that powers of cycles C_n^k are p-convex, for $p \geq 3$. We also have determined conditions on n and k, which determine whether a power of cycle is biconvex.

Finally, we mention that we have also considered the problem of deciding whether a graph has a cover into p convex sets. We define the convex cover of a graph G as a family of convex subsets of $V(G)$, such that the union of these sets is equal to $V(G)$ and none of this sets is contained in the union of other sets of the family. The concepts of convex partitions and convex covers are distinct. Figure 4 shows an example of a graph that has a cover into 2 convex sets but no partition into 2 convex sets. The results presented in this article are directly extensible for the convex cover problem. However, we do not know if there exists a linear-time algorithm to decide if a cograph has a cover into p convex sets.

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