On the Contour of Graphs[☆]

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Abstract

Let $G = (V, E)$ be a finite, simple and connected graph. Let $S \subseteq V$, its closed interval $I[S]$ is the set of all vertices lying on a shortest path between any pair of vertices of S. The set S is *geodetic* if $I[S] = V$. The eccentricity of a vertex v is the number of edges in the greatest shortest path between v and any vertex w of G. The contour $Ct(G)$ of G is the set formed by vertices v such that no neighbor of v has an eccentricity greater than v. We consider the problem of determining whether the contour of a graph class is geodetic. The diameter $diam(G)$ of G is the maximum eccentricity of the vertices in V. In this work we establish a relation between the diameter and the geodeticity of the contour of a graph. We prove that the contour is geodetic for graphs with diameter $k \leq 4$. Furthermore, for every $k > 4$, there is a graph with diameter k and whose contour is not geodetic. We show that the contour is geodetic for bipartite graphs with diameter $k \leq 7$, and for any $k > 7$ there is a bipartite graph with diameter k and non-geodetic contour. By applying these results, we solve the open problems mentioned by Cáceres et al. [3, 5] namely to decide whether the contour of cochordal graph, parity graph and bipartite graphs are geodetic.

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1. Introduction

In the last decades many concepts of continuous mathematics were applied to discrete mathematics, particularly, to graph theory. In this article we are interested in concepts related with the study of convexity in graphs.

Let $G = (V, E)$ be a finite, simple and connected graph. A family $\mathcal{C}$ of subsets of V is a *graph convexity*, or convexity, if it satisfies two properties: (i) $V, \emptyset$ belongs to $\mathcal{C}$; (ii) $\mathcal{C}$ is closed under intersections. The elements of $\mathcal{C}$ are called *convex sets*. Some of the early works on convexity of graphs are [14, 21, 23]. Afterwards, several papers on graph convexity have been published, e.g. [6, 8, 11, 12, 13, 15, 16, 17, 18, 20].

Many of the graph convexities are defined using interval functions. A well known convexity class is the *geodesic convexity*, where a set S is convex if it is formed by all vertices lying on a shortest path between any pair of vertices of S. Many authors considered geodesic convexity with different focus in subjects as convex partitions, geodetic sets, geodetic number, hull number and convexity number [1, 4, 9, 10]. For general information about convexity see [22].

In [5], the authors have investigated if it is possible to determine any convex set S of a graph using a subset of S, with a specific property, and a simple operation on these vertices. In particular, they have defined the *contour* $Ct(G)$ of a graph G as a subset of vertices of G formed by vertices with eccentricity greater than or equal its neighbors. They asked if $Ct(G)$ is always a geodetic set, i.e., if the shortest paths between vertices of $Ct(G)$ contain all vertices of G. It was exhibited a graph where this property is not valid, and it was proved that the contour of a distance-hereditary graph is geodetic. These results motivate the question posed in [5]: for which graph classes the contour of a graph is a geodetic set? In [3], it was proved that the contour of chordal graphs is geodetic and it was presented a scheme of subclasses of perfect graphs for which the contour is a geodetic set. Particularly, this problem has been left open for three graph classes: cochordal, parity and bipartite graphs.

The main contributions of this paper are the following results:

(i) Let G be any graph. If $diam(G) \leq 4$ then the contour of G is geodetic. Furthermore, for every $k > 4$, there is a graph having diameter k and whose contour is not geodetic.

(ii) Let G be a bipartite graph. If $diam(G) \leq 7$ then the contour of G is geodetic, otherwise for any $k > 7$, there is a bipartite graph having diameter k and whose contour is not geodetic.

In special, these two results solve all open problems of [3]. We also consider other graph classes and we prove that the contour of circulant graphs is geodetic (see Remark 5) and that result (i) remains true for planar graphs (see Corollary 17).

2. Preliminaries

Let $G = (V, E)$ be a graph with vertex set V and edge set E, where $|V| = n$ and $|E| = m$. In this work, all graphs are finite, simple and connected. We say that $G[S]$ is the subgraph of G *induced* by S. We denote by $\overline{G}$ the *complement* of the graph G.

A *geodesic* between v and w in G is a shortest path between v and w in the graph. The *closed interval* $I[v, w]$ is the set of all vertices lying on a geodesic between v and w. Given a set S, $I[S] = \bigcup_{u,v \in S} I[u, v]$. If $I[S] = S$, then S is a *convex set.* If $I[S] = V$, then S is *geodetic.*

The *length* of a path P between two vertices v and w, denoted by $|P|$, is the number of edges in P. The *distance* in $G = (V, E)$ between v and w, denoted by $d_G(v, w)$, is the length of a geodesic between v and w in G. The *eccentricity* of $v \in V$, denoted by $ecc_G(v)$ is the largest distance from v to any other vertex in G, i.e., $ecc(v) = max\{d_G(v, w)|w \in V\}$. The *diameter* of G, $diam(G)$, is equal to $max\{d_G(v, w)|\ v, w \in V\}$. The *radius* of G, $rad(G)$, is equal to $min\{ecc_G(v)|\ v \in V\}$. For simplicity, we omit G from the notation above. For basic concepts in graph theory see [2].

First, we recall two properties:

Remark 1. *Let G be a graph. Then $rad(G) \leq diam(G) \leq 2rad(G)$.*

Remark 2. *Let $G = (V, E)$ and $v, u \in V$. Then, $\mid ecc(v) - ecc(u) \mid \leq d(v, u)$. In particular, if v and w are adjacent vertices then $|ecc(v) - ecc(w)| \leq 1$.*

The following lemma is a direct consequence of the above remarks. It guarantees the existence of a path of length $rad(G)$ between any pair of vertex u_0, u_r such that $ecc(u_0) = rad(G)$ and $ecc(u_r) = 2rad(G) = diam(G)$.

Lemma 3. *Let G be a connected graph with $diam(G) = 2rad(G)$. Suppose that $ecc(u_0) = rad(G) = r$. Then, for any u_r such that $ecc(u_r) = diam(G)$, there exists a path $P = u_0, u_1, \cdots, u_r$ such that $ecc(u_0) < ecc(u_1) < \cdots < ecc(u_r)$. Moreover, $ecc(u_i) = ecc(u_{i-1}) + 1$, $1 \leq i \leq r$.*

A vertex v is an *eccentric vertex* of w if $d(w, v) = ecc(w)$. A vertex v is an *eccentric vertex* of G if it is the eccentric vertex of some $w \in V$. The *eccentric set* of G, $Ecc(G)$, is the set of all eccentric vertices of G [7].

$$Ecc(G) = \{v \in V(G) |\ \exists u \in V \text{ s.t. } ecc(u) = d(u, v)\}.$$

We define $N_G(v) = \{w \in V \mid d_G(v, w) = 1\}$ and $N_G[v] = \{v\} \cup N_G(v)$. Generalizing this concept, if $S \subseteq V$, then $N_G(S) = \cup_{v \in S} N_G(v)$ and $N_G[S] = \cup_{v \in S} N_G[v]$. Moreover, $N^i_G(v) = \{w \in V \mid d_G(v, w) = i\}$.

A set $S \subseteq V$ is an *independent set* if no two vertices of S are adjacent in G. A set $K \subseteq V$ is a *clique* if every two vertices of K are adjacent in G.

We say that $v \in V$ is a *simplicial* vertex of G if $N_G[v]$ is a clique. We say that $v \in V$ is a *universal* vertex of G if $N_G[v] = V$.

A vertex v is a *contour vertex* of G if no adjacent vertex of v has eccentricity greater than $ecc(v)$. The *contour* of G, $Ct(G)$, is the set formed by all contour vertices of G [5].

$$Ct(G) = \{v \in V | ecc(u) \leq ecc(v), \forall u \in N(v)\}$$

Remark 4. *Every simplicial vertex is a contour vertex.*

Next, we analyze graphs with the property that all vertices have the same eccentricity. Some relevant graph classes have this property, for example, circulant graphs. Let n, m and $a_1, \ldots, a_m$ be positive integers. A graph $G = (V, E)$ such that $V = \{0, \ldots, n-1\}$ and $E = \{\{i, i + a_j (mod\ n)\} |\ 0 \leq i \leq n-1, 1 \leq j \leq m\}$ is called *circulant*, and it is denoted by $C_n(a_1, \ldots, a_m)$.

It is easy to see that, by symmetry, all vertices of a circulant graph have the same eccentricity.

Remark 5. *If $G = C_n(a_1, \ldots, a_k)$, then $Ct(G)$ is geodetic.*

Theorem 6. *[19] If n is even and $a_1, \ldots, a_k$ are odd, then $C_n(a_1, \ldots, a_k)$ is bipartite.*

Consequently, the contour of the subclass of bipartite graphs defined by Theorem 6 is geodetic.

3. Graphs with diameter at most 4

The diameter of a graph G is a powerful parameter to decide whether $Ct(G)$ is geodetic. We start by showing basic properties.

The following remark establishes a relation between a vertex and its eccentric vertex.

Remark 7. *Let $G = (V, E)$ be a graph. If $e(v)$ is an eccentric vertex of $v \in V$, then $ecc(e(v)) \geq ecc(v)$.*

Lemma 8 guarantees the existence of a geodesic, between a vertex $u_t \in V$ and its eccentric vertex c, forming a sequence of vertices with increasing eccentricity.

Lemma 8. *[3] Let $G = (V, E)$ be a graph and let $u_0 \in V$. Suppose that $P = u_0, u_1, \ldots, u_t$ is a path in G such that $ecc(u_{i+1}) = ecc(u_i) + 1$, for each $i \in \{0, 1, \ldots, t-1\}$. Then, for each eccentric vertex x of u_t, there exists a geodesic between x and u_t that contains P.*

Lemma 8 is the main tool employed in our results.

Remark 9. *Let $\mathcal{P}$ be a geodesic between u_0 and u_t in G, formed by vertices of increasing eccentricity; and x an eccentric vertex of u_t. Then x is an eccentric vertex of each vertex of $\mathcal{P}$.*

Let $\mathcal{P}$ be a geodesic between a contour vertex of a graph G and any of its eccentric vertices. The next lemma identifies some vertices of $\mathcal{P}$ belonging to $I[Ct(G)]$. It follows directly from Lemma 8.

Lemma 10. *Let $G = (V, E)$ be a graph, $u_0 \in V$ and $P = u_0, u_1, \cdots, u_t$ be a path in G such that $ecc(u_{i+1}) = ecc(u_i) + 1$, for each $i \in \{0, 1, \cdots, t-1\}$. Let $z \in V$ be an eccentric vertex of u_t, and $\mathcal{P}$ a geodesic between z and u_t that contains P. Suppose that $u_t, z \in Ct(G)$, $z \in \mathcal{P} \setminus P$. Then, for each $u_i \in P$ we have that $u_i \in I[Ct(G)]$, $i \in \{0, 1, \cdots, t-1\}$.*

Lemma 11 states a property of some vertices that cannot be in $I[Ct(G)]$.

Lemma 11. *Let $G = (V, E)$ be a graph. If $v \in V$ does not belong to $I[Ct(G)]$, then $ecc(v) \leq diam(G) - 2$.*

Proof. Suppose that $v \notin I[Ct(G)]$ and $ecc(v) > diam(G) - 2$. Clearly, for any vertex v, if $ecc(v) = diam(G)$, then v is a contour vertex. If $ecc(v) = diam(G) - 1$, then either v is a contour vertex, or v is adjacent to a vertex w such that $ecc(w) = diam(G)$. So, $w \in Ct(G)$. Consequently, by Lemma 8, v lies in a geodesic between w and an eccentric vertex of w, $e(w)$. By Remark 7, $e(w)$ is a contour vertex. Thus, v belongs to $I[w, e(w)]$, which is a contradiction. □

The following results determine, for which integers k, the contour of every graph with diameter k is necessarily geodetic.

Theorem 12. *If G is a graph with $diam(G) \leq 4$, then $Ct(G)$ is geodetic.*

Proof. Let $G = (V, E)$ be a graph such that $diam(G) = 4$. Suppose that there exists a vertex $v_0 \in V$ such that $v_0 \notin I[Ct(G)]$. By Remark 1 and Lemma 11, we have that $ecc(v_0) = 2$.

Since $ecc(v_0) = 2$, by Lemma 3, for any vertex $v_2 \in V$ such that $ecc(v_2) = 4$, there exists a path v_0, v_1, v_2, such that $ecc(v_1) = 3$. Let $e(v_2)$ be an eccentric vertex of v_2. Since $ecc(v_2) = 4$, by Remark 7, $ecc(e(v_2)) = 4$ and, consequently, $e(v_2) \in Ct(G)$. Hence, by Lemma 8 there exists a geodesic between v_2 and $e(v_2)$ which contains v_0. Therefore, $v_0 \in I[Ct(G)]$, a contradiction.

The cases where $diam(G) < 4$ are trivial by Remark 1 and Lemma 11. □

We denote by P_n the graph consisting of a path with n vertices.

Corollary 13. *If G is P_6-free, then $Ct(G)$ is geodetic.*

To end this section we show that Theorem 12 is best as possible in the sense that, for each $k \geq 5$, there is a graph with diameter k and whose contour is not geodetic.

In fact, the work [3] describes the graph depicted in Figure 1, having diameter 5 and with no geodetic contour. We generalize this result and present a graph of arbitrary diameter greater than 5 and whose contour is not geodetic. See Figure 2.

A graph G is *chordal* if every cycle of length at least 4 has a chord. A graph G is *cochordal* if its complement is a chordal graph.

Corollary 14. *If G is a cochordal graph, then $Ct(G)$ is a geodetic set.*

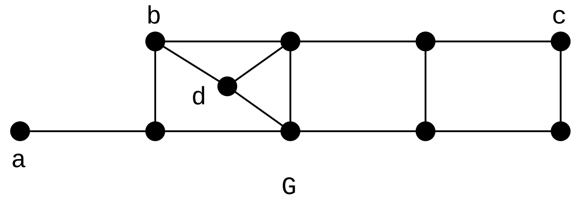

Figure 1: Graph G, with diameter 5, whose $Ct(G) = \{a, b, c\}$ and $d \notin I[Ct(G)]$.

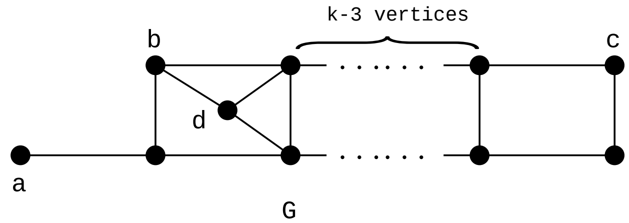

Figure 2: Graph G, with diameter $k \geq 5$, whose contour is not geodetic.

Proof. The complement of a P_5 has an induced cycle with 4 vertices. Hence, a cochordal graph G is P_5-free. Consequently, $diam(G) \leq 3$. □

Corollary 15. *For any $k \geq 5$, the graph of Figure 2 has diameter k and its contour is not geodetic.*

Proof. Let G be a graph of Figure 2. Then $Ct(G) = \{a, b, c\}$ and $d \notin I[Ct(G)]$. □

By Theorem 12 and Corollary 15 we conclude our first main result.

Theorem 16. *Let G be any graph. If $diam(G) \leq 4$ then the contour of G is geodetic. Furthermore, for every $k > 4$, there is a graph having diameter k and whose contour is not geodetic.*

Using Theorem 16, and observing that the graph of Figure 2 is a planar graph, we obtain the next corollary.

Corollary 17. *If a planar graph has diameter ≤ 4 then its contour is geodetic. Furthermore, for every $k > 4$, there is a planar graph having diameter k and whose contour is not geodetic.*

In the next sections, we focus our attention on bipartite graphs.

4. Bipartite graphs with diameter ≤ 7

In this section we consider bipartite graphs with diameter 5, 6 or 7. First, we describe useful properties of bipartite graphs.

Remark 18. *Let $G = (V, E)$ be a bipartite graph with bipartition $V = A \cup B$. Then for any $v \in V$ and for any positive integer i, either $N^i(v) \subset A$, or $N^i(v) \subset B$.*

Remark 19. *Let $G = (V, E)$ be a bipartite graph and u, w be two vertices of G such that $d(u, w) = k$. If w' is an adjacent vertex of w, then $d(u, w') = k-1$ or $d(u, w') = k + 1$. Particularly, if w is an eccentric vertex of u, then $d(u, w') = k - 1$.*

Lemma 20. *Let $G = (V, E)$ be a graph and let $u_0 \in V$. Suppose that $P = u_0, u_1, \ldots, u_k$, where $u_k = u$, is a path in G such that $ecc(u_{i+1}) = ecc(u_i) + 1$, for each $i \in \{0, 1, \ldots, k - 1\}$. Let w be an eccentric vertex of u_k and w' an adjacent vertex of w. Then, there exists a geodesic between w' and u_k that contains P.*

Proof. Since w is an eccentric vertex of all vertices in P we have that $d(u_0, w') = ecc(u_0) - 1$ and $d(u_k, w') = ecc(u_k) - 1$. By Remark 2, P is a geodesic, and this implies that $d(u_0, u_k) = k$. Since $ecc(u_k) = ecc(u_0) + k$, we conclude that $d(w', u_k) = d(w', u_0) + d(u_0, u_k)$. Hence, there exists a geodesic between w' and u containing P. $\square$

We apply Remarks 18 and 19 and Lemma 20 in the following theorems.

Theorem 21. *Let $G = (V, E)$ be a bipartite graph. Then, $diam(G) - ecc(v) \leq 2$ implies that $v \in I[Ct(G)]$, for any $v \in V \setminus Ct(G)$.*

Proof. The cases where $diam(G) - ecc(v) < 2$ follow from Lemma 11.

Let $u_0 \in V$ such that $ecc(u_0) = diam(G) - 2$. Then, there exists a path $P = u_0, u_1, \cdots, u_t$ between u_0 and $u_t \in Ct(G)$ such that $ecc(u_0) < ecc(u_1) < \cdots < ecc(u_t)$. Since $ecc(v) \leq diam(G)$, for any $v \in V$, we need to consider two different cases: $ecc(u_t) = diam(G)$ and $ecc(u_t) = diam(G) - 1$.

Suppose that $ecc(u_t) = diam(G)$. Then, $u_t \in Ct(G)$; every eccentric vertex z of u_t is such that $ecc(z) = diam(G)$, by Remark 7. Consequently, $z \in Ct(G)$ and $u_0 \in I[z, u_t] \subseteq I[Ct(G)]$, by Lemma 8.

Suppose that $ecc(u_t) = diam(G) - 1$ and let z be an eccentric vertex of u_t. Then by Lemma 8, there exists a geodesic between u_t and z through u_0.

Suppose that $z \in Ct(G)$. Then, $u_0 \in I[Ct(G)]$, by an analogous argument of that used in previous case. Suppose that $z \notin Ct(G)$. Then, there exists a vertex z' such that $\{z, z'\} \in E$ and $ecc(z') = diam(G)$. By Remark 18, $z' \in N^{ecc(u_t)-1}(u_t)$. Since $d(u_0, z) = diam(G) - 2$ and $ecc(u_0) = diam(G) - 2$, we have that $d(u_0, z') = diam(G) - 3$. Consequently, since $d(u_t, z') = diam(G) - 2$, $d(u_0, z') = diam(G) - 3$ and $d(u_0, u_t) = 1$, we have that there exists a geodesic between u_t and z' through u_0. Thus $u_0 \in I[Ct(G)]$. □

Next result is a consequence of Remark 1, Lemma 3 and Theorem 21.

Corollary 22. *The contour of every bipartite graph with diameter at most 6 is geodetic.*

In the next theorem we consider bipartite graphs with diameter 7.

Theorem 23. *Let $G = (V, E)$ be a bipartite graph with $diam(G) = 7$. Then, $I[Ct(G)] = V$.*

Proof. Let $u_0 \in V \setminus Ct(G)$. By Theorem 21, we may assume that $ecc(u_0) = 4$. Then, there exists a path $P = u_0, u_1, \cdots u_t$ between u_0 and $u_t \in Ct(G)$ such that $ecc(u_0) < ecc(u_1) < \cdots < ecc(u_t)$. Let z be an eccentric vertex of u_t. By Lemma 8, there exists a shortest path $\mathcal{P}'$ between u_t and z through u_0. We need to consider three different cases: $ecc(u_t) = diam(G)$; $ecc(u_t) = diam(G) - 1$; and $ecc(u_t) = diam(G) - 2$.

Suppose that $ecc(u_t) = diam(G)$. Then, $u_t, z \in Ct(G)$ and $u_0 \in I[Ct(G)]$.

Suppose that $ecc(u_t) = diam(G) - 1 = 6$. If $z \in Ct(G)$, then $u_0 \in I[Ct(G)]$. If $z \notin Ct(G)$, then there exists a vertex $z' \in Ct(G)$ such that $\{z, z'\} \in E$ and $ecc(z') = diam(G)$. By Remark 18, vertex z' belongs to $N^5(u_t)$ or $N^7(u_t)$. Since $ecc(u_t) = 6$, we have that $z' \in N^5(u_t)$. Since $d(u_0, u_t) = 2$ and u_0 lies on a geodesic between z and u_t, we conclude that $d(u_0, z) = 4$. Consequently, by Remark 18 and since $ecc(u_0) = 4$, we conclude that $d(u_0, z') = 3$. Hence, there exists a geodesic between u_t and z' through u_0, which implies that $u_0 \in I[Ct(G)]$.

Suppose that $ecc(u_t) = diam(G) - 2 = 5$. Let z be an eccentric vertex of u_t. If $z \in Ct(G)$, then $u_0 \in I[Ct(G)]$. Hence we may assume that $z \notin Ct(G)$. Consequently, we need to consider two possible cases, either $ecc(z) = 6$ or $ecc(z) = 5$.

Case 1: $ecc(z) = 6$.

Since $z \notin Ct(G)$ and $ecc(z) = 6$, there exists a vertex $z' \in Ct(G)$ such that $\{z, z'\} \in E$ and $ecc(z') = diam(G)$. By an analogous argument of that used in the case where $ecc(u_t) = 6$, we have $z' \in N^4(u_t)$. Since z is an eccentric vertex of u_0, we know that $d(u_0, z) = 4$. Consequently, using that G is bipartite and Remark 18, we conclude that $d(u_0, z') = 3$. Hence, there exists a geodesic between u_t and z' through u_0, which implies that $u_0 \in I[Ct(G)]$.

Case 2: $ecc(z) = 5$.

By a similar argument of that used in previous case, there exists a vertex $z' \in N^4(u_t)$ such that $\{z, z'\} \in E$, $ecc(z') = diam(G) - 1 = 6$ and $d(u_0, z') = 3$. If $z' \in Ct(G)$, then $u_0 \in I[Ct(G)]$, by Lemma 10. Suppose that $z' \notin Ct(G)$. Then, there exists a vertex z'' such that $\{z, z''\} \in E$ and $ecc(z'') = diam(G) = 7$. Since G is bipartite, either $z'' \in N^5(u_t)$ or $z'' \in N^3(u_t)$. Following, we analyze these cases.

Case 2-1: $z'' \in N^5(u_t)$.

Since $ecc(u_t) = 5$, vertex z'' is an eccentric vertex of u_t. Therefore, by Lemma 8, $u_0 \in I[Ct(G)]$.

Case 2-2: $z'' \in N^3(u_t)$.

Let z''' be an eccentric vertex of z'', which implies that $d(z'', z''') = 7$. By Remark 7, $z''' \in Ct(G)$. Since G is bipartite and $d(u_t, z'') = 3$ is odd, the vertices u_t and z'' are in different parts of the partition of V. The same is true for z'' and z'''. Hence, u_t and z''' are in the same part of the partition of V, which means that $d(u_t, z''')$ is even. Since $ecc(u_t) = 5$, then $z''' \in N^4(u_t)$ or $z''' \in N^2(u_t)$.

Case 2-2-1: $z''' \in N^4(u_t)$.

Since $u_0 \in N(u_t)$ and G is bipartite, $d(u_0, z''') = 3$. Clearly, $u_0 \in I[u_t, z'''] \subseteq I[Ct(G)]$.

Case 2-2-2: $z''' \in N^2(u_t)$.

Clearly, for some vertex $w \in V$ such that $ecc(w) = 6$, there exists a geodesic $\mathcal{P} = u_0, u_t, w, z'''$ such that $ecc(u_0) < ecc(u_1) < ecc(w) < ecc(z''') = diam(G)$. Let x be an eccentric vertex of z'''. By Remark 7, $ecc(x) = 7$, which means that $x \in Ct(G)$. By Lemma 8, we conclude that u_0 lies on a geodesic between z''' and x.

□

5. Bipartite graphs with diameter > 7

In [3] the authors have mentioned that it is unknown if there exists a bipartite graph such that the contour is not geodetic. The same question has been asked for parity graphs. A graph is a *parity* graph if any two induced paths, joining the same pair of vertices, have lengths of the same parity (odd or even). In this section we answer these questions. In Figure 3, we show a graph G together with the eccentricity of every vertex of G.

Proposition 24. *The graph of Figure 3 is bipartite and its contour is not geodetic.*

Proof. In Figure 3, the vertices marked with a square are contour vertices of G, but v is not in $I[Ct(G)]$. $\square$

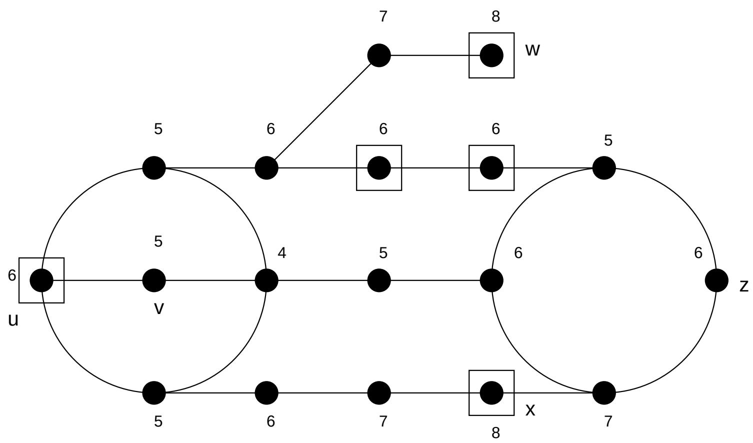

Figure 3: Bipartite graph G, with diameter 8, whose contour is not geodetic.

Furthermore, we extend the example of Figure 3 to show that, for any even $k \geq 8$, there exists a graph $H = (V, E)$ such that $diam(H) = k$ and $I[Ct(H)] \neq V$. The graph H in Figure 4 is a bipartite graph obtained from graph G (see Figure 3). This graph has diameter $8 + 2s$ and a vertex $v \notin I[Ct(H)]$.

Next, we exhibit, in Figure 5, a graph $J = (V, E)$ constructed from graph H, of Figure 4, deleting the vertex w. The graph J is bipartite, $Ct(J)$ is not geodetic and $diam(J) = 7 + 2s$, for $s \geq 1$. To verify that $I[Ct(J)] \neq V$, it is sufficient to see that $Ct(J) = \{u, w', x\}$ and $v \notin I[Ct(J)]$.

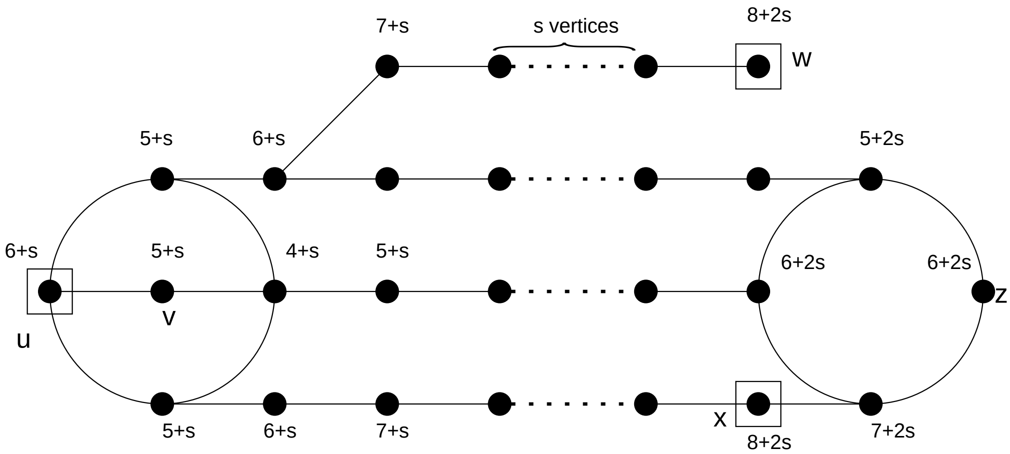

Figure 4: Bipartite graph H, with diameter $8+2s$, whose contour is not geodetic.

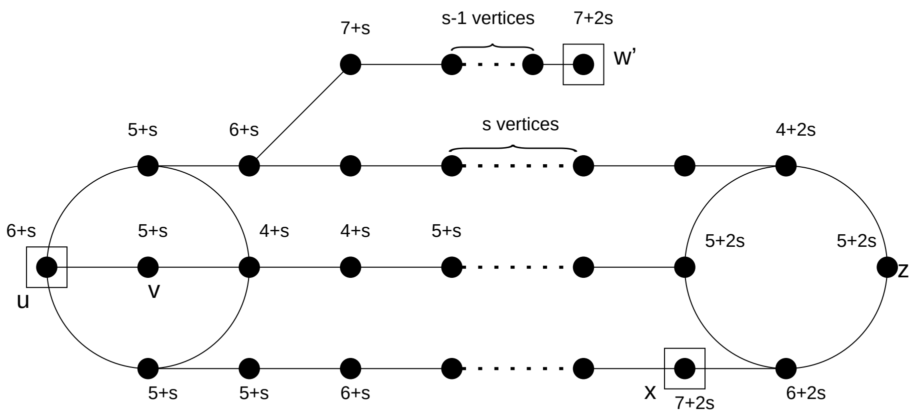

Figure 5: Bipartite graph J, with diameter $7+2s$, whose contour is not geodetic.

The above figures show that the contour of parity graphs is not geodetic since parity graphs is a superclass of bipartite graphs.

By Corollary 22, Theorem 23, Proposition 24 and Figures 4 and 5 we conclude our second main result.

Theorem 25. *Let G be a bipartite graph. If diam$(G) \leq 7$ then the contour of G is geodetic, otherwise for any $k > 7$, there is a bipartite graph having diameter k and whose contour is not geodetic.*

6. Conclusion

We have considered the problem of determining whether the contour of a graph is a geodetic set. In [3], the authors showed that there exists a graph for which the problem has answer NO. They left the problem open for 3 classes: cochordal; bipartite; and parity graphs. We have solved the problem for each of these classes. We have investigated the relation between the contour and the diameter of a graph. We have proved that if G is a graph such that $diam(G) \leq 4$, then $Ct(G)$ is geodetic, and shown that for every $k \geq 5$, there exists a graph G, with diameter k, such that the contour is not geodetic. We have proved that if G is a bipartite graph such that $diam(G) \leq 7$, then $Ct(G)$ is geodetic; and shown that for every $k \geq 8$, there exists a bipartite graph G with $diam(G) = k$ such that the contour is not geodetic.

We leave as one open question whether the contour of a bridged graph is geodetic. Finally, we mention the problem of characterizing the graphs for which the contour is geodetic.

References

[1] D. Artigas, S. Dantas, M.C. Dourado, J.L. Szwarcfiter, Partitioning a graph into convex sets, Discrete Mathematics 311 (2011) 1968–1977.

[2] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, 2008.

[3] J. Cáceres, C. Hernando, M. Mora, I.M. Pelayo, M. Puertas, C. Seara, Geodeticity of the contour of chordal graphs, Discrete Applied Mathematics 156 (2008) 1132 – 1142.

[4] J. Cáceres, M.C. Hernando, M. Mora, I.M. Pelayo, M.L. Puertas, C. Seara, On geodetic sets formed by boundary vertices, Discrete Mathematics 306 (2006) 188–198.

[5] J. Cáceres, A. Márquez, O.R. Oellermann, M.L. Puertas, Rebuilding convex sets in graphs, Discrete Mathematics 297 (2005) 26–37.

[6] M. Changat, J. Mathew, On triangle path convexity in graphs, Discrete Mathematics 206 (1999) 91 – 95.

[7] G. Chartrand, D. Erwin, G. Johns, P. Zhang, Boundary vertices in graphs, Discrete Mathematics 263 (2003) 25–34.

[8] V.D. Chepoi, V.P. Soltan, Conditions for invariance of set diameters under d-convexification in a graph, Cybernetics and Systems Analysis 19 (1983) 750–756.

[9] M.C. Dourado, J.G. Gimbel, J. Kratochvíl, F. Protti, J.L. Szwarcfiter, On the computation of the hull number of a graph, Discrete Mathematics 309 (2009) 5668–5674.

[10] M.C. Dourado, F. Protti, D. Rautenbach, J.L. Szwarcfiter, Some remarks on the geodetic number of a graph, Discrete Mathematics 310 (2010) 832–837.

[11] F.F. Dragan, F. Nicolai, A. Brandstädt, Convexity and HHD-free graphs, SIAM Journal on Discrete Mathematics 12 (1999) 119–135.

[12] P. Duchet, Convex sets in graphs, ii. minimal path convexity, Journal of Combinatorial Theory, Series B 44 (1988) 307–316.

[13] P.H. Edelman, R.E. Jamison, The theory of convex geometries, Geometriae Dedicata 19 (1985) 247–270.

[14] P. Erdős, E. Fried, A. Hajnal, E.C. Milner, Some remarks on simple tournments, Algebra Universalis 2 (1972) 238–245.

[15] M. Farber, Bridged graphs and geodesic convexity, Discrete Mathematics 66 (1987) 249–257.

[16] M. Farber, R.E. Jamison, Convexity in graphs and hypergraphs, SIAM J. Algebraic Discrete Methods 7 (1986) 433–444.

[17] F. Harary, J. Nieminen, Convexity in graphs, Journal of Differential Geometry 16 (1981) 185–190.

[18] M.A. Henning, M.H. Nielsen, O.R. Oellermann, Local steiner convexity, European Journal of Combinatorics 30 (2009) 1186–1193.

[19] C. Heuberger, On planarity and colorability of circulant graph, Discrete Mathematics 268 (2003) 153–169.

[20] F.M. Malvestuto, M. Mezzini, M. Moscarini, Computing simple-path convex hulls in hypergraphs, Information Processing Letters 111 (2011) 231–234.

[21] J.W. Moon, Embedding tournaments in simple tournaments, Discrete Mathematics 2 (1972) 389–395.

[22] M.J.L. Van de Vel, Theory of Convex Structures, North-Holland, Amsterdam, 1993.

[23] J.C. Varlet, Convexity in tournaments, Bull. Société Royale des Sciences de Liége 45 (1976) 570–586.