LECTURE NOTES ON
INTERMEDIATE FLUID MECHANICS

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Contents

Chapter 1

Mathematical preliminaries

1.1 Vectors

This is a short introduction to the algebra and calculus of scalar, vector and tensor fields. The scalar, vector or tensor quantity is considered to be a function of position $\mathbf{r}$ in three-dimensional space.

A scalar field is one in which a scalar $\phi(\mathbf{r})$ varies with position. In most areas of mechanics, including fluid mechanics, it is also common to talk of vectors which are quantities which have both magnitude as well as direction. Examples are velocities, accelerations and forces. Here we will represent vectors using bold-faced letters such as $\mathbf{a}$. Other commonly used notations are $\tilde{a}$, $\underline{a}$, and $\overrightarrow{a}$. A unit vector $\mathbf{n}$ is sometimes indicated as $\hat{n}$. If a vector $\mathbf{f}$ is a function of where it is evaluated, i.e. $\mathbf{f}(\mathbf{r})$ where $\mathbf{r}$ is the position vector of the location, $\mathbf{f}$ is referred to as a vector field.

Vectors in three-dimensional physical space have three components. The Cartesian components a_1, a_2 and a_3 of the vector $\mathbf{a}$ can be indicated explicitly in one of various ways:

$$\begin{aligned} \mathbf{a} &= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \\ &= [a_1 \; a_2 \; a_3]^T \\ &= (a_1, a_2, a_3) \\ &= a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \end{aligned}$$

Unit vectors in the x_1, x_2, and x_3 directions are $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. It is also common to use x, y, z for x_1, x_2, x_3 and $\mathbf{i}$,$\mathbf{j}$,$\mathbf{k}$ for $\mathbf{e}_1$,$\mathbf{e}_2$,$\mathbf{e}_3$.

1.1.1 Dot product

The scalar (also dot or inner) product of $\mathbf{a}$ and $\mathbf{b}$ is written as $\mathbf{a}\cdot\mathbf{b}$. and is defined by

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 \tag{1.1}$$

$$= \sum_{i=1}^{3} a_ib_i \tag{1.2}$$

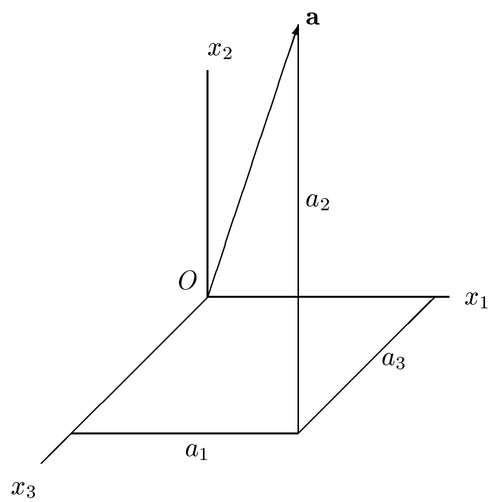


Figure 1.1: Vector **a** with its Cartesian components a_1, a_2, a_3.

The dot product is also equivalent to

$$\mathbf{a} \cdot \mathbf{b} = ab \cos\theta \tag{1.3}$$

where a and b are the magnitudes of **a** and **b** respectively, and θ is the angle between them. The vectors **a** and **b** are *orthogonal* if $\mathbf{a} \cdot \mathbf{b} = 0$.

1.1.2 Cross product

The cross (or vector) product **c** of **a** and **b** is written as $\mathbf{c} = \mathbf{a} \times \mathbf{b}$. The magnitude of **c** is $ab \sin\theta$; it is normal to both **a** and **b** in the direction given by the right-hand rule. The cross product is also equivalent to

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{1.4}$$

1.1.3 Line integrals

A line integral is of the form

$$I = \int_{\mathcal{C}} \mathbf{f} \cdot d\mathbf{l} \tag{1.5}$$

where **f** is a vector field, and $d\mathbf{l}$ is an element of curve $\mathcal{C}$.

In general the value of a line integral depends on the path. If, however, we take $\mathbf{f} = \nabla\phi$, where ϕ is a scalar field (see below for the gradient operator), then the integral I is independent of path. **f** is then called a *conservative* field, and ϕ is its *potential*.

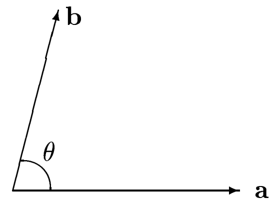


Figure 1.2: Arbitrary vectors **a** and **b**

1.1.4 Surface integrals

A surface integral is of the form

$$I = \int_{\mathcal{A}} \mathbf{f} \cdot \mathbf{n} \, d\mathcal{A} \tag{1.6}$$

where **f** is a vector field, A is an open or closed surface, dA is an element of this surface, and **n** is a unit vector normal to this element.

1.1.5 Differential operators

Surface integrals can be used for coordinate-independent definitions of the gradient, divergence and curl operators. Thus

$$\nabla\phi = \lim_{V\to 0} \frac{1}{V} \int_{\mathcal{A}} \mathbf{n}\phi \, d\mathcal{A} \tag{1.7}$$

$$\nabla\cdot\mathbf{f} = \lim_{V\to 0} \frac{1}{V} \int_{\mathcal{A}} \mathbf{n}\cdot\mathbf{f} \, d\mathcal{A} \tag{1.8}$$

$$\nabla\times\mathbf{f} = \lim_{V\to 0} \frac{1}{V} \int_{\mathcal{A}} \mathbf{n}\times\mathbf{f} \, d\mathcal{A} \tag{1.9}$$

where $\phi(\mathbf{r})$ is a scalar field, and $\mathbf{f}(\mathbf{r})$ is a vector field. V is the region enclosed within a closed surface S, and **n** is the unit normal to an element of the surface dA. The operators ∇, $\nabla\cdot$ and $\nabla\times$ are to be read as gradient of, divergence of, and curl of, respectively. The notation grad, div, and curl is also used for these operators.

A related operator is the Laplacian, ∇^2, where

$$\nabla^2 = \nabla\cdot\nabla \tag{1.10}$$

For a vector, the identity (1.21) below is often used. The Laplacian should not be confused with the operator $\nabla\nabla\cdot$ which is also possible for a vector, but is a different operator.

The directional derivative of $\phi(\mathbf{r})$ along a unit vector **a** is

$$\frac{\partial\phi}{\partial a} = \nabla\phi\cdot\mathbf{a} \tag{1.11}$$

where a is the coordinate in that direction. In the special case of **a** being any tangent to the surface $\phi =$ constant, we have $\partial\phi/\partial a = 0$. Thus $\nabla\phi$ is orthogonal to **a** and must be normal to the surface.

1.1.6 Identities

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b}) \quad (1.12)$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \quad (1.13)$$

$$\nabla\times(\nabla\phi) = \mathbf{0} \quad (1.14)$$

$$\nabla\cdot(\nabla\times\mathbf{f}) = 0 \quad (1.15)$$

$$\nabla\cdot(\phi\mathbf{f}) = \phi\nabla\cdot\mathbf{f}+\nabla\phi\cdot\mathbf{f} \quad (1.16)$$

$$\nabla\times(\phi\mathbf{f}) = \phi\nabla\times\mathbf{f}+\nabla\phi\times\mathbf{f} \quad (1.17)$$

$$\nabla\cdot(\mathbf{f}\times\mathbf{g}) = \mathbf{g}\cdot(\nabla\times\mathbf{f})-\mathbf{f}\cdot(\nabla\times\mathbf{g}) \quad (1.18)$$

$$\nabla\times(\mathbf{f}\times\mathbf{g}) = (\mathbf{g}\cdot\nabla)\mathbf{f}-(\mathbf{f}\cdot\nabla)\mathbf{g}+\mathbf{f}(\nabla\cdot\mathbf{g})-\mathbf{g}(\nabla\cdot\mathbf{f}) \quad (1.19)$$

$$\nabla(\mathbf{f}\cdot\mathbf{g}) = (\mathbf{f}\cdot\nabla)\mathbf{g}+(\mathbf{g}\cdot\nabla)\mathbf{f}+\mathbf{f}\times(\nabla\times\mathbf{g})+\mathbf{g}\times(\nabla\times\mathbf{f}) \quad (1.20)$$

$$\nabla\times(\nabla\times\mathbf{f}) = \nabla(\nabla\cdot\mathbf{f})-\nabla^2\mathbf{f} \quad (1.21)$$

$$\frac{d}{dt}(\mathbf{f}\cdot\mathbf{g}) = \mathbf{f}\cdot\frac{d\mathbf{g}}{dt}+\frac{d\mathbf{f}}{dt}\cdot\mathbf{g} \quad (1.22)$$

$$\frac{d}{dt}(\mathbf{f}\times\mathbf{g}) = \mathbf{f}\times\frac{d\mathbf{g}}{dt}+\frac{d\mathbf{f}}{dt}\times\mathbf{g} \quad (1.23)$$

Example 1.1

Prove by expanding into components that

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\begin{aligned}
\text{LHS} &= \begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\cdot\begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ c_1 & c_2 & c_3\\ d_1 & d_2 & d_3\end{vmatrix}\\
&= (a_2b_3-a_3b_2)(c_2d_3-c_3d_2)+(a_3b_1-a_1b_3)(c_3d_1-c_1d_3)+(a_1b_2-a_2b_1)(c_1d_2-c_2d_1)\\
&= a_2b_3c_2d_3+a_3b_2c_3d_2-a_2b_3c_3d_2-a_3b_2c_2d_3+a_3b_1c_3d_1+a_1b_3c_1d_3\\
&\quad -a_3b_1c_1d_3-a_1b_3c_3d_1+a_1b_2c_1d_2+a_2b_1c_2d_1-a_1b_2c_2d_1-a_2b_1c_1d_2\\
\text{RHS} &= (a_1c_1+a_2c_2+a_3c_3)(b_1d_1+b_2d_2+b_3d_3)-(a_1d_1+a_2d_2+a_3d_3)(b_1c_1+b_2c_2+b_3c_3)\\
&= (a_1c_1b_1d_1+a_1c_1b_2d_2+a_1c_1b_3d_3+a_2c_2b_1d_1+a_2c_2b_2d_2+a_2c_2b_3d_3\\
&\quad +a_3c_3b_1d_1+a_3c_3b_2d_2+a_3c_3b_3d_3)\\
&\quad -(a_1d_1b_1c_1+a_1d_1b_2c_2+a_1d_1b_3c_3+a_2d_2b_1c_1+a_2d_2b_2c_2+a_2d_2b_3c_3\\
&\quad +a_3d_3b_1c_1+a_3d_3b_2c_2+a_3d_3b_3c_3)\\
&= \text{LHS}
\end{aligned}$$

1.2 Special theorems

1.2.1 Green's theorem

Let $\mathbf{f}=f_x(x,y)\,\mathbf{i}+f_y(x,y)\,\mathbf{j}$ be a vector field, $\mathcal{C}$ a closed curve, and $\mathcal{A}$ the region enclosed by $\mathcal{C}$, all in the x-y plane. Then

$$\oint_{\mathcal{C}}\mathbf{f}\cdot d\mathbf{l} = \iint_{\mathcal{A}}\left(\frac{\partial f_y}{\partial x}-\frac{\partial f_x}{\partial y}\right)\,dx\;dy \quad (1.24)$$

1.2.2 Gauss's theorem

Let S be a closed surface, and V the region enclosed within it, then

$$\int_{\mathcal{A}} \mathbf{f} \cdot \mathbf{n} \, d\mathcal{A} = \int_{\mathcal{V}} \nabla \cdot \mathbf{f} \, d\mathcal{V} \tag{1.25}$$

where $d\mathcal{V}$ an element of volume, $d\mathcal{A}$ is an element of the surface, and $\mathbf{n}$ is the outward unit normal to it.

1.2.3 Green's identities

Applying Gauss's theorem to the vector $\mathbf{f} = \phi\nabla\psi$, we get

$$\int_{\mathcal{A}} \phi\nabla\psi \cdot \mathbf{n} \, d\mathcal{A} = \int_{\mathcal{V}} \nabla \cdot (\phi\nabla\psi) \, d\mathcal{V}$$

From this we get Green's first identity

$$\int_{\mathcal{A}} \phi\nabla\psi \cdot \mathbf{n} \, d\mathcal{A} = \int_{\mathcal{V}} (\phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi) \, d\mathcal{V} \tag{1.26}$$

Interchanging ϕ and ψ in the above and subtracting, we get Green's second identity

$$\int_{\mathcal{A}} (\phi\nabla\psi - \psi\nabla\phi) \cdot \mathbf{n} \, d\mathcal{A} = \int_{\mathcal{V}} (\phi\nabla^2\psi - \psi\nabla^2\phi) \, d\mathcal{V} \tag{1.27}$$

1.2.4 Stokes's theorem

Let S be an open surface, and the curve $\mathcal{C}$ its boundary. Then

$$\int_{\mathcal{A}} (\nabla \times \mathbf{f}) \cdot \mathbf{n} \, d\mathcal{A} = \oint_{\mathcal{C}} \mathbf{f} \cdot d\mathbf{l} \tag{1.28}$$

where $\mathbf{n}$ is the unit vector normal to the element dA, and $d\mathbf{l}$ an element of curve $\mathcal{C}$.

1.3 Cartesian coordinates

Consider the element of volume in Cartesian coordinates shown in Figure 1.3. The differential operations in this coordinate system can be deduced from the definitions. The ∇ operator is seen to have the Cartesian definition

$$\mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3} \tag{1.29}$$

1.3.1 Gradient of a scalar

We take the reference value of ϕ to be at the center of the element P. At the center of the two faces which are a distance $\pm\, dx_1/2$ away from P in the x_1-direction, it is $(\phi \pm \partial\phi/\partial x_1)dx_1/2$. Writing

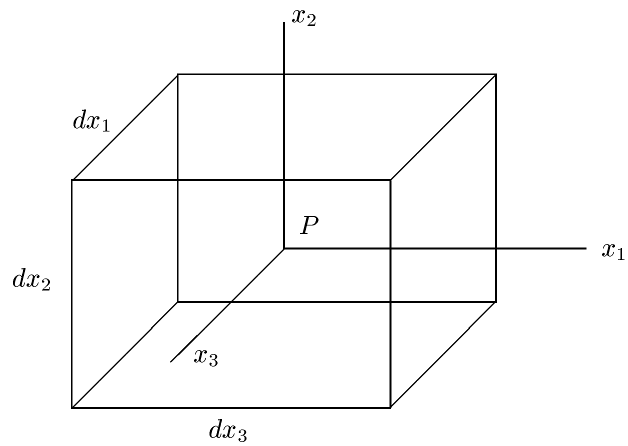


Figure 1.3: Element of volume in Cartesian coordinates

$V = dx_1\ dx_2\ dx_3$, equation (1.7) gives

$$\begin{aligned} \nabla\phi &= \lim_{V\to 0}\frac{1}{V}\Big[(\phi + \frac{\partial\phi}{\partial x_1}\frac{dx_1}{2})\mathbf{e}_1\ dx_2\ dx_3 - (\phi - \frac{\partial\phi}{\partial x_1}\frac{dx_1}{2})\mathbf{e}_1\ dx_2\ dx_3 \\ &\quad + \text{similar terms from the } x_2 \text{ and } x_3 \text{ faces}\Big] \\ &= \frac{\partial\phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial\phi}{\partial x_3}\mathbf{e}_3 \end{aligned} \tag{1.30}$$

1.3.2 Divergence of a vector

Equation (1.8) becomes

$$\begin{aligned} \nabla\cdot\mathbf{f} &= \lim_{V\to 0}\frac{1}{V}\Big[(f_1 + \frac{\partial f_1}{\partial x_1}\frac{dx_1}{2})\ dx_2\ dx_3 - (f_1 - \frac{\partial f_1}{\partial x_1}\frac{dx_1}{2})\ dx_2\ dx_3 \\ &\quad + \text{similar terms from the } x_2 \text{ and } x_3 \text{ faces}\Big] \\ &= \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \end{aligned} \tag{1.31}$$

1.3.3 Curl of a vector

From equation (1.9), we have

$$\nabla\times\mathbf{f} = \lim_{V\to 0}\frac{1}{V}\Big[(f_2 + \frac{\partial f_2}{\partial x_1}\frac{dx_1}{2})\mathbf{e}_3\ dx_2\ dx_3 - (f_3 + \frac{\partial f_3}{\partial x_1}\frac{dx_1}{2})\mathbf{e}_2\ dx_2\ dx_3$$

$$-(f_2 - \frac{\partial f_2}{\partial x_1}\frac{dx_1}{2})\mathbf{e}_3\ dx_2\ dx_3 + (f_3 - \frac{\partial f_3}{\partial x_1}\frac{dx_1}{2})\mathbf{e}_2\ dx_2\ dx_3$$

$$\left. + \text{ similar terms from the } x_2 \text{ and } x_3 \text{ faces}\right]$$

$$= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ f_1 & f_2 & f_3 \end{vmatrix} \tag{1.32}$$

1.3.4 Laplacian

The Laplacian of a scalar is

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} \tag{1.33}$$

and that of a vector

$$\nabla^2 \mathbf{f} = \frac{\partial^2 \mathbf{f}}{\partial x_1^2} + \frac{\partial^2 \mathbf{f}}{\partial x_2^2} + \frac{\partial^2 \mathbf{f}}{\partial x_3^2} \tag{1.34}$$

is similar.

1.4 Orthogonal curvilinear coordinates

Let the orthogonal curvilinear coordinates of a point be (q_1, q_2, q_3), where $q_i = q_i(x_1, x_2, x_3)$, the x_is being the Cartesian coordinates of the point. We can define scale factors h_1, h_2, h_3 such that

$$h_i = \sqrt{\left(\frac{\partial x_1}{\partial q_i}\right)^2 + \left(\frac{\partial x_2}{\partial q_i}\right)^2 + \left(\frac{\partial x_3}{\partial q_i}\right)^2} \tag{1.35}$$

The differential operators can be written using these scale factors.

Let $\phi(\mathbf{r})$ be a scalar field and $\mathbf{f}(\mathbf{r})$ a vector field, both being functions of the position vector $\mathbf{r}$. It turns out that:

$$\nabla \phi = \frac{1}{h_1}\frac{\partial \phi}{\partial q_1}\mathbf{e}_{q_1} + \frac{1}{h_2}\frac{\partial \phi}{\partial q_2}\mathbf{e}_{q_2} + \frac{1}{h_3}\frac{\partial \phi}{\partial q_3}\mathbf{e}_{q_3} \tag{1.36}$$

$$\nabla \cdot \mathbf{f} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(f_1 h_2 h_3) + \frac{\partial}{\partial q_2}(f_2 h_3 h_1) + \frac{\partial}{\partial q_3}(f_3 h_1 h_2)\right] \tag{1.37}$$

$$\nabla \times \mathbf{f} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \mathbf{e}_{q_1} & h_2 \mathbf{e}_{q_2} & h_3 \mathbf{e}_{q_3} \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ f_1 h_1 & f_2 h_2 & f_3 h_3 \end{vmatrix} \tag{1.38}$$

$$\nabla^2 \phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \phi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \phi}{\partial q_3}\right)\right] \tag{1.39}$$

where f_1, f_2, f_3 are the components of $\mathbf{f}$ in the q_1, q_2, q_3 directions respectively, and $\mathbf{e}_{q_1}, \mathbf{e}_{q_2}, \mathbf{e}_{q_3}$ are the unit vectors in these directions.

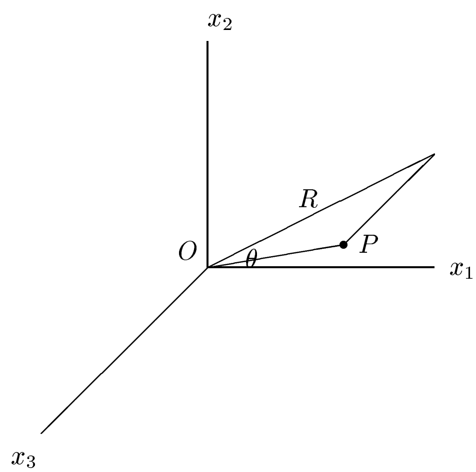


Figure 1.4: Cylindrical polar coordinates (R, θ, z) of point P.

1.4.1 Cylindrical polar coordinates

Coordinates:

$$\begin{aligned} q_1 &= R \\ q_2 &= \theta \\ q_3 &= z \end{aligned}$$

Relation to Cartesian coordinates:

$$\begin{aligned} x_1 &= R\cos\theta \\ x_2 &= R\sin\theta \\ x_3 &= z \end{aligned}$$

Scale factors:

$$\begin{aligned} h_1 &= 1 \\ h_2 &= R \\ h_3 &= 1 \end{aligned}$$

1.4.2 Spherical polar coordinates

Coordinates:

$$q_1 = r$$
$$q_2 = \theta$$
$$q_3 = \varphi$$

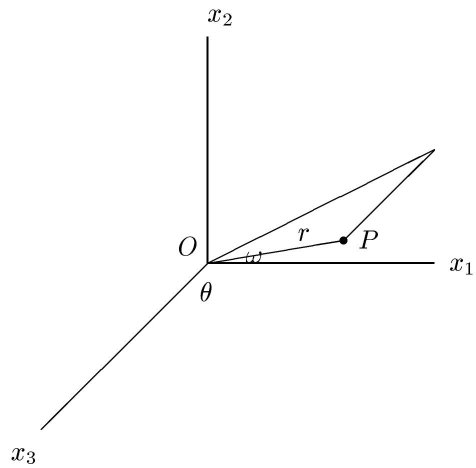


Figure 1.5: Spherical polar coordinates (r, θ, φ) of point P.

Relation to Cartesian coordinates:

$$\begin{aligned} x_1 &= r \sin\theta \cos\varphi \\ x_2 &= r \sin\theta \sin\varphi \\ x_3 &= r \cos\theta \end{aligned}$$

Scale factors:

$$\begin{aligned} h_1 &= 1 \\ h_2 &= r \\ h_3 &= r \sin\theta \end{aligned}$$

Example 1.2

Find expressions for the gradient, divergence, and curl in cylindrical coordinates (r, θ, z) where

$$\begin{aligned} x_1 &= r \cos\theta \\ x_2 &= r \sin\theta \\ x_3 &= z \end{aligned}$$

The 1,2 and 3 directions are associated with r, θ, and z, respectively. From equation (1.35) the scale factors are

$$\begin{aligned} h_r &= \sqrt{(\frac{\partial x_1}{\partial r})^2 + (\frac{\partial x_2}{\partial r})^2 + (\frac{\partial x_3}{\partial r})^2} \\ &= \sqrt{\cos^2\theta + \sin^2\theta} \\ &= 1 \\ h_\theta &= \sqrt{(\frac{\partial x_1}{\partial \theta})^2 + (\frac{\partial x_2}{\partial \theta})^2 + (\frac{\partial x_3}{\partial \theta})^2} \end{aligned}$$

$$
\begin{aligned}
&= \sqrt{r^2 \sin^2\theta + r^2\cos^2\theta} \\
&= r \\
h_z &= \sqrt{(\frac{\partial x_1}{\partial z})^2 + (\frac{\partial x_2}{\partial z})^2 + (\frac{\partial x_3}{\partial z})^2} \\
&= 1
\end{aligned}
$$

so that

$$
\begin{aligned}
\nabla\phi &= \frac{\partial \phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\mathbf{e}_\theta + \frac{\partial \phi}{\partial z}\mathbf{e}_z \\
\nabla\cdot\mathbf{f} &= \frac{1}{r}[\frac{\partial}{\partial r}(f_r r) + \frac{\partial}{\partial \theta}(f_\theta) + \frac{\partial}{\partial z}(f_z r)] \\
\nabla\times\mathbf{f} &= \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ f_r & f_\theta r & f_z \end{vmatrix}
\end{aligned}
$$

1.5 Index notation for Cartesian coordinates

Some of the vector relations can be written in a compact form by using the index notation. Let x_1, x_2, x_3 represent the three coordinate directions and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ the unit vectors in those directions. Then a vector $\mathbf{a}$ may be written as

$$\mathbf{a} = \sum_{i=1}^{3} a_i \mathbf{e}_i \tag{1.40}$$

where a_1, a_2, and a_3 are the three Cartesian components of $\mathbf{a}$. According to the Einstein summation convention, repetition of indices indicates summation, so that the Σ symbol can be left out. One has to take care that an index does not appear more than twice in a given product. Thus we can simply write

$$\mathbf{a} = a_i \mathbf{e}_i \tag{1.41}$$

It is also common to write $\mathbf{a} = a_i$, the single free index on the right side indicating that an $\mathbf{e}_i$ is assumed.

Two additional symbols are needed for later use. They are the Kronecker delta

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{1.42}$$

and the substitution symbol (or Levi-Civita density)

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if indices are in cyclical order 1,2,3,1,2,}\cdots \\ -1 & \text{if indices are not in cyclical order} \\ 0 & \text{if two or more indices are the same} \end{cases} \tag{1.43}$$

The identity

$$\epsilon_{ijk}\epsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{kl} \tag{1.44}$$

relates the two. The following identities are also easily shown:

$$\delta_{ii} = 3 \tag{1.45}$$
$$\delta_{ij} = \delta_{ji} \tag{1.46}$$
$$\delta_{ij}\delta_{jk} = \delta_{ik} \tag{1.47}$$
$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{1.48}$$
$$\epsilon_{ijk}\epsilon_{ljk} = 2\delta_{il} \tag{1.49}$$
$$\epsilon_{ijk}\epsilon_{ijk} = 6 \tag{1.50}$$

In this notation the scalar and vectors products can be written as

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i \tag{1.51}$$

and

$$\mathbf{a} \times \mathbf{b} = \epsilon_{ijk} a_i b_j \mathbf{e}_k \tag{1.52}$$

Furthermore

$$\nabla\phi = \frac{\partial\phi}{\partial x_i}\mathbf{e}_i \tag{1.53}$$
$$\nabla \cdot \mathbf{f} = \frac{\partial f_i}{\partial x_i} \tag{1.54}$$
$$\nabla \times \mathbf{f} = \epsilon_{ijk}\frac{\partial f_j}{\partial x_i}\mathbf{e}_k \tag{1.55}$$
$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x_i \partial x_i} \tag{1.56}$$
$$\nabla^2 f_j = \frac{\partial^2 f_j}{\partial x_i \partial x_i} \tag{1.57}$$

Gauss's theorem, equation (1.25), can be written as

$$\int_{\mathcal{A}} f_i n_i \; d\mathcal{A} = \int_{\mathcal{V}} \frac{\partial f_i}{\partial x_i} \; d\mathcal{V} \tag{1.58}$$

The index notation sometimes can greatly simplify the demonstration of vector relations, as in the example below.

Example 1.3

Prove using the index notation that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$\begin{aligned} \text{LHS} &= (\epsilon_{ijk} a_i b_j)(\epsilon_{lmk} c_l d_m) \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_i b_j c_l d_m \\ &= a_l b_m c_l d_m - a_m b_l c_l d_m \\ &= \text{RHS} \end{aligned}$$

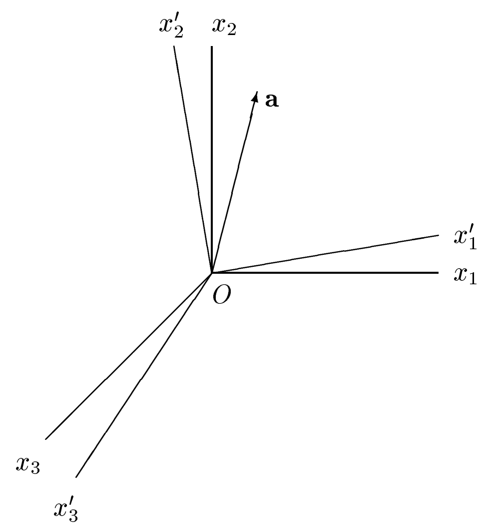


Figure 1.6: Cartesian coordinate systems $\mathcal{S}$ and $\mathcal{S}'$

1.6 Cartesian tensors

Consider the two right-handed Cartesian coordinate systems, S with coordinates x_1, x_2, x_3 and S' with x_1', x_2', x_3', shown in Figure 1.6. A scalar ϕ at the common origin O has the same value for either of the two systems. But now let us look at a vector **a** represented by the arrow. It has components a_1, a_2, a_3 in S and a_1', a_2', a_3' in S'. It is easy to show that the components are related by

$$a_j = A_{ij} a_i' \tag{1.59}$$

where $A_{ij} = \cos(x_i', x_j)$ is a transformation array. We may adopt the definition that a quantity with components that are transformed in this manner is a vector. The inverse transformation is

$$a_j' = A_{ji} a_i \tag{1.60}$$

The transformation array A_{ij} satisfies

$$A_{ki} A_{kj} = \delta_{ij} \tag{1.61}$$

This may shown from $x_i = A_{ki} x_k'$ and $x_k' = A_{kj} x_j$, differentiating the first with respect to x_j and then substituting the second. The matrix with elements A_{ij} is orthogonal.

A quantity **T** with components T_{ij} is called a tensor of second order if the components transform under the relations

$$T_{ij} = A_{ki} A_{lj} T_{kl}' \tag{1.62}$$
$$T_{ij}' = A_{ik} A_{jl} T_{kl} \tag{1.63}$$

Matrices are often used to represent vectors and tensors; a vector may be represented by a 3×1 column vector, and a tensor by a 3×3 matrix. For example the Kronecker delta, δ_{ij}, is the identity matrix **I**. If a matrix **A** is A_{ij}, its transpose $\mathbf{A}^T$ is A_{ji}.

The tensor product **ab** of two vectors **a** and **b** is the tensor $a_i b_j$. The multiplication **Aa** of a 3×3 matrix **A** and a 3×1 column vector **a** is a vector that is represented in index notation as $A_{ij} a_j$. Similerly the product **AB** of two 3×3 matrices **A** and **B** is $A_{ij} B_{jk}$.

Equation (1.61) is equivalent to

$$\mathbf{A}^T \mathbf{A} = \mathbf{I} \tag{1.64}$$

which means that **A** is an orthogonal matrix with

$$\mathbf{A}^{-1} = \mathbf{A} \tag{1.65}$$

where $\mathbf{A}^{-1}$ is the inverse of **A**. Thus

$$\mathbf{A}\mathbf{A}^T = \mathbf{I} \tag{1.66}$$

which can also be written as

$$A_{ik} A_{jk} = \delta_{ij} \tag{1.67}$$

The transformation equations can also be written as

$$\begin{aligned} \mathbf{a} &= \mathbf{A}^T \mathbf{a}' & (1.68) \\ \mathbf{a}' &= \mathbf{A}\mathbf{a} & (1.69) \\ \mathbf{T} &= \mathbf{A}^T \mathbf{T}' \mathbf{A} & (1.70) \\ \mathbf{T}' &= \mathbf{A}\mathbf{T}\mathbf{A}^T & (1.71) \end{aligned}$$

Example 1.4

Consider two Cartesian coordinate systems: $\mathcal{S}$ with unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, and $\mathcal{S}'$ with $(\mathbf{i}', \mathbf{j}', \mathbf{k}')$, where $\mathbf{i}' = \mathbf{i}$, $\mathbf{j}' = (\mathbf{j} - \mathbf{k})/\sqrt{2}$, $\mathbf{k}' = (\mathbf{j} + \mathbf{k})/\sqrt{2}$. The tensor **T** has the following components in S:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Find its components in S'.

The transformation matrix is

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

From equation (1.71) we get,

$$\mathbf{T}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 1 \end{pmatrix}$$

Tensors of higher order may be similarly defined. The components of a tensor of order three, S_{ijk}, transform as

$$S_{ijk} = A_{li} A_{mj} A_{nk} S'_{lmn} \tag{1.72}$$

and so on.

In summary, the order of a tensor is defined by the transformation rule of its components if the coordinate system is rotated.

Transformation	*Quantity*
$\phi = \phi$	ϕ is a tensor of order zero, or scalar
$a'_i = A_{ij} a_j$	a_i are components of a tensor of order one, or vector)
$T'_{ij} = A_{ik} A_{jl} T_{kl}$	T_{ij} are components of a tensor of order two
$S'_{ijk} = A_{il} A_{jm} A_{kn} S_{lmn}$	S_{ijk} are components of a tensor of order three

If there is no risk of confusion, we will refer to a tensor of order two as simply a tensor. We may also say that f_i is a vector and T_{ij} is a tensor, instead of being components of a vector and a tensor.

It can be shown that a tensor of order two $\mathbf{T}$ is an operator which transforms a vector $\mathbf{a}$ into another $\mathbf{b}$. Thus

$$b_i = T_{ij} a_j \tag{1.73}$$

where a_i and b_i are the components of $\mathbf{a}$ and $\mathbf{b}$ respectively.

Example 1.5

If T_{ij} and a_j are the components of a second order tensor and vector, respectively, show that $b_i = T_{ij} a_j$ are the components of a vector.

In the system S', the components are

$$\begin{aligned} b'_i &= T'_{ij} a'_j \\ &= \left[A_{ik} A_{jl} T_{kl}\right] \left[A_{jm} f_m\right] \\ &= A_{ik} \delta_{lm} T_{kl} a_m \\ &= A_{ik} T_{kl} a_l \\ &= A_{ik} b_k \end{aligned}$$

proving that g_i are the components of a vector.

Similarly it can be shown that $T_{ij} = a_i b_j$ are the components of a tensor of order two if a_i and b_j are the components of vectors. This product of vectors $\mathbf{a}$ and $\mathbf{b}$ is called a *dyad* and written as $\mathbf{a}$ $\mathbf{b}$. A linear combination of dyads, $\sum_{n=1}^{k} C_n\ \mathbf{a}_n\ \mathbf{b}_n$ is a *dyadic*.

Notice that $a_i b_i$ is a scalar while $a_i b_j$ is a tensor. The process of equating indices is called a contraction, also indicated by a dot, so that the contraction of $\mathbf{ab}$ is $\mathbf{a} \cdot \mathbf{b}$. For two second order tensors, a single contraction $\mathbf{R} \cdot \mathbf{S} = R_{ij} S_{jk}$ will give a tensor and a double contraction $\mathbf{R} : \mathbf{S} = R_{ij} S_{ji}$ a scalar.

The gradient operator increases the order of the tensor. Thus, for example, the gradient of a vector f_i is the second-order tensor $\partial f_i / \partial x_j$. The divergence operator decreases the order of a tensor (for this reason it cannot be applied to a scalar). The divergence of a tensor T_{ij} is the vector $\partial T_{ij} / \partial x_i$. The tensor version of Gauss's theorem, equation (1.58), is

$$\int_{\mathcal{A}} T_{ij} n_i\ dA = \int_{\mathcal{V}} \frac{\partial T_{ij}}{\partial x_i}\ d\mathcal{V} \tag{1.74}$$

An isotropic tensor is one that is invariant to rotation of the coordinate system.

Example 1.6

Show that the tensor represented by δ_{ij} is isotropic.

The tensor has components δ_{ij} in S. In S', the components of this tensor will be $A_{ik} A_{jl} \delta_{kl} = A_{il} A_{jl} = \delta_{ij}$. Thus this tensor has the same form in the rotated coordinate system.

A symmetric tensor is one for which $T_{ij} = T_{ji}$, while for an anti-symmetric tensor $T_{ij} = -T_{ji}$. Since

$$T_{ij} = \underbrace{\frac{1}{2}(T_{ij} + T_{ji})}_{symmetric} + \underbrace{\frac{1}{2}(T_{ij} - T_{ji})}_{anti-symmetric} \tag{1.75}$$

any tensor can be written as the sum of two tensors, one symmetric and the other anti-symmetric.

1.6.1 Symmetric tensors

For a any tensor or matrix $\mathbf{T}$ an eigenvalue λ is obtained from the equation

$$\mathbf{Tz} = \lambda \mathbf{z} \tag{1.76}$$

where $\mathbf{z}$ is a non-zero vector called eigenvector, and λ is an eigenvalue. There can be more than one eigenvalue and eigenvector for a given tensor. For a symmetric tensor the eigenvalues are real numbers. Taking $\mathbf{z} = [a\ b\ c]^T$, the scalar equations corresponding to the vector equation above are

$$\begin{aligned}
(T_{11} - \lambda)a + T_{12}b + T_{13}c &= 0 \\
T_{21}a + (T_{22} - \lambda)b + T_{23}c &= 0 \\
T_{31}a + T_{32}b + (T_{33} - \lambda)c &= 0
\end{aligned} \tag{1.77}$$

For a nontrivial solution for a, b, and c, we must have

$$\begin{vmatrix} T_{11} - \lambda & T_{12} & T_{13} \\ T_{21} & T_{22} - \lambda & T_{23} \\ T_{31} & T_{32} & T_{33} - \lambda \end{vmatrix} = 0 \tag{1.78}$$

The cubic equation in λ thus obtained is called a secular equation, and its three solutions give the eigenvalues $\lambda_1, \lambda_2, \lambda_3$. For each λ_i equations (1.77) provide a nonunique eigenvector $\mathbf{z}_i$.

The three eigenvectors $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ form an orthogonal coordinate system that is rotated with respect to the original. With unit vectors along these directions (called principal axes), the original tensor becomes diagonal

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{1.79}$$

There is thus considerable simplification obtained as a result of using principal axes as coordinate directions. For instance, the equation

$$T_{ij}x_i x_j = 1 \tag{1.80}$$

describing a quadric surface, is really equivalent to

$$T_{11}x_1^2 + T_{12}x_1x_2 + T_{13}x_1x_3 + T_{21}x_2x_1 + T_{22}x_2^2 + T_{23}x_2x_3 + T_{31}x_3x_1 + T_{32}x_3x_2 + T_{33}x_3^2 = 1 \tag{1.81}$$

with a number of cross terms $x_i x_j$ $(i \neq j)$. For a symmetric T_{ij}, one can change to new coordinates along its principal directions to get an equation for the surface of the form

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 = 1 \tag{1.82}$$

which can be interpreted geometrically as an ellipsoid, a paraboloid, or a hyperboloid.

Example 1.7

Find the eigenvalues and eigenvectors of

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

Eigenvalues are solutions of the equation

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{vmatrix} = 0$$

which gives

$$(1-\lambda)(-3+\lambda)(1+\lambda) = 0$$

so that $\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 3$.

Equations (1.77) become

$$\begin{aligned} (1-\lambda_i)a_i &= 0 \\ (1-\lambda_i)b_i + 2c_i &= 0 \\ 2b_i + (1-\lambda_i)c_i &= 0 \end{aligned}$$

For $\lambda_1 = -1$: $2a_1 = 0,\ 2b_1 + 2c_1 = 0,\ 2b_1 + 2c_1 = 0$. This gives $a_1 = 0,\ b_1 = -c_1$. Choose $b_1 = 1$ arbitrarily. The eigenvector is $\mathbf{e}_2 - \mathbf{e}_3$. To make it a unit vector, and remove some arbitrariness, we divide by its magnitude $\sqrt{2}$. So the unit eigenvector corresponding to this eigenvalue is $(\mathbf{e}_2 - \mathbf{e}_3)/\sqrt{2}$.

For $\lambda_2 = 1$: $2c_2 = 0,\ 2b_2 = 0$. This gives $b_2 = c_2 = 0$, while a_1 is arbitrary. Choose $a_1 = 1$ to make it a unit vector. Then the eigenvector corresponding to this eigenvalue is $\mathbf{e}_1$.

For $\lambda_3 = 3$: $-2a_3 = 0,\ -2b_3 + 2c_3 = 0,\ 2b_3 - 2c_3 = 0$. This gives $a_3 = 0,\ b_3 = c_3$. Choose $b_3 = 1$. Then the eigenvector is $\mathbf{e}_2 + \mathbf{e}_3$. To make it a unit vector, we divide by its magnitude $\sqrt{2}$. So the unit eigenvector corresponding to this eigenvalue is $(\mathbf{e}_2 + \mathbf{e}_3)/\sqrt{2})$.

Vectors in directions opposite to the ones chosen here would also be unit eigenvectors.

1.6.2 Anti-symmetric tensors

The diagonal terms of an anti-symmetric tensor T_{ij} are zero and it has only three independent terms as shown:

$$\mathbf{T} = \begin{bmatrix} 0 & -T_{21} & T_{13} \\ T_{21} & 0 & -T_{32} \\ -T_{13} & T_{32} & 0 \end{bmatrix} \tag{1.83}$$

The three terms can be made to constitute a vector. Thus, we can let

$$\mathbf{a} = T_{32}\mathbf{e}_1 + T_{13}\mathbf{e}_2 + T_{21}\mathbf{e}_3 \tag{1.84}$$

This vector has the property that

$$\mathbf{T}\ \mathbf{b} = \mathbf{a} \times \mathbf{b} \tag{1.85}$$

where $\mathbf{b}$ is any vector.

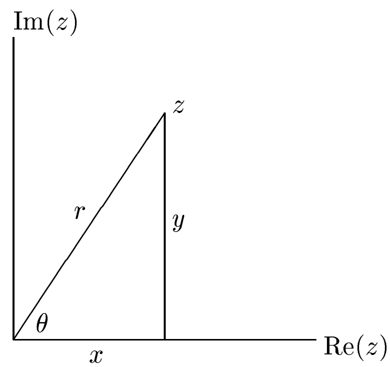


Figure 1.7: Complex number $z = x + iy$ in Argand plane.

1.7 Complex numbers and functions

A complex number z has the form

$$z = x + iy \tag{1.86}$$

where u and v are real, and $i = \sqrt{-1}$. x is the real part of z and y is its imaginary part. Keeping the definition of i in mind, these numbers can be added, subtracted, multiplied, and divided as usual. Two complex numbers are equal only if their real parts are equal and the imaginary parts are also equal.

A complex number can be geometrically represented in a plane (called the Argand plane) as shown in Fig. 1.7. The real part of z is plotted on the abscissa and the imaginary part on the ordinate.

The modulus or absolute value of the complex number is

$$|z| = +\sqrt{x^2 + y^2} \tag{1.87}$$

which is the distance r in the figure.

The conjugate of this number, $\bar{z}$, is defined by

$$\bar{z} = x - iy \tag{1.88}$$

so that

$$|z|^2 = z\bar{z} \tag{1.89}$$

1.7.1 Analytic functions

A real function $f(x)$ is said to be analytic at $x = x_0$ if f and all its derivatives exist at this point. An analytic function can be expanded in a Taylor series in the neighborhood of the point at which it is analytic.

A function of a complex variable, or complex function for short, defined in some domain is of the form $f(z)$ where z is a complex number. A derivative of a complex function, $f(z)$, at $z = z_0$ is defined as

$$\frac{df}{dz} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{1.90}$$

In contrast to the case of real functions, in this limiting process the approach $z \to z_0$ can be in different ways and the derivative exists only if all of these give the same value. A complex function is analytic at a point if its derivative exists at that point.

Let us suppose that the function $f(z)$ is analytic at $z = z_0$. Writing $z = x + iy$, we have f as a complex function of the real variables x and y. Thus

$$f' = \frac{df}{dz} \tag{1.91}$$

$$= \frac{df}{d(x+iy)} \tag{1.92}$$

As a special case the approach $z \to z_0$ can be by changing x only, which gives

$$f' = \frac{\partial f}{\partial x} \tag{1.93}$$

On the other hand we can approach z_0 by changing y only, giving

$$f' = -i\frac{\partial f}{\partial y} \tag{1.94}$$

If we separate f into its real and imaginary parts

$$f = u + iv \tag{1.95}$$

we get from the above that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1.96}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1.97}$$

These are the Cauchy-Riemann equations. If the partial derivatives exist, these are necessary and sufficient conditions for a complex function to be analytic at a point. For a complex function it can be shown that if $f'(z)$ exists, then so do all other derivatives.

From the above relations we can show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1.98}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{1.99}$$

The real and imaginary parts of a complex function are harmonic, i.e. they satisfy Laplace's equation.

It can happen that a function is analytic in a neighborhood of point z_0, but not at z_0 itself. A singular point is one at which the function is not analytic.

1.7.2 Polar form

A complex number may also be written in polar form using Euler's formula

$$z = re^{i\theta} \tag{1.100}$$

from which we can see that

$$\bar{z} = re^{-i\theta} \tag{1.101}$$

From Fig. 1.7 the relation between the Cartesian and polar forms is

$$x = r\cos\theta \tag{1.102}$$

$$y = r\sin\theta \tag{1.103}$$

Thus, we have Euler's relation

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.104}$$

and de Moivre's theorem

$$e^{in\theta} = \cos n\theta + i\sin n\theta \tag{1.105}$$

Examples of use of the polar form include the following:
(a) Write $e^{i(A+B)} = e^{iA}e^{iB}$ and expand both sides. The real and imaginary parts give the trigonometric relations for $\cos(A+B)$ and $\sin(A+B)$.
(b) The relation $z^n = r^n e^{in\theta}$ can be used to find the roots of complex numbers.
(c) The real and imaginary parts of the logarithm of a complex number may be found by using $\ln z = \ln r + i\theta$.

To find the Cauchy-Riemann conditions in polar coordinates, we write the derivative of a complex function as

$$f' = \frac{df}{d(re^{i\theta})} \tag{1.106}$$

Moving in the radial direction, this becomes

$$f' = e^{-i\theta}\frac{\partial f}{\partial r}$$

and in the circumferential direction it is

$$f' = -\frac{ie^{-i\theta}}{r}\frac{\partial f}{\partial \theta}$$

For a complex function of the form $f = u + iv$, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{1.107}$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} \tag{1.108}$$

1.7.3 Integrals

If the function $f(z)$ is analytic inside and on a closed contour $\mathcal{C}$, then according to Cauchy-Goursat's theorem

$$\int_{\mathcal{C}} f(z)\; dz = 0 \tag{1.109}$$

Furthermore, by Cauchy's theorem

$$\frac{d^n f}{dz^n}(z_0) = \frac{n!}{2\pi i}\int_{\mathcal{C}} \frac{f(z)}{(z-z_0)^{n+1}}\; dz \text{ for } n \geq 1 \tag{1.110}$$

and

$$f(z_0) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(z)}{(z-z_0)}\; dz \tag{1.111}$$

Problems

1. Using the identity

$$\epsilon_{ijk}\epsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{kl}$$

derive

$$\begin{aligned} \epsilon_{ijk}\epsilon_{lmk} &= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \\ \epsilon_{ijk}\epsilon_{ljk} &= 2\delta_{il} \\ \epsilon_{ijk}\epsilon_{ijk} &= 6 \end{aligned}$$

2. Prove by expanding into components that
   (a) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$
   (b) $\nabla \times \nabla\phi = 0$

3. Using $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ show that $\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$.

4. Show that

$$\nabla \times (\nabla \times \mathbf{f}) = \nabla(\nabla \cdot \mathbf{f}) - \nabla^2\mathbf{f}$$

5. Find the principal axes of the symmetric tensor

$$\begin{bmatrix} 2 & 4 & -6 \\ 4 & 2 & -6 \\ -6 & -6 & -15 \end{bmatrix}$$

Confirm using MAPLE.

6. Show that equation (1.85) holds for any vector $\mathbf{b}$, where $\mathbf{T}$ and $\mathbf{a}$ are given by equations (1.83) and (1.84) respectively.

7. Write the expressions for $\nabla\phi$, $\nabla \cdot \mathbf{f}$, and $\nabla \times \mathbf{f}$ in (a) cylindrical polar coordinates, and (b) spherical polar coordinates.

8. Find the principal axes of the tensor

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix}$$

9. Apply Stokes's theorem to the plane vector field $\mathbf{f}(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$ and a closed curve enclosing a plane region. What is the result called? Use this result to find $\oint_{\mathcal{C}} \mathbf{f} \cdot d\mathbf{l}$, where $\mathbf{f} = -y\mathbf{i} + x\mathbf{j}$ and the integration is counterclockwise along the sides $\mathcal{C}$ of the parallelogram with corners at (0,0), (1,0), (2,1), and (1,1).

10. Two-dimensional bipolar coordinates (ξ, η) are defined by

$$\begin{aligned} x &= \frac{\alpha \sinh\eta}{\cosh\eta - \cos\xi} \\ y &= \frac{\alpha \sin\xi}{\cosh\eta - \cos\xi} \end{aligned}$$

Find expressions for the gradient and divergence operators. Also computer generate some constant ξ and constant η lines.

11. Using the index notation, show that

$$\nabla \times (\mathbf{f} \times \mathbf{g}) = (\mathbf{g} \cdot \nabla)\mathbf{f} - (\mathbf{f} \cdot \nabla)\mathbf{g} + \mathbf{f}(\nabla \cdot \mathbf{g}) - \mathbf{g}(\nabla \cdot \mathbf{f})$$

where $\mathbf{f}$ and $\mathbf{g}$ are vector fields.

12. Consider two Cartesian coordinate systems: S with unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, and S' with $(\mathbf{i}', \mathbf{j}', \mathbf{k}')$, where $\mathbf{i}' = \mathbf{i}$, $\mathbf{j}' = (\mathbf{j} - \mathbf{k})/\sqrt{2}$, $\mathbf{k}' = (\mathbf{j} + \mathbf{k})/\sqrt{2}$. The tensor $\mathbf{T}$ has the following components in S:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Find its components in S'.

13. Find the matrix $\mathbf{A}$ that operates on any vector of unit length in the x-y plane and turns it through an angle θ around the z-axis without changing its length.

14. Prove the following identities using index notation:

    (a) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

    (b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$

    (c) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \cdot \mathbf{d}$

15. The position of a point is given by $\mathbf{R} = \mathbf{i}a\cos\omega t + \mathbf{j}b\sin\omega t$. Show that the path of the point is an ellipse. Find its velocity $\mathbf{V}$ and show that $\mathbf{R} \times \mathbf{V}$ = constant. Show also that the acceleration of the point is directed towards the origin and its magnitude is proportional to the distance from the origin.

16. Use Green's theorem to calculate $\oint_{\mathcal{C}} \mathbf{f} \cdot d\mathbf{l}$, where $\mathbf{f} = x^2\mathbf{i} + 2xy\mathbf{j}$, and $\mathcal{C}$ is the counterclockwise path around a rectangle with vertices at (0,0), (6,0), (0,4) and (6,4).

17. Derive an expression for the divergence of a vector in orthogonal paraboloidal coordinates

$$\begin{aligned} x &= uv\cos\theta \\ y &= uv\sin\theta \\ z &= \frac{1}{2}(u^2 - v^2) \end{aligned}$$

Determine the scale factors. Find $\nabla\phi$, $\nabla \cdot \mathbf{f}$, $\nabla \times \mathbf{f}$, and $\nabla^2\phi$ in this coordinate system.

18. Derive an expression for the gradient, divergence, curl and Laplacian operators in orthogonal parabolic cylindrical coordinates (u, v, w) where

$$\begin{aligned} x &= uv \\ y &= \frac{1}{2}(u^2 - v^2) \\ z &= w \end{aligned}$$

where $0 \le u < \infty$, $-\infty < v < \infty$, and $-\infty < w < \infty$.

19. Orthogonal elliptic cylindrical coordinates (u, v, z) are related to Cartesian coordinates (x, y, z) by

$$\begin{aligned} x &= a\cosh u\cos v \\ y &= a\sinh u\sin v \\ z &= z \end{aligned}$$

where $0 \le u < \infty$, $0 \le v < 2\pi$ and $-\infty < z < \infty$. Determine $\nabla\phi$, $\nabla \cdot \mathbf{f}$, $\nabla \times \mathbf{f}$ and $\nabla^2\phi$ in this system , where ϕ is a scalar field and $\mathbf{f}$ is a vector field.

20. Determine the unit vector normal to the surface $x^3 - xyz + z^3 = 1$ at the point (1,1,1).

21. Show using indicial notation that

$$\begin{aligned} \nabla \times \nabla\phi &= = 0 \\ \nabla \cdot \nabla \times \mathbf{f} &= 0 \\ \nabla(\mathbf{f} \cdot \mathbf{g}) &= (\mathbf{f} \cdot \nabla)\mathbf{g} + (\mathbf{g} \cdot \nabla)\mathbf{f} + \mathbf{f} \times (\nabla \times \mathbf{g}) + \mathbf{g} \times (\nabla \times \mathbf{f}) \\ \frac{1}{2}\nabla(\mathbf{f} \cdot \mathbf{f}) &= (\mathbf{f} \cdot \nabla)\mathbf{f} + \mathbf{f} \times (\nabla \times \mathbf{f}) \\ \nabla \cdot (\mathbf{f} \times \mathbf{g}) &= \mathbf{g} \cdot \nabla \times \mathbf{f} - \mathbf{f} \cdot \nabla \times \mathbf{g} \\ \nabla \times (\nabla \times \mathbf{f}) &= \nabla(\nabla \cdot \mathbf{f}) - \nabla^2\mathbf{f} \\ \nabla \times (\mathbf{f} \times \mathbf{g}) &= (\mathbf{g} \cdot \nabla)\mathbf{f} - (\mathbf{f} \cdot \nabla)\mathbf{g} + \mathbf{f}(\nabla \cdot \mathbf{g}) - \mathbf{g}(\nabla \cdot \mathbf{f}) \end{aligned}$$

22. Show that the Laplacian operator $\partial^2/\partial x_i \partial x_i$ has the same form in S and S'.

23. Show the identities (1.45)-(1.50).

24. For potential flow the velocity field is given by $\mathbf{u} = \nabla\phi$, where ϕ is the velocity potential. Consider the velocity potential

$$\phi(x, y, z) = x^2 + y^2 - 2z^2$$

    (a) Show that it satisfies Laplace's equation $\nabla^2\phi = 0$.

(b) Find the velocity field $\mathbf{u}$.

(c) Show that $\mathbf{u}$ has zero divergence.

25. Show that the following scalar field in cylindrical coordinates (which represents the stream function for flow around cylinders)

$$\psi = UR\left(1 - \frac{a^2}{R^2}\right)\sin\theta$$

where U and a are constants, satisfies Laplace's equation.

26. Show that the following scalar field in spherical coordinates (which represents the velocity potential for flow around spheres)

$$\phi = Ur\left(1 - \frac{1}{2}\frac{a^3}{r^3}\right)\cos\theta$$

where U and a are constants, satisfies Laplace's equation.

27. Show that the Cartesian vector field

$$\mathbf{u} = (y+z)\mathbf{i} + (z+x)\mathbf{j} + (x+y)\mathbf{k}$$

is divergence free and irrotational. Also find its gradient.

28. Find expressions for the gradient, divergence and curl operators in spherical coordinates.

29. Find expressions for the gradient, divergence and curl operators in cylindrical coordinates.

30. Show that the sum of two analytic functions is also analytic.

31. Determine the integral

$$\int_{\mathcal{C}} z^2 \, dz$$

where $\mathcal{C}$ is the path from (0,0) to (1,1) in the three different ways:
(a) straight lines from (0,0) to (1,0) to (1,1)
(b) straight line from (0,0) to (1,1)
(c) straight lines from (0,0) to (0,1) to (1,1).

32. If z and z' are the same complex number but in the unprimed and primed coordinate system, respectively, show that

$$z = z' e^{i\alpha}$$

33. Prove Cauchy-Goursat's theorem[1].

34. Show that

$$\phi(r,\theta) = a_0 + a_1 \ln r + a_2\theta + a_3\theta \ln r + \sum_{n=1}^{\infty}\left[\left(A_n r^n + \frac{B_n}{r^n}\right)\cos n\theta + \left(C_n r^n + \frac{D_n}{r^n}\right)\sin n\theta\right]$$

in polar coordinates is a harmonic function.

[1]Consult your favorite book on complex variable theory.

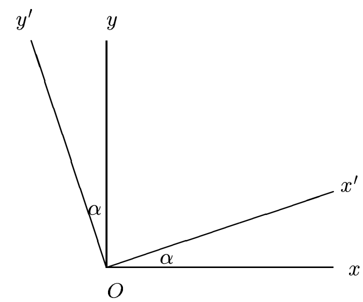


Figure 1.8: Coordinate systems $\mathcal{S}$ and $\mathcal{S}'$

Chapter 2

Flow, rate of deformation and stress

2.1 Flow lines

2.1.1 Streamlines

These are lines the tangents to which are in the direction of the velocity vector. Thus, in Cartesian coordinates

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{2.1}$$

is the equation of the streamlines, where $\mathbf{u} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$.

2.1.2 Pathlines

These are lines traced out by the fluid particles. Thus

$$\frac{dx}{dt} = u \tag{2.2}$$

$$\frac{dy}{dt} = v \tag{2.3}$$

$$\frac{dz}{dt} = w \tag{2.4}$$

where (x, y, z) is the position of a fluid particle at time t.

2.1.3 Streaklines

These are lines traced out by particles that have all passed through a given point at some previous time.

2.1.4 Timeline

This is the locus of a set of particles that initially defines a line.

Example 2.1

For a flow with velocity field $\mathbf{u} = U\cos\omega t\mathbf{i} + U\sin\omega t\mathbf{j}$, find (a) the streamline passing through the origin at time $t = \tau$, (b) pathline of particle that is at the origin when $t = 0$, and (c) streakline at time $t = \tau$ for a source at the origin.

(a) The velocity field at time $t = \tau$ is $\mathbf{u} = U\cos\omega\tau\mathbf{i} + U\sin\omega\tau\mathbf{j}$. The equation for the streamline is a solution of

$$\frac{dx}{U\cos\omega\tau} = \frac{dy}{U\sin\omega\tau}$$

This gives

$$Ux\sin\omega t = Uy\cos\omega t + a$$

Since $x = 0$ for $y = 0$, we get $a = 0$. Thus the required streamline is

$$y = x\tan\omega\tau$$

(b) The pathline is a solution of

$$\begin{aligned}\frac{dx}{dt} &= U\cos\omega t\\ \frac{dy}{dt} &= U\sin\omega t\end{aligned}$$

Thus

$$\begin{aligned}x &= \frac{U}{\omega}\sin\omega t + x_0\\ y &= -\frac{U}{\omega}\cos\omega t + y_0\end{aligned}$$

Since the particle is at (0,0) when $t = 0$, we have $x_0 = 0$, $y_0 = U/\omega$. The parametric form of the pathline is

$$\begin{aligned}x &= \frac{U}{\omega}\sin\omega t\\ y &= \frac{U}{\omega}(1 - \cos\omega t)\end{aligned}$$

Eliminating t, we get the equation

$$\frac{\omega^2}{U^2}x^2 + \left(\frac{\omega}{U}y - 1\right)^2 = 1$$

This is a circle of radius U/ω and center $(0, U/\omega)$.

(c) The particle position at time $t = \tau$ is given by

$$\begin{aligned}x &= \frac{U}{\omega}\sin\omega\tau + x_0\\ y &= -\frac{U}{\omega}\cos\omega\tau + y_0\end{aligned}$$

Since the particles that compose the streakline have been at the origin at a time $t = s$ (say), we have

$$\begin{aligned}0 &= \frac{U}{\omega}\sin\omega s + x_0\\ 0 &= -\frac{U}{\omega}\cos\omega s + y_0\end{aligned}$$

Using these values of x_0 and y_0, we get the parametric equation of the streakline at time $t = \tau$

$$\begin{aligned}x &= \frac{U}{\omega}(\sin\omega\tau - \sin\omega s)\\ y &= \frac{U}{\omega}(\cos\omega\tau + \cos\omega s)\end{aligned}$$

Eliminating s, we have

$$\left(\frac{\omega}{U}x - \sin\omega\tau\right)^2 + \left(\frac{\omega}{U}y + \cos\omega\tau\right)^2 = 1$$

which is a circle of radius $\frac{U}{\omega}$ and center $(\frac{U}{\omega}\sin\omega\tau, -\frac{U}{\omega}\cos\omega\tau)$.

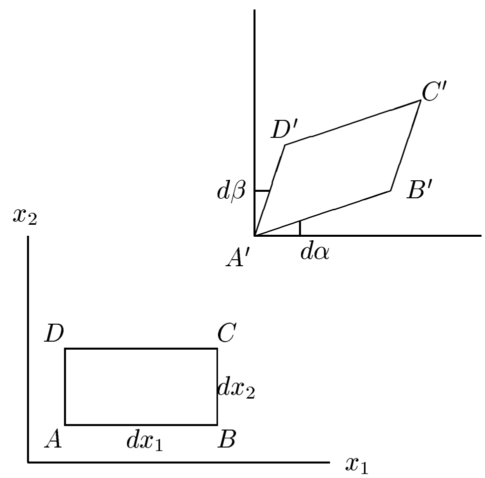


Figure 2.1: Deformation of fluid material

2.2 Rate of deformation

The gradient of the velocity vector is the deformation rate tensor **D**, where its components are

$$D_{ij} = \frac{\partial u_i}{\partial x_j} \tag{2.5}$$

We can split it up into the sum of an antisymmetric and a symmetric tensor, so that

$$D_{ij} = \Omega_{ij} + \varepsilon_{ij} \tag{2.6}$$

where

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{2.7}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.8}$$

Ω_{ij} and ε_{ij} are called the rotation rate and shear rate tensors respectively.

The vector corresponding to twice the anti-symmetric tensor Ω_{ij} is the vorticity given by

$$\mathbf{\Omega} = \nabla \times \mathbf{u} \tag{2.9}$$

A velocity field for which $\boldsymbol{\omega}$ is zero everywhere is said to be irrotational. Physically, the vorticity vector $\boldsymbol{\omega}$ is twice the rotation rate of infinitesimal fluid elements and the symmetric shear rate tensor ε_{ij} represents the rate at which these fluid elements are being deformed. This is shown below.

Consider the fluid in a material rectangle in Figure 2.1 of size $dx_1 \times dx_2$ moving with the flow. After an interval dt the points A, B, C, and D move to A', B', C', and D'. The instantaneous rates of rotation of sides AB and AD are

$$\frac{d\alpha}{dt} = \frac{\partial u_2}{\partial x_1} \tag{2.10}$$

$$\frac{d\beta}{dt} = \frac{\partial u_1}{\partial x_2} \tag{2.11}$$

Thus we have

$$\text{Rotation rate} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \tag{2.12}$$

$$\text{Shear rate} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) \tag{2.13}$$

The shear rate in all three coordinate planes are the off-diagonal components of the shear rate tensor $\frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$. The diagonal components of this tensor are the extensional strain rates $\partial u_1/\partial x_1$, $\partial u_2/\partial x_2$, and $\partial u_3/\partial x_3$.

2.3 Vortex lines and tubes

Tangents to vortex lines are in the direction of the vorticity vector. Thus

$$\frac{dx}{\omega_x} = \frac{dy}{\omega_y} = \frac{dz}{\omega_z} \tag{2.14}$$

is the equation of a vortex line, where

$$\boldsymbol{\omega} = \omega_x \mathbf{e}_x + \omega_y \mathbf{e}_y + \omega_z \mathbf{e}_z \tag{2.15}$$

2.4 Eulerian and Lagrangian descriptions

In the Eulerian approach, we describe the flow field by its properties as a function of space and time. In the Lagrangian approach, however, we follow a fluid "particle" and observe the change of properties with time as it is swept through the flow. Newton's laws are usually formulated in a Lagrangian sense, while the fluid equations as well as measurements are easier handled with the Eulerian description. Thus some relation between the two descriptions is necessary.

2.4.1 Material derivative

In this, as in other sections in this chapter, we will derive the appropriate expressions in Cartesian coordinates, and then generalize to other coordinate systems. Consider a property $\alpha(x_1, x_2, x_3, t)$, scalar, vector or tensor, which is function of position expressed in Cartesian coordinates x_1, x_2, x_3 and of time t. If we follow a fluid particle, the change in this property after a time dt will be $d\alpha$, where

$$d\alpha = \frac{\partial \alpha}{\partial t}\,dt + \frac{\partial \alpha}{\partial x_1}\,dx_1 + \frac{\partial \alpha}{\partial x_2}\,dx_2 + \frac{\partial \alpha}{\partial x_3}\,dx_3 \tag{2.16}$$

where dx_1, dx_2, dx_3 are the components of the displacement of the particle in the coordinate directions. Dividing by dt, we have

$$\frac{d\alpha}{dt} = \frac{\partial \alpha}{\partial t} + u_1 \frac{\partial \alpha}{\partial x_1} + u_2 \frac{\partial \alpha}{\partial x_2} + u_3 \frac{\partial \alpha}{\partial x_3} \tag{2.17}$$

where

$$\frac{dx_1}{dt} = u_1 \tag{2.18}$$

$$\frac{dx_2}{dt} = u_2 \tag{2.19}$$

$$\frac{dx_3}{dt} = u_3 \tag{2.20}$$

Frequently, the symbol D/Dt is used instead of d/dt to emphasize the fact that the time derivative is calculated following a fluid particle. It is called a material, total, or substantial derivative. Thus

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i} \tag{2.21}$$

The first term on the right hand side is called the local derivative, and the second the convective derivative. For other coordinates we can use

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \tag{2.22}$$

where $\mathbf{u}$ is the velocity vector.

2.5 Stress

The stress is defined as the force per unit area. It is a tensor since both force and area can be considered to be vectors (an elemental area has orientation). Consider an elemental volume of fluid as shown in Figure 2.2. There is a force on each one of the six surfaces. Let the stress tensor at the center of the element P be $\boldsymbol{\tau}$, so that those on the two faces normal to the x_1 direction are

$$\boldsymbol{\tau}^{\pm} = \boldsymbol{\tau} \pm \frac{\partial \boldsymbol{\tau}}{\partial x_1} \frac{dx_1}{2}$$

The components of the force per unit area on the face with normal in the $+x_1$ direction are marked as τ_{11}^+, τ_{12}^+, and τ_{13}^+, while those on the face with normal in the $-x_1$ direction are τ_{11}^-, τ_{12}^- τ_{13}^-. The stresses on the $+$ and $-$ faces may, in general, be different. $\boldsymbol{\tau}^-$ and $\boldsymbol{\tau}^+$ tend to $\boldsymbol{\tau}$ as the size of the element goes to zero. The components of the stress tensor at P can thus be represented as τ_{ij}, where the first index represents the face on which the force is acting, and the second the direction of the force.

If $\boldsymbol{\tau}$ is the state of stress at a point in the fluid, the force per unit area on an element of surface with unit normal $\mathbf{n}$, sometimes called the traction, is $\boldsymbol{\tau} \cdot \mathbf{n}$, or $\tau_{ij} n_j$. If the area of the element is dA, the force vector is $\boldsymbol{\tau} \cdot \mathbf{n}\, dA$, or $\tau_{ij} n_j\, dA$.

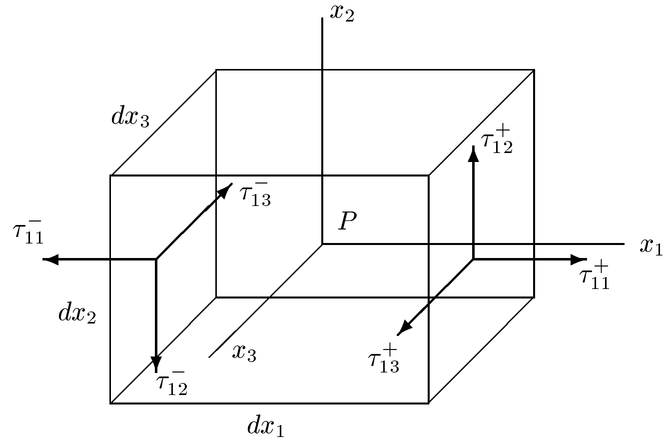


Figure 2.2: Forces per unit area on elemental volume

Problems

1. Find an expression for the acceleration of a particle $D\mathbf{u}/Dt$ for the two-dimensional velocity field
(a) $\mathbf{u} = x_1\mathbf{e}_1 - x_2\mathbf{e}_2$ in Cartesian coordinates, and
(b) $\mathbf{u} = U(1 - a^2/r^2)\cos\theta\mathbf{e}_r - U(1 + a^2/r^2)\sin\theta\mathbf{e}_\theta$ in cylindrical coordinates.
2. Show that $\frac{1}{2}$ curl $\mathbf{u}$ is the vector corresponding to the anti-symmetric vorticity tensor Ω_{ij}.
3. For the velocity field $\mathbf{u} = x_1\mathbf{e}_1 - x_2\mathbf{e}_2$, find the deformation-rate, vorticity, and strain-rate tensors.
4. For an axisymmetric flow in spherical coordinates (i.e. with $u_\omega = \partial/\partial\omega = 0$) show that the only nonzero component of the vorticity, ζ, is in the direction of the axis of symmetry, and that
$$\zeta = \frac{1}{r}\left[\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial\theta}\right]$$
5. Find an expression for the acceleration of a particle in the two-dimensional velocity field $\mathbf{u} = x_2^2\mathbf{e}_1 + x_1^2\mathbf{e}_2$.
6. For the two-dimensional velocity field $\mathbf{u} = x_1\mathbf{e}_1 - x_2\mathbf{e}_2$, find (a) the shape of a portion of fluid that is initially a rectangle with corners at (a, b), $(a + \delta a, b)$, $(a + \delta a, b + \delta b)$ and $(a, b + \delta b)$, where a, δa, b and δb are all non-negative, after an interval of time δt. Taking the appropriate limits, find (b) the rate of change in area, (c) the rates of rotation of the two diagonals of the rectangle and their average. Repeat for the velocity field $\mathbf{u} = x_2\mathbf{i}_1$.
7. For the velocity field $\mathbf{u} = (3x_1 + x_2 + 2x_3)\mathbf{e}_1 + (-x_1 + 3x_2 + 7x_3)\mathbf{e}_2 + (-2x_1 + x_2 + 3x_3)\mathbf{e}_3$ find the deformation rate, vorticity, and shear rate tensors. Find the principal axes of the shear rate tensor. Check using Maple.
8. Show that
$$\frac{D\mathbf{u}}{Dt} = \frac{\partial\mathbf{u}}{\partial t} + \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) + (\nabla\times\mathbf{u})\times\mathbf{u}$$
9. For the two-dimensional velocity field $\mathbf{u} = x_1\mathbf{e}_1 - x_2\mathbf{e}_2$, find the time derivative of the velocity determined by a relative-velocity sensor (i.e. the sensor measures the fluid velocity relative to it) moving with velocity $\mathbf{v} = t\mathbf{e}_1 + t^2\mathbf{e}_2$. The sensor is at the origin at time $t = 0$.
10. For the velocity field $\mathbf{u} = x_1\mathbf{e}_1 - x_2\mathbf{e}_2$, find the equations governing the motion of a skywriting airplane that is required to produce a perfect circle of unit radius centered at (1,1) at time $t = T$. The plane moves at a constant speed V and begins to write at $t = 0$ from the point $(2e^{-T}, e^T)$.

11. Find the force on an elemental area of size dA and unit normal $\mathbf{n}$, if the stress tensor at that location is $\boldsymbol{\tau}$.

12. A temperature sensor starts from the origin and moves with velocity $\mathbf{u}_s = t\mathbf{i} + t^2\mathbf{j}$, where t is time. Find its reading, $T_s(t)$, in a time-dependent temperature field $T(x, y, t) = txy$.

13. Find the shape of a material line at time t that is initially a circle with unit radius and center (2,2) moving with the flow field $\mathbf{u} = x\mathbf{i} - y\mathbf{j}$.

14. The surface $\tau_{ij}x_ix_j = 1$, where τ_{ij} is symmetric, is called the Cauchy stress quadric. Show that the normal component of the stress on any plane is inversely proportional to the square of the distance from the center of the quadric to its surface in the direction of the normal to the plane.

Chapter 3

Differential equations of motion

3.1 Approaches to balance equations

There are different ways in which the balance equations for fluid mechanics may be derived. The approach may be differential, i.e. applied to a element, or it may be integral, i.e. applied on a larger scale. Furthermore, the application may be to a fixed mass of fluid that is moving around and changing its shape in space, or to a fixed region in space through which the fluid flows in and out. So there are at least four ways which must be somehow related.

Consider a (scalar, vector or tensor) property α that is to be balanced.

3.1.1 Differential element

Though the following may be easily carried out for any coordinate system, we will analyze the situation in Cartesian coordinates. Consider the box-like element shown in Fig. 3.1.

Control volume

The box represents a region that is fixed in space and unchanging. Thus

$$\frac{d}{dt}(\rho\alpha \; dx_1 \; dx_2 \; dx_3) = \frac{\partial}{\partial t}(\rho\alpha) \; dx_1 \; dx_2 \; dx_3 \tag{3.1}$$

Control mass

Now the box is an element of fluid that is moving and changing its shape and volume. Thus we have

$$\frac{D}{Dt}(\rho\alpha \; dx_1 \; dx_2 \; dx_3) = \frac{D}{Dt}(\rho\alpha) \; dx_1 \; dx_2 \; dx_3 + \rho\alpha\frac{D}{Dt}(dx_1 \; dx_2 \; dx_3)$$

We also have

$$\begin{aligned}
\frac{D}{Dt}(dx_1 \; dx_2 \; dx_3) &= dx_2 \; dx_3 \; \frac{D}{Dt}(dx_1) + dx_3 \; dx_1 \; \frac{D}{Dt}(dx_2) + dx_1 \; dx_2 \; \frac{D}{Dt}(dx_3) \\
&= dx_2 \; dx_3 \left(\frac{\partial u_1}{\partial x_1} \; dx_1\right) + dx_3 \; dx_1 \left(\frac{\partial u_2}{\partial x_2} \; dx_2\right) + dx_1 \; dx_2 \left(\frac{\partial u_3}{\partial x_3} \; dx_3\right) \\
&= dx_1 \; dx_2 \; dx_3 \nabla \cdot \mathbf{u}
\end{aligned}$$

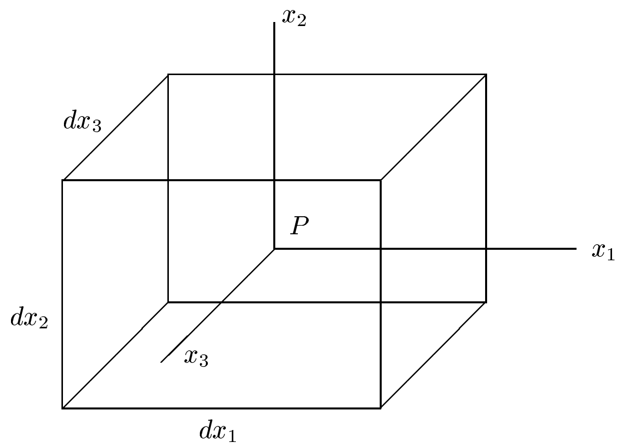


Figure 3.1: Element in Cartesian coordinates

from which

$$\frac{D}{Dt}(\rho\alpha\ dx_1\ dx_2\ dx_3) = \left[\frac{D}{Dt}(\rho\alpha) + \rho\alpha\nabla\cdot\mathbf{u}\right]\ dx_1\ dx_2\ dx_3 \tag{3.2}$$

3.1.2 Integral element

We consider now a volume $\mathcal{V}$, not necessarily small.

Control volume

The volume is a fixed region in space. Thus

$$\frac{d}{dt}\int_{\mathcal{V}} \rho\alpha\ d\mathcal{V} = \int_{\mathcal{V}} \frac{\partial}{\partial t}(\rho\alpha)\ d\mathcal{V} \tag{3.3}$$

is the rate of change of the total property within the volume.

Control mass

If V represents a fluid material, its shape, volume and position will change with time. The Reynolds transport theorem

$$\frac{D}{Dt}\int_{\mathcal{V}} \rho\alpha\ d\mathcal{V} = \int_{\mathcal{V}} \left[\frac{D}{Dt}(\rho\alpha) + \rho\alpha\nabla\cdot\mathbf{u}\right]\ d\mathcal{V} \tag{3.4}$$

describes the rate of change of the total property of this material as it moves around.

3.2 Mass balance

We will derive this equations in its different forms and show that they are all equivalent.

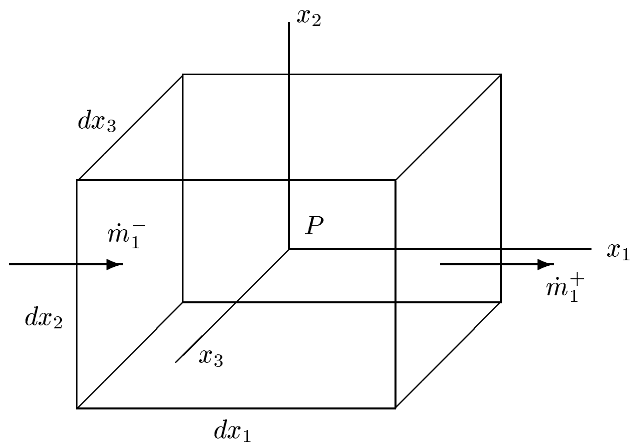


Figure 3.2: Mass flux in elemental volume

3.2.1 Elemental control volume

Consider an elemental control volume within the flow, Figure 3.2, of size $dx_1 \times dx_2 \times dx_3$ centered at $P(x_1, x_2, x_3)$ and aligned with the coordinate directions. There is mass flux through all six faces of this volume. The mass flux through each face is equal to the product of the local fluid density, the velocity normal to the face, and the area of the face. The fluxes in the two faces with normals in the $\pm\, x_1$ direction are

$$\dot{m}_1^+ = \left[\rho u_1 + \frac{\partial(\rho u_1)}{\partial x_1}\frac{dx_1}{2} + \ldots\right] dx_2 dx_3 \tag{3.5}$$

$$\dot{m}_1^- = \left[\rho u_1 - \frac{\partial(\rho u_1)}{\partial x_1}\frac{dx_1}{2} - \ldots\right] dx_2\ dx_3 \tag{3.6}$$

where ρ is the fluid density, and u_1 is the velocity component at P. The net gain of mass per unit time by the control volume due to flow in this direction is

$$\dot{m}_1^- - \dot{m}_1^+ = -\frac{\partial(\rho u_1)}{\partial x_1}\ dx_1\ dx_2\ dx_3 + \ldots \tag{3.7}$$

There are similar gains due to flow in the other two directions. The total is

$$-\left[\frac{\partial(\rho u_1)}{\partial x_1} + \frac{\partial(\rho u_2)}{\partial x_2} + \frac{\partial(\rho u_3)}{\partial x_3}\right] dx_1\ dx_2\ dx_3$$

On the other hand there is accumulation of mass within the control volume, the rate of which is

$$\frac{\partial \rho}{\partial t}\ dx_1\ dx_2\ dx_3$$

The rate of accumulation of mass should be equal to the total gain due to flow. Thus,

$$\frac{\partial \rho}{\partial t}\ dx_1\ dx_2\ dx_3 = -\left[\frac{\partial(\rho u_1)}{\partial x_1} + \frac{\partial(\rho u_2)}{\partial x_2} + \frac{\partial(\rho u_3)}{\partial x_3}\right] dx_1\ dx_2\ dx_3 \tag{3.8}$$

from which

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_1)}{\partial x_1} + \frac{\partial(\rho u_2)}{\partial x_2} + \frac{\partial(\rho u_3)}{\partial x_3} = 0 \tag{3.9}$$

In index notation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0 \tag{3.10}$$

For other coordinate systems

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{3.11}$$

Another way of writing this equation is

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_i}{\partial x_i} = 0 \tag{3.12}$$

or

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \tag{3.13}$$

The equation of conservation of mass is often called the continuity equation.

3.2.2 Elemental control mass

Following a fluid element that is initially $dx_1 \times dx_2 \times dx_3$, the mass of this element must be constant. Thus

$$\frac{D}{Dt}\left(\rho\ dx_1\ dx_2\ dx_3\right) = 0 \tag{3.14}$$

3.2.3 Integral control volume

The rate of change of mass with a control volume $\mathcal{V}$ plus the net mass flow out of it must be zero. Thus

$$\frac{d}{dt}\int_{\mathcal{V}} \rho\ d\mathcal{V} + \int_{\mathcal{A}} \rho \mathbf{u} \cdot \mathbf{n}\ d\mathcal{A} = 0 \tag{3.15}$$

3.2.4 Integral control mass

The rate of change of mass following a control mass of initial volume $\mathcal{V}$ is zero. So

$$\frac{D}{Dt}\int_{\mathcal{V}} \rho\ d\mathcal{V} = 0 \tag{3.16}$$

3.2.5 Equivalence

(a) Using equation (3.2) in (3.14), we get equation (3.13).
(b) Using the Stokes theorem on the second term of equation (3.15), we get

$$\int_{\mathcal{V}} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] \, d\mathcal{V} = 0 \tag{3.17}$$

Since this equation holds for any $\mathcal{V}$, the integrand must be zero everywhere. This gives equation (3.11).
(c) Using the Reynolds transport theorem, equation (3.16) becomes

$$\int_{\mathcal{V}} \left[\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} \right] \, d\mathcal{V} = 0 \tag{3.18}$$

Again since this equation holds for any $\mathcal{V}$, the integrand must be zero everywhere. This gives equation (3.13).

3.3 Linear momentum equation

We will use only an elemental control mass instantaneously of size $dx_1 \times dx_2 \times dx_3$. According to Newton's second law in an inertial frame of reference, the product of its mass and its acceleration is equal to the force on it. Let $\mathbf{f}$ be the force on the fluid per unit volume. Then, writing this equation per unit volume

$$\rho \frac{D\mathbf{u}}{Dt} = \mathbf{f} \tag{3.19}$$

The force per unit volume on a fluid element can be considered to be the sum of body forces $\mathbf{f}_b$ (such as gravity or electromagnetic forces) which act over the entire volume of the element, and surface forces per unit volume $\mathbf{f}_s$ (such as stress) which act through the surface.

A conservative body force is one for which

$$\mathbf{f}_b = \nabla G \tag{3.20}$$

where G is a scalar potential. If gravity is the only body force, then

$$\mathbf{f}_b = \rho \mathbf{g} \tag{3.21}$$

where $\mathbf{g}$ is the gravity force per unit mass.

The surface forces can be understood by considering a control mass shown in Figure 3.3. If the components of $\boldsymbol{\tau}$ on the two faces with normals in the $\pm\, x_1$ direction were the same, their net contribution would be zero since they would be in opposite directions. So a uniform stress field produces no force on the fluid element. For an inhomogeneous stress field in which $\boldsymbol{\tau}$ is a function of position, there is a net surface force. For the moment let us consider the forces in the $\pm\, x_1$ direction only. Each one of the six faces has a component in this direction. Thus

$$\tau_{11}^{+} = \tau_{11} + \frac{\partial \tau_{11}}{\partial x_1} \frac{dx_1}{2} \tag{3.22}$$

$$\tau_{11}^{-} = \tau_{11} - \frac{\partial \tau_{11}}{\partial x_1} \frac{dx_1}{2} \tag{3.23}$$

(3.24)

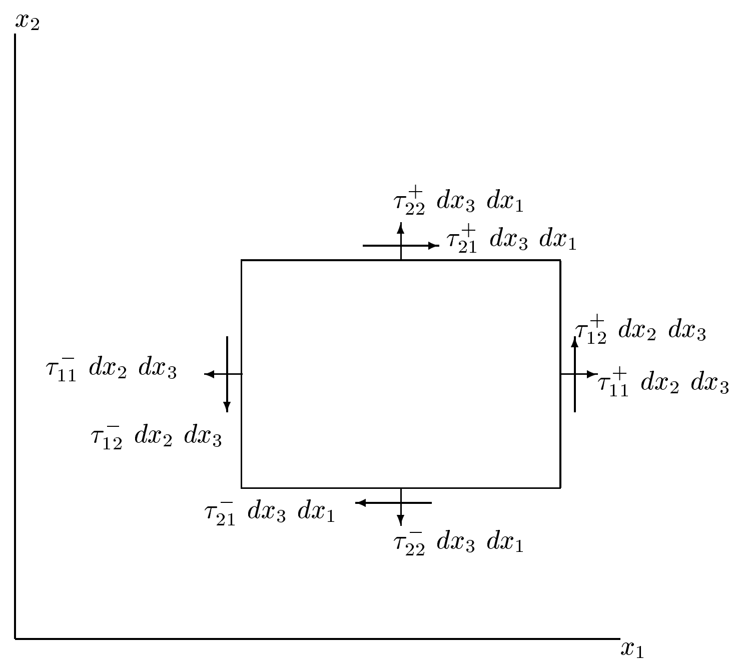


Figure 3.3: Forces for linear and angular momentum equations

The pair of faces with normals in the $\pm\ x_1$ direction contributes a force $(\partial\tau_{11}/\partial x_1)\ dx_1\ dx_2\ dx_3$. Similarly, on including also the contributions from the other two pairs of faces, we have the total surface force in this direction. Per unit volume, we get

$$f_{s1} = \frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} \tag{3.25}$$

The linear momentum in the x_1 direction gives

$$\rho \frac{Du_1}{Dt} = \frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + \rho f_1 \tag{3.26}$$

Generalizing this expression to include components of linear momentum in the other directions also, we have

$$\rho \frac{Du_j}{Dt} = \frac{\partial \tau_{ij}}{\partial x_i} + \rho f_j \tag{3.27}$$

which, for other coordinate systems, is

$$\rho \frac{D\mathbf{u}}{Dt} = \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f} \tag{3.28}$$

As a reminder, the left hand side represents the product of the mass times the acceleration of an element of fluid, while the two terms on the right are the surface and body forces respectively on this element, all per unit volume.

Multiplying the continuity equation, equation (3.10), by u_i and adding to equation (3.27), we get

$$\frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_i}(\rho u_i u_j) = \frac{\partial \tau_{ij}}{\partial x_i} + \rho f_j \tag{3.29}$$

or

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f} \tag{3.30}$$

which is another form of the momentum equation.

The linear momentum equation is also called the Cauchy equation.

3.4 Linear momentum equation in a non-inertial frame

If the coordinate system with respect to which the velocities and positions are measured is non-inertial, i.e. rotating and/or linearly accelerating, other terms have to be included in the momentum equation.

Consider a non-inertial frame characterized by $\mathbf{r}_0$, $\mathbf{u}_0$, and $\mathbf{a}_0$ which are the position, velocity, and acceleration of its origin, and $\boldsymbol{\Omega}$, and $\dot{\boldsymbol{\Omega}}$ which are its rotation rate, and angular acceleration, all with respect to an inertial frame. The positions of a particle measured in the two frames are then related by

$$\mathbf{r}' = \mathbf{r}_0 + \mathbf{r} \tag{3.31}$$

where $\mathbf{r}'$ and $\mathbf{r}$ are in the inertial and non-inertial frames, respectively. Similarly velocities are related by

$$\mathbf{u}' = \mathbf{u}_0 + \mathbf{u} + \boldsymbol{\Omega} \times \mathbf{r} \tag{3.32}$$

and accelerations by

$$\mathbf{a}' = \mathbf{a}_0 + \mathbf{a} + \dot{\boldsymbol{\Omega}} \times \mathbf{r} + 2\boldsymbol{\Omega} \times \mathbf{u} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \tag{3.33}$$

The linear momentum equation is then

$$\rho \left[\frac{Du_i}{Dt} + a_i + \epsilon_{jki}\Omega_j r_k + 2\epsilon_{jki}\Omega_j u_k + \epsilon_{lmi}\epsilon_{njm}\Omega_l \Omega_n r_j \right] = \frac{\partial \tau_{ij}}{\partial x_i} + \rho f_j \tag{3.34}$$

or

$$\rho \left[\frac{D\mathbf{u}}{Dt} + \mathbf{a} + \dot{\boldsymbol{\Omega}} \times \mathbf{r} + 2\boldsymbol{\Omega} \times \mathbf{u} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \right] = \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f} \tag{3.35}$$

$2\dot{\boldsymbol{\Omega}} \times \mathbf{u}$ is the Coriolis and $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$ the centripetal acceleration term.

3.5 Angular momentum equation

Consider an elemental control mass of instantaneous size $dx_1 \times dx_2 \times dx_3$. Figure 3.3 shows the stresses on a section of a fluid element normal to the x_3 axis. We can use Euler's equation which govern the angular dynamics of a rigid body[1]. For the x_3 axis

$$M_3 = I_3 \frac{d\Omega_3}{dt} + (I_2 - I_1)\Omega_1 \Omega_2 \tag{3.36}$$

where M_i, I_i, Ω_i are the total moment, the moment of inertia and angular velocity of rotation about the principal axis x_i. We will assume that the fluid is non-polar, i.e. that there are no body moments, so that the only contribution to the moment comes from the stresses.

[1]See, for instance, H. Goldstein, *Classical Mechanics*, Addison Wesley, Reading, MA, 1950.

The moment of inertia about the x_3 axis is

$$
\begin{aligned}
I_3 &= \frac{1}{12}(\text{mass of fluid element})(dx_1^2 + dx_2^2) \\
&= \frac{\rho}{12}(dx_1 \ dx_2 \ dx_3)(dx_1^2 + dx_2^2)
\end{aligned}
\tag{3.37}
$$

Similarly for I_1 and I_2. Equation (3.36) becomes

$$
\begin{aligned}
& (\tau_{12} - \frac{\partial \tau_{12}}{\partial x_1}\frac{dx_1}{2}) \ dx_2 \ dx_3 \frac{dx_1}{2} + (\tau_{12} + \frac{\partial \tau_{12}}{\partial x_1}\frac{dx_1}{2}) \ dx_2 \ dx_3 \ \frac{dx_1}{2} \\
- & (\tau_{21} - \frac{\partial \tau_{21}}{\partial x_2}\frac{dx_2}{2}) \ dx_1 \ dx_3 \frac{dx_2}{2} - (\tau_{21} + \frac{\partial \tau_{21}}{\partial x_2}\frac{dx_2}{2}) \ dx_1 \ dx_3 \ \frac{dx_2}{2} \\
= & \frac{\rho}{12} \ dx_1 \ dx_2 \ dx_3 (dx_1^2 + dx_2^2)\frac{d\Omega_3}{dt} \\
& + \frac{\rho}{12} \ dx_1 \ dx_2 \ dx_3 [(dx_1^2 + dx_3^2) - (dx_2^2 + dx_3^2)]\Omega_1 \Omega_2
\end{aligned}
\tag{3.38}
$$

We divide by $dx_1 \ dx_2 \ dx_3$ and take the limit as dx_1, dx_2, and dx_3 go to zero, to obtain

$$
\tau_{12} = \tau_{21} \tag{3.39}
$$

This can be repeated for the other two coordinate axes giving $\tau_{13} = \tau_{31}$ and $\tau_{32} = \tau_{23}$. Thus, the conservation of angular momentum of a fluid element implies that the stress tensor is symmetric:

$$
\tau_{ij} = \tau_{ji} \tag{3.40}
$$

or

$$
\boldsymbol{\tau} = \boldsymbol{\tau}^T \tag{3.41}
$$

3.6 Energy equation

We consider the elemental control mass shown in Figure 3.4. The total energy per unit mass of the fluid, e_t, is the sum of its internal and kinetic energy. Thus

$$
e_t = e + \frac{1}{2}u_i u_i \tag{3.42}
$$

where e is the specific internal energy. The rate of inflow of energy in the form of heat and work is equal to the rate of increase of total energy of the fluid. Thus, per unit volume, we have

$$
\rho \frac{De_t}{Dt} = \dot{Q} + \dot{W} \tag{3.43}
$$

where $\dot{Q}$ and $\dot{W}$ are the inflow of heat and work respectively per unit volume, the sign convention being that energy coming into the system is considered positive.

The heat inflow is due to the spatial variation in the heat flux $\dot{\mathbf{q}}$, which is a vector in the direction of the heat flow with units of energy per unit time per unit area. Let $\dot{q}_1^-$ and $\dot{q}_1^+$ be the heat flux components in the two faces which have normals in the $\pm \ x_1$ direction. The flow of heat through these faces are $\dot{q}_1^- \ dx_2 \ dx_3$ and $\dot{q}_1^+ \ dx_2 \ dx_3$, respectively. Since

$$
\dot{q}_1^+ = \dot{q}_1^- + \frac{\partial \dot{q}_1}{\partial x_1} \ dx_1 + \ldots \tag{3.44}
$$

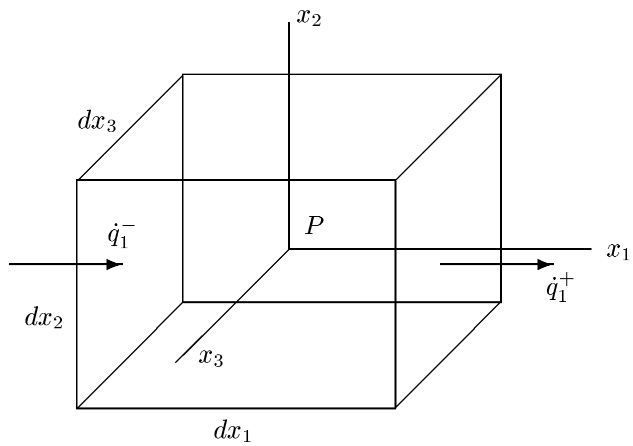


Figure 3.4: Energy fluxes in elemental volume

the net energy gain due to heat flow in the x_1-direction is

$$-\frac{\partial \dot{q}_1}{\partial x_1}\ dx_1\ dx_2\ dx_3$$

Similarly, one can obtain the contributions from the heat flow in the other two directions. Summing them up we get

$$\dot{Q} = -\frac{\partial \dot{q}_i}{\partial x_i} \tag{3.45}$$

on a unit volume basis.

Now let us consider the rate of work inflow $\dot{W}$. We know that the rate of work done by a force is the dot product of the force and velocity of its point of application. Let us look at the work done by the surface forces first. The rate of work done on the two $-$ face in the x_1-direction is

$$-(\tau_{11}^- u_1^- + \tau_{12}^- u_2^- + \tau_{13}^- u_3^-)\ dx_2\ dx_3 = -\tau_{1j}^- u_j^-\ dx_2\ dx_3$$

where the negative sign comes from the fact that the force and the velocity are in opposite directions. The work done on the + face in the same direction is

$$(\tau_{11}^+ u_1^+ + \tau_{12}^+ u_2^+ + \tau_{13}^+ u_3^+)\ dx_2\ dx_3 = \tau_{1j}^+ u_j^+\ dx_2\ dx_3$$

Since

$$\tau_{1j}^+ u_j^+ = \tau_{1j}^- u_j^- + \frac{\partial(\tau_{ij} u_j)}{\partial x_1}\ dx_1$$

the work inflow is

$$\frac{\partial(\tau_{1j} u_j)}{\partial x_1}\ dx_1\ dx_2\ dx_3$$

On including the contributions from the other two pairs of faces also, we have the rate of work inflow due to the surface forces $\dot{W}_s$, where

$$\dot{W}_s = \frac{\partial(\tau_{ij}u_j)}{\partial x_i} \tag{3.46}$$

per unit volume. In addition the work done by the body force $\dot{W}_b$ is

$$\dot{W}_b = \rho u_i f_i \tag{3.47}$$

per unit volume. So

$$\dot{W} = \dot{W}_s + \dot{W}_b \tag{3.48}$$

Using equations (3.45)–(3.48), equation (3.43) becomes

$$\rho\frac{De_t}{Dt} = -\frac{\partial \dot{q}_i}{\partial x_i} + \frac{\partial(\tau_{ij}u_j)}{\partial x_i} + \rho u_i f_i \tag{3.49}$$

or

$$\rho\frac{De_t}{Dt} = -\nabla\cdot\dot{\mathbf{q}} + \nabla\cdot(\boldsymbol{\tau}\cdot\mathbf{u}) + \rho\mathbf{u}\cdot\mathbf{f} \tag{3.50}$$

As a reminder, the left hand side represents the rate of change of total energy; the first term on the right is the heat inflow, the second is the work done by the surface forces, and the third the work done by gravity, all per unit volume.

The above is the equation of conservation of total energy. It can be simplified by subtracting out the mechanical energy part. Take the dot product of the velocity and the momentum equation (3.27)

$$\rho u_j\frac{Du_j}{Dt} = u_j\frac{\partial \tau_{ij}}{\partial x_i} + \rho u_j f_j \tag{3.51}$$

This is the mechanical energy equation; subtract this from the total energy equation to get

$$\rho\frac{De}{Dt} = -\frac{\partial \dot{q}_i}{\partial x_i} + \tau_{ij}\frac{\partial u_j}{\partial x_i} \tag{3.52}$$

or

$$\rho\frac{De}{Dt} = -\nabla\cdot\dot{\mathbf{q}} + \boldsymbol{\tau} : \nabla\mathbf{u} \tag{3.53}$$

This is the thermal energy equation, sometimes simply referred to as the energy equation.

3.7 Entropy equation

In one form the second law of thermodynamics states that the change in entropy of a system, δS, is governed by

$$\delta S \geq \frac{\delta Q}{T} \tag{3.54}$$

where δQ is the heat input to the system, and T is its absolute temperature. The equality holds for a reversible process.

Considering an elemental control mass of size $dx_1 \times dx_2 \times dx_3$, we find that

$$\rho\frac{Ds}{Dt} \geq -\frac{\partial}{\partial x_i}\left(\frac{\dot{q}_i}{T}\right) \tag{3.55}$$

or

$$\rho \frac{Ds}{Dt} \geq -\nabla \cdot \left(\frac{\dot{\mathbf{q}}}{T} \right) \tag{3.56}$$

per unit volume, where s is the specific entropy.

3.8 Additional relations

To complete the mathematical formulation of the problem, additional equations which involve properties specific to a particular fluid material are needed.

3.8.1 Equations of state

Incompressible fluid

In many occasions the density of fluid following a particle is taken to be constant. In this case

$$\frac{D\rho}{Dt} = 0 \tag{3.57}$$

In general the density may be different for different particles, though the homogeneous fluid is a special case of this.

Ideal gas

This is a commonly used approximation for the behavior of gases:

$$p = \rho R T \tag{3.58}$$

where R is the particular constant for the gas.

3.8.2 Constitutive equations

Inviscid fluid

The constitutive relation for a zero viscosity fluid is

$$\tau_{ij} = -p\delta_{ij} \tag{3.59}$$

or

$$\boldsymbol{\tau} = -p\mathbf{I} \tag{3.60}$$

The stress tensor is diagonal.

Newtonian fluid

A constitutive relation for a fluid is that which relates the stress and strain rate tensors. A Newtonian fluid has the following properties:

1. For a fluid at rest the stress is hydrostatic, and the pressure is the thermodynamic pressure.
2. The stress tensor $\boldsymbol{\tau}$ is linearly related to the deformation-rate tensor $\mathbf{D}$, and does not depend on the rate of rotation tensor.

3. There are no preferred directions in the fluid properties.

Under these conditions the constitutive relation for a Newtonian fluid can be shown to be

$$\tau_{ij} = \left(-p + \lambda \frac{\partial u_k}{\partial x_k}\right) \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{3.61}$$

or

$$\boldsymbol{\tau} = (-p + \lambda \nabla \cdot \mathbf{u})\,\mathbf{I} + \mu \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right] \tag{3.62}$$

where μ and λ are the first and second coefficients of viscosity.

Non-Newtonian fluids

A fluid which does not have a shear stress–rate of deformation relation given by equation (3.61) is non-Newtonian. In a broad sense, the behavior of non-magnetic continua can be divided into several categories:
(I) *Purely viscous fluids:* The shear rate depends only on the shear stress. The fluid can be Newtonian or non-Newtonian.
(II) *Time-dependent fluids:* The shear rate depends not only on the shear stress but also on the duration of the stress.
(III) *Viscoelastic materials:* The shear rate depends on the imposed stress as well as the strain.
(IV) *Complex rheological bodies:* Materials displaying combinations of the characteristics above.

Fourier's law

The heat flux due to conduction heat transfer in a fluid is governed by the relation

$$\dot{q}_i = -k \frac{\partial T}{\partial x_i} \tag{3.63}$$

or

$$\dot{\mathbf{q}} = -k \nabla T \tag{3.64}$$

where T is the fluid temperature, and $k(T)$ is the coefficient of thermal conductivity.

There are fluids in which a heat flux is produced by a concentration gradient also (diffusion-thermo or Dufour effect).

3.9 Governing equations for special cases

3.9.1 Mass balance

For an incompressible fluid, equation (3.10) reduces to

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{3.65}$$

or

$$\nabla \cdot \mathbf{u} = 0 \tag{3.66}$$

3.9.2 Linear momentum equation

Inviscid fluid

The momentum equation becomes

$$\rho \frac{Du_j}{Dt} = -\frac{\partial p}{\partial x_j} + \rho f_j \tag{3.67}$$

or

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{f} \tag{3.68}$$

This is called the Euler equation.

Newtonian fluid

Substituting this into the momentum equation (3.27) gives the so-called Navier-Stokes equation

$$\rho \frac{Du_j}{Dt} = -\frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\lambda \frac{\partial u_k}{\partial x_k}\right) + \frac{\partial}{\partial x_i}\left[\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] + \rho f_j \tag{3.69}$$

or

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \nabla\left(\lambda \nabla \cdot \mathbf{u}\right) + \nabla \cdot \left[\mu \left\{\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right\}\right] + \rho \mathbf{f} \tag{3.70}$$

Because of the continuity equation (3.65), the constitutive relation for an incompressible Newtonian fluid reduces to

$$\tau_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{3.71}$$

or

$$\boldsymbol{\tau} = -p\mathbf{I} + \mu \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right] \tag{3.72}$$

The second coefficient of viscosity does not play a role in the mechanics of incompressible fluids. Taking μ to be constant, the Navier-Stokes equation, equation (3.69), becomes

$$\rho \frac{Du_j}{Dt} = -\frac{\partial p}{\partial x_j} + \mu \frac{\partial^2 u_j}{\partial x_i x_i} + \rho f_j \tag{3.73}$$

or

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{f} \tag{3.74}$$

Often the equation is divided out by ρ and written as

$$\frac{Du_j}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_j} + \nu \frac{\partial^2 u_j}{\partial x_i x_i} + f_j \tag{3.75}$$

or

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f} \tag{3.76}$$

where

$$\nu = \frac{\mu}{\rho} \tag{3.77}$$

3.9.3 Energy equation

Inviscid fluid

With Fourier's law the energy equation is

$$\rho \frac{De}{Dt} = k \frac{\partial^2 T}{\partial x_i x_i} - p \frac{\partial u_i}{\partial x_i} \tag{3.78}$$

or

$$\rho \frac{De}{Dt} = k \nabla^2 T - p \nabla \cdot \mathbf{u} \tag{3.79}$$

where k has been taken to be constant. The last term on the right is the rate of work due to the pressure.

Newtonian fluid

For a Fourier-Newtonian fluid with constant properties, we have

$$\rho \frac{De}{Dt} = k \frac{\partial^2 T}{\partial x_i \partial x_i} - p \frac{\partial u_i}{\partial x_i} + \Phi \tag{3.80}$$

or

$$\rho \frac{De}{Dt} = k \nabla^2 T - p \nabla \cdot \mathbf{u} + \Phi \tag{3.81}$$

where

$$\Phi = \lambda \left(\frac{\partial u_i}{\partial x_i} \right)^2 + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_j}{\partial x_i} \tag{3.82}$$

or

$$\Phi = \lambda \left(\nabla \cdot \mathbf{u} \right)^2 + \mu \left[\nabla \mathbf{u} + \left(\nabla \mathbf{u} \right)^T \right] : \nabla \mathbf{u} \tag{3.83}$$

Φ is a positive quantity called the dissipation function; physically it represents the rate of change of energy per unit volume from mechanical to thermal form.

For an incompressible fluid, we can write

$$\frac{DT}{Dt} = \alpha \nabla^2 T + \frac{\Phi}{\rho c} \tag{3.84}$$

where $e = cT$ and $\alpha = k/\rho c$, where c is the specific heat and α is the thermal diffusivity.

3.9.4 Boundary conditions

Hydrodynamic and thermal boundary conditions are needed for the differential equations.

- For a viscous fluid the velocity at a solid wall is usually taken to be that of the wall itself. For an inviscid fluid only the normal velocities are equated.
- At a free surface the stress is continuous across the surface if surface tension is not considered. Otherwise there is a pressure differential given by

$$\Delta p = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \tag{3.85}$$

  where R_1 and R_2 are the principal radii of curvature of the free surface, and γ is the coefficient of surface tension.

- The temperature at the boundary is that of the wall itself. Sometimes, however, it is more convenient to to prescribe the normal derivative of the temperature at the wall; for an adiabatic wall it is zero, while for a wall with prescribed heat flux, it is a given quantity.

3.10 Nondimensionalization

The purpose of nondimensionalization is to normalize the variables and to bring out the relative importance of the different terms in the equations. Often the procedure depends on the problem being considered. Careful use of a method based on the dimensions of the physical quantities being considered will often provide considerable qualitative and quantitative information of the flow.

As an example of nondimensionalization let us look at incompressible Navier-Stokes and energy equations. Using asterisks for nondimensional quantities, we can choose the following dimensionless variables

$$t^* = tU/L,\ \mathbf{x}^* = \mathbf{x}/L,\ \mathbf{u}^* = \mathbf{u}/U,\ p^* = p/\rho U^2,\ T^* = (T - T_0)/\Delta T \tag{3.86}$$

where U and L are characteristic velocity and length scales; T_0 is a reference temperature and ΔT is a characteristic temperature difference. Writing $\mathbf{f} = g\mathbf{e}_g$, where $\mathbf{e}_g$ is the dimensionless unit vector in the direction of gravity, the equations become

$$\frac{D\mathbf{u}^*}{Dt^*} = -\nabla^* p^* + \frac{1}{Re}\nabla^{*2}\mathbf{u}^* + \frac{1}{Fr}\mathbf{e}_g \tag{3.87}$$

$$\frac{DT^*}{Dt^*} = \frac{1}{Re\ Pr}\nabla^{*2}T^* + \frac{Ec}{Re}\Phi^* \tag{3.88}$$

where ∇^* and ∇^{*2} are the nondimensional gradient and Laplacian operators respectively, and Φ^* is the nondimensional dissipation function. In this case the important parameters are Re, Pr, and Ec. These and some other nondimensional groups that commonly occur in fluid mechanics and heat transfer are listed below along with their physical significance.

Name	*Symbol*	*Definition*	*Physical significance*
Biot number	Bi	hL/k_s	ratio of solid to fluid thermal resistance
Eckert number	Ec	$U^2/c\Delta T$	ratio of dissipation to internal energy change
Fourier number	Fo	$\alpha t/L^2$	dimensionless time
Grashof number	Gr	$g\beta\Delta T L^3/\nu^2$	ratio of buoyancy to viscous forces
Nusselt number	Nu	hL/k	dimensionless heat transfer coefficient
Peclet number	Pe	$\rho c U L/k$	ratio of convection to conduction heat transfer
Prandtl number	Pr	$\mu c/k$	ratio of momentum to thermal diffusivities
Rayleigh number	Ra	$g\beta\Delta T L^3/\alpha\nu$	$= Gr\ Pr$
Reynolds number	Re	$\rho U L/\mu$	ratio of inertia to viscous forces

Additional nomenclature:

g	acceleration due to gravity
h	convective heat transfer coefficient
k	thermal conductivity of fluid
k_s	thermal conductivity of solid
α	thermal diffusivity
β	coefficient of thermal expansion
ν	kinematic viscosity $= \mu/\rho$

3.11 Molecular approach

See I. Michelson, *Molecular Basis of Fluid Mechanics.*

Problems

1. Show that ρ = constant, $\mathbf{u} = x_1\mathbf{e}_1 - x_2\mathbf{e}_2$ satisfy the continuity equation.
2. Show that the following velocity field in polar coordinates
$$\mathbf{u} = U(1 - \frac{a^2}{r^2}) \cos\theta \mathbf{e}_r - U(1 + \frac{a^2}{r^2}) \sin\theta \mathbf{e}_\theta .$$
is incompressible and irrotational.
3. Show that the two-dimensional velocity field
$$\mathbf{u} = -\frac{Kx_2}{x_1^2 + x_2^2}\mathbf{e}_1 + \frac{Kx_1}{x_1^2 + x_2^2}\mathbf{e}_2$$
where K is a constant, is incompressible and irrotational.
4. From an elemental control volume in cylindrical coordinates, derive the continuity equation in cylindrical coordinates.
5. Show that the continuity equation can also be written as
$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{u} = 0.$$
6. Using equations (2.21) and (3.10), show that the difference between equations (3.49) and (3.51) is (3.52).
7. Show that the dissipation function in equation (3.82) can also be written as
$$\Phi = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$
8. Show that the continuity equation for axisymmetric flow in spherical coordinates can be satisfied by the Stokes stream function $\psi(r, \theta)$, where
$$u_r = \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta},\ u_\theta = -\frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r}$$
9. Coordinates (ξ, η, ζ) in an elliptic cylindrical coordinate system are related to the Cartesian coordinates (x, y, z) through $x = a \cosh\xi \cos\eta,\ y = a \sinh\xi \sin\eta,\ z = \zeta$. If u_ξ, u_η and u_ζ are the fluid velocities in the directions of increasing ξ, η, and ζ respectively, write the continuity equation in elliptic cylindrical coordinates for an incompressible fluid.
10. For a Newtonian fluid with velocity field $\mathbf{u} = x_1\mathbf{e}_1 - x_2\mathbf{e}_2$, find the stress tensor and, by substituting into the governing equations, the pressure distribution $p(x, y)$.
11. Find the stress tensor for a Newtonian fluid with the velocity field $\mathbf{u} = x_2^2\mathbf{e}_1 + x_1^2\mathbf{e}_2$.
12. Neglecting viscous forces, find the diameter of a laminar water jet coming down from a faucet as a function of the distance downstream.
13. Find the stress tensor for a Newtonian fluid with velocity field $\mathbf{u} = x_1\mathbf{e}_1 - x_2\mathbf{e}_2$ in Cartesian coordinates.
14. For the toroidal coordinates (r, ϕ, θ) in the figure (the cross section has been enlarged to indicate r and ϕ) show that
$$x = (R + r\cos\phi)\cos\theta,\ y = (R + r\cos\phi)\sin\theta,\ z = r\sin\phi$$
where (x, y, z) are suitably chosen Cartesian coordinates.
Find expressions for the gradient, divergence and curl in this coordinate system.
15. Given a two-dimensional velocity field $(x_1 + x_2)\mathbf{e}_1 + (3x_1 + 4x_2)\mathbf{e}_2$, show that the flow is compressible. Find the strain rate and vorticity tensors, and (c) the principal axes of the strain rate tensor.
16. For steady flow and a divergence-free velocity field, show that the flow lines are also constant density lines.

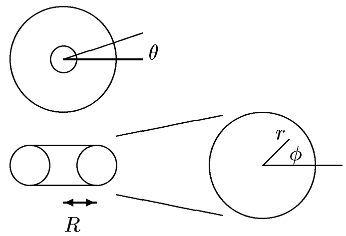


Figure 3.5: Toroidal coordinate system (r, ϕ, θ)

17. For the velocity field $\mathbf{u} = x_1\mathbf{e}_1 - x_2\mathbf{e}_2$, find (a) the acceleration of the elemental volume of size $dx_1 \times dx_2 \times dx_3$ at (x_1, x_2, x_3), and (b) from Newton's second law, the net force on the volume required to produce this acceleration.
18. Derive the linear momentum equation in cylindrical coordinates.
19. Show that
$$\frac{\partial}{\partial t}(\rho u_i) = \frac{\partial}{\partial x_i}\left(\tau_{ji} - \rho u_j u_i\right)$$
20. Show the theorem of stress means:
$$\int_A F\tau_{ij} n_j \; dA = \int_{\mathcal{V}} \left[\tau_{ij}\frac{\partial F}{\partial x_j} + \rho F\left(\frac{Du_i}{Dt} - f_i\right)\right] \; d\mathcal{V}$$
for any function $F(x_i, t)$.
21. Starting from an elemental *control volume* in Cartesian coordinates, prove the momentum equation.
22. For each one of the N chemical species in a fluid mixture, m_i is the mass of one mole, n_i is the number of moles per unit volume of fluid, $\mathbf{u}_i$ is the velocity, and $\dot{r}_i$ is the generation by chemical reaction per unit volume per unit time, where $i = 1, \ldots, N$. Show that the mass balance equation for each species is
$$\frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i \mathbf{u}_i) = \dot{r}_i$$
where $\rho_i = m_i n_i$ (no sum). Show also that
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$
where $\sum_{i=1}^N \rho_i = \rho$ and $\rho\mathbf{u} = \sum_{i=1}^N \rho_i \mathbf{u}_i$.
23. Use the Reynolds's transport theorem to derive the mass, momentum and total energy equations.
24. The governing equations for mass, momentum and internal energy balance for any fluid may be combined and written in the form
$$\frac{\partial \mathbf{f}}{\partial t} + \frac{\partial \mathbf{p}}{\partial x} + \frac{\partial \mathbf{q}}{\partial y} + \frac{\partial \mathbf{r}}{\partial z} = \mathbf{s}$$
in Cartesian coordinates. Find the column matrices $\mathbf{f}$, $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, and $\mathbf{s}$ in terms of the density and components of the velocity vector, stress tensor, and body force vector.
25. For a Newtonian fluid with velocity field $\mathbf{u} = x_1\mathbf{e}_1 - x_2\mathbf{e}_2$, find, by substituting into the governing equations, the pressure distribution $p(x_1, x_2)$ and the stress tensor.
26. Show that the tensors
$$\delta_{ij}$$
and
$$\alpha\delta_{ij}\delta_{pq} + \beta\left(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}\right) + \gamma\left(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}\right)$$
are isotropic, where α, β, and γ are arbitrary scalars.
27. Write down the parameters that might affect the period of a swinging pendulum and find, by dimensional analysis, the period as a function of these parameters.

28. If **r**' and **r** are the position vectors of a particle in an inertial and non-inertial frame of reference, show that

$$\frac{d^2\mathbf{r}'}{dt^2} = \frac{d^2\mathbf{r}}{dt^2} + \mathbf{a} + \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r} + 2\boldsymbol{\Omega} \times \frac{d\mathbf{r}}{dt} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$$

where **a** and $\boldsymbol{\Omega}$ are the linear acceleration and rotation vectors of the non-inertial frame[2].

29. The momentum equation for an incompressible, Newtonian fluid in a coordinate system rotating at an angular speed $\boldsymbol{\Omega}$ is

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}$$

where **r** is a position vector. Nondimensionalize this equation, find the nondimensional groups, and from this determine if the rotation of the earth is important in the study of oceanographic phenomena? Choose appropriate scales.

30. Show that for a steady flow, the no-slip condition at a fixed wall requires that $\nabla \cdot \mathbf{u}$ also vanish at the wall.

31. For an incompressible, Newtonian fluid with constant properties in a closed container of volume V, show that the rate of decrease of total kinetic energy of the fluid is

$$\mu \int_{\mathcal{V}} \boldsymbol{\omega} \cdot \boldsymbol{\omega} \; d\mathcal{V}$$

where $\boldsymbol{\omega}$ is the vorticity vector. The body force is conservative. In a similar way, find the rate of change of total internal energy of the fluid.

32. For steady, inviscid, barotropic flow show that Bernoulli's equation holds along a vortex line.

33. Using the vorticity equation for an inviscid fluid, show that if the vorticity of a fluid particle is once zero it is always zero.

34. Find the stress tensor for a Newtonian fluid corresponding to the velocity field $\mathbf{u} = x_1\mathbf{e}_1 - x_2\mathbf{e}_2$. Find also the components of the same tensor in a coordinate system that is rotated 45° about the x_3-axis.

35. The momentum equation for an incompressible, Newtonian fluid in a coordinate system rotating at a constant angular speed $\boldsymbol{\Omega}$ is

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}$$

where **r** is a position vector. Show that the vorticity equation has an additional term $2(\boldsymbol{\Omega} \cdot \nabla)\mathbf{u}$ as compared to the non-rotating case[3].

36. Applying the momentum equation

$$\frac{d}{dt}\int_{CV} \rho \mathbf{u} \; d\mathcal{V} + \int_{CS} \mathbf{u}\,(\rho \mathbf{u} \cdot \mathbf{n}) \; dA = \mathbf{F}$$

over an elemental control volume in Cartesian coordinates, derive the momentum equation in differential form.

37. Write the vorticity equation for an incompressible, Newtonian fluid in component form using polar coordinates (r, θ) and the notation

$$\begin{aligned} \mathbf{u} &= u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta \\ \boldsymbol{\omega} &= \omega_z \mathbf{e}_z \end{aligned}$$

38. (a) Show that the moment of the surface forces $\boldsymbol{\tau}(\mathbf{r})$ about a point O within an arbitrary volume $\mathcal{V}$ with surface S is

$$\int_S \epsilon_{ijk} r_j \tau_{km} n_m \; dA$$

where **r** is the position vector relative to O of an elemental surface area dA with normal **n**.

(b) Using Gauss's theorem write this as

$$\int_{\mathcal{V}} \epsilon_{ijk} \frac{\partial (r_j \tau_{km})}{\partial x_m} \; d\mathcal{V} = \int_{\mathcal{V}} \epsilon_{ijk} \left[\tau^0_{kj} + r_j \frac{\partial \tau^0_{km}}{\partial x_m} \right] d\mathcal{V}$$

where $\boldsymbol{\tau}^0$ is the stress at O.

(c) As $\mathcal{V} \to 0$, show that this reduces to $\epsilon_{ijk}\tau^0_{kj} = 0$, from which $\tau^0_{ij} = \tau^0_{ji}$.

[2]Consult your favorite book on classical mechanics.

[3]Hint: You may use Cartesian coordinates for some of the steps.

39. Show that the incompressible Navier-Stokes equation with constant properties can be written as

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla P - \nu\nabla \times (\nabla \times \mathbf{u})$$

where

$$\begin{aligned} P &= p - \rho G \\ \mathbf{f} &= \nabla G \end{aligned}$$

40. Consider the incompressible Navier-Stokes equation for a fluid in coordinates that are rotating at a rate $\mathbf{\Omega} = \Omega\,\mathbf{k}$, where Ω is a constant. Neglect the body force. Use the following scalings: U for velocity, L for length, L/U for time, and ρU^2 for pressure to non-dimensionlize the equation. Show that the Rossby number $\epsilon = U/\Omega L$ and the Ekman number $E = \nu/\Omega L^2$ appear as nondimensional parameters.

41. For flow in a coordinate system fixed to the earth, find numerical values for the Rossby and Ekman numbers for (a) 1 m/s water velocity on a scale of 30 cm and (b) 100 km/hr wind at a scale of 100 km. For each indicate if the rotation of the earth is important.

42. Using the Gibbs relation

$$T\ ds = de + p\ d\left(\frac{1}{\rho}\right)$$

and the definition of enthalpy

$$h = e + \frac{p}{\rho}$$

show that the energy equation for a Fourier-Newtonian fluid can be written as

$$\rho T\frac{Ds}{Dt} = \nabla\cdot(k\nabla T) + \Phi$$

or

$$\rho\frac{Dh}{Dt} = \nabla\cdot(k\nabla T) + \frac{Dp}{Dt} + \Phi$$

where

$$\Phi = \lambda\,(\nabla\cdot\mathbf{u})^2 + \mu\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right] : \nabla\mathbf{u}$$

43. Show that the 2nd law of thermodynamics for a Fourier-Newtonian fluid can be written as

$$\Phi + \frac{k}{T}\nabla T\cdot\nabla T \geq 0$$

Chapter 4

Special theorems

4.1 Circulation

The circulation Γ around a closed curve $\mathcal{C}$ is defined as

$$\Gamma = \oint_{\mathcal{C}} u_i \; dl_i \tag{4.1}$$

or

$$\Gamma = \oint_{\mathcal{C}} \mathbf{u} \cdot d\mathbf{l} \tag{4.2}$$

From Stokes's theorem, we have

$$\oint \mathbf{u} \cdot d\mathbf{l} = \int_{\mathcal{A}} \nabla \times \mathbf{u} \cdot \mathbf{n} \; d\mathcal{A} \tag{4.3}$$

so that

$$\Gamma = \int_{\mathcal{A}} \boldsymbol{\omega} \cdot \mathbf{n} \; d\mathcal{A} \tag{4.4}$$

where $\mathcal{A}$ is any surface with the boundary $\mathcal{C}$.

4.2 Kelvin's theorem

We consider an inviscid, barotropic fluid (one for which the pressure is only a function of density) with a conservative body force. The circulation around a closed material curve $\mathcal{C}$ is

$$\Gamma = \oint_{\mathcal{C}} u_j \; dl_j \tag{4.5}$$

so that

$$\frac{D\Gamma}{Dt} = \oint_{\mathcal{C}} \left[\frac{Du_j}{Dt} \; dl_j + u_j \frac{D(dl_j)}{Dt} \right] \tag{4.6}$$

But

$$\oint_{\mathcal{C}} u_j \frac{D(dl_j)}{Dt} = \oint_{\mathcal{C}} u_j \; du_j$$

$$= \oint_C d\left(\frac{1}{2}u_j u_j\right) \tag{4.7}$$
$$= 0 \tag{4.8}$$

Also

$$\frac{Du_j}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_j} + \frac{\partial G}{\partial x_j} \tag{4.9}$$

Since

$$\oint_C \frac{1}{\rho}\frac{\partial p}{\partial x_j}\, dx_j = \oint_C \frac{dp}{\rho} \tag{4.10}$$

and

$$\begin{aligned} \oint_C \frac{\partial G}{\partial x_j}\, dx_j &= \oint_C dG \\ &= 0 \end{aligned} \tag{4.11}$$

the only non-zero term is

$$\frac{D\Gamma}{Dt} = -\oint_C \frac{dp}{\rho} \tag{4.12}$$

For a barotropic flow for which ρ and p are uniquely related, this is also zero. Thus

$$\frac{D\Gamma}{Dt} = 0 \tag{4.13}$$

4.3 Vorticity equation

For a Newtonian fluid with constant density and properties and conservative body force, the momentum equation is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla\left(\frac{p}{\rho}\right) + \nu + \nabla^2\mathbf{u} + \nabla G \tag{4.14}$$

Because of the identity

$$\nabla(\mathbf{u}\cdot\mathbf{u}) = 2\mathbf{u}\cdot\nabla\mathbf{u} + 2\mathbf{u}\times\nabla\times\mathbf{u} \tag{4.15}$$

it can be written as

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) - \mathbf{u}\times\boldsymbol{\omega} = -\nabla\left(\frac{p}{\rho}\right) + \nu\nabla^2\mathbf{u} \tag{4.16}$$

Since

$$\nabla\times\nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{p}{\rho} - G\right) = 0 \tag{4.17}$$
$$\nabla\times(\mathbf{u}\times\omega) = \mathbf{u}\nabla\cdot\boldsymbol{\omega} - \boldsymbol{\omega}\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\nabla\mathbf{u} \tag{4.18}$$
$$= -\mathbf{u}\cdot\nabla\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\nabla\mathbf{u} \tag{4.19}$$

the curl of the momentum equation gives us

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\nabla\mathbf{u} + \nu\nabla^2\boldsymbol{\omega} \tag{4.20}$$

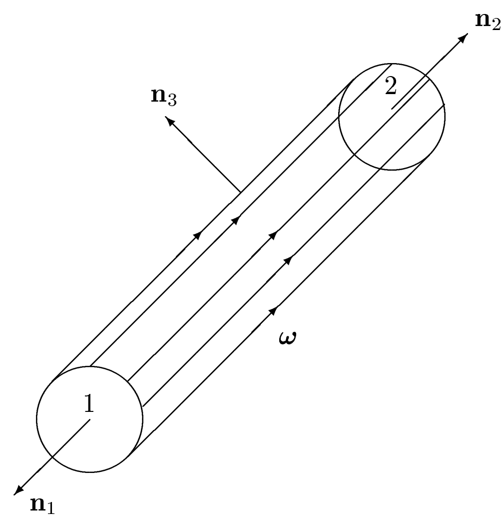


Figure 4.1: Portion of vortex tube.

The term $\boldsymbol{\omega} \cdot \nabla \mathbf{u}$ represents stretching and tilting of the vortex lines.

For a two-dimensional flow in the x-y plane, the vorticity

$$\boldsymbol{\omega} = \zeta \mathbf{e}_z \tag{4.21}$$

is in the z-direction, so that

$$\frac{\partial \zeta}{\partial t} + \mathbf{u} \cdot \nabla \zeta = \nu \nabla^2 \zeta \tag{4.22}$$

4.4 Helmholtz's theorem

Many of the ideas regarding vorticity, vortex lines and vortex tubes are summarized in the following theorems. A portion of a vortex tube is shown in Fig. 4.1.

- *Vortex lines move with the fluid:*

  Consider a small area $d\mathcal{A}$ on the side of a vortex tube. Since the vorticity vector and the unit normal to this area are perpendicular, the circulation around $d\mathcal{A}$ is zero. After an interval to time, the fluid forming this area has moved to a new position, but its circulation, by Kelvin's theorem, is still zero. This can be said for all $d\mathcal{A}$ lying on the surface of the vortex tube so that the fluid elements forming the new tube also form a vortex tube. Thus vortex tubes move with the fluid. In the limit an infinitesimal vortex tube is a line moving with the fluid.

- *Circulation around a vortex tube is constant:*

  The sides of a vortex tube are vortex lines. Since

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} \tag{4.23}$$

we have that

$$\nabla \cdot \boldsymbol{\omega} = 0 \tag{4.24}$$

We integrate this over the volume of a vortex tube

$$\int_{\mathcal{V}} \nabla \cdot \boldsymbol{\omega} \; d\mathcal{V} = 0 \tag{4.25}$$

Using Gauss's theorem, we have

$$\int_{\mathcal{A}} \boldsymbol{\omega} \cdot \mathbf{n} \; d\mathcal{A} = 0 \tag{4.26}$$

where $\mathbf{n}$ is the unit normal vector to the surface $\mathcal{A}$. Since the integrand vanishes on the sides of the vortex tube, we have that

$$\int_{\mathcal{A}_1} \boldsymbol{\omega} \cdot \mathbf{n}_1 \; d\mathcal{A} + \int_{\mathcal{A}_2} \boldsymbol{\omega} \cdot \mathbf{n}_2 \; d\mathcal{A} = 0 \tag{4.27}$$

from which

$$\Gamma_1 = \Gamma_2 \tag{4.28}$$

- *A vortex tube cannot end with a fluid; it must end at a boundary or form a closed loop:*

  Since the circulation is constant along a vortex tube, the tube cannot end within a fluid.

- *Circulation around a vortex tube is constant in time:*

  This follows from Kelvin's theorem since the vortex tubes are made of material lines.

4.5 Bernoulli's theorem

For an inviscid fluid with a conservative body force, the momentum equation is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{\rho} \nabla p + \nabla G \tag{4.29}$$

We have the identity

$$\mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2} \nabla \left(\mathbf{u} \cdot \mathbf{u} \right) - \mathbf{u} \times \nabla \times \mathbf{u} \tag{4.30}$$

Furthermore, if the flow is barotropic

$$\nabla \int \frac{dp}{\rho} = \frac{1}{\rho} \nabla p \tag{4.31}$$

Thus

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \left(\int \frac{dp}{\rho} + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} - G \right) = \mathbf{u} \times \boldsymbol{\omega} \tag{4.32}$$

There are two special cases.

4.5.1 Steady flow

Defining

$$B = \int \frac{dp}{\rho} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} - G \tag{4.33}$$

we see that

$$\nabla B = \mathbf{u} \times \boldsymbol{\omega} \tag{4.34}$$

Since

$$\begin{aligned} \frac{DB}{Dt} &= \frac{\partial B}{\partial t} + \mathbf{u}\cdot\nabla B \\ &= 0 \end{aligned} \tag{4.35}$$

B is a constant along a pathline which, for steady flow, is also a streamline. The constant may be different for different streamlines.

4.5.2 Irrotational flow

Since $\boldsymbol{\omega} = 0$, we can take $\mathbf{u} = \nabla\phi$. Thus, from equation (4.32), we have

$$\nabla\left(\frac{\partial\phi}{\partial t} + B\right) = 0 \tag{4.36}$$

Thus,

$$\frac{\partial\phi}{\partial t} + \int \frac{dp}{\rho} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} - G \tag{4.37}$$

is constant everywhere, though it may vary with time.

Problems

1. For an inviscid, barotropic fluid with a conservative body force in a coordinate system rotating at a constant rate $\mathbf{\Omega}$, show that Kelvin's theorem is

$$\frac{D}{Dt}\left(\oint_C \mathbf{u}\cdot\, d\mathbf{l} + 2\int_A \mathbf{\Omega}\cdot\mathbf{n}\, dA\right) = 0$$

where A is an area bounded by the closed curve C. [Hint: Start with $\mathbf{u}' = \mathbf{u} + \mathbf{\Omega}\times\mathbf{r}$, where $\mathbf{u}'$ and $\mathbf{u}$ are the fluid velocities in non-rotating and rotating frames, and $\mathbf{r}$ is the position vector in the rotating frame.]

2. For steady, inviscid, barotropic flow show that Bernoulli's equation holds along a vortex line.
3. Using the vorticity equation for an inviscid fluid, show that if the vorticity of a fluid particle is once zero it is always zero.
4. Derive the vorticity equation for a compressible Newtonian fluid with constant properties and with a conservative body force.

Chapter 5

Ideal flow

In this chapter we will discuss the motion of an incompressible, inviscid fluid without vorticity.

For an inviscid fluid we know that a fluid particle, once irrotational, is always irrotational. Thus the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is zero everywhere for such a flow, so that

$$\nabla \times \mathbf{u} = 0 \tag{5.1}$$

everywhere. This is satisfied by

$$\mathbf{u} = \nabla \phi \tag{5.2}$$

where ϕ is called the velocity potential.

In addition let us consider the fluid to be incompressible, so that the continuity equation is

$$\nabla \cdot \mathbf{u} = 0 \tag{5.3}$$

from which

$$\nabla^2 \phi = 0 \tag{5.4}$$

For an inviscid fluid the normal velocity at a solid or impermeable boundary must be zero. Thus

$$\left. \frac{\partial \phi}{\partial n} \right|_w = 0 \tag{5.5}$$

is the boundary condition for equation (5.4), where n is the coordinate normal to the wall.

5.1 Two-dimensional flows

Now, we will also, for simplicity, confine ourselves to plane, two-dimensional flow. Thus, we can write the continuity equation as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.6}$$

This is satisfied by

$$u = \frac{\partial \psi}{\partial y} \tag{5.7}$$

$$v = -\frac{\partial \psi}{\partial x} \tag{5.8}$$

where $\psi(x, y)$ is the stream function.

In two dimensions the vorticity $\boldsymbol{\omega} = \omega_z \mathbf{k} = \zeta \mathbf{k}$, where

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{5.9}$$

For an irrotational flow $\zeta = 0$, so that we can write

$$u = \frac{\partial \phi}{\partial x} \tag{5.10}$$

$$v = \frac{\partial \phi}{\partial y} \tag{5.11}$$

which corresponds to equation (5.2).

The stream function and velocity potential both satisfy Laplace's equations

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{5.12}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{5.13}$$

5.2 Properties

(a) Consider a line with $\psi =$ constant. Along this line

$$0 = d\psi \tag{5.14}$$

$$= \frac{\partial \psi}{\partial x}\, dx + \frac{\partial \psi}{\partial y}\, dy \tag{5.15}$$

$$= -v\ dx + u\ dy \tag{5.16}$$

from which

$$\frac{dx}{u} = \frac{dy}{v} \tag{5.17}$$

Since this is the equation of a streamline, it follows that the stream function is constant along a stream line.

(b) Let Q be the volume flow rate per unit depth between two stream lines with stream functions ψ_1 and ψ_2, respectively. Then

$$dQ = u\ dy - v\ dx \tag{5.18}$$

$$= \frac{\partial \psi}{\partial y}\, dy + \frac{\partial \psi}{\partial x}\, dx \tag{5.19}$$

$$= d\psi \tag{5.20}$$

so that we have

$$Q = \psi_2 - \psi_1 \tag{5.21}$$

(c) The ψ = constant lines have a slope

$$\left(\frac{dy}{dx}\right)_\psi = \frac{v}{u} \tag{5.22}$$

On the other hand, the ϕ = constant lines have a slope

$$\left(\frac{dy}{dx}\right)_\phi = -\frac{u}{v} \tag{5.23}$$

from which

$$\left(\frac{dy}{dx}\right)_\psi \left(\frac{dy}{dx}\right)_\phi = -1 \tag{5.24}$$

The constant ψ and constant ϕ lines are thus orthogonal to each other.

5.3 Complex representation

Comparing equations (5.10) and (5.11) with (5.7) and (5.8), we have

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \tag{5.25}$$

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial y} \tag{5.26}$$

These are the Cauchy-Riemann conditions for the complex function

$$F(z) = \phi(x,y) + i\psi(x,y) \tag{5.27}$$

to be analytic everywhere. $z = x + iy$ is the position coordinate, and $F(z)$ is the complex potential.

The derivative of the complex potential is

$$W(z) = \frac{dF}{dz} \tag{5.28}$$

where W is referred to as the complex velocity. Since $F(z)$ is an analytic function, we can take its derivative in any direction. Choosing the x-direction, we get

$$\begin{aligned} W(z) &= \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} \\ &= u - iv \end{aligned} \tag{5.29}$$

Furthermore

$$W\overline{W} = u^2 + v^2 \tag{5.30}$$

5.4 Polar form

We will frequently have to use polar coordinates (r, θ), and their corresponding velocity components (u_r, u_θ) as shown in Fig. 5.1. The velocity vector can be written as

$$\mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta \tag{5.31}$$

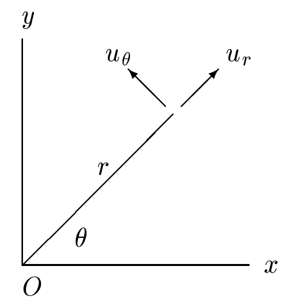


Figure 5.1: Velocity components u_r and u_θ in polar coordinates.

The incompressibility condition is

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}(ru_r) = 0 \tag{5.32}$$

so that we can define the stream function as

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \tag{5.33}$$

$$u_\theta = -\frac{\partial \psi}{\partial r} \tag{5.34}$$

$$\tag{5.35}$$

The irrotationality condition is

$$\zeta = \frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial \theta}\right] \tag{5.36}$$

$$= 0 \tag{5.37}$$

from which

$$u_r = \frac{\partial \phi}{\partial r} \tag{5.38}$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \tag{5.39}$$

$$\tag{5.40}$$

Laplace's equations for ψ and ϕ are

$$\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 \psi}{\partial \theta^2} = 0 \tag{5.41}$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 \phi}{\partial \theta^2} = 0 \tag{5.42}$$

Since $F = \phi + i\psi$, we have

$$W = \frac{dF}{dz} \tag{5.43}$$

$$= e^{-i\theta}\left(\frac{\partial\phi}{\partial r} + i\frac{\partial\psi}{\partial r}\right) \tag{5.44}$$

$$= (u_r - iu_\theta)e^{-i\theta} \tag{5.45}$$

Referring to Fig.5.1 the relation between the Cartesian and polar components of the velocity is found to be

$$u = u_r\cos\theta - u_\theta\sin\theta \tag{5.46}$$

$$v = u_r\sin\theta + u_\theta\cos\theta \tag{5.47}$$

and

$$u_r = u\cos\theta + v\sin\theta \tag{5.48}$$

$$u_\theta = -u\sin\theta + v\cos\theta \tag{5.49}$$

The polar unit vectors can be written in terms of the Cartesian unit vectors as

$$\mathbf{e}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \tag{5.50}$$

$$\mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \tag{5.51}$$

Notice that the polar unit vectors depend on θ. Thus

$$\frac{\partial\mathbf{e}_r}{\partial\theta} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \tag{5.52}$$

$$= \mathbf{e}_\theta \tag{5.53}$$

$$\frac{\partial\mathbf{e}_\theta}{\partial\theta} = -\cos\theta\mathbf{i} - \sin\theta\mathbf{j} \tag{5.54}$$

$$= -\mathbf{e}_r \tag{5.55}$$

5.5 Summary of equations

	Cartesian	*Polar*
Incompressibility	$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$	$\frac{1}{r}\left[\frac{\partial}{\partial r}(ru_r)+\frac{\partial u_\theta}{\partial\theta}\right]=0$
Irrotationality	$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0$	$\frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\theta)-\frac{\partial u_r}{\partial\theta}\right]=0$
Velocity potential	$u=\frac{\partial\phi}{\partial x}$ $v=\frac{\partial\phi}{\partial y}$	$u_r=\frac{\partial\phi}{\partial r}$ $u_\theta=\frac{1}{r}\frac{\partial\phi}{\partial\theta}$
Stream function	$u=\frac{\partial\psi}{\partial y}$ $v=-\frac{\partial\psi}{\partial x}$	$u_r=\frac{1}{r}\frac{\partial\psi}{\partial\theta}$ $u_\theta=-\frac{\partial\psi}{\partial r}$
Cauchy-Riemann eqns.	$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y}$ $\frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x}$	$\frac{\partial\phi}{\partial r}=\frac{1}{r}\frac{\partial\psi}{\partial\theta}$ $\frac{1}{r}\frac{\partial\phi}{\partial\theta}=-\frac{\partial\psi}{\partial r}$
Laplace's eqn.	$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$ $\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$	$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r})+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0$ $\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\psi}{\partial r})+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}=0$
Complex potential	$F=\phi+i\psi$	$F=\phi+i\psi$
Complex velocity	$W=u-iv$	$W=(u_r-iu_\theta)e^{-i\theta}$

5.6 Simple flows

5.6.1 Uniform flow

$$F = Uz \tag{5.56}$$
$$\psi = Uy \tag{5.57}$$
$$\phi = Ux \tag{5.58}$$
$$W = U \tag{5.59}$$
$$u = U \tag{5.60}$$
$$v = 0 \tag{5.61}$$

5.6.2 Source or sink

$$F = \frac{m}{2\pi}\ln z \tag{5.62}$$
$$\psi = \frac{m}{2\pi}\theta \tag{5.63}$$
$$\phi = \frac{m}{2\pi}\ln r \tag{5.64}$$

$$W = \frac{m}{2\pi z} \tag{5.65}$$

$$u_r = \frac{m}{2\pi} \tag{5.66}$$

$$u_\theta = 0 \tag{5.67}$$

5.6.3 Vortex

$$F = -i\frac{\Gamma}{2\pi}\ln z \tag{5.68}$$

$$\psi = -\frac{\Gamma}{2\pi}\ln r \tag{5.69}$$

$$\phi = \frac{\Gamma}{2\pi}\theta \tag{5.70}$$

$$W = -i\frac{\Gamma}{2\pi z} \tag{5.71}$$

$$u_r = 0 \tag{5.72}$$

$$u_\theta = \frac{\Gamma}{2\pi r} \tag{5.73}$$

5.6.4 Sector with angle π/n

$$F = Uz^n \tag{5.74}$$

$$\psi = Ur^n \sin n\theta \tag{5.75}$$

$$\phi = Ur^n \cos n\theta \tag{5.76}$$

$$W = nUz^{n-1} \tag{5.77}$$

$$u_r = nUr^{n-1}\cos n\theta \tag{5.78}$$

$$u_\theta = -nUr^{n-1}\sin n\theta \tag{5.79}$$

5.7 Combined flows

Since the Laplace's equation is linear, solutions may be added together to produce other solutions.

5.7.1 Doublet

$$F = \frac{\mu}{z} \tag{5.80}$$

$$\psi = -\mu\frac{\sin\theta}{r} \tag{5.81}$$

$$\phi = \mu\frac{\cos\theta}{r} \tag{5.82}$$

$$W = -\frac{\mu}{z^2} \tag{5.83}$$

$$u_r = -\mu\frac{\cos\theta}{r^2} \tag{5.84}$$

$$u_\theta = -\mu\frac{\sin\theta}{r^2} \tag{5.85}$$

5.7.2 Cylinder without circulation

$$F = Uz\left(1 + \frac{a^2}{z^2}\right) \tag{5.86}$$

$$\psi = Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta \tag{5.87}$$

$$\phi = Ur\left(1 + \frac{a^2}{r^2}\right)\cos\theta \tag{5.88}$$

$$W = U\left(1 - \frac{a^2}{z^2}\right) \tag{5.89}$$

$$u_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta \tag{5.90}$$

$$u_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta \tag{5.91}$$

5.7.3 Cylinder with circulation

$$F = Uz\left(1 + \frac{a^2}{z^2}\right) + i\frac{\Gamma}{2\pi}\ln\frac{z}{a} \tag{5.92}$$

$$\psi = Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi}\ln\frac{r}{a} \tag{5.93}$$

$$\phi = Ur\left(1 + \frac{a^2}{r^2}\right)\cos\theta - \frac{\Gamma}{2\pi}\theta \tag{5.94}$$

$$W = U\left(1 - \frac{a^2}{z^2}\right) + i\frac{\Gamma}{2\pi z} \tag{5.95}$$

$$u_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta \tag{5.96}$$

$$u_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi a} \tag{5.97}$$

5.8 Forces on a submerged body

For steady, incompressible, irrotational flow we can use Bernoulli's equation

$$\frac{p}{\rho} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} = B \tag{5.98}$$

to determine the pressure, where B is a constant.

On the surface of a body the only force is due to pressure. The total force on the body is given by

$$\mathbf{F} = \int_{\mathcal{A}} p\mathbf{n}\; d\mathcal{A} \tag{5.99}$$

where $d\mathcal{A}$ is an elemental area of the surface $\mathcal{A}$ with normal $\mathbf{n}$.

5.8.1 Cylinder with circulation

The pressure on the surface of the cylinder is given by

$$p = p_\infty + \frac{1}{2}\rho U^2 - \rho\left(2U\sin\theta + \frac{\Gamma}{2\pi a}\right)^2 \tag{5.100}$$

where p_∞ and U are the pressure and velocity of the flow far from the cylinder. The components of the force on the cylinder are

$$\begin{aligned} F_x &= -al\int_0^{2\pi} p\cos\ d\theta \\ &= 0 \end{aligned} \tag{5.101}$$

$$\begin{aligned} F_y &= -al\int_0^{2\pi} p\sin\ d\theta \\ &= \rho U\Gamma l \end{aligned} \tag{5.102}$$

where l is the length of the cylinder. This is the Kutta-Joukowsky theorem.

5.9 Conformal transformation

An analytic function $\zeta = f(z)$ that maps the z-plane to the ζ-plane is called a conformal transformation. Let $z = x + iy$ and $\zeta = \xi + i\eta$. Then it can be shown that if

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{5.103}$$

then

$$\frac{\partial^2\phi}{\partial \xi^2} + \frac{\partial^2\phi}{\partial \eta^2} = 0 \tag{5.104}$$

also. We can also show that ratios of length and angles are preserved in the transformation. The complex velocity in the two planes is related by

$$\begin{aligned} W(z) &= \frac{dF}{dz} \\ &= \frac{d\zeta}{dz}\frac{dF}{d\zeta} \\ &= \frac{d\zeta}{dz}W(\zeta) \end{aligned} \tag{5.105}$$

5.9.1 Joukowski transformation

This is

$$z = \zeta + \frac{c^2}{\zeta} \tag{5.106}$$

where c is a constant. The function is analytic for $\zeta \neq 0$. The reverse mapping

$$\zeta = \frac{z}{2} \pm \sqrt{\frac{z^2}{4} - c^2} \tag{5.107}$$

is non-unique. As an example the Joukowski transformation can be applied to a uniform flow to obtain flow around a cylinder.

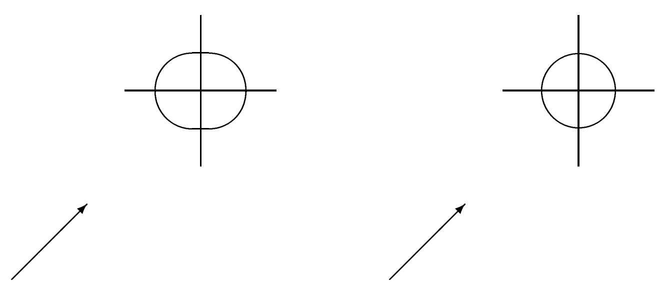


Figure 5.2: Mapping from a circle to an ellipse

Flow around an ellipse

Let c be real and positive. Then any point on a circle with center at the origin, of radius a where $a < c$, in the ζ-plane can be written as $\zeta = ae^{i\nu}$, or

$$\xi = a\cos\nu \tag{5.108}$$
$$\eta = a\sin\nu \tag{5.109}$$

Substituting in the transformation and separating the real and imaginary parts, we have

$$x = \left(a + \frac{c^2}{a}\right)\cos\nu \tag{5.110}$$
$$y = \left(a - \frac{c^2}{a}\right)\sin\nu \tag{5.111}$$

Eliminating ν, we have

$$\left(\frac{x}{a + c^2/a}\right)^2 + \left(\frac{y}{a - c^2/a}\right)^2 = 1 \tag{5.112}$$

which is the equation for an ellipse. Figure 5.2 shows the circle and its elliptic image.

The complex potential for a uniform flow at an angle α around a cylinder of radius a is

$$F(\zeta) = U\left(\zeta e^{-i\alpha} + \frac{a^2}{\zeta}e^{i\alpha}\right) \tag{5.113}$$

Substituting equation (5.107), we get the complex function

$$F(z) = U\left[\left(\frac{z}{2} \pm \sqrt{\frac{z^2}{4} - c^2}\right)e^{-i\alpha} + \left(\frac{a^2}{\frac{z}{2} \pm \sqrt{\frac{z^2}{4} - c^2}}\right)e^{i\alpha}\right] \tag{5.114}$$

for flow around an ellipse.

Flow around a flat plate

For a body with sharp trailing edge, the rear stagnation point is at the trailing edge. This is the Kutta condition. Consider a flat plate $-2a \leq x \leq 2a$, where $a = c$. Add a circulation of strength

$$\Gamma = 4\pi U a \sin\alpha \tag{5.115}$$

and complex potential

$$F = \frac{i\Gamma}{2\pi} \ln \frac{z}{a} \tag{5.116}$$

to satisfy the Kutta condition. Thus

$$F(\zeta) = U\left(\zeta e^{-i\alpha} + \frac{a^2}{\zeta} e^{i\alpha}\right) + i2Ua\sin\alpha \ \ln\frac{\zeta}{a} \tag{5.117}$$

Substituting equation (5.107), we get the complex potential in the z-plane.

Determing the force on the flat plate, we find that

$$\begin{aligned} F_y &= \rho U \Gamma l \\ &= 4\pi\rho U^2 a l \sin\alpha \end{aligned} \tag{5.118}$$

Interms of the area $A = 4al$, the coefficient of lift is

$$\begin{aligned} C_L &= \frac{F_y}{\frac{1}{2}\rho U^2 A} \\ &= 2\pi\sin\alpha \end{aligned} \tag{5.119}$$

Flow around a symmetrical Joukowski airfoil

This is obtained by a transformation of a circle that has its center on the real axis but displaced from the origin, as shown in Fig. 5.3.

Flow around a circular-arc airfoil

The circle is now on the imaginary axis, but not at the origin as shown in Fig. 5.4.

Flow around a Joukowski airfoil

This is shown in Fig. 5.5.

5.10 Three-dimensional axisymmetric flow

Either cylindrical or spherical coordinates may be used to deal with axisymmetric problems.

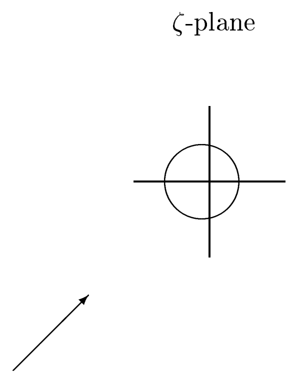


Figure 5.3: Mapping from a circle to a symmetrical airfoil

5.10.1 Cylindrical coordinates

5.10.2 Spherical coordinates

For axisymmetric flow there is zero velocity and dependence in the φ-drection. The vorticity vector is thus given by

$$\boldsymbol{\omega} = \frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial \theta}\right]\mathbf{e}_\varphi \tag{5.120}$$

For irrotational flow this is zero, so that the velocity potential is defined by

$$u_r = \frac{\partial \phi}{\partial r} \tag{5.121}$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \tag{5.122}$$

The mass conservation equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(u_\theta \sin\theta) = 0 \tag{5.123}$$

This can be satisfied by the Stokes stream function

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial \psi_s}{\partial \theta} \tag{5.124}$$

$$u_\theta = -\frac{1}{r\sin\theta}\frac{\partial \psi_s}{\partial r} \tag{5.125}$$

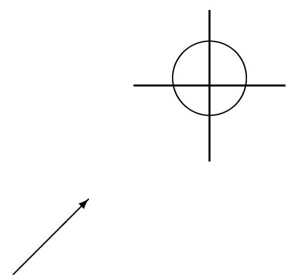


Figure 5.4: Mapping from a circle to a circular arc

Uniform flow

$$\phi = Ur\cos\theta \tag{5.126}$$

$$\psi_s = \frac{1}{2}Ur^2\sin^2\theta \tag{5.127}$$

$$u_r = U\cos\theta \tag{5.128}$$

$$u_\theta = -U\sin\theta \tag{5.129}$$

Source and sink

$$\phi = -\frac{m}{4\pi r} \tag{5.130}$$

$$\psi_s = -\frac{m}{2\pi}(1+\cos\theta) \tag{5.131}$$

$$u_r = \frac{m}{4\pi r^2} \tag{5.132}$$

$$u_\theta = 0 \tag{5.133}$$

Doublet

$$\phi = -\frac{\mu}{4\pi r}\cos\theta \tag{5.134}$$

$$\psi_s = -\frac{\mu}{4\pi r}\sin^2\theta \tag{5.135}$$

$$u_r = -\frac{\mu}{2\pi r^3}\cos\theta \tag{5.136}$$

$$u_\theta = -\frac{\mu}{2\pi r^3}\sin^2\theta \tag{5.137}$$

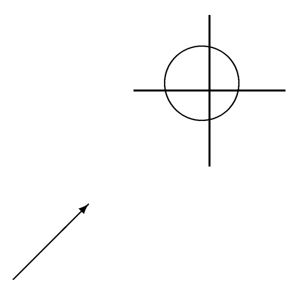


Figure 5.5: Mapping from a circle to a Joukowski airfoil

Flow around a sphere

$$\phi = Ur\left(1+\frac{a^3}{2r^3}\right)\cos\theta \tag{5.138}$$

$$\psi_s = \frac{1}{2}Ur^2\left(1-\frac{a^3}{r^3}\right)\sin^2\theta \tag{5.139}$$

$$u_r = U\left(1-\frac{a^3}{r^3}\right)\cos\theta \tag{5.140}$$

$$u_\theta = -U\left(1+\frac{a^3}{2r^3}\right)\sin\theta \tag{5.141}$$

Problems

1. Show that the complex potential, $F(z)$, that satisfies $z = c\cosh F$ represents flow through an aperture.
2. Computer generate plots of the flow around a symmetrical Joukowsky airfoil.
3. If $F^2 = U^2(z^2+c^2)$, show that the stream function satisfies
$$y^2 = \frac{\psi^2\left[U^2(x^2+c^2)+\psi^2\right]}{U^2(U^2x^2+\psi^2)}$$
and that it represents a stream of velocity U past a thin obstacle of length c projecting perpendicularly from a straight boundary.
4. State and prove the circle theorem.
5. A source is symmetrically placed near a corner. Find the force on the walls due to potential flow from the source.
6. Find the velocity and pressure fields for flow over a wedge of angle β.

Chapter 6

Incompressible viscous flow: exact solutions

6.1 Flow between flat plates

Consider fluid between two infinite flat plates separated by a distance L. The coordinate system is as shown. The lower plate is stationary while the upper one is moving with a velocity U parallel to itself. The plates are also maintained at different temperatures. Neglect gravity and assume a steady flow in the x-direction, with $u = u(y),\ v = w = 0,\ T = T(y)$. Of the governing equations, equation (A.1) is identically satisfied. Equations (A.3) and (A.4) become

$$0 = -\frac{\partial p}{\partial y} \tag{6.1}$$

$$0 = -\frac{\partial p}{\partial z} \tag{6.2}$$

from which we know that the pressure is a function of x alone. Let us assume that a pressure gradient is imposed externally so that dp/dx is a known constant. Equation (A.2) becomes

$$0 = -\frac{dp}{dx} + \mu\frac{d^2u}{dy^2} \tag{6.3}$$

Equation (A.5) simplifies to

$$0 = k\frac{d^2T}{dy^2} + \mu\left(\frac{du}{dy}\right)^2 \tag{6.4}$$

6.1.1 Alternative derivation of governing equations

Consider the forces in the x-direction on an element of fluid of dimension $dx \times dy$ as shown. They are due to pressure on each one of the vertical faces and to shear stress at the horizontal faces. These forces will be calculated per unit length in the z-direction. The net force due to pressure is

$$F_p = -dp\ dy = -\frac{dp}{dx}\ dx\ dy \tag{6.5}$$

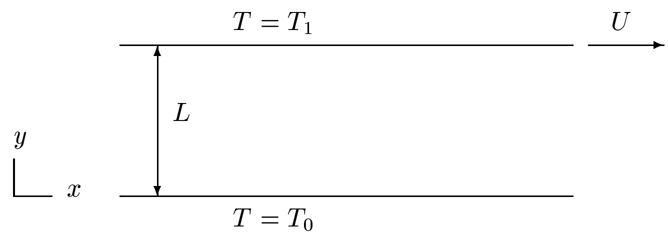


Figure 6.1: Flow between flat plates

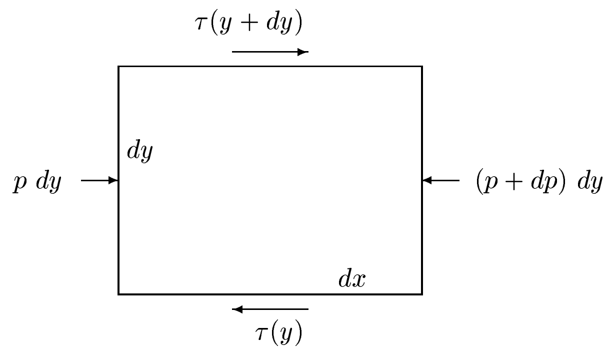


Figure 6.2: Force balance in flow between flat plates

The net force due to shear stress is

$$F_\tau = \Big[\tau(y + dy) - \tau(y)\Big]\ dx = \frac{d\tau}{dy}\ dy\ dx \tag{6.6}$$

Since the fluid element is moving at constant velocity and not accelerating, the sum of the forces on it must be zero. Thus

$$F_p + F_\tau = 0 \tag{6.7}$$

from which, on dividing by $dx\ dy$, we get the momentum equation

$$\frac{dp}{dx} + \frac{d\tau}{dy} = 0 \tag{6.8}$$

The constitutive relation for a Newtonian fluid, equation (3.61), gives the stress field as

$$\begin{bmatrix} -p & \mu\frac{du}{dy} & 0 \\ \mu\frac{du}{dy} & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

Thus, with the present notation

$$\tau = \mu\frac{du}{dy} \tag{6.9}$$

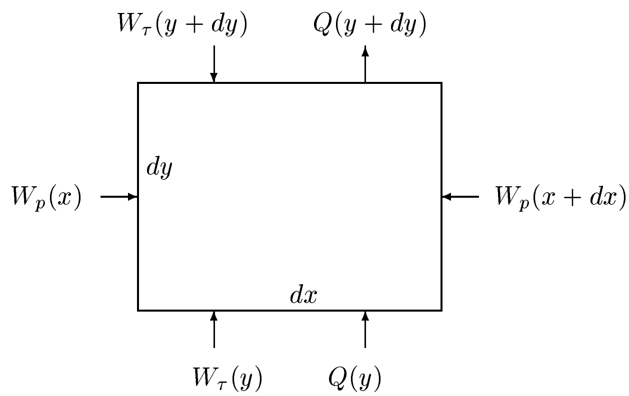


Figure 6.3: Energy balance in flow between flat plates

so that

$$-\frac{dp}{dx} + \mu\frac{d^2u}{dy^2} = 0 \tag{6.10}$$

This is the same as equation (6.3).

Let us now consider the rate at which energy is flowing into the fluid element, again per unit length in the z-direction.

There is no heat conduction through the vertical faces since the temperature does not vary in the x-direction. The pressure, however, does work on the fluid element.

$$W_p(x) = pu\ dy \tag{6.11}$$

$$W_p(x+dx) = -\left[p + \frac{dp}{dx}\ dx\right]\ u\ dy \tag{6.12}$$

remembering that u does not vary in the x-direction and that the pressure force and velocity are in opposite directions for $W_p(x+dx)$. $W_\tau(y)$ and $W_\tau(y+dy)$ are the work done by the shear forces. Referring to Fig. 6.3, we have

$$W_\tau(y) = \tau u\ dx \tag{6.13}$$

$$W_\tau(y+dy) = \left[\tau u + \frac{d(\tau u)}{dy}\ dy\right]\ dx \tag{6.14}$$

The heat fluxes are $Q(y)$ and $Q(y+dy)$ where

$$Q(y) = -k\frac{dT}{dy}\ dx \tag{6.15}$$

$$Q(y+dy) = \left[-k\frac{dT}{dy} + \frac{d}{dy}\left(-k\frac{dT}{dy}\right)\ dy\right]\ dx \tag{6.16}$$

Summing up, conservation of energy gives the relation

$$Q(y) - Q(y+dy) + W_p(x) + W_p(x+dx) + W_\tau(y) + W_\tau(y+dy) = 0 \tag{6.17}$$

Substituting for the different terms and dividing by $dx\ dy$, we get

$$k\frac{d^2T}{dy^2} - u\frac{dp}{dx} + \frac{d(\tau u)}{dy} = 0 \tag{6.18}$$

This is the total energy equation. If we subtract u times the momentum equation (6.1.1) from this, we have

$$k\frac{d^2T}{dy^2} + \tau\frac{du}{dy} = 0 \tag{6.19}$$

which is the thermal energy equation. Using equation (6.9) for the shear stress, we obtain

$$k\frac{d^2T}{dy^2} + \mu\left(\frac{du}{dy}\right)^2 = 0 \tag{6.20}$$

which is equation (6.4).

6.1.2 Velocity and temperature profiles

The solution to equation (6.3) is

$$u(y) = \frac{1}{\mu}\frac{dp}{dx}\left(\frac{y^2}{2} + Ay + B\right) \tag{6.21}$$

where A and B are the constants of integration. The boundary conditions are

$$u = 0 \text{ at } y = 0 \tag{6.22}$$
$$u = U \text{ at } y = L \tag{6.23}$$

The first condition gives $B = 0$. From the second we have

$$A = \frac{\mu U}{L\ dp/dx} - \frac{L}{2} \tag{6.24}$$

The velocity profile is

$$u(y) = \frac{y}{L}U - \frac{L^2}{2\mu}\frac{dp}{dx}\frac{y}{L}\left(1 - \frac{y}{L}\right) \tag{6.25}$$

Nondimensionally

$$u^* = \eta + P\eta(1-\eta) \tag{6.26}$$

where the dimensionless velocity, distance and pressure gradient are

$$u^* = \frac{u}{U} \tag{6.27}$$
$$\eta = \frac{y}{L} \tag{6.28}$$
$$P = \frac{L^2}{2\mu U}\left(-\frac{dp}{dx}\right) \tag{6.29}$$

respectively. The velocity of the upper plate U can be used to scale the fluid velocity only if it is nonzero. Otherwise, it can be some other characteristic velocity, like the mean flow velocity for instance.

Substituting the velocity profile from equation (6.25) into the energy equation (6.4), we have

$$\frac{d^2T}{dy^2} = -\frac{\mu}{k}\left[\left(\frac{U}{L}\right)^2 - \frac{U}{\mu}\frac{dp}{dx}\left(1-2\frac{y}{L}\right) + \frac{L^2}{4\mu^2}\left(\frac{dp}{dx}\right)^2\left(1-2\frac{y}{L}\right)^2\right] \tag{6.30}$$

the solution to which is

$$T = A + By - \frac{\mu}{k}\left[\frac{1}{2}\left(\frac{Uy}{L}\right)^2 - \frac{Uy^2}{2\mu}\frac{dp}{dx}\left(1-\frac{2y}{3L}\right) + \frac{L^2y^2}{8\mu^2}\left(\frac{dp}{dx}\right)^2\left(1-\frac{4y}{3L}+\frac{2y^2}{3L^2}\right)\right] \tag{6.31}$$

where A and B are constants. Introducing the boundary conditions

$$T = T_0 \text{ at } y = 0 \tag{6.32}$$

$$T = T_1 \text{ at } y = L \tag{6.33}$$

we can evaluate A and B and get the temperature profile

$$\begin{aligned} T = T_0 + (T_1 - T_0)\frac{y}{L} &+ \frac{\mu U^2}{2k}\frac{y}{L}\left(1-\frac{y}{L}\right) - \frac{UL^2}{6k}\frac{p}{x}\frac{y}{L}\left(1-3\frac{y}{L}+2\frac{y^2}{L^2}\right) \\ &+ \frac{L^4}{24\mu k}\left(\frac{dp}{dx}\right)^2\frac{y}{L}\left(1-3\frac{y}{L}+4\frac{y^2}{L^2}-2\frac{y^3}{L^3}\right) \end{aligned} \tag{6.34}$$

In terms of nondimensional temperature $T^* = (T - T_0)/(T_1 - T_0)$ and distance $\eta = y/L$, this can be written as

$$T^* = \eta + Br\left[\frac{1}{2}\eta(1-\eta)\frac{1}{3}P\eta(1-3\eta+2\eta^2) + \frac{1}{6}P^2\eta(1-3\eta+4\eta^2-2\eta^3)\right] \tag{6.35}$$

where the Brinkman number $Br = Ec\ Pr$, with

$$\text{Eckert number } Ec = \frac{U^2}{c(T_1 - T_0)} \tag{6.36}$$

$$\text{Prandtl number } Pr = \frac{\mu c}{k} \tag{6.37}$$

The temperature difference $T_1 - T_0$ can be used as a scale only if it is nonzero.

6.1.3 Couette flow

If no pressure gradient is imposed on the flow, $P = 0$ and the dimensionless velocity and temperature profiles become

$$u^* = \eta \tag{6.38}$$

$$T^* = \eta + \frac{1}{2}Ec\ Pr\ \eta(1-\eta) \tag{6.39}$$

The maximum temperature occurs within the flow at $\eta = \eta_m$ where

$$\eta_m = \frac{1}{2} + \frac{1}{Ec\ Pr} \tag{6.40}$$

if $0 \leq \eta \leq 1$. Then the maximum temperature is

$$T_m^* = \frac{1}{2}\left(1 + \frac{1}{Ec\ Pr} + \frac{1}{8}Ec\ Pr\right) \tag{6.41}$$

6.1.4 Poiseuille flow

On the other hand, if the velocity of the upper plate is zero, the flow is driven solely by the pressure gradient, $U = 0$. The velocity profile is then

$$u(y) = -\frac{L^2}{2\mu}\frac{dp}{dx}\frac{y}{L}\left(1 - \frac{y}{L}\right) \tag{6.42}$$

The maximum velocity is at the center $y = L/2$, so that

$$u_{max} = \frac{L^2}{8\mu}\frac{dp}{dx} \tag{6.43}$$

The mean velocity $\bar{U}$ is defined by the volume flow rate per unit area. In this case it is given by

$$\bar{U} = \frac{1}{L}\int_0^L u(y)\ dy \tag{6.44}$$

$$= -\frac{L^2}{12\mu}\frac{dp}{dx} \tag{6.45}$$

$$= \frac{2}{3}u_{max} \tag{6.46}$$

Using this to get a nondimensional velocity u^*, we have

$$u^* = \frac{u}{\bar{U}} \tag{6.47}$$

$$= 6\eta(1-\eta) \tag{6.48}$$

6.1.5 Heat generation

The heat flux through either of the plates is $-k\ dT/dy$ per unit area. For simplicity, let us consider the Couette flow problem with zero dp/dx. We can write a heat balance for the control volume shown. Since

$$\frac{dT}{dy} = \frac{T_1 - T_0}{L} + \frac{\mu U^2}{2kL}\left(1 - 2\frac{y}{L}\right) \tag{6.49}$$

we have

$$Q_0 = -kA\left(\frac{T_1 - T_0}{L} + \frac{\mu U^2}{2kL}\right) \tag{6.50}$$

$$Q_1 = -kA\left(\frac{T_1 - T_0}{L} - \frac{\mu U^2}{2kL}\right) \tag{6.51}$$

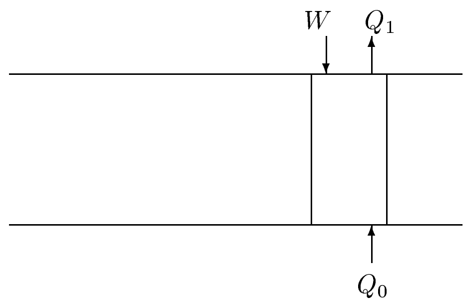


Figure 6.4: Heat generation in flow between flat plates

where A is the cross sectional area of the control volume. So the heat generated within the control volume is

$$Q_G = Q_1 - Q_0 \tag{6.52}$$
$$= \frac{\mu U^2 A}{L} \tag{6.53}$$

This is due to the work done on the fluid by the shear stress. There is no pressure work; also there is no work done at the lower plate due to shear stress since the fluid velocity there is zero. The rate of work done at the upper plate is

$$W = AU\tau\Big|_{y=h} \tag{6.54}$$
$$= \frac{\mu U^2 A}{L} \tag{6.55}$$

which is converted into heat.

6.2 Flow between coaxial rotating cylinders

Fluid is contained within two infinitely long cylinders as shown in Fig. 6.5. The inner and outer cylinders have radii r_0 and r_1, rotate counterclockwise at angular speeds of ω_0 and ω_1, and are kept at temperatures T_0 and T_1, respectively. We use cylindrical coordinates with the velocity components $u_r = u_z = 0$, $u_\theta = u_\theta(r)$, the pressure $p = p(r)$, and the temperature $T = T(r)$. Gravity is neglected.

The continuity equation (A.6) is identically satisfied. Equations (A.6) - (A.6) give

$$\rho\left(-\frac{u_\theta^2}{r}\right) = -\frac{dp}{dr} \tag{6.56}$$
$$0 = \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)\frac{u_\theta}{r^2}\right] \tag{6.57}$$
$$0 = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] + \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right]^2 \tag{6.58}$$

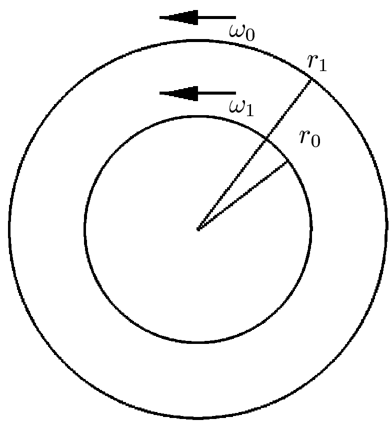


Figure 6.5: Flow between rotating cylinders

Equation (6.57) can be simplified to

$$\frac{d^2u_\theta}{dr^2} + \frac{d}{dr}\left(\frac{u_\theta}{r}\right) = 0 \tag{6.59}$$

On integrating twice, we have

$$u_\theta = Ar + \frac{B}{r} \tag{6.60}$$

where A and B are constants. The boundary conditions are

$$u_\theta = \omega_0 r_0 \text{ at } r = r_0 \tag{6.61}$$

$$u_\theta = \omega_1 r_1 \text{ at } r = r_1 \tag{6.62}$$

The constants can be determined to give the velocity profile

$$u_\theta = \frac{1}{r_1^2 - r_0^2}\left[(\omega_1^2 r_1^2 - \omega_0^2 r_0^2)r - (\omega_1 - \omega_0)\frac{r_0^2 r_1^2}{r}\right] \tag{6.63}$$

The pressure distribution from equation (6.56) is

$$p = p_0 \quad + \quad \frac{\rho}{(r_1^2 - r_0^2)^2}\left[(\omega_1^2 r_1^2 - \omega_0^2 r_0^2)^2 \frac{r^2}{2}\right. \tag{6.64}$$

$$\left. - \quad 2(\omega_1 - \omega_0)(\omega_1^2 r_1^2 - \omega_0^2 r_0^2)\ln r - (\omega_1 - \omega_0)^2 \frac{r_0^4 r_1^4}{4r}\right] \tag{6.65}$$

where p_0 is a constant of integration.

The temperature distribution

$$\frac{T - T_0}{T_1 - T_0} = Br\ \frac{r_1^4(1 - \omega_1/\omega_0)}{r_1^4 - r_0^4}\left(1 - \frac{r_0^2}{r^2}\right)\left[1 - \frac{\ln(r/r_0)}{\ln(r_1/r_0)}\right] + \frac{\ln(r/r_0)}{\ln(r_1/r_0)} \tag{6.66}$$

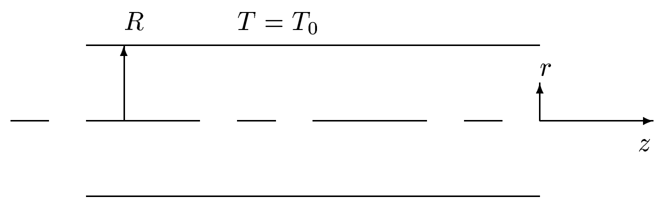


Figure 6.6: Flow in a circular pipe

is obtained by solving equation (6.58). The Brinkman number is given by

$$Br = \frac{\mu r_0^2 \omega_0^2}{k(T_1 - T_0)} \tag{6.67}$$

6.3 Flow in a circular pipe

Flow is through a long pipe with a pressure gradient dp/dz as in Fig. 6.6. We neglect gravity and let $u_z = u_z(r)$, $u_r = u_\theta = 0$. The continuity equation (A.6) is identically satisfied. Equations (A.6)–(A.6) reduce to

$$0 = -\frac{\partial p}{\partial r} \tag{6.68}$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial q} \tag{6.69}$$

so that $p = p(z)$ only. Equations (B.7) and (B.8) become

$$0 = -\frac{dp}{dz} + \mu \left[\frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr} \right) \right] \tag{6.70}$$

The solution of this equation is

$$u_z = \frac{R^2}{\mu}\left(-\frac{dp}{dz} \right)\left[-\frac{1}{4}\left(\frac{r}{R} \right)^2 + A\ln\left(\frac{r}{R} \right) + B \right] \tag{6.71}$$

At the centerline the velocity must be finite, so that $A = 0$. The other condition is that

$$u_z = 0 \text{ at } r = R \tag{6.72}$$

After calculating B, we obtain the velocity profile as

$$u_z = \frac{1}{4\mu}\left(-\frac{dp}{dz} \right)\left(R^2 - r^2 \right) \tag{6.73}$$

The maximum velocity U_{max} is at the centerline $r = 0$, where

$$U_{max} = \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right) \tag{6.74}$$

The mean velocity is

$$\bar{U} = \frac{1}{\pi R^2}\int_0^R u_z 2\pi r\ dr \tag{6.75}$$

$$= \frac{R^2}{8\mu}\left(-\frac{dp}{dz}\right) \tag{6.76}$$

$$= \frac{1}{2}U_{max} \tag{6.77}$$

so that

$$u_z = 2\bar{U}\left(1 - \frac{r^2}{R^2}\right) \tag{6.78}$$

Equation (6.71) can be written as

$$k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] = -\frac{16\mu\bar{U}^2 r^2}{R^4} \tag{6.79}$$

6.3.1 Isothermal wall

The wall is at a constant temperature T_0. Since $T = T(r)$, the energy equation becomes

$$0 = k\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)\right] + \mu\left(\frac{du_z}{dr}\right)^2 \tag{6.80}$$

Integrating, and using the conditions that T is finite at $r = 0$, and $T = T_0$ at $r = R$, we get

$$T = T_0 + \frac{\mu\bar{U}^2}{k}\left(1 - \frac{r^4}{R^4}\right) \tag{6.81}$$

6.4 Flow over a porous wall

A uniform flow of velocity U exists over a porous wall as shown in Fig. 6.7. There is suction through the porous wall resulting in a normal velocity of V. The pressure p is constant. The two-dimensional velocity field is of the form $u = u(y)$. From the continuity equation $\partial v/\partial y = 0$, so that V is independent of y. Since $v = -V$ at $y = 0$, it is $v = -V$ everywhere. The x-momentum equation is

$$-\rho V\frac{du}{dy} = \mu\frac{d^2u}{dy^2} \tag{6.82}$$

with boundary conditions

$$u = 0 \text{ at } y = 0 \tag{6.83}$$

$$u = U \text{ at } y \to \infty \tag{6.84}$$

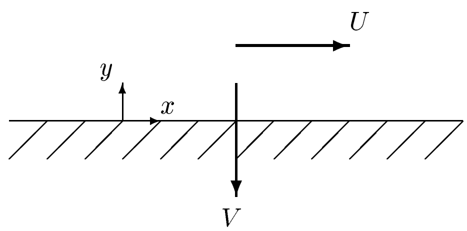


Figure 6.7: Flow over a porous wall

The solution is

$$u = U(1 - e^{-Vy/\nu}) \tag{6.85}$$

where the kinematic viscosity $\nu = \mu/\rho$.

6.5 Natural convection between vertical flat plates

In natural convection, body forces due to changes in density are taken into account. For a small change in the temperature, the following linear relationship may be used

$$\rho = \rho_0 \left[1 - \beta(T - T_0) \right] \tag{6.86}$$

where β is the coefficient of volumetric expansion, and ρ_0 is the density at the reference temperature T_0. Under the Boussinesq approximation we keep the properties of the fluid constant, but use a variable density only in the body force term of the momentum equation. Viscous dissipation may usually be neglected.

A simple solution is obtained for the case of fully developed flow between flat plates at different temperatures as shown in Fig. 6.8. The reference temperature is taken to be the average of the two. The pressure gradient is hydrostatic, i.e. $dp/dx = \rho_0 g$.

For $u = u(y),\ v = w = 0,\ T = T(y)$, the governing equations are

$$0 = \mu \frac{d^2u}{dy^2} + \rho_0 g \beta (T - T_0) \tag{6.87}$$

$$0 = k \frac{d^2T}{dy^2} \tag{6.88}$$

with boundary conditions

$$u = 0,\ T = T_0 - \Delta T \text{ at } y = -L \tag{6.89}$$

$$u = 0,\ T = T_0 + \Delta T \text{ at } = L \tag{6.90}$$

Solutions are

$$u = \frac{\rho_0 g \beta \Delta T L^2}{6\mu} \frac{y}{L} \left(\frac{y^2}{L^2} - 1 \right) \tag{6.91}$$

$$T = T_0 + \frac{y}{L} \Delta T \tag{6.92}$$

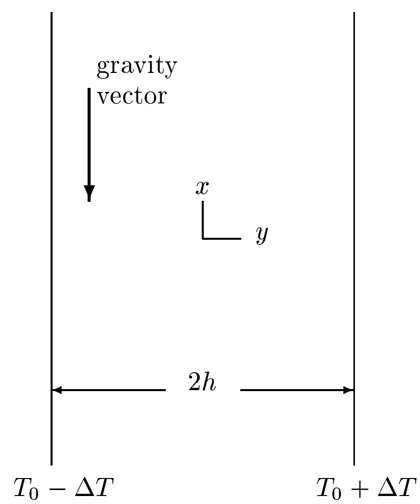


Figure 6.8: Natural convection between parallel walls

Nondimensionally, if

$$u^* = \frac{\rho_0 L u}{\mu} \tag{6.93}$$

$$T^* = \frac{T - T_0}{\Delta T} \tag{6.94}$$

$$\eta = \frac{y}{L} \tag{6.95}$$

then

$$u^* = \frac{Gr}{6}\eta(\eta^2 - 1) \tag{6.96}$$

$$T^* = \eta \tag{6.97}$$

where the Grashof number is $Gr = g\beta\Delta T L^3/\nu^2$.

Problems

1. A 5 cm diameter shaft rotates at 3000 rpm within a bearing of length 5 cm. The clearance is 0.1 mm and filled with oil of viscosity 0.01 N s/m^2. Approximating the flow of oil to be that between flat plates, find the rate of heat generation within the bearing.
2. For Poiseuille flow between flat plates, find the principal axes of the stress tensor at one of the plates.
3. For Couette flow between flat plates, we can define the Nusselt number as $Nu = h_c L/k$, where $q_{wall} = h_c(T_1 - T_0)$. Find the Nusselt numbers at the walls in terms of the Brinkman number.
4. Find the stress and strain rate tensors for the problem of flow between rotating cylinders.
5. Fluid of viscosity μ is contained in the small gap of width δ, between two cones. Find the torque necessary to rotate the inner cone at an angular velocity Ω, keeping the outer one stationary. Find also the heat generated within the fluid. Use a flat plate approximation for the flow in the gap.

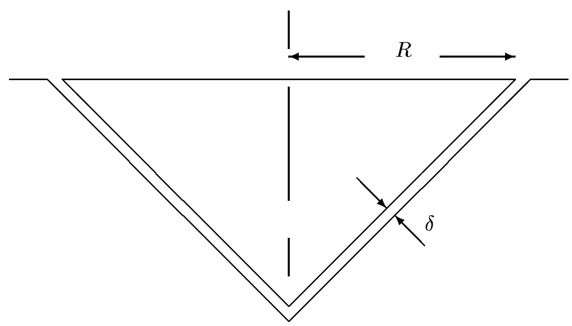


Figure 6.9: Flow between cones

6. An incompressible Newtonian liquid flows down an infinite vertical plane in a film of constant thickness δ under the action of gravity. Determine the velocity profile, and write the simplified differential equation governing the temperature profile.
7. Using a force balance on the elemental volume shown, derive equation (6.70).
8. Two porous plates are separated by a distance $2L$; the flow through each plate is as indicated. The flow is driven by a constant pressure gradient dp/dx. Find the velocity field.
9. Show that the volume flow rate Q for laminar flow in a circular pipe is given by
$$Q = \frac{\pi R^4}{8\mu}\left(-\frac{dp}{dx}\right)$$
10. The flow rate of glycerine through the apparatus shown is measured to be 0.66 cm^3/min. Given that the density of glycerine is 1260 kg/m^3, determine its viscosity.
11. Determine the wall shear stress τ_w and the coefficient of friction corresponding to this, $C_f = \tau_w/\frac{1}{2}\rho U^2$, for flow over an infinite porous wall with suction.
12. SAE 30 motor oil (thermal conductivity = 0.145 W/m K) occupies the space between two parallel plates which are kept at 20°C and 2 mm apart. One plate is stationary, while the other is moving parallel to itself at 20 m/s. What is the maximum temperature in the oil?
13. A 5 cm diameter shaft rotates at 3000 rpm within a bearing of length 5 cm. The clearance is 0.1 mm and filled with SAE 30 motor oil (thermal conductivity = 0.145 W/m K). Approximating the flow of oil to be that between flat plates, find the maximum temperature in the oil if the shaft and bearing are both kept at 20°C.
14. Show, by taking the curl of the two-dimensional momentum equation for an incompressible Newtonian fluid, that
$$\frac{D\zeta}{Dt} = \nu\nabla^2\zeta$$
where ζ is the magnitude of the vorticity normal to the plane of the flow.
15. Determine the velocity and temperature profiles in the pressure-driven flow in the annular space between two long cylinders.
16. A viscous, incompressible liquid between two infinite, parallel plates falls due to gravity alone. Find the velocity profile.

17. Two vertical, porous plates are separated by a distance $2L$; the flow through each plate, U, is to the right. The left and right plates have temperatures $T_0 - \Delta T$ and $T_0 + \Delta T$, respectively. The flow is driven by natural convection. Neglect viscous dissipation.

    (a) Find the simplified equations for the velocity and temperature fields.

    (b) Solve the temperature field.

    (c) Solve the velocity field in terms of constants of integration. Indicate how you would determine the constants.

18. An incompressible, viscous fluid occupies the annular space between two infinite concentric cylinders. The inner cylinder of radius r_1 is pulled in an axial direction with constant velocity U, while the outer one of radius r_2 is stationary. Find the velocity profile in the fluid.

19. Find the velocity field in a layer of liquid of constant thickness flowing down an inclined plane due to gravity.

20. Find the heat dissipation rate for a journal bearing carrying a 150 mm diameter shaft rotating at 720 rpm. The bearing is 200 mm long and there is an average 0.15 mm clearance between the shaft and the bearing. Lubrication is by turbine oil of viscosity 0.0012 Ns/m^2. Assume Couette flow in the clearance.

21. Find the fully developed velocity field in an axisymmetric film of liquid flowing down due to gravity outside a vertical, circular rod. The radius of the rod is a, and the film thickness is δ.

Chapter 7

Incompressible viscous flow: negligible inertia

7.1 Stokes's flow

At low Reynolds numbers, the inertia forces may be neglected. Under steady flow conditions then, we have

$$\text{div } \mathbf{u} = 0 \tag{7.1}$$

$$-\text{ grad } p + \mu \nabla^2 \mathbf{u} = 0 \tag{7.2}$$

This is a linear set of equations. On taking the divergence of the momentum equation, and using the mass conservation equation, we have

$$\nabla^2 p = 0 \tag{7.3}$$

On taking the curl, however, we have

$$\nabla^2 \boldsymbol{\omega} = 0 \tag{7.4}$$

since the curl of any gradient is zero, and $\boldsymbol{\omega} = \text{ curl } \mathbf{u}$.

7.2 Uniform flow past a sphere

The governing equations (7.1) and (7.2) can be solved analytically for steady uniform flow around a sphere, as shown in Fig. 7.1. In a mixture of spherical and Cartesian coordinates

$$u = U\left[\frac{3}{4}\frac{ax^2}{r^3}\left(\frac{a^2}{r^2} - 1\right)\frac{1}{4}\frac{a}{r}\left(3 + \frac{a^2}{r^2}\right) + 1\right] \tag{7.5}$$

$$v = U\frac{3}{4}\frac{axy}{r^3}\left(\frac{a^2}{r^2} - 1\right) \tag{7.6}$$

$$w = U\frac{3}{4}\frac{axy}{r^3}\left(\frac{a^2}{r^2} - 1\right) \tag{7.7}$$

$$p - p_0 = -\frac{3}{2}\frac{\mu U ax}{r^3} \tag{7.8}$$

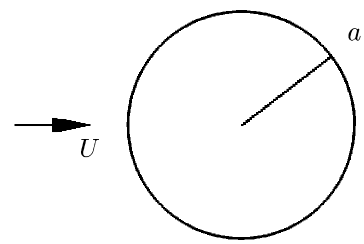


Figure 7.1: Flow past a sphere

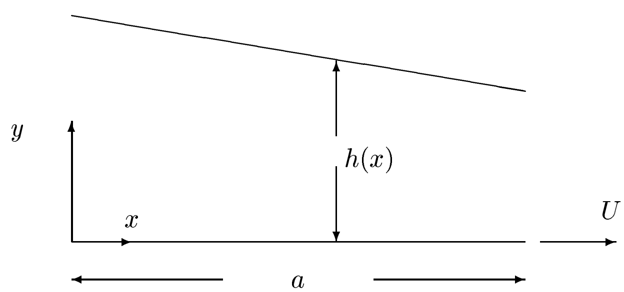


Figure 7.2: Lubrication flow

where

$$r^2 = x^2 + y^2 + z^2 \tag{7.9}$$

This velocity field is zero at $r = a$, and becomes $u = U,\ v = w = 0$ at $r \to \infty$.

The drag due to the pressure is $2\pi\mu a U$, and that due to viscous stress is $4\pi\mu a U$. The total drag force F_d on the sphere is thus

$$F_d = 6\pi\mu a U \tag{7.10}$$

This result is valid only for small Reynolds number $Re = \rho U(2a)/\mu < 1$.

7.3 Lubrication theory

Fluid is contained within the space between two surfaces sliding against each other as in Fig. 7.2. The lower surface has a horizontal motion of velocity U; the upper one which is slightly inclined is stationary. The flow is mostly in the x-direction so that $v \ll u$. Since $w = 0$ from two-dimensionality, the y- and z-momentum equations are negligible compared to the x-momentum equation. Thus, $p = p(x)$. The viscous terms in the x-momentum equation are of order $\mu\partial^2 u/\partial y^2 \sim \mu U/h^2$, the other viscous terms being smaller since $\partial^2 u/\partial x^2 \ll \partial^2 u/\partial y^2$. The inertia term is of order $\rho u \partial u/\partial x \sim \rho U^2/a$. The ratio, called a reduced Reynolds number $Re^* = (\rho U a/\mu)(h^2/a^2)$, is normally small in lubrication problems so that inertia can be neglected altogether.

So the x-momentum equation becomes

$$\frac{dp}{dx} = \mu\frac{\partial^2 u}{\partial y^2} \tag{7.11}$$

Since the gradient dp/dx is a function of x, this can be integrated twice to give

$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + A(x)y + B(x) \tag{7.12}$$

The boundary conditions to be applied are

$$y = 0,\ u = U \tag{7.13}$$
$$y = h(x),\ u = 0 \tag{7.14}$$

so that we get

$$u = U\left(1 - \frac{y}{h(x)}\right) - \frac{1}{2\mu}\frac{dp}{dx}y\left(h(x) - y\right) \tag{7.15}$$

The continuity equation can be written as

$$Q = \int_0^{h(x)} u\ dy \tag{7.16}$$

where Q, the volume flow rate per unit length, is a constant. From this

$$Q = \frac{h}{2}\left(U - \frac{h^2}{6\mu}\frac{dp}{dx}\right) \tag{7.17}$$

At both ends, $x = 0$ and $x = a$, the pressure must be atmospheric. Integrating this equation, and using the first condition at $x = 0$, we have

$$p = p_0 + 6\mu U \int_0^x \frac{1}{h(x)^2}\ dx - 12\mu Q \int_0^x \frac{1}{h(x)^3}\ dx \tag{7.18}$$

The condition at the other end $x = a$ gives

$$Q = \frac{1}{2}UH \tag{7.19}$$

where the characteristic thickness H is given by

$$H = \frac{\int_0^a \frac{1}{h(x)^2}\ dx}{\int_0^a \frac{1}{h(x)^3} dx} \tag{7.20}$$

Per unit length the total force on the lower surface has the two components

$$F_x = -\int_0^a \mu\left(\frac{\partial u}{\partial y}\right)\bigg|_{y=0} dx \tag{7.21}$$
$$F_y = \int_0^a p(x)\ dx \tag{7.22}$$

7.4 Flow in porous media

The continuity equation for incompressible flow in a porous medium is

$$\text{div } \mathbf{u} = 0 \tag{7.23}$$

For the momentum equation, the simplest model is that due to Darcy in which the flow velocity is taken proportional to the imposed pressure gradient. Thus

$$\mathbf{u} = -\frac{K}{\mu}\left(\text{grad } p - \rho\mathbf{f}\right) \tag{7.24}$$

Here K is called the permeability of the medium. Thus, in the incompressible Navier-Stokes equation with constant properties, the inertia terms are dropped and the viscous force per unit volume is represented by $-(\mu/K)\mathbf{u}$. The condition on the velocity is that of zero normal velocity at a boundary, allowing for slip in the tangential direction.

From equations (7.23) and (7.24), we get

$$\nabla^2 p = 0 \tag{7.25}$$

from which the pressure distribution can be determined.

The energy equation is

$$\rho_f c_f \left[\sigma \frac{\partial T}{\partial t} + \mathbf{u}\cdot \text{ grad } T\right] = k_{eff}\nabla^2 T \tag{7.26}$$

where k_{eff} is the effective thermal conductivity of the porous medium, and

$$\sigma = \frac{\phi\rho_f c_f + (1-\phi)\rho_s c_s}{\rho_f c_f} \tag{7.27}$$

Subscripts f and s refer to the fluid and solid respectively, and ϕ is the porosity of the material.

Problems

1. A 2 mm diameter stainless steel ball (density = 8000 kg/m^3) is dropped in glycerine (viscosity = 0.8 Ns/m^2). Find the terminal velocity of the ball, i.e. the velocity reached after the initial transient has died down.
2. In the lubrication problem, find the components of the force on the moving lower plate if the gap is $h(x) = b - cx$ for $0 \le x \le a$, where $a < b/c$, $b > 0$, $c > 0$.
3. For Stokes flow around a sphere, show that the drag coefficient is given by $C_d = 24/Re$, where $C_d = F_d/(\pi R^2 \frac{1}{2}\rho U^2)$, $Re = 2\rho U R/\mu$.
4. Find the terminal velocity of fall in air of a 10 mm diameter aluminum particle. Density of aluminum = 2700 kg/m^3, density of air = 1.2 kg/m^3, viscosity of air = 1.9×10^{-5} Ns/m^2. Check that the Reynolds number is small enough for Stokes flow.
5. Show that the velocity and pressure fields given by equation (7.5)–(7.8) are solutions of the governing equations (7.1) and (7.2).
6. In the lubrication problem, show that the special case of a wedge-like shape with $h(x) = b(c-x)$, gives $H = 2bc(c-a)/(2c-a)$. Find the force per unit length $\int_0^a p(x)\,dx$.
7. If the entrance and exit gap widths for the previous problem are h_1 and h_2 respectively, show that $H = 2h_1h_2/(h_1+h_2)$. Find the pressure distribution, the pressure and the viscous forces.
8. A glass-fiber porous filter is placed within a duct with a square section 30 cm × 30 cm. If the permeability of the filter is 0.5×10^{-9} m^2, find the pressure drop for 0.2 m^3/s of air flow through the filter. Viscosity of air = 1.9×10^{-5} N s/m^2.

9. Find the radial volume flow rate of fluid through an annular porous material. The inner and outer pressures are known. Find also the temperature distribution if the inner and outer surfaces are kept at different temperatures.
10. Show that $\nabla^2 p = 0$ for flow of an incompressible fluid through a porous medium.
11. A viscous fluid occupies the space between an upper plate of shape $h(x) = a/(1 + bx)$ and a flat lower plate moving with velocity U, where a, b and U are constants. Given that $U = 5$ m/s, $\mu = 0.01$ Ns/m^2 and the distances $h_1 = 2$ mm, $h_2 = 1$ mm, $L = 10$ cm, find (a) the characteristic thickness, and (b) the pressure distribution.
12. A cylindrical, compressed-air tank (inner radius a, outer radius b, length L) leaks air (viscosity μ, density ρ) through its cylindrical, porous walls to the atmosphere. If the permeability of the wall is K, find the air pressure, $p(t)$, in the tank as a function of time. Assume an ideal gas law for the air and isothermal conditions.
13. Estimate the net pressure force on a shaft rotating at speed ω in a journal bearing. The shaft radius is a, the bearing radius is $a + \delta$, and the distance between the shaft and bearing centers is ϵ. Assume that $\delta/a \ll 1$, and make any related approximations.

Chapter 8

Incompressible viscous flow: flat plate boundary layer

8.1 Prandtl boundary layer equations

For high Reynolds number, the entire flow region can be divided into two parts for hydrodynamic analysis: an outer region where the effect of viscosity is not important, and an inner region near boundaries and solid walls where it is. This is shown in Fig. 8.1 The viscous region is called a hydrodynamic boundary layer. A similar division can be made for thermal analysis, as shown in Fig. 8.2. The region near a solid wall where temperature gradients are steep and conduction is important is the thermal boundary layer. Outside this region conduction effects can be neglected.

We take the leading edge of the plate to be the origin of coordinates, with x being along and y normal to the plate. The thicknesses of the layers $\delta(x)$ and $\delta_T(x)$ have been exaggerated in the figure. For large Reynolds number, they are usually very thin, so that we can make the following approximations within the boundary layers: $v \ll u$, $\partial/\partial x \ll \partial/\partial y$. Under these conditions, the governing equations can be simplified.

According to the y-momentum equation, the pressure does not depend on y. The pressure $p(x)$ is thus determined by the fluid mechanics outside the boundary layer where viscous effects are not important. Using the Bernoulli equation, we have

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = U\frac{dU}{dx} \tag{8.1}$$

where $U(x)$ is the free-stream velocity outside the boundary layer.

The boundary layer continuity, x-momentum, and energy equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{8.2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{8.3}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{8.4}$$

Viscous dissipation has been neglected in the energy equation.

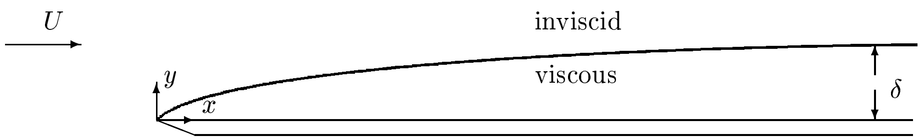


Figure 8.1: Hydrodynamic boundary layer over a flat plate

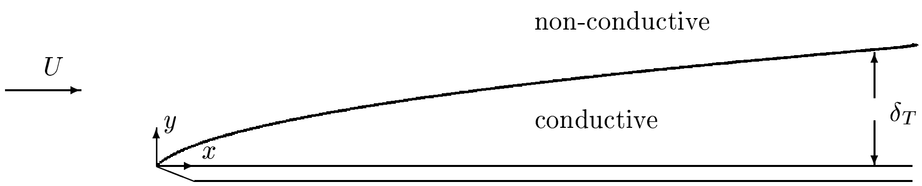


Figure 8.2: Thermal boundary layer over a flat plate

The boundary conditions for these equations are

$$u = v = 0,\ T_w \text{ at } y = 0 \tag{8.5}$$
$$u = U,\ T = T_\infty \text{ as } y \to \infty \tag{8.6}$$

These equations can be written in terms of a stream function $\psi(x, y)$ where

$$u = \frac{\partial \psi}{\partial y} \tag{8.7}$$
$$v = -\frac{\partial \psi}{\partial x} \tag{8.8}$$

satisfying the continuity equation (8.2). The x-momentum and energy equations, (8.3) and (8.4) respectively, become

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = U\frac{dU}{dx} + \nu\frac{\partial^3 \psi}{\partial y^3} \tag{8.9}$$
$$\frac{\partial \psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{8.10}$$

with the boundary conditions

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = 0,\ T = T_w \text{ at } y = 0 \tag{8.11}$$
$$\frac{\partial \psi}{\partial y} = U,\ T = T_\infty \text{ as } y \to 0 \tag{8.12}$$

The thickness of the boundary layer can be defined in several ways. One of the most common is δ where $u(\delta) = 0.99U$. Another, the displacement thickness, is defined by

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) \, dy \tag{8.13}$$

The momentum thickness is

$$\theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) \, dy \tag{8.14}$$

8.2 Blasius solution

Consider a flat plate where the velocity U is a constant. We will have a zero pressure gradient $dp/dx = 0$, so that the pressure is constant everywhere. Now let

$$\eta = y\sqrt{\frac{U}{\nu x}} \tag{8.15}$$

$$\psi = \sqrt{\nu x U} f(\eta) \tag{8.16}$$

The momentum equation becomes the Blasius's equation

$$2f''' + ff'' = 0 \tag{8.17}$$

with

$$f = f' = 0 \text{ at } \eta = 0 \tag{8.18}$$

$$f' = 1 \text{ as } \eta \to \infty \tag{8.19}$$

This problem can be integrated numerically; the results of $f(\eta)$ and $f'(\eta)$ are shown in Fig. 8.3. Since $u = Uf'$, the f' curve also represents u/U.

According to the numerical solution $u(\eta) = 0.99U$ at $\eta = 5$. Thus from equation (8.15) we have

$$5 = \delta\sqrt{\frac{U}{\nu x}} \tag{8.20}$$

This can be written as

$$\frac{\delta}{x} = \frac{5}{\sqrt{Re_x}} \tag{8.21}$$

where the local Reynolds number $Re_x = Ux/\nu$. The local shear stress at the wall

$$\tau_{wx} = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} \tag{8.22}$$

can be written as

$$\tau_{wx} = \mu\sqrt{\frac{U^3}{\nu x}} f''(0) \tag{8.23}$$

Since $f''(0)$ is numerically found to be 0.332, we can obtain the local coefficient of friction defined by $C_{fx} = \tau_{wx}/\frac{1}{2}\rho U^2$ to be

$$C_{fx} = \frac{0.664}{\sqrt{Re_x}} \tag{8.24}$$

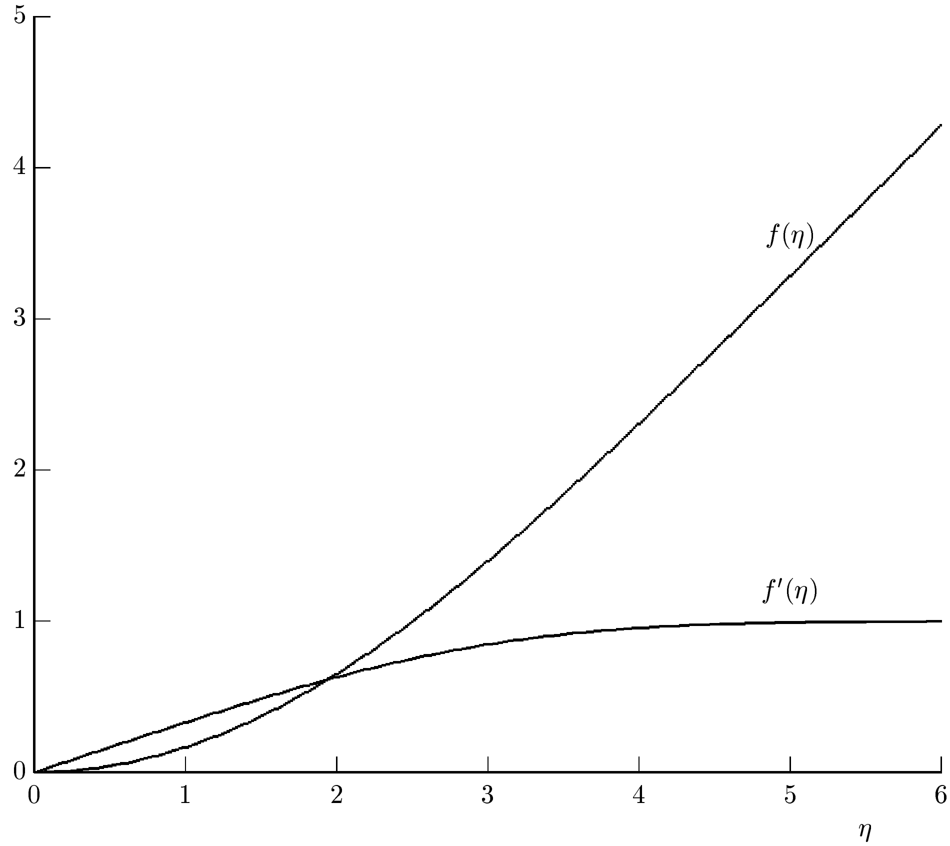


Figure 8.3: Solution of Blasius's equation

Using the similarity variable η, and $T = T(\eta)$, the energy equation becomes

$$T'' + \frac{1}{2}\ Pr\ fT' = 0 \tag{8.25}$$

To obtain simple boundary conditions, the temperature can be nondimensionalized as

$$T^* = \frac{T - T_w}{T_\infty - T_w} \tag{8.26}$$

The temperature profile is then given by

$$T^{*\prime\prime} + \frac{1}{2}\ Pr\ fT^{*\prime} = 0 \tag{8.27}$$

with the boundary conditions

$$T^* = 0 \text{ at } \eta = 0 \tag{8.28}$$

$$T^* = 1 \text{ as } \eta \to \infty \tag{8.29}$$

Again this problem can be solved numerically for a given Prandtl number. For $Pr > 0.6$, it is seen that a good approximation for the slope of the temperature profile at the wall is

$$T^{*\prime}(0) = 0.332\ Pr^{1/3} \text{ for } Pr > 0.6 \tag{8.30}$$

The local convective heat transfer coefficient may be expressed as

$$h(x) = \frac{-k(\partial T/\partial y)_{y=0}}{T_w - T_\infty} \tag{8.31}$$

The local Nusselt number defined as $Nu_x = h(x)x/k$ comes out to be

$$Nu_x = 0.332\ Re_x^{1/2}\ Pr^{1/3} \text{ for } Pr > 0.6 \tag{8.32}$$

In addition it is found that

$$\frac{\delta}{\delta_T} = Pr^{1/3} \text{ for } Pr > 0.6 \tag{8.33}$$

8.3 Momentum integral method

Equation (8.3) can be written as

$$\frac{\partial(u^2)}{\partial x} + \frac{\partial(uv)}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{8.34}$$

which, on integrating across the boundary layer from $y = 0$ to $y = \delta$, becomes

$$\int_0^\delta \frac{\partial(u^2)}{\partial x}\ dy + Uv(x,\delta) = -\nu\left.\frac{\partial u}{\partial y}\right|_{y=0} \tag{8.35}$$

since $u(x,\delta) = U,\ (\partial u/\partial y)_{y=\delta} = 0$. Integrating the continuity equation (8.2) within the same limits, we get

$$v(x,\delta) = -\int_0^\delta \frac{\partial u}{\partial x}\ dy \tag{8.36}$$

Applying the Leibnitz formula

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} f(x,y)\ dy = \int_{\alpha(x)}^{\beta(x)} \frac{\partial f(x,y)}{\partial x}\ dy + f(x,\beta)\frac{d\beta}{dx} - f(x,\alpha)\frac{d\alpha}{dx} \tag{8.37}$$

we can write

$$\frac{d}{dx}\int_0^\delta u(U-u)\ dy = U\left[\int_0^\delta \frac{\partial u}{\partial x}\ dy + U\frac{d\delta}{dx}\right] - \left[\int_0^\delta \frac{\partial(u^2)}{\partial x}\ dy + U^2\frac{d\delta}{dx}\right] \tag{8.38}$$

Substituting equations (8.35) and (8.36) into the right hand side, we get the momentum integral equation

$$\frac{d}{dx}\int_0^\delta u(U-u)\ dy = \nu\left.\frac{\partial u}{\partial y}\right|_{y=0} \tag{8.39}$$

For an approximate solution, assume a cubic polynomial for the velocity profile

$$\frac{u}{U} = a_0 + a_1\frac{y}{\delta} + a_2\left(\frac{y}{\delta}\right)^2 \tag{8.40}$$

Using the conditions

$$u(x,0) = 0,\ u(x,\delta) = U \text{ and } \left.\frac{\partial u}{\partial y}\right|_{y=\delta} = 0 \tag{8.41}$$

the constants can be determined as

$$a_0 = 0,\ a_1 = 2 \text{ and } a_2 = -1 \tag{8.42}$$

so that

$$\frac{u}{U} = 2\eta - \eta^2 \tag{8.43}$$

where

$$\eta = \frac{y}{\delta} \tag{8.44}$$

Substituting into the momentum integral equation (8.39) gives

$$\frac{d}{dx}\left(\frac{2}{15}\delta U^2\right) = \frac{2\nu U}{\delta} \tag{8.45}$$

Using the initial condition $\delta(0) = 0$, the solution of this equation is

$$\delta = \sqrt{30}\sqrt{\frac{\nu x}{U}} \tag{8.46}$$

which in nondimensional form is

$$\frac{\delta}{x} = \frac{5.48}{\sqrt{Re_x}} \tag{8.47}$$

The local coefficient of friction is found to be

$$C_{fx} = \frac{0.73}{\sqrt{Re_x}} \tag{8.48}$$

These can be compared with the numerical results of equations (8.21) and (8.24).

8.4 Vertical plate natural convection

Let us consider a flat plate at temperature T_w which is higher than the temperature T_∞ of the surroundings. Fluid flow is then upwards in the vicinity of the plate. The Boussinesq approximation can be used for the governing equations. We make the usual boundary layer assumptions and take $u \gg v, \partial/\partial y \gg \partial/\partial x$, where y is normal to the plate and x is upwards.

The boundary layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{8.49}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta(T - T_0) \tag{8.50}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 u}{\partial y^2} \tag{8.51}$$

where β is the coefficient of thermal expansion (for example, for a perfect gas, $\beta = 1/T$, where T is the absolute temperature.) The boundary conditions are

$$u = v = 0,\ T = T_w \text{ at } y = 0 \tag{8.52}$$

$$u = 0,\ T = T_\infty \text{ at } y \to \infty \tag{8.53}$$

Similarity solutions can be obtained by using the variables

$$\eta = \frac{1}{\sqrt{2}} \, Gr_x^{1/4} \, \frac{y}{x} \tag{8.54}$$

$$u = 2 \, Gr_x^{1/2} \, \frac{\nu}{x} \frac{df}{d\eta} \tag{8.55}$$

$$v = \frac{1}{\sqrt{2}} \, Gr_x^{1/4} \, \frac{\nu}{x} \left(\eta \frac{df}{d\eta} - 3f \right) \tag{8.56}$$

$$T^* = \frac{T - T_\infty}{T_w - T_\infty} \tag{8.57}$$

where the local Grashof number is

$$Gr_x = \frac{\beta g (T_w - T_\infty) x^3}{\nu^2} \tag{8.58}$$

The continuity equation (8.49) is satisfied. The y-momentum equation (8.50) and the energy equation (8.51) become

$$\frac{d^3 f}{d\eta^3} + 3f \frac{d^2 f}{d\eta^2} - 2 \left(\frac{df}{d\eta} \right)^2 + T^* = 0 \tag{8.59}$$

$$\frac{d^2 T^*}{d\eta^2} + 3 \; Pr \; f \frac{dT^*}{d\eta} = 0 \tag{8.60}$$

The boundary conditions are

$$f = f' = 0, \; T^* = 1 \text{ at } \eta = 0 \tag{8.61}$$

$$f' = T^* = 0 \text{ as } \eta \to \infty \tag{8.62}$$

This problem can be numerically solved.

The local Nusselt number is

$$Nu_x = -\frac{x}{T_w - T_\infty} \left(\frac{\partial T}{\partial y} \right)_{y=0} \tag{8.63}$$

$$= -\frac{1}{\sqrt{2}} \, Gr_x^{1/4} \left(\frac{\partial T^*}{\partial \eta} \right)_{\eta=0} \tag{8.64}$$

Numerical computations show that the values approximate

$$Nu_x = \left[\frac{Gr_x \; Pr^2}{2.435 + 4.884 \; Pr^{1/2} + 4.953 \; Pr} \right]^{1/4} \tag{8.65}$$

Problems

1. Show that the total drag per unit length $F_D = \int_0^L \tau_{wx} \; dx$ on a flat plate of length L can be expressed as a coefficient of drag $C_D = 1.328/\sqrt{Re_L}$.
2. Show that the average heat transfer coefficient $h = \frac{1}{L} \int_0^L \; h(x) \; dx$ on a flat plate of length L can be expressed as a Nusselt number $Nu_L = 0.664 Re_L^{1/2} \; Pr^{1/3}$ for $Pr > 0.6$.

3. Using

$$u(x,y) = U(x) f'(\eta)$$
$$v(x,y) = -\frac{\partial}{\partial x}\int_0^y u(x,y)\, dy$$

where $\eta = \eta(x,y)$ and $U(x)$ is the inviscid x-velocity outside the boundary layer, show that the boundary layer equations reduce to a form

$$f''' + g_1 f'' = g_2 f f'' + g_3(f'^2 - f f'' - 1)$$

Find g_1, g_2 and g_3.

4. Use the momentum integral method with the velocity profile $u = U\sin(\pi y/2\delta)$ to determine δ/x and C_{fx}.

5. A square flat plate of side 30 cm, maintained at temperature 50°C, is exposed to a flow of air (density = 1.2 kg/m^3, viscosity = 1.9×10^{-5} Ns/m^2, Prandtl number = 0.71) at 20°C parallel to it. Find the drag force on one side of the plate as well as the rate of heat transfer from it. Determine also the thicknesses of the hydrodynamic and thermal boundary layers at the trailing edge of the plate.

6. Find the extra term in the flat plate boundary layer problem with viscous dissipation.

7. Show that $\exp(\frac{1}{2} Pr \int f(\eta)\, d\eta)$ is an integrating factor for equation (8.27). Solve for T^* in terms of $f(\eta)$.

8. Given $T^* = (T - T_\infty)/(T_w - T_\infty)$ and the similarity variables

$$\eta = \frac{1}{\sqrt{2}}\, Gr_x^{1/4} \frac{y}{x}$$
$$f(\eta) = \frac{\sqrt{2}}{4\nu\, Gr_x^{1/4}} \psi$$

where $\psi(x,y)$ is the stream function, reduce the boundary layer equations (8.50)–(8.53) for natural convection in a vertical flat plate to ordinary differential equations (8.59)–(8.60). Transform also the boundary conditions.

9. Determine the extra term in the equation (8.60) corresponding to viscous dissipation, if it is included in the vertical flat plate natural convection analysis.

10. Show that the momentum integral equation corresponding to the vertical flat plate natural convection problem is

$$\beta g \int_0^\delta (T - T_\infty)\, dy - \frac{d}{dx}\int_0^\delta u^2\, dy = \nu \left.\frac{\partial u}{\partial y}\right|_{y=0} \qquad (8.66)$$

11. An incompressible fluid enters the space between two flat plates at velocity U. Estimate the distance downstream at which the two boundary layers meets. State your assumptions.

12. The one-seventh power law for the velocity profile $u = U(y/\delta)^{1/7}$ is often used for a turbulent flat plate boundary layer. Find δ/x and C_{fx} using the momentum integral. Wrong?

13. Using a cubic velocity profile for a laminar boundary layer over a flat plate, find the boundary layer thickness.

14. In the flat plate boundary layer problem, use the momentum integral method with a linear velocity profile to determine the (a) boundary layer thickness δ as a function of distance downstream x, and (b) the local coefficient of friction.

15. Find the rate of heat transfer by natural convection from one side of a vertical 10 cm × 10 cm flat plate which is 10°C hotter than the ambient air. The coefficient of expansion of air $\beta = 3.4 \times 10^{-3}$ K^{-1}.

Chapter 9

Hydrodynamic stability and turbulence

9.1 Hydrodynamic stability

We will consider the stability of the parallel flow of an incompressible fluid with constant properties. The field velocity for this flow is $u = \overline{u}(y)$, $v = w = 0$, the pressure field is $p = \overline{p}(x)$, and the flow domain is $y > 0$ with a wall at $y = 0$. For this the governing equations are

$$0 = -\frac{1}{\rho}\,\frac{d\overline{p}}{dx} + \nu\,\frac{\partial^2 \overline{u}}{\partial y^2} \tag{9.1}$$

where the bars represent the basic flow. Let us apply small two-dimensional time-dependent perturbations so that

$$\begin{aligned} u(x,y,t) &= \overline{u}(y) + u'(x,y,t) &(9.2)\\ v(x,y,t) &= v'(x,y,t) &(9.3)\\ p(x,y,t) &= \overline{p}(x) + p'(x,y,t) &(9.4)\end{aligned}$$

where the primed quantities are small. Substitute in the governing equations, neglect terms with products of primed terms, and use equation (9.1) to get

$$\begin{aligned} \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} &= 0 &(9.5)\\ \frac{\partial u'}{\partial t} + \overline{u}\frac{\partial u'}{\partial x} + v'\frac{d\overline{u}}{dy} &= -\frac{1}{\rho}\frac{\partial p'}{\partial x} + \nu\left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2}\right) &(9.6)\\ \frac{\partial v'}{\partial t} + \overline{u}\frac{\partial v'}{\partial x} &= -\frac{1}{\rho}\frac{\partial p'}{\partial y} + \nu\left(\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2}\right) &(9.7)\end{aligned}$$

Boundary conditions on the velocity components are

$$\begin{aligned} u' = v' = 0 \quad &\text{for} \quad y = 0 &(9.8)\\ u' = v' \to 0 \quad &\text{for} \quad y \to \infty &(9.9)\end{aligned}$$

Using the perturbation stream function defined by

$$u' = \frac{\partial \psi}{\partial y} \tag{9.10}$$

$$v' = -\frac{\partial \psi}{\partial x} \tag{9.11}$$

$$\tag{9.12}$$

and eliminating p', we get

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) - \frac{d^2 \overline{u}}{dy^2}\frac{\partial \psi}{\partial x} = \nu\left(\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4}\right) \tag{9.13}$$

We make a normal mode expansion of the form

$$\psi(x, y, t) = \Psi(y)e^{i\alpha(x-ct)} \tag{9.14}$$

where α is real and $c = c_r + ic_i$ is complex quantity. Thus we can write

$$\psi(x, y, t) = \Psi(y)e^{\alpha c_i t}e^{i\alpha(x-c_r t)} \tag{9.15}$$

Physically, Ψ is the amplitude of the perturbation wave, c_i is its temporal growth rate, α is its wave number, and αc_r is its frequency.

With these variables we get the Orr-Sommerfeld equation

$$(\overline{u} - c)\left(\frac{d^2 \Psi}{dy^2} - \alpha^2 \Psi\right) - \frac{d^2 \overline{u}}{dy^2}\Psi = \frac{\nu}{i\alpha}\left(\frac{d^4 \Psi}{dy^4} - 2\alpha^2 \frac{d^2 \Psi}{dy^2} + \alpha^4 \Psi\right) \tag{9.16}$$

Boundary conditions are

$$\Psi = \frac{d\Psi}{dy} = 0 \quad \text{for} \quad y = 0 \tag{9.17}$$

$$\Psi = \frac{d\Psi}{dy} \to 0 \quad \text{for} \quad y \to \infty \tag{9.18}$$

The problem can be nondimensionalized using

$$u^* = \frac{u}{U} \tag{9.19}$$

$$c^* = \frac{c}{U} \tag{9.20}$$

$$\Psi^* = \frac{\Psi}{UL} \tag{9.21}$$

$$\alpha^* = \alpha L \tag{9.22}$$

$$y^* = \frac{y}{L} \tag{9.23}$$

where the velocity scale is U and the length scale is L. The nondimensional Orr-Sommerfeld equation is

$$(\overline{u}^* - c^*)\left(\frac{d^2 \Psi^*}{dy^{*2}} - \alpha^{*2}\Psi^*\right) - \frac{d^2 \overline{u}^*}{dy^{*2}}\Psi^* = \frac{1}{i\alpha Re}\left(\frac{d^4 \Psi^*}{dy^{*4}} - 2\alpha^{*2}\frac{d^2 \Psi^*}{dy^{*2}} + \alpha^4 \Psi^*\right) \tag{9.24}$$

where

$$Re = \frac{UL}{\nu} \tag{9.25}$$

and with

$$\Psi^* = \frac{d\Psi^*}{dy^*} = 0 \quad \text{for} \quad y^* = 0 \tag{9.26}$$

$$\Psi^* = \frac{d\Psi^*}{dy^*} \to 0 \quad \text{for} \quad y^* \to \infty \tag{9.27}$$

From equation (9.15) we see that the imaginary part of c^* determines the stability of the flow; if the imaginary part is negative it is stable, but if it is positive it is unstable. For a given flow field $u^*(y)$ and values of α^* and Re, we have an eigenvalue problem for the determination of c^* that will satisfy the boundary conditions. This is usually done numerically.

9.2 Turbulence

At low Reynolds number a flow is usually laminar. As Re is increased, it becomes unstable and disturbances whicha re naturally present begin to grow. For higher Re, the flow becomes turbulent with "random" fluctuations of the velocity and pressure fields.

For fully turbulent flow we can write the velocty and pressure fields as

$$u_i = \overline{u}_i + u_i' \tag{9.28}$$

$$p = \overline{p} + p' \tag{9.29}$$

where the bars indicate the time-averaged value and the primes the fluctuationg components. Note that the time average of the primed quantities is zero.

Substituting into the governing equations for an incompressible fluid with constant properties

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{9.30}$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{9.31}$$

and averaging over time, we get

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0 \tag{9.32}$$

$$\rho\left(\overline{u}_j\frac{\partial \overline{u}_i}{\partial x_j}\right) = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu\frac{\partial \overline{u}_i}{\partial x_j} - \rho\overline{u_i'u_j'}\right) \tag{9.33}$$

which resemble the equations for steady laminar flow in the time-averaged quantities except for the additional stress term $-\rho\overline{u_i'u_j'}$ due to turbulence which is called the Reynolds stress. Additional equations must be introduced for these terms before the governing equations can be solved. One common method is to model it as

$$-\overline{u_i'u_j'} = \epsilon\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) \tag{9.34}$$

where ϵ is called the eddy viscosity.

There are various turbulence models for determining ϵ. For example, one of them is the mixing length theory of von Kármán for flow next to a wall which proposes that

$$\epsilon = \rho\kappa^2 y^2 \left|\frac{d\overline{u}}{dy}\right| \tag{9.35}$$

where $\kappa = 0.41$.

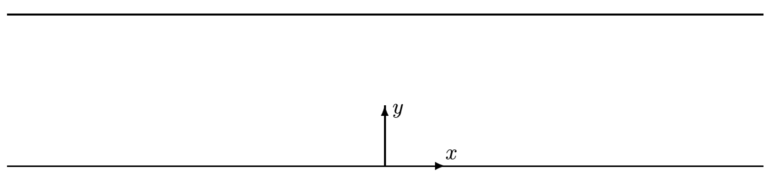
Problems

1. For turbulent flow of an incompressible fluid with constant properties, show that the Navier-Stokes equation becomes

$$\rho\left(\overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j}\right) = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial \overline{u}_i}{\partial x_j} - \rho\overline{u_i' u_j'}\right)$$

where the overbars indicate a time average.

2. Show the following for the problem of two-dimensional stability of *inviscid* parallel flow between flat plates.



(a) For the velocity field $u = U(y) + u'(x, y, t)$, $v = v'(x, y, t)$, $w = 0$, neglect the products of primed quantities and show from the governing equations that

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 v' - \frac{d^2 U}{dy^2}\frac{\partial v'}{\partial x} = 0$$

with $v' = 0$ at $y = 0, 1$, where 1 is the distance between the plates.

(b) Substitute

$$v'(x, y, t) = \hat{v}(y)e^{i\alpha(x-ct)}$$

where α is real, and $\hat{v}$ and c are complex. Show that

$$(U - c)\left(\frac{d^2\hat{v}}{dy^2} - \alpha^2\hat{v}\right) - \frac{d^2U}{dy^2}\hat{v} = 0$$

with $\hat{v} = 0$ at $y = 0, 1$.

(c) Divide by $U - c$, multiply by the complex conjugate $\hat{v}^*$, integrate by parts, and use the boundary conditions to show that

$$\int_0^1 \left(\left|\frac{d\hat{v}}{dy}\right|^2 + \alpha^2|\hat{v}|^2\right) dy + \int_0^1 \frac{d^2U}{dy^2}\frac{|\hat{v}|^2}{U-c}\, dy = 0$$

(d) Show that the imaginary part of this is

$$c_i \int_0^1 \frac{d^2U}{dy^2}\frac{|\hat{v}|^2}{(U-c_r)^2 + c_i^2}\, dy = 0$$

where $c = c_r + ic_i$, which can only be satisfied if d^2U/dy^2 changes sign in the interval $y = 0, 1$.

Chapter 10

Compressible flow in gases

10.1 Basic equations

10.1.1 Equations of state for real gases

Perfect or ideal gas

$$p = \rho R T \tag{10.1}$$

Van der Waals

$$p = \frac{RT}{\frac{1}{\rho} - b} - a\rho^2 \tag{10.2}$$

Redlich-Kwong

$$p = \frac{RT}{\frac{1}{\rho} - b} - \frac{a\rho^2}{(1 + b\rho)T^{1/2}} \tag{10.3}$$

Virial equation

$$p = \rho R T + a(T)\rho^2 + b(T)\rho^3 + \dots \tag{10.4}$$

Beattie-Bridgeman

$$p = \rho^2 R T \left(\frac{1}{\rho} + b\right)(1 - \epsilon) - A\rho^2 \tag{10.5}$$

Berthelot

$$p = \frac{\rho R T}{1 - b\rho} - \frac{a\rho^2}{T} \tag{10.6}$$

10.1.2 Thermodynamics of perfect gases

We will assume for the most part that the flow is that of a (thermally) perfect gas which satisfies the relation

$$p = \rho R T \tag{10.7}$$

where R is the particular gas constant.

The specific internal energy e and specific enthalpy $h = e + p/\rho$ are given by

$$de = c_v \, dT \tag{10.8}$$

$$dh = c_p \, dT \tag{10.9}$$

where c_v and c_p are the specific heats at constant volume and constant pressure respectively. The difference is

$$c_p - c_v = R \tag{10.10}$$

and the ratio is

$$\frac{c_p}{c_v} = \gamma \tag{10.11}$$

From these

$$c_p = \frac{\gamma R}{\gamma - 1} \tag{10.12}$$

$$c_v = \frac{R}{\gamma - 1} \tag{10.13}$$

In this chapter the specific heats will be taken to be constant (i.e. calorically perfect gas). Thus the integrated versions of equations (10.8) and (10.9) are

$$e_2 - e_1 = c_v(T_2 - T_1) \tag{10.14}$$

$$h_2 - h_1 = c_p(T_2 - T_1) \tag{10.15}$$

The thermodynamic properties are related by the Gibbs equations

$$T \, ds = de - \frac{p}{\rho^2} \, d\rho \tag{10.16}$$

$$= dh - \frac{dp}{\rho} \tag{10.17}$$

Changes in specific entropy are determined from

$$ds = c_v \frac{dT}{T} - R \frac{d\rho}{\rho} \tag{10.18}$$

$$= c_p \frac{dT}{T} - R \frac{dp}{p} \tag{10.19}$$

which can be integrated to give

$$s_2 - s_1 = c_v \ln \frac{T_2}{T_1} - R \ln \frac{\rho_2}{\rho_1} \tag{10.20}$$

$$= c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} \tag{10.21}$$

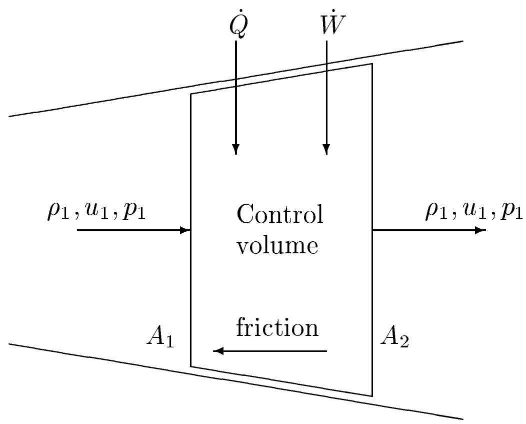


Figure 10.1: One-dimensional flow

For an isentropic process $s_2 - s_1 = 0$, from which

$$\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1} \tag{10.22}$$

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma} \tag{10.23}$$

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} \tag{10.24}$$

10.2 One-dimensional steady flow equations

Consider the flow of a compressible fluid through a variable area duct. Take a control volume as indicated in Fig. 10.1 where the conditions upstream and downstream are shown. The one-dimensional governing equations for steady flow conditions in a finite control volume can be written as follows.

Mass:

$$\dot{m} = \rho_1 u_1 A_1 = \rho_2 u_2 A_2 \tag{10.25}$$

Momentum:

$$\sum F = \dot{m}(u_2 - u_1) \tag{10.26}$$

Energy:

$$\dot{Q} + \dot{W} + \dot{m}\left(h_1 + \frac{u_1^2}{2} - h_2 - \frac{u_2^2}{2}\right) = 0 \tag{10.27}$$

where the subscripts 1 and 2 denote quantities at the inlet and outlet of the control volume, respectively; $\sum F$ is the sum of all forces on the control volume including pressure and frictional forces; $\dot{m}$ is the mass flow rate through the duct; $\dot{Q}$ and $\dot{W}$ are the heat flow rate and work flow rate to the control volume.

10.2.1 Differential form

Differential relations can also be written for an infinitesimal control volume. In this case $\rho_1 = \rho$, $u_1 = u$, $p_1 = p$ and $A_1 = A$ at section 1, and $\rho_2 = \rho + d\rho$, $u_2 = u + du$, $p_2 = p + dp$ and $A_2 = A + dA$ at section 2.

The mass conservation equation is

$$uA\ d\rho + \rho A\ du + \rho u\ dA = 0 \tag{10.28}$$

where products of differentials have been neglected. Dividing by $\rho u A$, we have

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0 \tag{10.29}$$

This could also have been obtained by taking the logarithm of $\rho u A =$ constant, and differentiating.

In the momentum equation the forces are

$$\sum F = pA + (p + \frac{dp}{2})\ dA - (p + dp)(A + dA) - \tau_w P\ dx \tag{10.30}$$

P is the perimeter of the section and τ_w is the shear stress at the wall. The pressure on the lateral surface is assumed to be $p + dp/2$. The wall shear stress is often approximated by

$$\tau_w = \frac{1}{8} f \rho u^2 \tag{10.31}$$

where f is the Darcy friction factor that is a function of the local Reynolds number and wall roughness. Thus, the momentum equation is

$$dp + \rho u\ du + f \frac{\rho u^2}{2D_h}\ dx = 0 \tag{10.32}$$

where products of differentials have been neglected again. D_h is the hydraulic diameter of the duct defined by $D_h = 4A/P$.

In the energy equation, we neglect work transfer $\dot{W}$ and write the heat transfer rate as $\dot{Q} = q'\ dx$, where q' is the heat inflow per unit length. Thus

$$-\frac{q'}{\dot{m}}\ dx + dh + u\ du = 0 \tag{10.33}$$

where $\dot{m} = \rho u A$.

10.2.2 Constant area without friction, heat or work transfer

If the area is constant, the mass conservation equation (10.25) is

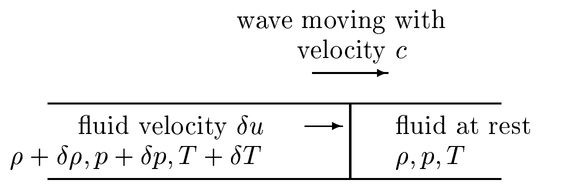
$$\rho_1 u_1 = \rho_2 u_2 \tag{10.34}$$

The constant area momentum equation (10.26) without friction at the wall becomes

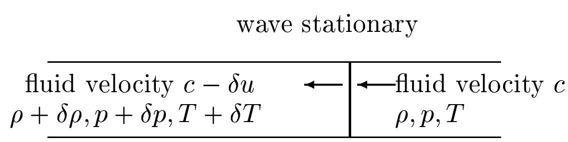
$$p_1 + \rho_1 u_1 = p_2 + \rho_2 u_2 \tag{10.35}$$

Without heat and work transfer the energy equation (10.27) reduces to

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \tag{10.36}$$



(a) Moving wave



(b) Fixed wave

Figure 10.2: Pressure wave

10.3 Speed of sound

Consider the propagation of a small pressure, temperature, density, etc. disturbance with a velocity c. The equations of conservation can be easily written if we consider the steady problem in a frame moving with the wave as shown in Fig. 10.2.

The mass conservation equation (10.25) is

$$\rho c A = (\rho + \delta\rho)(c - \delta u)A \tag{10.37}$$

where A is the cross-sectional area. Thus

$$\delta u = c\frac{\delta\rho}{\rho + \delta\rho} \tag{10.38}$$

The momentum equation (10.26) is

$$pA - (p + \delta p)A = \rho A c\left[(c - \delta u) - c\right] \tag{10.39}$$

from which

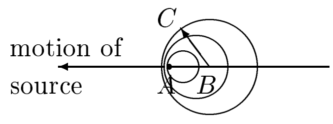
$$\delta p = \rho c \; \delta u \tag{10.40}$$

Combining equations (10.38) and (10.40)

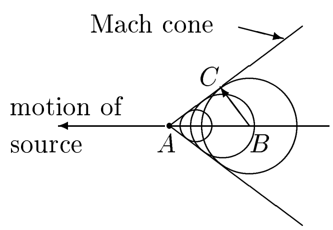
$$c^2 = \frac{\delta p}{\delta\rho}\left(1 + \frac{\delta\rho}{\rho}\right) \tag{10.41}$$

As $\delta\rho \to 0$, we have

$$c = \sqrt{\frac{dp}{d\rho}} \tag{10.42}$$



(a) subsonic case



(b) supersonic case

Figure 10.3: Propagation of sound

This is the speed of an infinitesimal pressure wave, commonly termed an acoustic wave or sound.

For an isentropic process

$$p = \text{constant } \rho^\gamma \tag{10.43}$$

from which $dp/d\rho = \text{constant } \gamma\rho^{\gamma-1} = \gamma p/\rho$. Thus

$$c = \sqrt{\frac{\gamma p}{\rho}} \tag{10.44}$$

$$= \sqrt{\gamma R T} \tag{10.45}$$

For air at 20°C, $c = 343.2$ m/s. Equation (10.42) can be directly used for liquids or solids; typical values are $c = 1400$ m/s for water, and $c = 6000$ m/s for steel.

A sound pattern from a moving source depends on whether it is moving faster (supersonic) or slower than the speed of sound (subsonic). Figures 10.3 (a) and (b) represent the position at an instant of time of the spherical wave fronts emitted when the source was at different points along its line of travel. A is the current position of the source, B its position a time Δt ago, and BC the radius of the wave front which grew from B. Thus, if U is the speed of the source, $\overline{AB} = U\ \Delta t$ and $\overline{BC} = c\ \Delta t$. In the supersonic case, the sound is kept confined to a Mach cone, the half angle of which is $\alpha = \angle CAB$, where

$$\sin\alpha = \frac{c}{U} \tag{10.46}$$

$$= \frac{1}{M} \tag{10.47}$$

where the Mach number $M = U/c$.

10.4 Stagnation and critical properties

The thermodynamic properties obtained on reducing the velocity of a flow through a frictionless, adiabatic process are defined as the stagnation properties (indicated by the subscript 0). Thus, in the case of enthalpy

$$h_0 = h + \frac{u^2}{2} \tag{10.48}$$

The stagnation temperature is given by

$$T_0 = T + \frac{u^2}{2c_p} \tag{10.49}$$

To write this in terms of the Mach number

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2 \tag{10.50}$$

since $u^2/c_pT = u^2(\gamma-1)/\gamma RT = (\gamma-1)u^2/c^2 = (\gamma-1)M^2$.

In addition, because the process is isentropic

$$\begin{aligned}\frac{p_0}{p} &= \left(\frac{T_0}{T}\right)^{\gamma/(\gamma-1)} \\ &= \left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)} \end{aligned} \tag{10.51}$$

$$\begin{aligned}\frac{\rho_0}{\rho} &= \left(\frac{T_0}{T}\right)^{1/(\gamma-1)} \\ &= \left(1 + \frac{\gamma-1}{2}M^2\right)^{1/(\gamma-1)} \end{aligned} \tag{10.52}$$

Critical properties (indicated by *) are those obtained when the flow is accelerated or decelerated to sonic velocity in an isentropic manner. At $M = 1$, equations (10.50), (10.51) and (10.52) become

$$\frac{T_0}{T^*} = \frac{\gamma+1}{2} \tag{10.53}$$

$$\frac{p_0}{p^*} = \left(\frac{\gamma+1}{2}\right)^{\gamma/(\gamma-1)} \tag{10.54}$$

$$\frac{\rho_0}{\rho^*} = \left(\frac{\gamma+1}{2}\right)^{1/(\gamma-1)} \tag{10.55}$$

10.5 Normal shocks

Consider Fig. 10.2 (b) in which the wave is of large amplitude, i.e. δp, $\delta\rho$, δT and δu are not necessarily small. In the absence of heat and work transfer, equations (10.27) and (10.48) indicate that the stagnation enthalpy and hence stagnation temperature are the same on either side of the shock. Thus

$$T_{01} = T_{02} \tag{10.56}$$

Using equation (10.50)

$$\frac{T_2}{T_1} = \frac{2 + (\gamma - 1)M_1^2}{2 + (\gamma - 1)M_2^2} \tag{10.57}$$

Since $T_2/T_1 = (\rho_1/\rho_2)(p_2/p_1) = (u_2/u_1)(p_2/p_1) = (M_2/M_1)(p_2/p_1)\sqrt{(T_2/T_1)}$, we have

$$\frac{p_2}{p_1} = \frac{M_1}{M_2}\sqrt{\frac{2 + (\gamma - 1)M_1^2}{2 + (\gamma - 1)M_2^2}} \tag{10.58}$$

Also, since $\rho_2/\rho_1 = (p_2/p_1)(T_1/T_2)$,

$$\frac{\rho_2}{\rho_1} = \frac{M_1}{M_2}\sqrt{\frac{2 + (\gamma - 1)M_2^2}{2 + (\gamma - 1)M_1^2}} \tag{10.59}$$

For the stagnation pressures

$$\frac{p_{02}}{p_{01}} = \left(\frac{p_{02}/p_2}{p_{01}/p_1}\right)\frac{p_2}{p_1} \tag{10.60}$$

so that

$$\frac{p_{02}}{p_{01}} = \frac{M_1}{M_2}\left[\frac{2 + (\gamma - 1)M_1^2}{2 + (\gamma - 1)M_2^2}\right]^{(\gamma+1)/[2(\gamma-1)]} \tag{10.61}$$

Since $\rho u^2 = (p/RT)u^2 = \gamma p u^2/c^2$, the momentum equation (10.35) reduces to

$$p_1 + \gamma p_1 M_1^2 = p_2 + \gamma p_2 M_2^2 \tag{10.62}$$

from which

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \tag{10.63}$$

Combining with equation (10.58)

$$\frac{M_1\sqrt{2 + (\gamma - 1)M_1^2}}{1 + \gamma M_1^2} = \frac{M_2\sqrt{2 + (\gamma - 1)M_2^2}}{1 + \gamma M_2^2} \tag{10.64}$$

This can be solved to give either $M_1 = M_2$ (no shock) or

$$M_2 = \sqrt{\frac{2 + (\gamma - 1)M_1^2}{2\gamma M_1^2 - (\gamma - 1)}} \tag{10.65}$$

Equation (10.65) can be substituted into equations (10.57)–(10.61) to get T_2/T_1, p_2/p_1, ρ_2/ρ_1 and p_{02}/p_{01} in terms of either M_1 or M_2. Thus, we have

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}\left(M_1^2 - 1\right) \tag{10.66}$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)M_1^2}{(\gamma - 1)M_1^2 + 2} \tag{10.67}$$

The shock is a process with entropy change. Equations (10.20) and (10.21) can be reduced to

$$\frac{s_2 - s_1}{R} = \frac{\gamma}{\gamma - 1}\ln\left[\frac{2 + (\gamma - 1)M_1^2}{(\gamma + 1)M_1^2}\right] + \frac{1}{\gamma - 1}\ln\left[\frac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1}\right] \tag{10.68}$$

Figure 10.4 shows that $s_2 - s_1 < 0$ if $M_1 < 1$, which would violate the second law of thermodynamics. So a subsonic to supersonic shock is not physically possible.

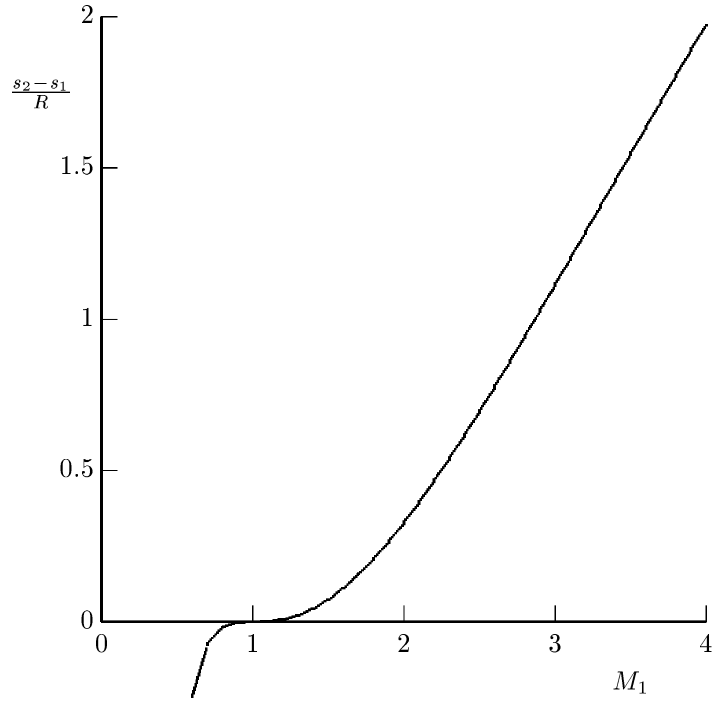


Figure 10.4: Entropy change as function of incoming Mach number

10.6 Oblique shocks

Figure 10.5 shows an oblique shock in which the flow is deflected through an angle δ. The tangential velocity is the same on either side of the shock, so that

$$u_1 \cos\beta = u_2 \cos(\beta - \delta) \tag{10.69}$$

The normal velocity undergoes a shock as described in the previous section. We can use the normal shock relations as long as we substitute $M_1 \sin\beta$ for the Mach number on the upstream side and $M_2 \sin(\beta - \delta)$ on the downstream side. Thus

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2 \sin^2\beta}{(\gamma-1)M_1^2 \sin^2\beta + 2} \tag{10.70}$$

From continuity

$$\rho_1 u_1 \sin\beta = \rho_2 u_2 \sin(\beta - \delta) \tag{10.71}$$

and equation (10.69) we get

$$\frac{\rho_2}{\rho_1} = \frac{\tan\beta}{\tan(\beta - \delta)} \tag{10.72}$$

Equating the two expressions for ρ_2/ρ_1 and simplifying, we get

$$M_1^2 = \frac{2\cos(\beta - \delta)}{\sin\beta\,[\sin(2\beta - \delta) - \gamma \sin\delta]} \tag{10.73}$$

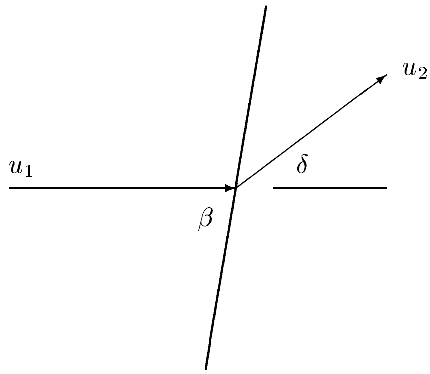


Figure 10.5: Oblique stationary shock

For given M_1 and δ, two values of β are possible. One is called a strong shock and the other a weak shock.

The downstream Mach number is

$$M_2 = \frac{1}{\sin(\beta-\delta)}\sqrt{\frac{2+(\gamma-1)M_1^2\sin^2\beta}{2\gamma M_1^2\sin^2\beta-(\gamma-1)}} \tag{10.74}$$

The pressure, density, temperature, stagnation temperature and stagnation pressure ratios are given by

$$\frac{p_2}{p_1} = 1+\frac{2\gamma}{\gamma+1}\left(M_1^2\sin^2\beta-1\right) \tag{10.75}$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2\sin^2\beta}{(\gamma-1)M_1^2\sin^2\beta+2} \tag{10.76}$$

$$\frac{T_2}{T_1} = \left[2+(\gamma-1)M_1^2\sin^2\beta\right]\frac{2\gamma M_1^2\sin^2\beta-(\gamma-1)}{(\gamma+1)^2M_1^2\sin^2\beta} \tag{10.77}$$

$$\frac{T_{02}}{T_{01}} = 1 \tag{10.78}$$

$$\frac{p_{02}}{p_{01}} = \left[\frac{(\gamma+1)M_1^2\sin^2\beta}{2+(\gamma-1)M_1^2\sin^2\beta}\right]^{\gamma/(\gamma-1)}\left[\frac{\gamma+1}{2\gamma M_1^2\sin^2\beta-(\gamma-1)}\right]^{1/(\gamma-1)} \tag{10.79}$$

10.7 Flow in ducts

The momentum equation (10.32), divided by p becomes

$$dp+\rho u\,du+f\frac{\rho u^2}{2D_h}\,dx=0 \tag{10.80}$$

To summarize, the governing equations for a variable area duct with heat transfer and friction are

$$p = \rho RT \tag{10.81}$$

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0 \quad (10.82)$$

$$dp + \rho u\ du = -f\frac{\rho u^2}{2D_h}\ dx \quad (10.83)$$

$$d\left(h + \frac{u^2}{2}\right) = \frac{q'\ dx}{\dot{m}} \quad (10.84)$$

In addition we will use

$$c = \sqrt{\gamma R T} \quad (10.85)$$

$$M = \frac{u}{c} \quad (10.86)$$

$$c_v = \frac{R}{\gamma - 1} \quad (10.87)$$

$$c_p = \frac{\gamma R}{\gamma - 1} \quad (10.88)$$

10.7.1 Adiabatic, frictionless flow in variable area ducts

Thermodynamically, the process is isentropic. Since

$$\begin{aligned}
\frac{du}{u} + \frac{dA}{A} &= -\frac{d\rho}{\rho} \\
&= u\frac{du}{dp}\ d\rho \text{ from equation (10.83)} \\
&= u^2\frac{d\rho}{dp}\frac{du}{u} \\
&= \frac{u^2}{c^2}\frac{du}{u} \\
&= M^2\frac{du}{u}
\end{aligned}$$

Thus

$$\frac{du}{u} = \frac{1}{M^2 - 1}\frac{dA}{A} \quad (10.89)$$

Also $du/u = -dp/\rho u^2$, so that

$$\frac{dp}{\rho u^2} = \frac{1}{1 - M^2}\ \frac{dA}{A} \quad (10.90)$$

Furthermore, since $dp/\rho u^2 = (c^2/u^2)(d\rho/\rho) = (1/M^2)(d\rho/\rho)$, we have

$$\frac{d\rho}{\rho} = \frac{M^2}{1 - M^2}\frac{dA}{A} \quad (10.91)$$

Similarly

$$\frac{dM}{M} = \frac{2 + (\gamma - 1)M^2}{2(M^2 - 1)}\ dA \quad (10.92)$$

From the equation of state

$$\frac{dT}{T} = \frac{dp}{p} - \frac{d\rho}{\rho} \quad (10.93)$$

	$M < 1$ (subsonic)	$M > 1$ (supersonic)
$dA < 0$ (converging)	$du > 0$ $dM > 0$ $dp < 0$ $d\rho < 0$ $dT < 0$ $dp_0 = 0$ $d\rho_0 = 0$ $dT_0 = 0$ $ds = 0$	$du < 0$ $dM < 0$ $dp > 0$ $d\rho > 0$ $dT > 0$ $dp_0 = 0$ $d\rho_0 = 0$ $dT_0 = 0$ $ds = 0$
$dA > 0$ (diverging)	$du < 0$ $dM < 0$ $dp > 0$ $d\rho > 0$ $dT > 0$ $dp_0 = 0$ $d\rho_0 = 0$ $dT_0 = 0$ $ds = 0$	$du > 0$ $dM > 0$ $dp < 0$ $d\rho < 0$ $dT < 0$ $dp_0 = 0$ $d\rho_0 = 0$ $dT_0 = 0$ $ds = 0$

Table 10.1: Duct flow with variable area

Since $\rho u^2/p = \rho\gamma RTM^2/p = \gamma M^2$, we have

$$\frac{dT}{T} = (\gamma - 1)\frac{M^2}{1 - M^2}\frac{dA}{A} \tag{10.94}$$

Table 10.1 shows the changes in velocity, pressure, density and Mach number which take place in converging ($dA < 0$) and diverging ($dA > 0$) ducts. The behavior of subsonic and supersonic flows is seen to be different. A duct that decreases the fluid velocity is called a diffuser and that which increases it is a nozzle.

Equating the mass flow rate to that at critical conditions, $\rho u A = \rho^* u^* A^*$, so that

$$\begin{aligned}
\frac{A}{A^*} &= \frac{\rho^*}{\rho}\frac{u^*}{u} = \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\frac{u^*}{u} \\
&= \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\frac{\sqrt{\gamma R T^*}}{u} \\
&= \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\frac{\sqrt{\gamma R T^*}}{u}\sqrt{\frac{T^*}{T_0}}\sqrt{\frac{T_0}{T}} \\
&= \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2}M^2\right)\right]^{(\gamma+1)/[2(\gamma-1)]}
\end{aligned} \tag{10.95}$$

where we have used equations (10.50), (10.52), (10.55), and (10.53). As shown in Fig. 10.6 this has a minimum at $M = 1$ where $A_{min} = A^*$.

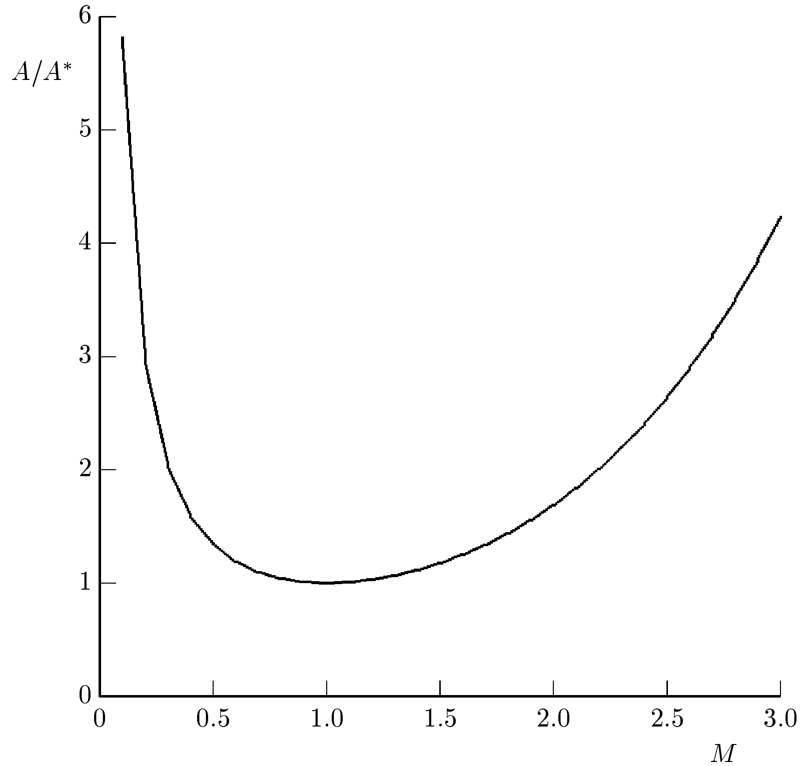


Figure 10.6: Variation of area with Mach number

From

$$\dot{m} = \rho u A = \frac{p}{RT}\frac{u}{c}\sqrt{\gamma RT}A = pM\sqrt{\frac{\gamma}{RT}}A$$
$$= p_0\sqrt{\frac{\gamma}{RT_0}}\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/[2(\gamma-1)]}A^* \quad (10.96)$$

we get

$$\dot{m}_{max} = \sqrt{\gamma\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}}A^*\rho_0\sqrt{RT_0} \quad (10.97)$$

10.7.2 Frictionless flow with heat transfer in constant area ducts

1 and 2 are two sections along the length of a duct that is frictionless, but with heating. The momentum equation gives

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \quad (10.98)$$

Since $\rho u^2 = (p/RT)u^2 = \gamma p M^2$, we have

$$\frac{p_2}{p_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2} \quad (10.99)$$

Using equation (10.51) for the stagnation pressure, we have

$$\frac{p_{02}}{p_{01}} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)\left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right)^{\gamma/(\gamma-1)} \tag{10.100}$$

Similarly, it can be shown that

$$\frac{T_2}{T_1} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2\left(\frac{M_2}{M_1}\right)^2 \tag{10.101}$$

$$\frac{T_{02}}{T_{01}} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2\left(\frac{M_2}{M_1}\right)^2\left(\frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2}\right) \tag{10.102}$$

$$\frac{\rho_2}{\rho_1} = \left(\frac{1+\gamma M_2^2}{1+\gamma M_1^2}\right)\left(\frac{M_1}{M_2}\right)^2 \tag{10.103}$$

$$\frac{u_2}{u_1} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2\left(\frac{M_2}{M_1}\right)^3 \tag{10.104}$$

$$\frac{s_2-s_1}{R} = \ln\left[\left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^{(\gamma+1)/(\gamma-1)}\left(\frac{M_2}{M_1}\right)^{2\gamma/(\gamma-1)}\right] \tag{10.105}$$

Table 10.2 shows the changes in flow variables for subsonic or supersonic flow with either heating or cooling.

The energy equation is

$$h_1 + \frac{u_1^2}{2} + \dot{Q} = h_2 + \frac{u_2^2}{2} \tag{10.106}$$

from which

$$h_{02} - h_{01} = \dot{Q} \tag{10.107}$$

and

$$c_p(T_{02} - T_{01}) = \dot{Q} \tag{10.108}$$

Thus

$$\frac{\dot{Q}}{c_p T_1} = \frac{T_{01}}{T_1}\left(\frac{T_{02}}{T_{01}} - 1\right) \tag{10.109}$$

$$= \left(1+\frac{\gamma-1}{2}M_1^2\right)\left[\left(\frac{M_2^2}{M_1^2}\right)\left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2\left(\frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2}\right) - 1\right] \tag{10.110}$$

The maximum heat is transferred when $M_2 = 1$. Thus

$$\frac{\dot{Q}_{max}}{c_p T_{in}} = \frac{(M_{in}^2 - 1)^2}{2M_{in}^2(\gamma+1)} \tag{10.111}$$

where T_{in} and M_{in} are the inlet temperature and Mach number. respectively. A $T(s)$ graph is called the Rayleigh line.

Example 10.1

Air flows along a frictionless duct of diameter 5 cm with heat transfer. The mass flow rate and the inlet temperature are 1 kg/s and 300 K, respectively. The duct is long enough for the flow to reach sonic velocity at

	$M < 1$ (subsonic)	$M > 1$ (supersonic)
$\dot{Q} < 0$ (cooling)	$du < 0$	$du > 0$
	$dM < 0$	$dM > 0$
	$dp > 0$	$dp < 0$
	$d\rho > 0$	$d\rho < 0$
	*	$dT < 0$
	$dp_0 > 0$	$dp_0 > 0$
	$d\rho_0 > 0$	$d\rho_0 > 0$
	$dT_0 < 0$	$dT_0 < 0$
	$ds < 0$	$ds < 0$
$\dot{Q} > 0$ (heating)	$du > 0$	$du < 0$
	$dM > 0$	$dM < 0$
	$dp < 0$	$dp > 0$
	$d\rho < 0$	$d\rho > 0$
	**	$dT > 0$
	$dp_0 < 0$	$dp_0 < 0$
	$d\rho_0 < 0$	$d\rho_0 < 0$
	$dT_0 > 0$	$dT_0 > 0$
	$ds > 0$	$ds > 0$

Table 10.2: Duct flow with heat transfer; * $dT < 0$ for $M < \gamma^{-1/2}$, $dT > 0$ for $M > \gamma^{-1/2}$; ** $dT > 0$ for $M < \gamma^{-1/2}$, $dT < 0$ for $M > \gamma^{-1/2}$

its end. The heat rate in is 1000 W/m. Find the Mach number, velocity, temperature, pressure, density and entropy at different distances along the duct, if (a) $M_{in} = 0.1$, and (b) $M_{in} = 2$. Take s_{in} as the reference entropy. Also draw the Rayleigh line.

We march from $M = M_{in}$ to $M = 1$ in steps of 0.1, using the following equations:

$$
\begin{aligned}
T &= T_{in}\left(\frac{1+\gamma M_{in}^2}{1+\gamma M^2}\right)\left(\frac{M}{M_{in}}\right)^2 \\
T_{0,in} &= T_{in}\left(1+\frac{\gamma-1}{2}M_{in}^2\right) \\
T_0 &= T_{0,in}\left(\frac{1+\gamma M_{in}^2}{1+\gamma M^2}\right)^2\left(\frac{M}{M_{in}}\right)^2\left(\frac{2+(\gamma-1)M^2}{2+(\gamma-1)M_{in}^2}\right) \\
u &= M\sqrt{\gamma R T} \\
\rho &= \frac{4\dot{m}}{\pi D_h^2 u} \\
p &= \rho R T \\
x &= \frac{\gamma R}{(\gamma-1)q'}(T_0 - T_{0,in}) \\
s - s_{in} &= R\ln\left[\left(\frac{1+\gamma M_{in}^2}{1+\gamma M^2}\right)^{(\gamma+1)/(\gamma-1)}\left(\frac{M}{M_{in}}\right)^{2\gamma/(\gamma-1)}\right]
\end{aligned}
$$

where $D_h = 0.05$ m, $\dot{m} = 1$ kg/s, $T_{in} = 300$ K, $q' = 1000$ W. For air, $R = 287$ J/kg s, and $\gamma = 1.4$.

(a) Subsonic case: $M_{in} = 0.2$. See Table 10.3.

x (m)	M	u (m/s)	T (K)	p (Pa)	ρ (kg/m^3)	$s - s_{in}$ (J/kg K)
0	.2	69.43774	300	631506.8	7.334573	0
303.3285	.3	146.5223	593.6832	592247.9	3.475897	704.0557
622.1643	.4	239.6283	893.1949	544829.3	2.125359	1138.304
906.2821	.5	339.4734	1147.259	493978.7	1.500254	1417.877
1129.551	.6	438.7874	1331.055	443398.4	1.16069	1598.144
1286.332	.7	532.7679	1441.684	395534.5	.9559439	1711.128
1383.38	.8	618.787	1488.992	351725.3	.8230557	1777.251
1432.606	.9	695.8091	1487.597	312498.2	.7319481	1810.247
1446.48	1	763.8152	1452	277863	.6667794	1819.632

Table 10.3: Subsonic case

x (m)	M	u (m/s)	T (K)	p (Pa)	ρ (kg/m^3)	$s - s_{in}$ (J/kg K)
0	2	694.3774	300	63150.68	.7334573	0
14.18464	1.9	683.1945	321.7892	68846.13	.745463	45.64732
29.32387	1.8	670.5458	345.3839	75288.03	.7595249	91.05396
45.34278	1.7	656.1908	370.8109	82598.99	.7761405	135.8114
62.07742	1.6	639.8452	398.0154	90923.75	.7959679	179.3697
79.23067	1.5	621.1749	426.8109	100432.4	.8198918	220.9881
96.30785	1.4	599.7907	456.8078	111323.3	.8491232	259.6678
112.5285	1.3	575.244	487.3127	123824.9	.8853568	294.0566
126.7066	1.2	547.0294	517.1887	138194.5	.9310216	322.3155
137.1034	1.1	514.597	544.6774	154712.2	.9896992	341.9308
141.258	1	477.3844	567.1876	173664.4	1.066847	349.4442

Table 10.4: Supersonic case

(b) Supersonic case: $M_{in} = 2$. See Table 10.4.

Figure 10.7 is the Rayleigh line. Notice that for the supersonic part the curve has been shifted to the right so that the entropy for $M < 1$ and $M > 1$ match at the maximum point.

10.7.3 Adiabatic flow with friction in constant area ducts

Let 1 and 2 be two sections along the length of a duct that with frictionless, but adiabatic. The energy equation is

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \tag{10.112}$$

so that

$$\frac{T_{02}}{T_{01}} = 1 \tag{10.113}$$

Thus

$$\frac{T_2}{T_1} = \frac{T_{01}/T_1}{T_{02}/T_2}$$

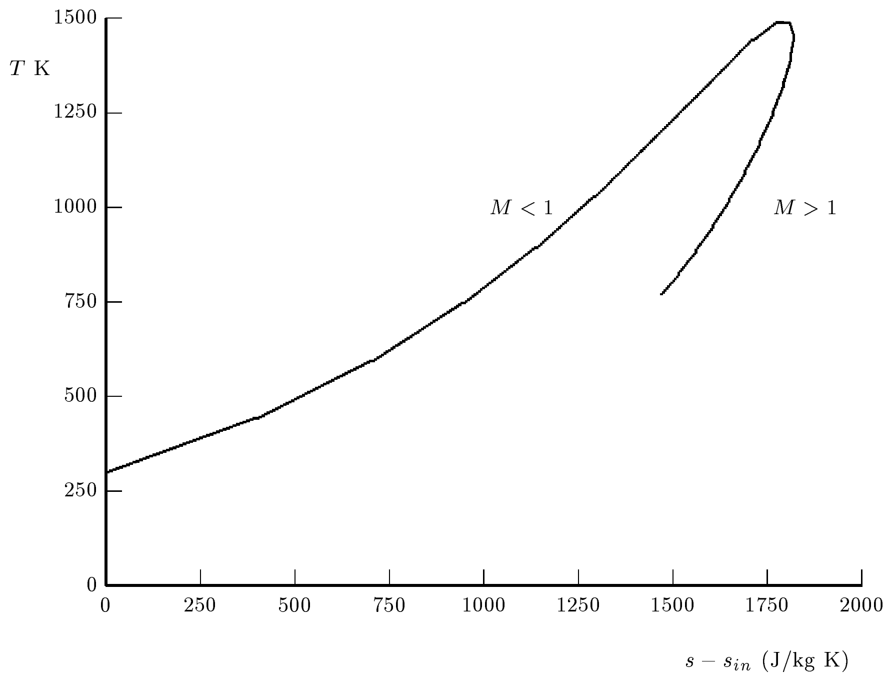


Figure 10.7: Rayleigh line

$$= \frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2} \qquad (10.114)$$

on using equation (10.50).

Similarly, we can derive

$$\frac{p_2}{p_1} = \left(\frac{M_1}{M_2}\right)\left(\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}\right)^{1/2} \qquad (10.115)$$

$$\frac{p_{02}}{p_{01}} = \left(\frac{M_1}{M_2}\right)\left(\frac{2+(\gamma-1)M_2^2}{2+(\gamma-1)M_1^2}\right)^{(\gamma+1)/[2(\gamma-1))]} \qquad (10.116)$$

$$\frac{\rho_2}{\rho_1} = \left(\frac{M_1}{M_2}\right)\left(\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}\right)^{-1/2} \qquad (10.117)$$

$$\frac{s_2-s_1}{R} = \ln\left[\left(\frac{M_2}{M_1}\right)\left(\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}\right)^{(\gamma+1)/[2(\gamma-1)]}\right] \qquad (10.118)$$

Table 10.5 shows the changes in flow variables in the flow direction in subsonic or supersonic flow. The change in Mach number with distance x from the inlet is given by

$$\frac{f\,dx}{D_h} = \frac{4}{\gamma M^2}\frac{1-M^2}{2+(\gamma-1)M^2}\,\frac{dM}{M} \qquad (10.119)$$

which can be integrated to give

$$\frac{\bar{f}\,x}{D_h} = \left[-\frac{1}{\gamma M^2} - \frac{\gamma+1}{2\gamma}\ln\left(\frac{2M^2}{2+(\gamma-1)M^2}\right)\right]_{M_{in}}^{M} \qquad (10.120)$$

$M < 1$ (subsonic)	$M > 1$ (supersonic)
$du > 0$	$du < 0$
$dM > 0$	$dM < 0$
$dp < 0$	$dp > 0$
$d\rho < 0$	$d\rho > 0$
$dT < 0$	$dT > 0$
$dp_0 < 0$	$dp_0 < 0$
$d\rho_0 < 0$	$d\rho_0 < 0$
$dT_0 = 0$	$dT_0 = 0$
$ds > 0$	$ds > 0$

Table 10.5: Duct flow with friction

where $\bar{f}$ is the average value of f, and $M = M_{in}$ at $x = 0$. If L_{max} is the length to reach sonic velocity, then

$$\frac{\bar{f} L_{max}}{D_h} = \frac{1 - M_{in}^2}{\gamma M_{in}^2} + \frac{\gamma + 1}{2\gamma} \ln \frac{(\gamma + 1) M_{in}^2}{2 + (\gamma - 1) M_{in}^2} \tag{10.121}$$

The $T(s)$ graph is called the Fanno line.

Example 10.2

Air flows along an adiabatic duct of diameter 5 cm. The mass flow rate and the inlet temperature are 1 kg/s and 300 K, respectively. The duct is long enough for the flow to reach sonic velocity at its end. The average friction factor is 0.02. Find the Mach number, velocity, temperature, pressure, density and entropy at different distances along the duct, if (a) $M_{in} = 0.1$, and (b) $M_{in} = 2$. Take s_{in} as the reference entropy. Also draw the Fanno line.

We march from $M = M_{in}$ to $M = 1$ in steps of 0.1, using the following equations:

$$\begin{aligned}
T &= T_{in} \left(\frac{2 + (\gamma - 1) M_{in}^2}{2 + (\gamma - 1) M^2} \right) \\
u &= M \sqrt{\gamma R T} \\
\rho &= \frac{4 \dot{m}}{\pi D_h^2 u} \\
p &= \rho R T \\
x &= \frac{D_h}{\bar{f}} \left[-\frac{1}{\gamma M^2} - \frac{\gamma + 1}{2\gamma} \ln \left(\frac{2 M^2}{2 + (\gamma - 1) M^2} \right) \right]_{M_{in}}^{M} \\
s - s_{in} &= R \ln \left[\left(\frac{M}{M_{in}} \right) \left(\frac{2 + (\gamma - 1) M_{in}^2}{2 + (\gamma - 1) M^2} \right)^{(\gamma+1)/[2(\gamma-1)]} \right]
\end{aligned}$$

where $D_h = 0.05$ m, $\dot{m} = 1$ kg/s, $T_{in} = 300$ K, $\bar{f} = 0.02$. For air, $R = 287$ J/kg s, and $\gamma = 1.4$.

(a) Subsonic case: $M_{in} = 0.2$. See Table 10.6.

(b) Supersonic case: $M_{in} = 2$. See Table 10.7.

Figure 10.8 is the Fanno line. Notice that for the supersonic part the curve has been shifted to the right so that the entropy for $M < 1$ and $M > 1$ match at the maximum point.

x (m)	M	u (m/s)	T (K)	p (Pa)	ρ (kg/m^3)	$s - s_{in}$ (J/kg K)
0	.2	69.43774	300	631506.8	7.334573	0
23.08504	.3	103.6438	297.053	418931.6	4.91391	107.869
30.56193	.4	137.2511	293.0233	312060.3	3.710688	178.6735
33.66051	.5	170.087	288	247499.1	2.994327	227.8277
35.10611	.6	201.9992	282.0895	204121.9	2.521278	262.3004
35.81282	.7	232.8589	275.4098	172877.7	2.187145	285.9084
36.15244	.8	262.5617	268.0851	149242.9	1.93972	301.0229
36.29688	.9	291.0284	260.2409	130705.1	1.749988	309.258
36.33316	1	318.2037	252	115757.1	1.600535	311.7904

Table 10.6: Subsonic case

x (m)	M	u (m/s)	T (K)	p (Pa)	ρ (kg/m^3)	$s - s_{in}$ (J/kg K)
0	2	694.3774	300	63150.68	.7334573	0
.0766702	1.9	674.4331	313.5889	67963.24	.7551471	23.42138
.1577754	1.8	653.1241	327.6699	73331.94	.7797847	45.72262
.2429831	1.7	630.3724	342.2053	79349.09	.8079291	66.6891
.3315991	1.6	606.1022	357.1429	86128.81	.8402811	86.07614
.422366	1.5	580.2419	372.4138	93814.3	.8777308	103.6034
.5131462	1.4	552.7264	387.9311	102588	.9214255	118.9502
.6004115	1.3	523.5005	403.5875	112686.8	.9728668	131.7471
.6783964	1.2	492.5214	419.2547	124424.3	1.034059	141.5664
.737654	1.1	459.7626	434.7827	138226.4	1.107737	147.9067
.7624915	1	425.2175	450	154687	1.197731	150.1722

Table 10.7: Supersonic case

10.8 Multi-dimensional flow

10.8.1 Stagnation enthalpy

Taking the dot product of the momentum equation

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p \tag{10.122}$$

by the vector $\mathbf{u}$ we get

$$\rho \frac{D}{Dt}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) = -\mathbf{u}\cdot\nabla p \tag{10.123}$$

The energy equation without heat conduction is

$$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} \tag{10.124}$$

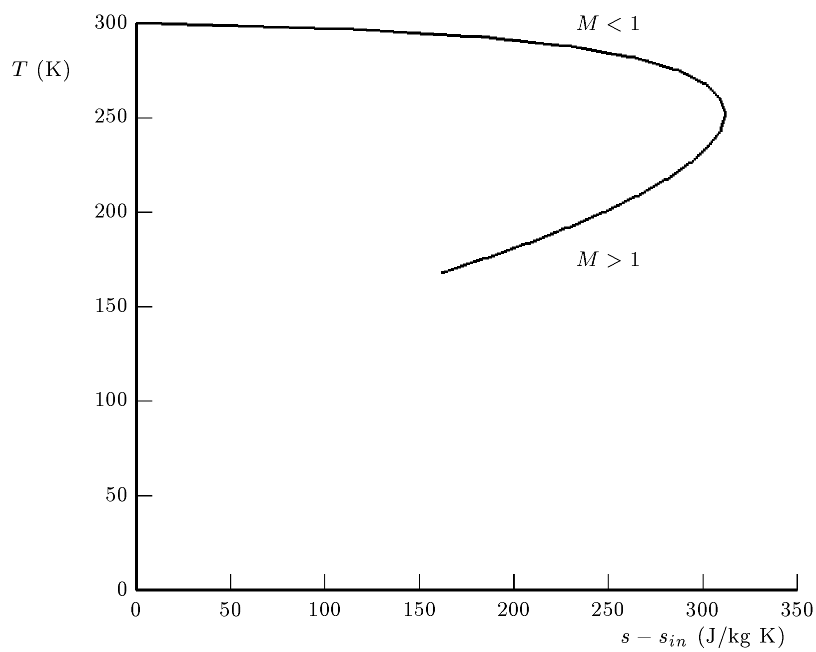


Figure 10.8: Fanno line

Adding the two, we get

$$\rho \frac{D}{Dt}\left(h + \frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) = \frac{Dp}{Dt} - \mathbf{u}\cdot\nabla p \tag{10.125}$$

which is equivalent to

$$\rho \frac{Dh_0}{Dt} = \frac{\partial p}{\partial t} \tag{10.126}$$

For steady flow, we have

$$\frac{Dh_0}{Dt} = 0 \tag{10.127}$$

which means that h_0 is constant along a streamline.

10.8.2 Crocco's theorem

Equation (10.122) is

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) - \mathbf{u}\times\boldsymbol{\omega} = -\frac{1}{\rho}\nabla p \tag{10.128}$$

since

$$\mathbf{u}\cdot\nabla\mathbf{u} = \nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) - \mathbf{u}\times\boldsymbol{\omega} \tag{10.129}$$

Using the Gibbs relation

$$T\ ds = dh - \frac{1}{\rho}\ dp \tag{10.130}$$

which is equivalent to

$$T\ \nabla s = \nabla h - \frac{1}{\rho}\nabla p \tag{10.131}$$

we get Crocco's equation

$$\mathbf{u} \times \boldsymbol{\omega} + T\ \nabla s = \nabla\left(h + \frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) + \frac{\partial \mathbf{u}}{\partial t} \tag{10.132}$$

For steady flow this becomes

$$\mathbf{u} \times \boldsymbol{\omega} + T\ \nabla s = 0 \tag{10.133}$$

Thus irrotationality implies that the entropy is constant everywhere, and vice-versa.

10.8.3 Irrotational flow

If $\boldsymbol{\omega} = \nabla \times \mathbf{u} = 0$, we have

$$\mathbf{u} = \nabla\phi \tag{10.134}$$

It can be shown from the governing equations that

$$\nabla\cdot\mathbf{u} = \frac{1}{c^2}\left\{\mathbf{u}\cdot[(\mathbf{u}\cdot\nabla)\mathbf{u}] + \frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial t} + \mathbf{u}\cdot\mathbf{u}\right)\right\} \tag{10.135}$$

from which

$$\nabla^2\phi = \frac{1}{c^2}\left\{\nabla\phi\cdot[(\nabla\phi\cdot\nabla)\nabla\phi] + \frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial t} + \nabla\phi\cdot\nabla\phi\right)\right\} \tag{10.136}$$

The pressure, obtained from the energy equation, is

$$p = p_\infty\left[1 + \frac{\gamma-1}{2c_\infty^2}\left(U^2 - \mathbf{u}\cdot\mathbf{u}\right)\right]^{\gamma/(\gamma-1)} \tag{10.137}$$

where at infinity the pressure and velocity are p_∞ and U. The pressure coefficient defined by

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U^2} \tag{10.138}$$

becomes

$$C_p = \frac{2}{\gamma M_\infty^2}\left\{\left[1 + \frac{\gamma-1}{2c_\infty^2}\left(U^2 - \mathbf{u}\cdot\mathbf{u}\right)\right]^{\gamma/(\gamma-1)} - 1\right\} \tag{10.139}$$

10.8.4 Small-perturbation theory

For steady flow over a slender body, we can write

$$\mathbf{u} = U\mathbf{e}_x + \mathbf{u}' \tag{10.140}$$

or

$$\phi = Ux + \Phi \tag{10.141}$$

where $\mathbf{u}' = \nabla\Phi$. Equation (10.136) becomes

$$\nabla^2\Phi = \frac{1}{c^2}\left(U\mathbf{e}_x + \nabla\Phi\right)\cdot\left[\left(U\mathbf{e}_x + \nabla\Phi\right)\cdot\nabla\right]\left(U\mathbf{e}_x + \nabla\Phi\right) \tag{10.142}$$

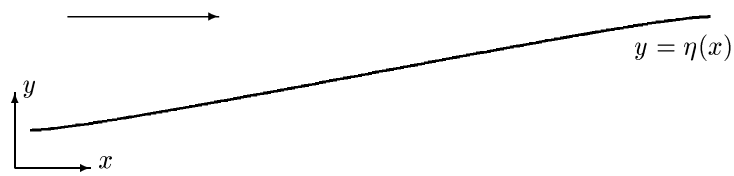


Figure 10.9: Subsonic flow past wall

We assume that the devation from uniform flow is small, so that $|\mathbf{u}'| \ll U$. Under this condition, we get

$$\nabla^2 \Phi = \frac{U^2}{c^2} \frac{\partial^2 \Phi}{\partial x^2} \tag{10.143}$$

This can be further approximated by

$$\nabla^2 \Phi = \frac{U^2}{c_\infty^2} \frac{\partial^2 \Phi}{\partial x^2} \tag{10.144}$$

which gives

$$\left(1 - M_\infty^2\right) \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{10.145}$$

In the small-perturbation approximation, the pressure coefficient becomes

$$\begin{aligned} C_p &= -2\frac{u'}{U} \\ &= -\frac{2}{U}\frac{\partial \Phi}{\partial x} \end{aligned} \tag{10.146}$$

where u' is the x-component of $\mathbf{u}'$.

10.8.5 Subsonic flow

Consider two-dimensional flow next to a wall of shape $y = \eta(x)$ as in Fig. 10.9. The small-perturbation approximation gives

$$\left(1 - M_\infty^2\right) \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \tag{10.147}$$

with boundary conditions

$$\begin{aligned} \frac{\partial \Phi}{\partial y} &= U\frac{d\eta}{dx} \quad \text{at wall} \\ \frac{\partial \Phi}{\partial x} &= \text{finite as } y \to \infty \end{aligned}$$

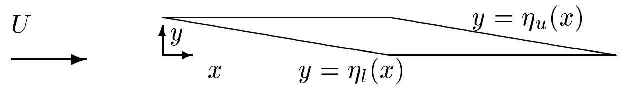


Figure 10.10: Airfoil in supersonic flow

With new variables defined by

$$\Phi' = \sqrt{1 - M_\infty^2}\Phi$$
$$y' = \sqrt{1 - M_\infty^2}\, y$$

we have

$$\frac{\partial^2 \Phi'}{\partial x^2} + \frac{\partial^2 \Phi'}{\partial y'^2} = 0 \qquad (10.148)$$

with boundary conditions

$$\frac{\partial \Phi'}{\partial y'} = U\frac{d\eta}{dx} \quad \text{at wall}$$
$$\frac{\partial \Phi'}{\partial x} = \text{finite as } y' \to \infty$$

Thus

$$\frac{C_p}{C_p'} = \frac{\partial \Phi/\partial x}{\partial \Phi'/\partial x} \qquad (10.149)$$
$$= \frac{1}{\sqrt{1 - M_\infty^2}} \qquad (10.150)$$

which is the Prandtl-Glauert rule; C_p is the coefficient of pressure in the compressible flow, and C_p' is the corresponding value for incompressible flow.

10.8.6 Supersonic flow

Consider flow around an airfoil with upper and lower surfaces $y = \eta_u(x)$ and $y = \eta_l(x)$, respectively, as shown in Fig. 10.10. Thus

$$\frac{\partial^2 \Phi}{\partial x^2} - \frac{1}{M_\infty^2 - 1}\frac{\partial^2 \Phi}{\partial y^2} = 0 \qquad (10.151)$$

with

$$\frac{\partial \Phi}{\partial y} = U\frac{d\eta_{u,l}}{dx} \quad \text{at wall}$$
$$\frac{\partial \Phi}{\partial x} = \text{finite as } y \to \infty$$

This is a wave equation with a solution of the type

$$\Phi = f(x \pm \sqrt{M_\infty^2 - 1}\, y) \qquad (10.152)$$

The solutions for the upper and lower parts of the flow field are different, so that

$$\Phi = \begin{cases} f_u(x - \sqrt{M_\infty^2 - 1}\; y) & \text{upper} \\ f_l(x + \sqrt{M_\infty^2 - 1}\; y) & \text{lower} \end{cases} \tag{10.153}$$

The boundary condition at the wall gives

$$\begin{aligned} -\sqrt{M_\infty^2 - 1}\; f_u' &= U\frac{d\eta_u}{dx} \qquad \text{upper surface} \\ \sqrt{M_\infty^2 - 1}\; f_l' &= U\frac{d\eta_l}{dx} \qquad \text{lower surface} \end{aligned}$$

Thus, from equation (10.146), we get

$$C_p = -\frac{2}{U} f_{u,l}' \tag{10.154}$$

which gives

$$C_p = \begin{cases} \frac{2}{\sqrt{M_\infty^2 - 1}} \frac{d\eta_u}{dx} & \text{upper surface} \\ -\frac{2}{\sqrt{M_\infty^2 - 1}} \frac{d\eta_l}{dx} & \text{lower surface} \end{cases} \tag{10.155}$$

Lift coefficient

The coefficient of lift is

$$C_L = \frac{1}{\frac{1}{2}\rho_\infty U^2 L} \int_0^L (p_l - p_u)\; dx \tag{10.156}$$

where L is the chord and p_u and p_l are the pressures at the upper and lower surfaces. Substituting

$$p = p_\infty + \frac{1}{2}\rho_\infty U^2 C_p \tag{10.157}$$

we get

$$C_L = -\frac{2}{\sqrt{M_\infty^2 - 1}} \left[\eta_l + \eta_u\right]_{x=0}^{x=L} \tag{10.158}$$

Taking $\eta_l(L) = \eta_u(L) = 0$ and $\eta_l(0) = \eta_u(0) = \alpha L$, where α is the angle of attack

$$C_L = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}} \tag{10.159}$$

Drag coefficient

The coefficient of drag is

$$C_L = \frac{1}{\frac{1}{2}\rho_\infty U^2 L} \int_0^{\alpha L} (p_l - p_u)\; dy \tag{10.160}$$

which becomes

$$C_D = \frac{2}{\sqrt{M_\infty^2 - 1}}\; \frac{1}{L} \int_0^L \left[\left(\frac{d\eta_l}{dx}\right)^2 + \left(\frac{d\eta_u}{dx}\right)^2\right]\; dx \tag{10.161}$$

Problems

1. An airplane is flying at a height of 5 km at a Mach number of 1.5. Find L.
2. Show that for $M \ll 1$, the equation
$$\frac{p_0}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma-1)}$$
can be expanded to
$$p_0 = p + \frac{1}{2}\rho u^2 + \ldots$$
3. For a normal shock wave, show that $(s_2 - s_1)/R = -\ln(p_{02}/p_{01})$.
4. An air stream with temperature 300 K undergoes a normal shock. The downstream Mach number is half that upstream. Determine the air velocities upstream and downstream of the shock.
5. Show from equation (10.95) that A/A^* is a minimum at $M = 1$.
6. A converging-diverging air nozzle has a throat area of 1 cm^2. Find the inlet and exit areas if the Mach numbers there are 0.1 and 2, respectively.
7. A large pressurized tank containing air at 1 MPa pressure, 100 K temperature floating in space develops a small leak through a hole of area 0.01 mm^2. Find the mass flow rate through the leak.
8. A stationary normal shock exists in a flow of a gas for which the ratio of specific heats is 1.3. If the upstream Mach number is very large, find (a) the downstream Mach number, and (b) the density ratio.
9. Conditions upstream of a shock in a flow of air are $M_1 = 3$, $T_1 = 300$ K, and $p_1 = 100$ kPa. The downstream pressure is 1 MPa. Find if the shock is normal or oblique and the downstream temperature and Mach number.
10. Air flows at 60 m/s into an adiabatic commercial steel pipe of diameter 5 cm at a temperature and pressure of 100°C and 1 MPa. Determine (a) the Mach number at a distance of 50 m down the pipe, and (b) the length of the pipe for choked flow. Take the average friction factor to be 0.02.
11. Air at a stagnation pressure and stagnation temperature of 300 kPa, 400 K flows isentropically through a converging nozzle from an inlet area of 100 cm^2 to an exit area of 50 cm^2. Find the mass flow rate for a back pressure of 100 kPa.
12. Air enters a heated, frictionless duct of square cross-section at a pressure of 100 kPa, temperature 300 K, and Mach number 0.2. Find the amount of heat added per unit mass necessary to choke the flow.
13. A supersonic wind tunnel is formed at the divergent section of a CD nozzle. The flow in a 15 cm × 15 cm square test section of such a tunnel is at a Mach number of 3 at temperature −20°C and pressure 50 kPa. Calculate the mass flow rate.
14. The inlet to a frictional, adiabatic, 10 cm diameter, 8.2 m long pipe is air at 100 kPa, 300 K, and Mach number of 2.5. If the friction factor is 0.002, find the temperature at the outlet.
15. Air at a stagnation pressure of 150 kPa flows isentropically through a converging nozzle. Find the back pressure for which the nozzle is just choked.
16. Air flows through a diverging duct. Inlet conditions are: $T = 500$ K, $p = 1000$ kPa, $A = 0.01$ m^2, $u = 1000$ m/s; outlet conditions are $T = 1000$ K, $p = 100$ kPa, $A = 0.08$ m^2. There is heat transfer between the surroundings and the air in the pipe. Determine (a) the velocity at the outlet, (b) the mass flow rate, (c) direction and magnitude of the heat transfer, and (d) change in specific entropy between inlet and outlet.
17. Find the speed of travel of a shock wave in air given that the pressure and temperature ahead are 0.1 MPa, 300 K, respectively, and the pressure behind is 0.5 MPa.
18. Air flows without friction through a duct. At the inlet the temperature is 300 K, the pressure is 10^5 Pa, and the velocity is 120 m/s. At the outlet the pressure is 0.5×10^5 Pa. Determine the change in Mach number and entropy from the inlet to the outlet.
19. Using
$$\begin{aligned} \rho u A &= \dot{m} \\ h + \frac{u^2}{2} &= h_0 \\ p &= \rho R T \\ h &= c_p T \\ s - s_1 &= c_v \log\left[\frac{p}{p_1}\left(\frac{\rho_1}{\rho}\right)^\gamma\right] \end{aligned}$$

show that the equation of the Fanno line is

$$\frac{s - s_1}{c_v} = \log\left[h(h_0 - h)^{(\gamma-1)/2}\right] + \log\left[\frac{\rho_1^\gamma}{p_1}\frac{R}{c_p}\left(\frac{2A^2}{\dot{m}^2}\right)^{(\gamma-1)/2}\right]$$

where the subscript 1 refers to inlet conditions.

20. Using

$$\begin{aligned} \rho u A &= \dot{m} \\ h + \frac{u^2}{2} &= h_0 \\ p &= \rho R T \\ h &= c_p T \\ s - s_1 &= c_v \log\left[\frac{p}{p_1}\left(\frac{\rho_1}{\rho}\right)^\gamma\right] \end{aligned}$$

show that the equation of the Fanno line is

$$\frac{s - s_1}{c_v} = \log\left[h(h_0 - h)^{(\gamma-1)/2}\right] + \log\left[\frac{\rho_1^\gamma}{p_1}\frac{R}{c_p}\left(\frac{2A^2}{\dot{m}^2}\right)^{(\gamma-1)/2}\right]$$

where the subscript 1 refers to inlet conditions.

21. Air flow steadily through a round tube of diameter D with both friction and heat transfer. Conditions T_1, p_1, and M_1 at the inlet and p_2 and M_2 at the outlet are known. Calculate the heat added per unit mass and the friction force exerted by the gas on the tube.

22. Find the drag coefficient for the double-wedge airfoil in supersonic flow at zero angle of attack using the slender body approximation.

Chapter 11

Compressible effects in liquids and two-phase flow

11.1 Waterhammer

Compressibility in liquids leads to the effect known as waterhammer which is due to the rapid rapid opening or closing of valves or other changes in a liquid pipeline. The effect is due to the finite speed of sound resulting from finite compressibility. The bulk modulus K represents the ratio of the pressure change on a given material and the relative density change that it produces. Thus we have

$$K = \rho \frac{dp}{d\rho} \tag{11.1}$$

The speed of sound c in any material is given by

$$c = \sqrt{\frac{K}{\rho}} \tag{11.2}$$

For an ideal incompressible fluid, $K \to \infty$, so that $c \to \infty$. Another factor that is important is that the material of which the wall of the pipe carrying liquid is not absolutely rigid. As an acoustic wave travels through it, the high and low pressures cause not only a compression and expansion of the liquid but also deformation of the pipe wall. This affects the overall compressibility of the liquid-pipe system, and must be taken into account in the speed of sound. In a thin-walled pipe the speed of sound is

$$c = \sqrt{\frac{K}{\rho(1 + \frac{K\alpha}{E})}} \tag{11.3}$$

where E is the Young's modulus of the material of the wall, and a is a constant which depends on the kind of restraints on the pipe. As a special case $E \to \infty$ for a rigid pipe material, which gives equation (11.2). Table 11.1 gives the values of a for specific cases of pipe restraints.

where D and e are the diameter and wall thickness of the pipe, and μ is the Poisson's ratio of the material of the wall.

Consider now a long pipe of length L with a constant pressure liquid inlet at one end and a valve at the other. Liquid flows through the pipe at velocity U and approximately uniform pressure p_0. At

Restraint	α
Anchored against longitudinal movement	$\frac{D}{e}(1-\mu^2)$
Anchored at upper end	$\frac{D}{e}$
With frequent expansion joints	$\frac{D}{e}(1.25-\mu)$

Table 11.1: α for specific restraints

time $t = 0$, the valve is suddenly closed leading to the progression of events schematically indicated in Fig. 11.1; liquid velocity and pressure and pipe diameter are shown at different instants of time. The liquid that is coming towards the valve builds up a pressure which compresses the liquid near the valve and expands a portion of the pipe. The time for this compression to travel the length of the pipe as a wave is L/c so that we will look at time intervals of $L/2c$. At $t = L/2c$, the valve half of the pipe has expanded and the liquid has come to a stop there. The liquid in the inlet half is still coming in. At $t = L/c$, the wave has reached the inlet; the entire pipe is expanded and the liquid velocity is zero. The compression wave reflects back from the inlet; at $t = 3L/2c$ the inlet half of the pipe contracts to its normal size squeezing the liquid towards the inlet. At $t = 2L/c$, the entire pipe is at normal size with liquid velocity towards the inlet. This liquid motion reduces the pipe diameter so that at $t = 5L/2c$ the valve half of the pipe is at less than normal size. In other words the compression wave coming from the inlet has reflected from the valve as an expansion wave. At $t = 3L/c$, this wave reaches the inlet and the entire pipe is at reduced area with zero liquid velocity. At $7L/2c$ the expansion wave has reflected from the inlet. At $t = 4L/c$ it reaches the valve. This completes the cycle, with the whole process repeating itself until damped out.

The frequency of the pressure wave set up by the sudden closing of the valve in this example is $c/4L$. Since $2L/c$ is the time taken for the compression wave to start from the valve and return, any valve closing time smaller than this would cause waterhammer effects to be felt in the pipe. The peak pressure rise is due to the conversion of the entire kinetic energy of the fluid, i.e. $\Delta p = \rho U^2/2$.

11.2 Two-phase systems

Multiphase flows in pipelines involving solids, liquids or gases are fairly common. Here we will look at some aspects of two-phase liquid-gas flows.

11.2.1 Homogeneous two-phase mixture

Consider a homogeneous dispersion of liquid droplets in a gas, or gas bubbles in a liquid as shown in Fig. 11.2. The speed of sound is given by equation (11.2) where the bulk modulus K is that of the mixture. This has to be determined in terms of those of the pure liquid K_l and pure gas K_g. If α is the volume fraction of gas, and $1-\alpha$ is that of the liquid, then the density ρ is given by

Since $\Delta V = V\Delta p/K = \alpha V \Delta p/K_g + (1-\alpha)V\Delta p/K_l$ we have

$$K = \frac{1}{\frac{\alpha}{K_g} + \frac{1-\alpha}{K_l}}(?) \quad (11.4)$$

$$c_2 = \frac{1}{\frac{\alpha}{K_g} + \frac{1-\alpha}{K_l}} \frac{1}{\alpha\rho_g + (1-\alpha)\rho_l} \quad (11.5)$$

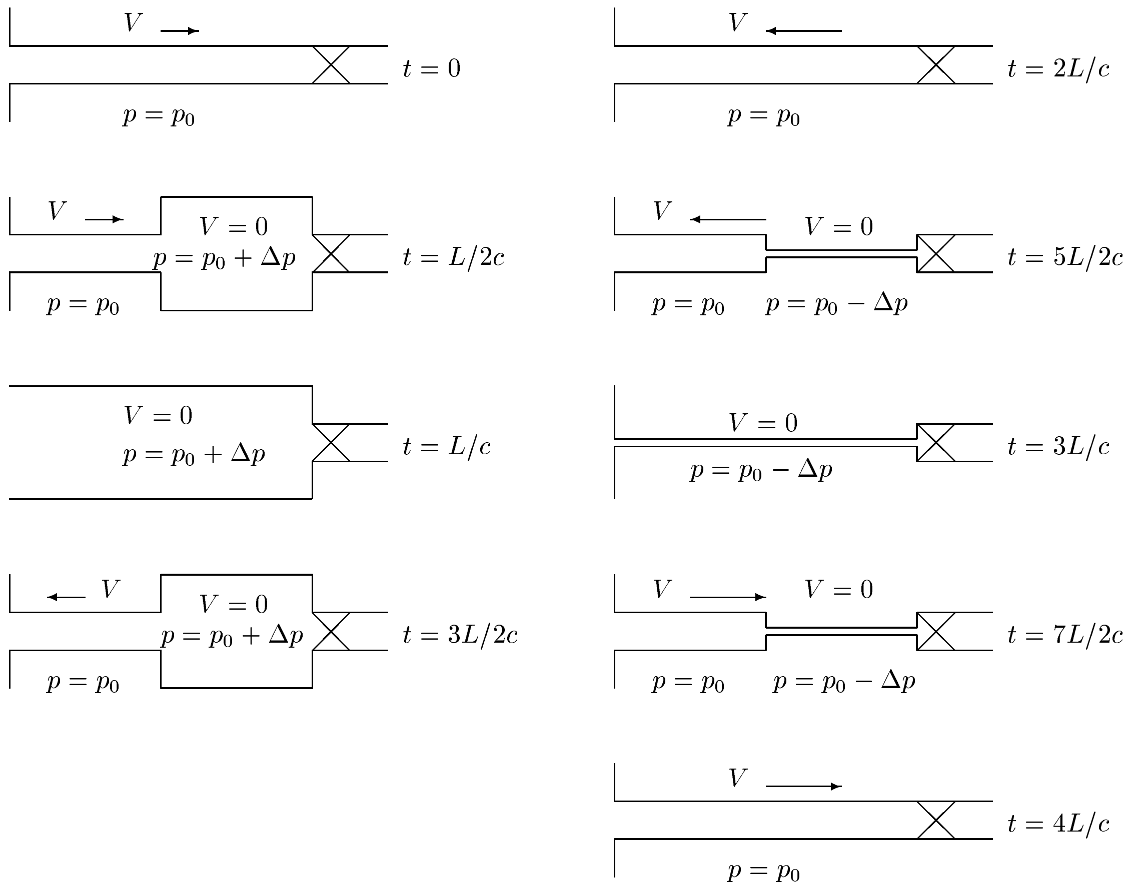


Figure 11.1: Sequence of events in waterhammer

$$= \frac{1}{\frac{\alpha}{\rho_g c_g^2} + \frac{1-\alpha}{\rho_l c_l^2}} \frac{1}{\alpha\rho_g + (1-\alpha)\rho_l} \tag{11.6}$$

where

$$c_g^2 = \frac{K_g}{\rho_g} \tag{11.7}$$

$$c_l^2 = \frac{K_l}{\rho_l} \tag{11.8}$$

In the limit of $\alpha \to 1$, $c^2 \to c_g^2$, and for $\alpha \to 0$, $c^2 \to c_l^2$. We can take $K_l \gg K_g$, $\rho_l \gg \rho_g$ as approximations, so that

$$c^2 = \frac{K_g}{\rho_l} \frac{1}{\alpha(1-\alpha)} \tag{11.9}$$

$$\rho = \alpha\rho_g + (1-\alpha)\rho_l \tag{11.10}$$

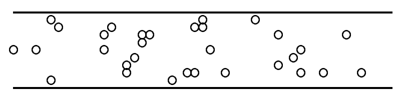


Figure 11.2: Homogeneous two-phase flow

The minimum value is at $\alpha = 1/2$

$$c^2_{min} = 2\rho\frac{K_g}{\rho_l} \tag{11.11}$$

11.2.2 Two-phase flows in horizontal pipes

The following are patterns in a horizontal pipe:

Bubble of froth flow	dispersed bubbles of gas throughout liquid
Plug flow	alternate plugs of liquid and gas along upper part of pipe
Stratified flow	liquid at bottom, gas on top with smooth interface
Wavy flow	stratified with interfacial waves
Slug flow	frothy slug moves at velocity larger than liquid velocity
Annular flow	liquid film, gas core
Spray or dispersed flow	liquid as fine droplets

Flow pattern prediction

This can be done from the accompanying diagram (not yet included) where
$l = (\rho_G/\rho_{air})(\rho_L/\rho_{water})$
$y = (\rho_{water}/\rho_L)[(\mu_L/\mu_{water})]^{1/3}(\rho_{water}/\rho_L)^{2/3}$
$G = \dot{m}/A$ is a mass velocity
Air and water properties are at 20°C, 101.3 kPa.

Lockhart-Martinelli pressure drop prediction

To determine the pressure drop in a horizontal pipe, find

1. Reynolds numbers Re_L, Re_G and flow regime (laminar or turbulent) for each single-phase fluid.
2. Pressure drops Δp_L, Δp_G for each phase using $\Delta p_i = f(L/D)\left[\frac{1}{2}\rho_i U_i^2\right]$.
3. $X = \sqrt{\Delta p_L/\Delta p_G}$.
4. Y_L or Y_G from graph
5. $\Delta p_{TP} = Y_L \Delta p_L = Y_G \Delta p_G$

Critical condition in flashing flow

As p decreases due to friction, the saturation temperature also decreases and the liquid vaporizes. Below a certain backpressure p_b the mass flow rate $\dot{m}$ does not change, This is the critical mass flow rate $\dot{m}_c$ and depends on the thermodynamics state of the fluid and the flow regime. The critical or maximum mass velocity is given in the graph[1] (not yet included), where $E = h_{01} + q = h_{02}$, $p_c =$ pressure p_2 inside tube at 2 at critical condition, and $x =$ vapor weight fraction (quality) at 2.

Problems

1. Saturated water flows in a 2.5 cm diameter boiler tube where there is vaporization due to pressure drop and heat addition. The outlet quality and pressure are 0.8 and 2 atmospheres, respectively. What is the maximum mass flow rate?
2. Water is flowing through a cast iron pipe of length 10 m anchored against longitudinal movement. The wall thickness is 1 mm and the pipe diameter is 1 cm. A valve at the end of the pipe is suddenly closed. How fast must the pipe be closed for waterhammer effects to be felt? What is the frequency of the pressure wave? Use a thin wall approximation.
3. Water flows at 5 m/s through a 30 cm diameter PVC pipe of length 50 m and thickness 5mm. The modulus of elasticity and Poisson's ratio of PVC are 2.8×10^9 Pa and 0.45, respectively. The bulk modulus of water is 2.19×10^9 Pa. The pipe can be assumed to be anchored against longitudinal movement. There is a valve at the end of the pipe that is being closed. (a) How slowly must this valve be closed so that waterhammer effects are not produced? If the valve were closed suddenly, find (b) the time it takes for the pressure wave to travel upstream to the inlet of the pipe, (c) the frequency of oscillation of the pressure within the pipe, and (d) an estimate of the pressure rise within the pipe.
4. A mixture of gaseous CO_2 and liquid gasoline flows along a horizontal pipe of 100 m length, inner diameter 20 cm, and roughness 0.2 mm. The mass velocity of the CO_2 is 10 kg/sm^2, and that of the gasoline is 250 kg/sm^2. Find (a) the flow pattern of the two-phase flow, and (b) the pressure drop. Use the following property values:

Fluid	*Density* (kg/m^3)	*Viscosity* (Ns/m^2)	*Surface tension* (N/m)
Gasoline	680	2.92×10^{-4}	2.16×10^{-2}
Water (at 20°C,1 atm)	998	10^{-3}	7.28×10^{-2}
Air (at 20°C, 1 atm)	1.2	1.8×10^{-5}	
CO_2	1.82	1.48×10^{-5}	

5. Air and water flow in a 50 cm diameter horizontal pipe at mass flow rates of 0.1 and 0.2 kg/s respectively. determine the flow pattern.
6. Water under saturated conditions enters a long 10 cm diameter tube. Find the critical mass flow rate if the exit quality and pressure are 0.4 and 700 kPa.
7. Crude oil (density 800 kg/m^3, viscosity 60×10^{-5} Ns/m^2) and natural gas (density 0.8 kg/m^3, viscosity 1.4×10^{-5} Ns/m^2) flow along a 5 cm diameter, 300 m long pipe with roughness 0.15 mm. Find the pressure drop for crude oil and natural gas flow rates of 0.64 kg/s and 0.04 kg/s.

[1] Hodge, B.K., Analysis and Design of Energy Systems, Prentice-Hall, Englewood Cliffs, NJ, 1990.

Chapter 12

Numerical methods

In this chapter we discuss some of the basic ideas regarding numerical methods as they relate to the solution of the ordinary and partial differential equations which occur in fluid dynamics.

12.1 Ordinary differential equations

If $x(t)$, $y(t)$ and $z(t)$ are the Cartesian coordinates of a fluid particle moving in a velocity field $u\mathbf{i} + v\mathbf{j} + w\mathbf{j}$, then

$$\frac{dx}{dt} = u(x, y, t) \tag{12.1}$$

$$\frac{dy}{dt} = v(x, y, t) \tag{12.2}$$

$$\frac{dz}{dt} = w(x, y, t) \tag{12.3}$$

These equations can be integrated using any numerical procedure to determine the flow lines.

12.1.1 Fourth-order Runge-Kutta integration

This is a popular integration technique for ordinary differential equations with given initial values. Choose an appropriately small time step Δt, for example $\Delta t = 0.01$. Let

$$x_i = x(i\ \Delta t)$$
$$y_i = y(i\ \Delta t)$$

Then x_0, y_0 is the initial condition which is known. Knowing any x_n, y_n we can find x_{n+1}, y_{n+1} in the following manner. First determine the six numbers

$$\begin{aligned} p_0 &= \Delta t\ u(x_n, y_n, t) \\ q_0 &= \Delta t\ v(x_n, y_n, t) \\ p_1 &= \Delta t\ u(x_n + \frac{1}{2}p_0, y_n + \frac{1}{2}q_0, t + \frac{1}{2}\Delta t) \end{aligned}$$

$$
\begin{aligned}
q_1 &= \Delta t\ v(x_n + \frac{1}{2}p_0, y_n + \frac{1}{2}q_0, t + \frac{1}{2}\Delta t) \\
p_2 &= \Delta t\ u(x_n + \frac{1}{2}p_1, y_n + \frac{1}{2}q_1, t + \frac{1}{2}\Delta t) \\
q_2 &= \Delta t\ v(x_n + \frac{1}{2}p_1, y_n + \frac{1}{2}q_1, t + \frac{1}{2}\Delta t) \\
p_3 &= \Delta t\ u(x_n + p_2, y_n + q_2, t + \Delta t) \\
q_3 &= \Delta t\ v(x_n + p_2, y_n + q_2, t + \Delta t)
\end{aligned}
$$

In the right-hand sides, the functions $u(x,y,t)$ and $v(x,y,t)$ are evaluated at the values of x, y and t indicated in each expression.

Then

$$
\begin{aligned}
x_{n+1} &= x_n + \frac{1}{6}(p_0 + 2p_1 + 2p_2 + p_3) \\
y_{n+1} &= y_n + \frac{1}{6}(q_0 + 2q_1 + 2q_2 + q_3)
\end{aligned}
$$

From x_0, y_0 get x_1, y_1, then x_2, y_2, and so on as long as necessary.

12.2 Partial differential equations

There are different ways in which the incompressible flow problem in two dimensions with constant properties may be written. As an example we will consider the problem of incompressible, two-dimensional flow without body force.

12.2.1 Primitive variables

In nondimensional terms, we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{12.4}$$

$$\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \tag{12.5}$$

$$\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] \tag{12.6}$$

$$\left[\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right] = \frac{1}{Re\ Pr}\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right] + \frac{Ec}{Re}\Phi \tag{12.7}$$

where

$$\Phi = \frac{1}{Re}\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right]$$

12.2.2 Stream function-vorticity

The stream function $\psi(x,y)$ defined by

$$u = \frac{\partial \psi}{\partial y} \tag{12.8}$$

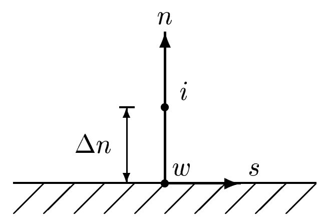


Figure 12.1: Boundary condition on vorticity

$$v = -\frac{\partial \psi}{\partial x} \tag{12.9}$$

satisfies the continuity equation (12.4). The vorticity, ω, has only the z-component, ζ, which is defined by

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{12.10}$$

and which satisfies

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \zeta = 0 \tag{12.11}$$

The vorticity equation

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \zeta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial y} = \frac{1}{Re}\left[\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right] \tag{12.12}$$

is obtained by taking the curl of the momentum equation.

Boundary conditions

At a solid boundary the value of ψ may be specified. In Fig. 12.1 the value of ζ at a solid boundary, for example ζ_w at the point w, can be related to the ψ at an interior point, ψ_i, in the following way. At the boundary, let s and n be the coordinates along and normal to it, respectively. Then, from a Taylor series expansion, we have

$$\psi_i = \psi_w + \left.\frac{\partial \psi}{\partial n}\right|_w \Delta n + \frac{1}{2}\left.\frac{\partial^2 \psi}{\partial n^2}\right|_w (\Delta n)^2 + \ldots \tag{12.13}$$

where Δn is the distance between w and i along the coordinate n normal to the wall. The normal and tangential velocity components at the wall must vanish. The first of these gives

$$\left.\frac{\partial \psi}{\partial s}\right|_w = 0 \tag{12.14}$$

indicating that ψ_w must be a constant along the wall. The other velocity component gives

$$\left.\frac{\partial \psi}{\partial n}\right|_w = 0 \tag{12.15}$$

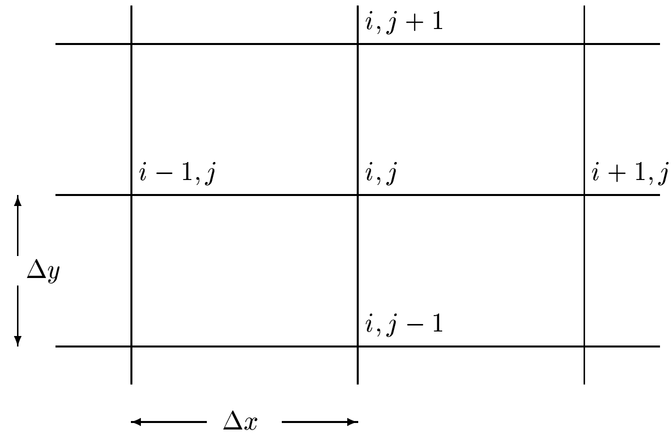


Figure 12.2: Computational mesh

Since

$$\zeta_w = -\left.\frac{\partial^2 \psi}{\partial s^2}\right|_w - \left.\frac{\partial^2 \psi}{\partial n^2}\right|_w \tag{12.16}$$

and ψ_w is constant along the wall, we get from equation (12.13) that

$$\zeta_w = 2\frac{\psi_w - \psi_i}{(\Delta n)^2} \tag{12.17}$$

12.3 Numerical methods

There are many numerical methods that have been developed for the governing equations of fluid mechanics: finite-difference, finite-element, boundary-element, spectral methods, among others. At this point we will describe only the first.

12.3.1 Finite differences

There are several variations of the finite difference method, the following being one of them. With reference to Fig. 12.2 he x-derivatives of a function $\phi(x, y)$ are written in the approximate forms shown:

$$\frac{\partial \phi}{\partial x} = \frac{\phi_{i,j} - \phi_{i-1,j}}{\Delta x} \text{ backward difference} \tag{12.18}$$

$$\frac{\partial \phi}{\partial x} = \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta x} \text{ central difference} \tag{12.19}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{(\Delta x)^2} \text{ central difference} \tag{12.20}$$

The derivatives in the y-direction may be similarly written.

The steady state may be obtained by starting from certain initial conditions and integrating forward in time. One way of doing this with the vorticity-stream function formulation is to first change equation (12.11) to a fictitious form

$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \zeta \tag{12.21}$$

which reduces to the correct form under steady-state conditions. The time derivatives in this and equation (12.12) are then approximated by

$$\frac{\partial \phi_{i,j}}{\partial t} = \frac{\phi_{i,j}^{k+1} - \phi_{i,j}^{k}}{\Delta t} \tag{12.22}$$

where $k+1$ and k refer to instants in time $t + \Delta t$ and t, respectively. The rest of the terms in these equations are assumed to be at time t and are hence known. The integration process is halted when the dependent variables are seen not to change very much with time.

Example 12.1

Compute the developing flow in a plane channel with uniform inflow.

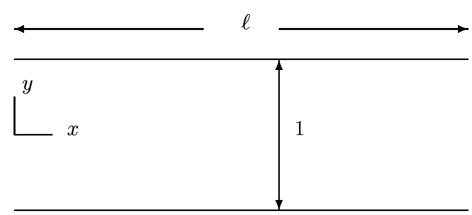


Figure 12.3: Flow between flat plates

The characteristic velocity and length are taken to be the uniform inlet velocity and the channel width, respectively. In nondimensional terms they are both unity, while the length of the channel is ℓ. Suitable boundary conditions are

$$u = 1,\ v = 0 \text{ at } x = 0 \tag{12.23}$$

$$u = v = 0 \text{ at } y = \pm\frac{1}{2} \tag{12.24}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 \text{ at } x = \ell \tag{12.25}$$

The first condition refers to the uniform inlet velocity, the second to no slip at the walls, and the third to the fully developed nature of the flow field at the exit section.

Finite differencing may be applied on a regular mesh that is obtained by dividing the region into N and M parts in the x and y-directions, respectively, so that $\Delta x = \ell/N$ and $\Delta y = 1/M$. Either the primitive or the vorticity-stream function formulation may be employed. Symmetry around the $y = 0$ plane may be used, in which case $\partial u/\partial y = v = 0$ at $y = 0$, instead of condition (12.24) at $y = -1/2$.

Problems

1. By numerical integration, determine where the fluid particles originally within a rectangle of size 0.1×0.1 centered at (1,1) are after 50 time units. The velocity field is two-dimensional:

$$\begin{aligned} u &= y \\ v &= x - x^3 + \gamma \cos t \end{aligned}$$

   Take (a) $\gamma = 0$, and (b) $\gamma = 1$.

2. Solve the following problem numerically. A viscous fluid enters a channel with a uniform velocity profile. The channel is long enough for the flow to become fully developed. Using suitable geometrical and flow parameters, compute the velocity field.

   Your report should be comprehensive; include details of how you set up the problem, description of the numerical method that you used, graphs of the results that you obtained, and comparisons with available analytical solutions.

3. In the problem above, determine the temperature field if the entering fluid and the walls of the channel are all at different temperatures.

4. For the two-dimensional, unsteady velocity field $u\mathbf{i} + v\mathbf{j}$, where

$$\begin{aligned} u &= y \\ v &= x - x^3 + \gamma \cos t \end{aligned}$$

   determine, by numerical integration, where the fluid particles originally within a square of size 0.1×0.1 centered at (1,1) are after 50 time units. Consider two cases: (a) $\gamma = 0$, and (b) $\gamma = 1$. Take as initial conditions a large number of different points within the square, say for example 100 evenly-spaced points, and integrate up to $t = 50$. Store the final coordinates in a file. Plot the initial as well as the final states of all the points. The particles should end up well-mixed (chaotically) for $\gamma = 1$, but not for $\gamma = 0$.

Appendix A

Governing equations

A.1 Integral form

Assumptions: nondeformable, inertial control volume

Conservation of mass

$$\frac{d}{dt}\int_{CV}\rho\ d\mathcal{V}+\int_{CS}\rho\mathbf{V}\cdot\mathbf{n}\ d\mathcal{A}=0$$

Newton second law for linear momentum

$$\frac{d}{dt}\int_{CV}\rho\mathbf{V}\ d\mathcal{V}+\int_{CS}\mathbf{V}\left(\rho\mathbf{V}\cdot\mathbf{n}\right)\ d\mathcal{A}=\mathbf{F}$$

Newton second law for angular momentum

$$\frac{d}{dt}\int_{CV}\rho\left(\mathbf{r}\times\mathbf{V}\right)\ d\mathcal{V}+\int_{CS}\left(\mathbf{r}\times\mathbf{V}\right)\left(\rho\mathbf{V}\cdot\mathbf{n}\right)\ d\mathcal{A}=\mathbf{T}$$

First law of thermodynamics

$$\frac{d}{dt}\int_{CV}\rho e\ d\mathcal{V}+\int_{CS}\left(e+\frac{p}{\rho}\right)\left(\rho\mathbf{V}\cdot\mathbf{n}\right)\ d\mathcal{A}=\dot{Q}+\dot{W}_s$$

Second law of thermodynamics

$$\frac{d}{dt}\int_{CV}\rho s\ d\mathcal{V}+\int_{CS}s\left(\rho\mathbf{V}\cdot\mathbf{n}\right)\ d\mathcal{A}\geq\int_{CS}\frac{1}{T}\left(\frac{\dot{Q}}{\mathcal{A}}\right)\ d\mathcal{A}$$

A.2 Differential form

Conservation of mass

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0 \\ \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} &= 0\end{aligned}$$

or

$$\begin{aligned}\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} &= 0 \\ \frac{D\rho}{Dt} + \rho \frac{\partial u_i}{\partial x_i} &= 0\end{aligned}$$

Newton second law for linear momentum

$$\begin{aligned}\rho \frac{D\mathbf{u}}{Dt} &= \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f} \\ \rho \frac{Du_j}{Dt} &= \frac{\partial \tau_{ij}}{\partial x_i} + \rho f_j\end{aligned}$$

Newton second law for angular momentum

$$\begin{aligned}\boldsymbol{\tau} &= \boldsymbol{\tau}^T \\ \tau_{ij} &= \tau_{ji}\end{aligned}$$

First law of thermodynamics

$$\begin{aligned}\rho \frac{De}{Dt} &= -\nabla \cdot \dot{\mathbf{q}} + \boldsymbol{\tau} : \nabla \mathbf{u} \\ \rho \frac{De}{Dt} &= -\frac{\partial \dot{q}_i}{\partial x_i} + \tau_{ij} \frac{\partial u_j}{\partial x_i}\end{aligned}$$

Second law of thermodynamics

$$\rho \frac{Ds}{Dt} \geq -\frac{\partial}{\partial x_i} \left(\frac{\dot{q}_i}{T} \right)$$

$$\rho \frac{Ds}{Dt} \geq -\nabla \cdot \left(\frac{\dot{\mathbf{q}}}{T} \right)$$

A.3 Cartesian coordinates

Velocity vector

$$\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$$

Material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Equations of motion (incompressible, Newtonian fluid with constant properties)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{A.1}$$

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = -\frac{\partial p}{\partial x} + \mu\nabla^2 u + \rho f_x \tag{A.2}$$

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right] = -\frac{\partial p}{\partial y} + \mu\nabla^2 v + \rho f_y \tag{A.3}$$

$$\rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right] = -\frac{\partial p}{\partial z} + \mu\nabla^2 w + \rho f_z \tag{A.4}$$

$$\rho c\left[\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right] = k\nabla^2 T + \Phi \tag{A.5}$$

where

$$\Phi = 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \mu\left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\right]$$

A.4 Cylindrical coordinates

Velocity vector

$$\mathbf{u} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_z\mathbf{e}_z$$

Material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}$$

Laplacian

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

Equations of motion (incompressible, Newtonian fluid with constant properties)

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

$$\rho\left[\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z\frac{\partial u_r}{\partial z}\right] = -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right) + \rho f_r$$

$$\rho\left[\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right] = -\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left(\nabla^2 u_\theta-\frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right)+\rho f_\theta$$

$$\rho\left[\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial\theta}+u_z\frac{\partial u_z}{\partial z}\right] = -\frac{\partial p}{\partial z}+\mu\nabla^2 u_z+\rho f_z$$

$$\rho c\left[\frac{\partial T}{\partial t}+u_r\frac{\partial T}{\partial r}+\frac{u_\theta}{r}\frac{\partial T}{\partial\theta}+u_z\frac{\partial T}{\partial z}\right] = k\nabla^2 T+\Phi$$

where

$$\begin{aligned}\Phi &= 2\mu\left[\left(\frac{\partial u_r}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u_\theta}{\partial\theta}+u_r\right)^2+\left(\frac{\partial u_z}{\partial z}\right)^2\right]\\ &+ \mu\left[\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial\theta}\right)^2+\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)^2+\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right)^2\right]\end{aligned}$$

A.5 Spherical coordinates

Velocity vector

$$\mathbf{u}=u_r\mathbf{e}_r+u_\theta\mathbf{e}_\theta+u_\phi\mathbf{e}_\phi$$

Material derivative

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r}\frac{\partial}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

Laplacian

$$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Equations of motion (incompressible, Newtonian fluid with constant properties)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta\sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} = 0$$

$$\begin{aligned}\rho\left[\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{u_\theta^2+u_\phi^2}{r}\right] &= -\frac{\partial p}{\partial r}+\mu\Bigg(\nabla^2 u_r-\frac{2u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\\ &\quad -\frac{2u_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\phi}\frac{\partial u_\phi}{\partial\phi}\Bigg)+\rho f_r\end{aligned}$$

$$\rho\left[\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right] = -\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left(\nabla^2 u_\theta-\frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right)+\rho f_\theta$$

$$\rho\left[\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial\theta}+u_z\frac{\partial u_z}{\partial z}\right] = -\frac{\partial p}{\partial z}+\mu\nabla^2 u_z+\rho f_z$$

$$\rho c\left[\frac{\partial T}{\partial t}+u_r\frac{\partial T}{\partial r}+\frac{u_\theta}{r}\frac{\partial T}{\partial\theta}+u_z\frac{\partial T}{\partial z}\right] = k\nabla^2 T+\Phi$$

where

$$\begin{aligned}\Phi &= 2\mu\left[\left(\frac{\partial u_r}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u_\theta}{\partial\theta}+u_r\right)^2+\left(\frac{\partial u_z}{\partial z}\right)^2\right]\\ &+ \mu\left[\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial\theta}\right)^2+\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)^2+\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right)^2\right]\end{aligned}$$

Appendix B

Use of MATLAB

Matrix manipulation and plotting can be done using MATLAB[1] which is available for several machines including Macintosh and UNIX systems. In a SPARCstation, enter MATLAB by typing

```
matlab
```

on the command line. The MATLAB prompt

```
>>
```

will be displayed.

B.1 Graphing

The following steps will enable you to read and plot a data file of points.

- To load data the command is

  `>> load` *nameofdatafile*

  where *nameofdatafile* is a file containing your data.

- To plot the data as points the command is

  ```
  >> plot(nameofdatafile(:,1),nameofdatafile(:,2),'c')
  ```

  The values in column one of your data will be plotted against the corresponding values in column two. The character `c` specifies which character you will use to mark points with, and must be between single quotes. Choices are

  ```
  .
  o
  x
  +
  *
  ```

  If this character is not included in the plot command, the points will be connected by a line.

[1]See University of Notre Dame document number U4910, *Using MATLAB 4.2 on the SPARCstations.*

- The commands to title your graph are

```
>> title('Your title')
>> xlabel('Your x-axis label')
>> ylabel('Your y-axis label')
```

- To add more data to your graph you must first type

```
>> hold
```

  Otherwise a new graph will be made.

B.2 Plotting streamlines

As an example of stremline plotting we will consider flow around a Joukowski airfoil. The Joukowski tranformation is given by

$$z = \zeta + \frac{c^2}{\zeta},$$

where the ζ-plane corresponds to the tranformed space. Once the solution is obtained in this space, the inverse mapping is required to obtain the solution in the z-plane. To this end, the above equation can be solved for ζ to give

$$\zeta = \frac{z}{2} \pm \sqrt{\left(\frac{z}{2}\right)^2 - c^2}.$$

Since it is desired to have $\zeta \to z$ for large values of z, the positive root of the inverse mappling must be chosen. This conclusion is valid for z expressed in polar form (i.e., $z = \exp i\theta$); however, care must be taken when expressing z in Cartesian form (i.e., $z = x + iy$). For instance, for $z = -b$ where b is a large real number, the above inverse mapping (with + root taken) predicts that $\zeta \sim 0$; thus the negative root should be used to preserve the desired mapping.

Therefore, when plotting the streamlines for the Joukowski airfoil in which $z = x + iy$, it is necessary to change from the positive to the negative root in the above inverse transform for $x < 0$ in order to assure that $\zeta \sim z$ for large z.

The following are hints for plotting the streamlines in MATLAB.

- Generate a computational grid in the z-plane:

```
>> x = xmin :  dx :  xmax;
>> y = ymin :  dy :  ymax;
>> [X,Y] = meshgrid(x,y);
>> Z = X + i*Y;
```

- Use the FOR and IF commands to create a loop in which ζ at each $z = x + i * y$ location can be computed. Remember to specify the $(-)$ root of the inverse transform for $x < 0$. (also, type “help for” or “help if” at MATLAB prompt for information on how to create a loop).

- Compute the complex potential $F(z)$.

- Determine the values of the stream function at each (x, y) location:

```
>> psi = imag(F);
```

- Contour plot the stream function values to obtain the streamlines (type “help contour” to get online help information concerning this plot procedure):

```
>> contour(X,Y,psi,50)
```

Appendix C

Use of MAPLE

On the Sparcstations, type

`maple;`

to load MAPLE,

`with(linalg);`

to include the linear algebra packet, and

`help(`*topic*`);`

to get help on *topic* and examples.

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