The Polytrope Star: A Didactic Numerical Model

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Abstract

Based on the polytropic gas sphere which model a star, this can be used in order to evaluate through one–dimensional equations the variation of pressure, density and mass as a function of the distance from the center of the star. In order to understand all mathematical implications a sequence of steps are easily explained which also contribute to build a didactic model for better understanding of this Polytrope star model.

1 Introduction

The purpose of the present paper is to investigate a very simple star model. It should be noted that a polytrope is not a star but a sphere of gas having a very specific type of equilibrium. Also, it does not simulate important physical process and quantities such nuclear reactions, chemical mixing, energy transport, or mass loss by stellar wind. Therefore, as a very simple model, it comes as a first step in order to understand more complex star models [3].

What intended is to establish first of all the general equations in one–dimensional space for our self–gravitating polytropic gas sphere and so, by the adoption of some hypothesis reduce such equations to the well known Lane, Kelvin, Eddington, and Emden simple stellar model. Following this a sequence of steps are described in order to establish a logical procedure to simulate the pressure, density and mass by numerical means as a function of the distance from the centre of the polytrope star.

2 Fundamental Equations

The equations governing the spherically symmetrical flow of a polytropic gas of **adiabatic index** γ under the influence of its own gravitation are [1]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{Gm}{r^2} \tag{1}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} = -\rho(\frac{\partial u}{\partial r} + 2\frac{u}{r}) \tag{2}$$

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$$\frac{\partial m}{\partial r} = 4\pi r^2 \rho \tag{3}$$

$$\frac{\partial p\rho^{-\gamma}}{\partial t} + u\frac{\partial p\rho^{-\gamma}}{\partial r} = 0 \tag{4}$$

Where $m(r,t)$, $u(r,t)$, $p(r,t)$ and $\rho(r,t)$ denote respectively the mass inside a sphere of radius r, the gas velocity, pressure and density at a distance r from the centre at time t. By means of equation(4), equation(2) transforms into

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r} = -p\gamma\left(\frac{\partial u}{\partial r} + 2\,\frac{u}{r}\right) \tag{5}$$

The expression $p = k\rho^{\gamma}$ relate the pressure and density to an adiabatic process in thermodynamic equilibrium (adiabatic–reversible or isentropic) and is know as law of **Poisson**.

For the equilibrium state where $u = 0$, $\frac{\partial}{\partial t} = 0$, one can write: $p = p^{'}(R)$; $\rho = \rho^{'}(R)$; $m = m^{'}(R)$; $r = R$. then equations(1,3) and equation(4) are reduced to:

$$\frac{1}{\rho^{'}}\frac{dp^{'}}{dR} + \frac{Gm^{'}}{R^2} = 0 \tag{6}$$

$$\frac{dm^{'}}{dR} = 4\pi R^2 \rho^{'} \tag{7}$$

$$p^{'}\rho^{'1-\gamma} = p_0\rho_0^{-\gamma} \tag{8}$$

While the equation(2) and equation(5) give merely the relations $\rho = \rho^{'}$ and $p = p^{'}$ in the present case.

It is necessary to make a few remarks about the value of the *ratio of specific heat* γ or *adiabatic index*. From the kinematic theory of gases, one can obtain an expression for the specific heat c_v as [2].

$$c_v = \frac{f}{2}\frac{R}{m^g} \tag{9}$$

where R is the universal gas constant.

In the equation(9) f is the number of degrees of freedom of a molecule, which has to be considered as a rigid union of atoms. Using the known relation for the specific heat c_p defined as:

$$c_p - c_v = \frac{R}{m^g} \tag{10}$$

where $\Updownarrow$ is the dimensionless mole mass[1]. Thus, the expression for the *ratio of the specific heat* is

$$\gamma = \frac{c_p}{c_v} = 1 + \frac{2}{f} \tag{11}$$

From which the following values for γ are obtained:

As stated in the introduction of this work the polytropes are not real stars and also are no longer used in professional astrophysics. However, in the earlier times polytropes gave very important contributions for the astronomical and astrophysical development. As a simple model was used as step to understand the more complicated models in time to come. The following are shown some those classical polytropes:

[1] The numerical value of m^g corresponds to the mass of the gas measured in grams, whose volume is 22.4 l at 0°C and 760mm Hg pressure.

Relation of f with γ				
Value of f	**3**	**5**	**6**	**10**
$\gamma = \frac{c_p}{c_v}$	5/3	7/5	4/3	12/10

Table 1: Relation of f with γ

- $f \to 3$ This case was used to model a fully convective star where radiation pressure is not important, also used for non–relativistic degenerate white dwarf. For mono atomic gases(for example, inert gases). Here, only the three translational degrees of freedom are considered.
- $f \to 5$ For diatomic molecules (for example, H_2, O_2, N_2 and air). It is a dumb–bell model in which, in addition to the three translational degrees of freedom, the two rotational degrees of freedom are also considered.
- $f \to 6$ This case was used to state the standard model of Eddington used to describe a fully radiative star or a white dwarf with relativistic degeneration. For a general arrangement of molecules (for example, for all multiple atom gases). In this case, three translational and three rotational degrees of freedom are considered.
- $f \to 10$ This case was an approximation for the distribution of star in a globular cluster. The model calculation is done from zero to a certain distance from the star core since for this value of f, the **zero point** is located at infinity as will be discussed in the next sections.

In case, however, one go to extreme states ($f > 6$), then there is a strong dependence of c_p, c_v on the temperature. The reason is the internal energy of the gases contain not only contributions from translation and rotation, but at higher temperature (at stellar core) there are also contributions from vibration; as well as from dissociation and ionization. These contributions can be determinated from the quantum theory. The choice of f depends on the type of star one wants to study. The f parameter is the **polytrope index**, and can be bigger than 6 in order to represent such situations.

3 The Lane–Emden Equation

The purpose of this paper is, therefore, by means of a numerical simulation to compute the structure of the polytrope star given as result the variation of pressure, density and mass as a function of the distance of the centre to the surface. Such calculations is done for a given total mass, radius and for a given polytrope index f. Considering the set of equations(6,7 and 8) for a position r, one can now rewrite them as:

$$\frac{dp}{dr} = -\frac{Gm}{r^2}\rho \tag{12}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{13}$$

$$p\rho^{-\gamma} = p_0 \rho_0^{-\gamma} \tag{14}$$

The idea is to merge all three equations into only one ordinary differential equation which describe the polytrope star structure. Following this, one can introduce a function $\Phi(r)$ defined as

$$\Phi(r) = \left(\frac{\rho(r)}{\rho_0}\right)^{\gamma-1} \tag{15}$$

Using the relation stated by equation(14) one can combine it with the last equation(15) and obtain

$$\Phi(r) = \left(\frac{p(r)}{p_0}\right)^{\frac{\gamma}{\gamma-1}} \tag{16}$$

Where p_0, ρ_0 are respectively the pressure and density at the centre of the star, where $r = 0$. By the use of the equations(12,15 and 16) one can write the following ordinary differential equation:

$$\frac{\gamma}{\gamma-1} r^2 \frac{p_0}{\rho_0} \frac{d\Phi(r)}{dr} = -Gm(r) \tag{17}$$

doing $x = \frac{r}{r_n}$ then $r = xr_n$, so

$$\frac{\gamma}{\gamma-1} x^2 r_n^2 \frac{p_0}{\rho_0} \frac{d\Phi}{dx} = -Gm(x) r_n \tag{18}$$

When both sides of equation(18) are differentiated to x and using $\frac{dm(x)}{dx}$ definition given by equation(13) after change r by x as defined before, one get

$$\frac{2}{x}\frac{d\Phi}{dx} + \frac{d^2\Phi}{dx^2} = -G\,4\,\pi r_n^2 \frac{\rho_0^2}{p_0} \frac{\gamma-1}{\gamma} \Phi^{\frac{1}{\gamma-1}} \tag{19}$$

In order to make the last coefficient on the right hand side as unit, one define

$$r_n^2 = \frac{p_0 \gamma}{G 4 \pi \rho_0^2 (\gamma-1)} \tag{20}$$

then

$$\Phi''(x) = -\Phi^{\frac{1}{\gamma-1}}(x) - \frac{2}{x}\Phi'(x) \tag{21}$$

As was stated, this final equation (21) is known as the Lane–Emden equation which describes the complete structure of a polytrope star.

3.1 Considerations on the $\Phi(x)$ Function

For each value of the γ a different Lane–Emden equation is obtained as well as another solution of $\Phi(x)$ is also obtained. The initial conditions of the equation(21) are independent of the **adiabatic index** γ as can be seem by the following relations:

$$\begin{array}{llll} \text{for} & x = 0 & \rightarrow & \Phi(0) = 1 \quad (\rho(0) = \rho_c) \\ \text{for} & x = 0 & \rightarrow & \Phi'(0) = 0 \quad (\rho(0) = \rho_c) \end{array} \tag{22}$$

As defined before when $x = 0$ implies to $r = 0$, which means the stellar center, so ρ_c is the density at the center. Considering equation(15) one can observe that $\Phi(x)$ is a decreasing function, starting from 1 at $x = 0$, until it crosses the x axis at a certain **zero point** for $x(\gamma)$. The smaller the value of γ, the further the zero crossing point will be from the origin. Now, is ease to discuss the physical meaning of $x(\gamma)$ at that point when $\Phi(x)$ becomes zero. If one recall the equations(15 and 16) is clear that density and pressure are zero since $\Phi(x)$ is also zero, or in other words, one have reached the star surface.

Considering values bigger than $f = 10$, the solution for $\Phi(x)$ will not be possible and no zero point, or no intersection would happens. This situation shows that the solution is unable to find the star surface. The behavior of the density, pressure is a decreasing function until reach a minimum value and after start an increasing variation which reveal a physically meaningless solution.

4 Numerical Integration Procedure

The main goal in this section is seek a numerical procedure in order to integrate the equation(21) under the initial conditions established by the relations given in equation(22). Adopting an iteration algorithm to solve equation(21) in such way when $\Phi(x)$ becomes negative would show that the intersection will happens in some where along the x axis. So, is possible to refine the solution and determine in accurate sense the star surface or where the $\Phi(x)$ becomes zero.

Just to remind, some input parameters are chosen as the total mass m, the total radius R, and the polytrope index f, but it doesn't have to be an integer. Once one have established such parameters four steps can be defined in order to solve by iteration the equation(21):

1. Integrate from $x = 0$ to $x = dx$,
2. Integrate from $x = dx$ until $\Phi(x)$ becomes negative,
3. Calculate the exact location where $\Phi(x) = 0$, and
4. Calculation of the core pressure and density values.

As equation(21) is a second order ordinary differential equation is suitable reduce the order by introducing a new variable, say $\Omega(x)$. This will reduce the order, but as result, two first order ordinary differential equations will have to be solved together. Doing as follow:

$$\Phi'(x) = \Omega(x) \tag{23}$$

the equation(21) is then rewritten as

$$\Omega'(x) = -\Phi^{\frac{1}{\gamma-1}}(x) - \frac{2}{x}\Omega(x) \tag{24}$$

Under this situation the initial condition given by relations (22) are now written as

$$\begin{array}{lcl} \text{for} \quad x = 0 & \rightarrow & \Phi(0) = 1 \\ \text{for} \quad x = 0 & \rightarrow & \Omega(0) = 0 \end{array} \tag{25}$$

Now the set of equations given by (23) and (24) they need to be solved by the some numerical scheme. The finite difference midpoint scheme is adopted, following a two stage numerical scheme as:

- *First Step*

$$x_{i+1/2} = x_i + 0.5dx \tag{26}$$

$$\Phi_{i+1/2} = \Phi_i + 0.5\Omega_i \tag{27}$$

$$\Omega_{i+1/2} = \Omega_i + 0.5dx\left(-\Phi_i^{\frac{1}{\gamma-1}} - \frac{2}{x_i}\Omega_i\right) \tag{28}$$

- *Second Step*

$$x_{i+1} = x_i + dx \tag{29}$$

$$\Phi_{i+1} = \Phi_i + dx\,\Omega_{i+1/2} \tag{30}$$

$$\Omega_{i+1} = \Omega_i + dx\left[-\Phi_{i+1/2}^{\frac{1}{\gamma-1}} - \frac{2}{x_{i+1/2}}\Omega_{i+1/2}\right] \tag{31}$$

As it is ease to be seem these discrete equations are coupled, so the final solution of Φ_{i+1} carries the variation of Ω_i and some additional procedure need to be established in order to refine the solution for $\Phi_{i+1} = 0$ which implies in having determinated the star surface for a given problem. Other important point which need to be put in attention is about the singular point for $x = 0$ as can be noted in the equation(28). This is discussed in the next subsection.

4.1 The Singular Point Fix Solution

At the beginning of the iterations is need to calculate the solution for $x = 0$, but by inspection of equation(28) and equation(31) is clear the singular division operation at this point. In order to get a solution for such problem one need define a especial procedure for $i = 1$. Equation(15) is known for $x = 0$, as defined by the initial condition given by relations (22) and (25). So, one manner of finding out this solution is seek an analytic function, lets say, $F(x)$ which represent $\Phi(x)$ near $x = 0$.

Based on this fact, $F(x)$ could be represented by the following polynomial; $F(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + ...$ which will represent $\Phi(x)$ at the vicinity of $x = 0$ under the following conditions:

$$\begin{array}{ccc} F(x) & = & 1 \\ F^{'}(0) & = & 0 \\ \text{and also satisfying} & & \\ F^{"}(x) & = & -F^{\frac{1}{\gamma-1}}(x) - 2\frac{F^{'}(x)}{x} \end{array} \tag{32}$$

Doing the necessary algebraic calculation in order to determine the polynomial coefficients one get

$$\Phi(x) = F(x) = 1 - \frac{x^2}{6} + \frac{x^4}{120(\gamma-1)} \tag{33}$$

and by direct way

$$\Omega(x) = F'(x) = -\frac{x}{3} + \frac{x^3}{30(\gamma-1)} \tag{34}$$

Then, for $i = 1$ the numerical procedure is now established as:

$$x_1 = dx \tag{35}$$

$$\Phi_1 = 1 - \frac{(dx)^2}{6} + \frac{(dx)^4}{120(\gamma-1)} \tag{36}$$

$$\Omega_1 = -\frac{dx}{3} + \frac{(dx)^3}{30(\gamma-1)} \tag{37}$$

In order to keep the solution as much accurate as possible dx never should be larger than 0.18 since then the error of the polynomial expansion will increase too much. Once Φ_1 is determinated, the values of the density and pressure at x_i for $i = 1$ are obtained from:

$$\rho_i = \rho_0\, \Phi_i^{\frac{1}{\gamma-1}} \tag{38}$$

$$p_i = p_0\, \Phi_i^{\frac{\gamma-1}{\gamma}} \tag{39}$$

which both comes from equations(15 and 16) respectively.

One can define an auxiliary variable which relates the discrete mass. Using the equation(12) and equation(16) and doing some algebraic manipulations is possible to get

$$L\, m(x) \;=\; -x^2 \Omega \tag{40}$$

where $L = \frac{1}{4\pi r_n^3 \rho_0}$ which multiplied by the solar mass will normalize L as a function of solar mass. The three parameters ρ_0, p_0, and L will be calculated at the end of the calculations when the exact location of the zero point of $\Phi(x)$ and the value of $\Omega(x)$ are known.

4.2 The Refinement Solution for the Zero Point

The numerical process described in the item 4 need to be followed in a such way that for each step one need to check the signal change on the $\Phi_{i+1/2}$ and Φ_{i+1} functions. If a signal change is detected the iteration process is stopped, which is saying that the Φ function has already intercepted the x axis. At this stage is suitable to calculate the exact location of the zero point and also the exact value of Φ using for it the last point where $\Phi(x)$ was still positive. Considering the figure 1

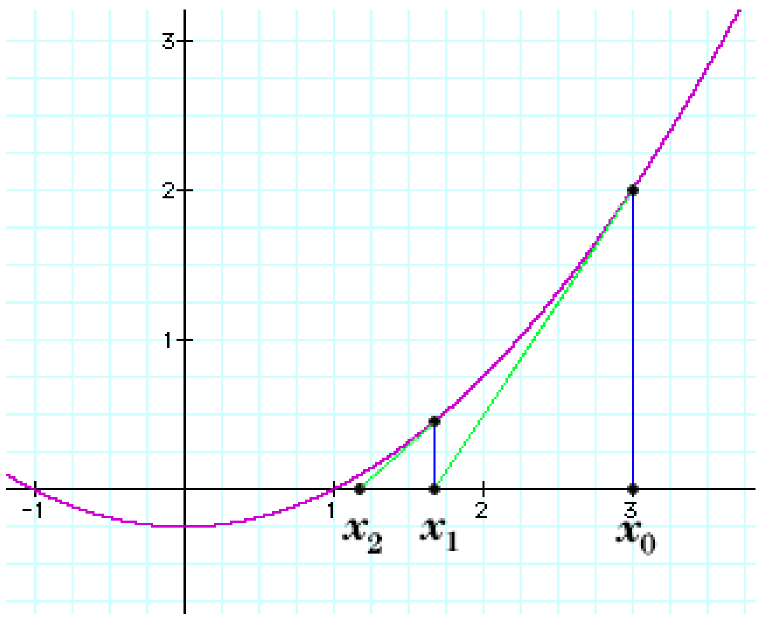

Figure 1: Newton's Method

draw the tangent to the curve $y = \Phi(x)$ at a point near a root of $\Phi(x) = 0$ and use the intercepted x of this tangent as an approximation to the root. This process is known as Newton's method which is essentially an iterative process based on the following formula:

$$x_n \;=\; x_{n-1} \;-\; \frac{\Phi(x_{n-1})}{\Omega(x_{n-1})} \tag{41}$$

Relating to the equation(41) x_{n-1} is the last point (for the last positive value of Φ) and lets call Φ_{n-1} and Ω_{n-1} the values of Φ and Ω calculated at $(n-1)$. Ω_n is also calculated since one have reached, with sufficient accuracy, the value of x_n. Using the Lane–Emden equation at this point is possible to determine Ω function as:

$$\Omega_n \;=\; \Omega_{n-1} \;+\; (x_n \;-\; x_{n-1}) \left[-\Phi_{n-1}^{\frac{1}{\gamma-1}} \;-\; \frac{2}{x_{n-1}} \Omega_{n-1} \right] \tag{42}$$

As was said before, once has been determinated the value of x and Ω the rest of missing parameters can be calculated from these results and from the initial data chosen at the beginning. Using express the mass and radius as solar units the density, pressure and mass in the core of the polytrope star are calculated by the following equations

$$\text{mean density} \qquad \rho_m = \frac{1.42M}{R^3} \qquad (\frac{g}{cm^3}) \tag{43}$$

$$\text{core density} \qquad \rho_0 = -\frac{\rho_m\, x}{3\Omega} \qquad (\frac{g}{cm^3}) \tag{44}$$

$$\text{core pressure} \quad p_0 = \frac{9.048\, 10^{14} M^2(\gamma-1)}{\Omega^2 R^4 \gamma} \qquad (\frac{dyne}{cm^2}) \tag{45}$$

$$\text{parameter} \qquad L = -\frac{x^2\Omega}{M} \tag{46}$$

$$\text{parameter} \qquad r_n = \frac{R}{x} \tag{47}$$

All the ideas discussed until this point are based on the fact that one knows as input data the **mass, radius** and the **polytrope index**. The final results are the pressure, density and mass which are calculated by equations(43–47) at the core. However, another situation is; given the core pressure, density and the polytrope index one would be able to calculate at each point the pressure, density and mass. Such case is unusual since it is easer to have a general idea on stellar masses and radii rather than on central pressures and densities. Based on this intuitive fact was decided to adopt the presented approach which was specifying the mass, radius, and polytrope index as input data.

5 Application

In order to compute the structure of a polytrope star two cases are numerically simulated.

Numerical simulation of Case 1

```
INITIAL CONDITIONS (Star Model)

  f   =>  3.0
  dx  =>  0.05
  M   =>  2.0
  R   =>  3.0
```

The polytrope solution for this initial data as shown in figure 2, figure 3 and figure 4 .

```
FINAL RESULTS for CASE -- 1 --

 Central Pressure =>  4.3296406828568E+014
 Average Density  =>  1.0518518200627E-001
 Central Density  =>  6.3058269894623E-001
 Mass parameter   =>  1.35641023207190
 Distance unit    =>  8.2100283372169E-001
 x-final          =>  3.65406777757480       exact = 3.654
 -x2ome (final)   =>  2.71282046414380       exact = 2.714
```

where -x2ome $=$ Ωx-final^2.

Comparing the present results with the exact values of x-final and Ω which can be found in [3] one conclude that these results are indeed good.

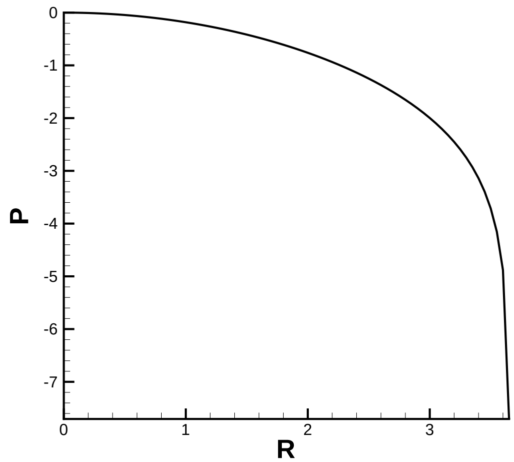

Figure 2: Pressure

Numerical simulation of Case 2

```
INITIAL CONDITIONS (Star Model)

   f   =>  5.0
   dx  =>  0.10
   M   =>  3.0
   R   =>  4.0
```

The polytrope solution for this initial data as shown in figure 5, figure 6 and figure 7 .

```
   FINAL RESULTS for CASE -- 2 --

Central Pressure =>  1.5742960359148E+015
Average Density  =>  6.6562497988343E-002
Central Density  =>  1.56592360552000
Mass parameter   =>  7.2828979458985E-001
Distance unit    =>  7.4592888879470E-001
x-final          =>  5.36244146069120           exact = 5.355
-x2ome (final)   =>  2.18486938376960           exact = 2.187
```

6 Acknowledgments

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References

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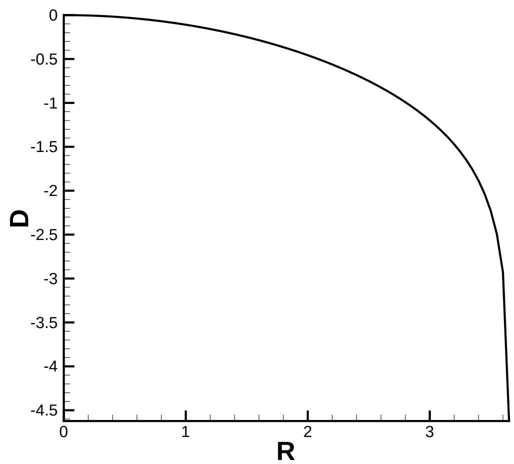

Figure 3: Density

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[3] Hellings, P., *Astrophysics with a PC*, Willmann–Bell Inc., USA, 1994.

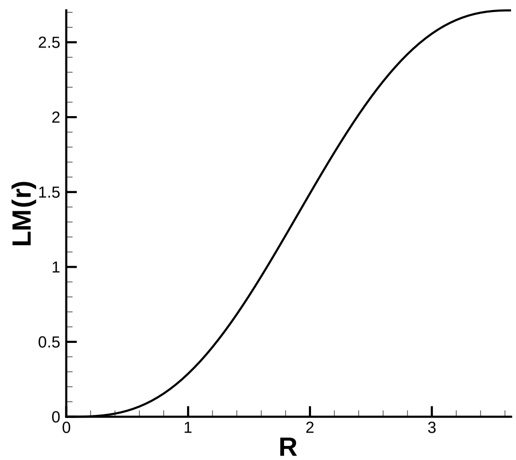

Figure 4: LM(r) function

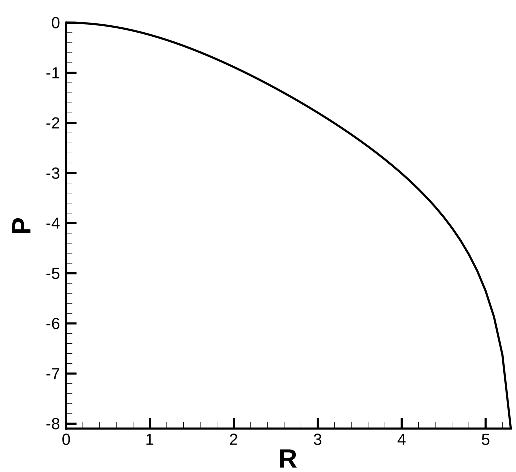

Figure 5: Pressure

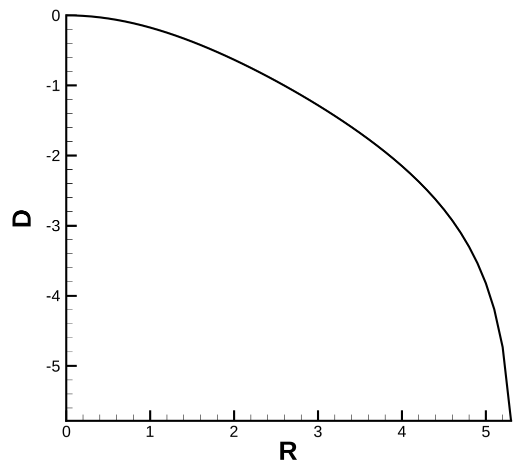

Figure 6: Density

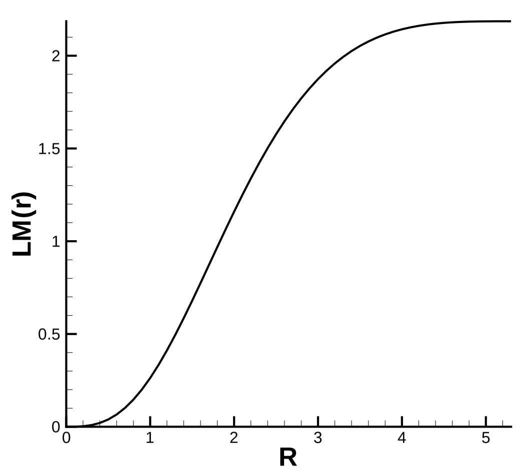

Figure 7: LM(r) function