The Homogeneous Star Model Consisting of Convective and Radiative Envelop - Part II

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Abstract

Following the model proposed by the polytropic gas sphere which model a simple star structure, the heavy stars ($M > M\odot$)[1] that constitute the upper main sequence is studied by a proposed composite model consisting of a convective core and a radiative envelope extending right up to the star surface. In order to understand well the mathematical implications involving the numerical solution of such model a sequence of steps are easily explained which also contribute to build up a didactic exposition for a better understanding of the present model.

1 Introduction

The purpose of the present paper is to investigate a more elaborate star interior model as has been done in the article [1]. So, the present work follow as part II of that article and still keeps the main objective which is make easy the understanding of the numerical simulation for the two process: the convective core and radiative envelop, as part of the present star model.

The heavy stars ($M > M\odot$) derive their energy from the CNO[2] cycle and, as result, develop convective core. These stars, in their early main sequence stage, at least, may conform to a composite model consisting of a convective core and a radiative envelop which extend till to the star surface.

One shall assume that the chemical composition is uniform throughout the model, it can then represent only the initial state of these main sequence stars, the state in which they just begin to generate energy through thermonuclear reactions.

This assumptions do not take into account the energy released in the form of neutrinos, which escape from the star without contributing to the energy balance considered. In the present work will be discussed, from the proposed model, the numerical simulation procedure which describe the thermonuclear proprieties of the present star model. In this study the temperature range is considered as given by two energy mechanisms cycles which are the "pp" and "CNO" cycles.

2 The Physical Characteristics of the Model

As were discussed in [1], here called as part I, the basic equations which describe the polytropic star has already been presented and as shown are the so called Lane-Emden equations. However, in the present model

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[1] M is the star mass and $M\odot$ is the solar mass.

[2] Carbon-Nitrogen-Oxygen, temperature between about $12x10^6$K and $50x10^6$K.

the pressure in the star interior consists of gas pressure and the radiation pressure, which differ a little of the model described in [1]. Also the temperature is so high that the corresponding radiation pressure represents a important contribution for the total pressure value. Based on these facts, the total pressure is written as

$$P = \frac{R.\rho.T}{\mu} + \frac{1}{3}.a.T^4 \tag{1}$$

where

- μ = the mean molecular weight and has the same value in the star core as in the envelop as well
- R = the gas constant = 8.314 x10^7 (erg/mol/K)
- a = the Stefan-Boltzmann constant = 7.56464 x 10^{-15} (erg/cm^3/K^4)
- T = temperature in Kelvin (K)

As one can write $P = Pg + Pr$ it is convenient to express the gas pressure (Pg) and radiation pressure (Pr) in units of the total pressure (P) by making use of a new parameter called (β) as:

$$\begin{aligned} Pg &= \beta P \\ Pr &= (1-\beta)P \\ P &= \frac{R\rho T}{\mu\beta} \end{aligned}$$

Then

- for M considered low as compared with $M\odot$ β is close to 1
- for M considered high as compared $M\odot$ β is close to 0.95

In the star interiors the atoms are almost completely ionized and one can assume, in first approximation, that is dealing with a state of complete ionization. Let shall X be the mass in grams of H (hydrogen) contained (before ionization) in one gram of the mixture, and Y be the mass in grams of the He (helion) contained (before ionization) in one gram of the mixture. Finally, let $Z = 1 - X - Y$ be the mass of all elements other than H and He contained in one gram of the mixture. Lets shall apply the galactic abundance normally used in almost all models as :

$$\begin{aligned} X &= 0.70 \rightarrow \text{hydrogen} \\ Y &= 0.27 \rightarrow \text{helium} \\ Z &= 0.03 \rightarrow \text{other elements} \end{aligned}$$

The mean molecular weight is given by

$$\mu = \frac{1}{0.25\,Y + 1.5\,X + 0.5} \tag{2}$$

If in the core only hydrogen were present $Y = Z = 0$ and $X = 1$ which implies in $\mu = 0.5$ or $m = mH/2$. This result is known; which say the equivalent mass of hydrogen plasma is the half of the mass of hydrogen nucleus. If were present in the core only He $\mu = 1.33$. Other situation could be as all component were metals then μ will be $\mu = 2$. But in real conditions one can have $0.7 \leq \mu \leq 2$.

Based on the fact of the present study is related to the polytropes, the temperature can be computed by the equation (1). Starting from initial value for T which is given by:

$$T_0 = \frac{\mu P}{R\rho} \tag{3}$$

The temperature is calculated by iteration process using the recursive relation

$$T_{(i+1)} = \frac{\mu}{R\rho}\left(P - \frac{1}{3}aT_i^4\right) \tag{4}$$

So, for given values of μ, ρ and P is possible calculate the temperature by this iterative algorithm.

The energy production, or luminosity, depends essentially on the composition, the temperature T, and the mass density ρ of the star mixture at a distance r from de centre. It also can be considered that for temperatures of the order of 10^7K [2] (precisely between 4x10^6 and 50x10^6K), the only long-term equations possible under the present star conditions are:

- the pp - chain (4x10^6 and 25x10^6K)
- the CNO cycle (12x10^6 and 50x10^6K)

The pp chain (proton-proton) and CNO (carbon-nitrogen-oxygen) cycle are two different models of fusion of protons into α-particles [2].

Following the approximated calculation procedure given in [3] the energy production involving the pp chain and CNO cycle is written as:

$$\begin{aligned}
t &= \left(\frac{T}{10^9}\right)^{1/3} \\
P_1 &= 1 + 0.133t + 1.09t^2 + 0.938t^3 \\
P_2 &= 1 + 0.027t - 0.788t^2 - 0.149t^3 + 0.261t^4 + 0.127t^5 \\
E_{pp} &= 2.37\text{x}10^4 t^{-2} P_1 \exp(\frac{-3.38}{t}) \\
E_{CNO} &= 8.66651\text{x}10^2 5t^{-2} P_2 \exp(\frac{-15.228}{t} - \frac{t^6}{9.5481})
\end{aligned}$$

and the final relation for the energy production is

$$E = \rho X^2 E_{pp} + 0.02\,\rho X E_{CNO} \tag{5}$$

Where E is the energy production of 1 gram of star matter, the energy increase in a shell with size dr at distance r from the stellar centre is given by

$$dL_r = 4\pi E r^2 dr \tag{6}$$

The total energy production of the star (the Luminosity) is the sum of the contributions dL_r of all the subsequent shells throughout the star.

3 The Polytrope Fitting Scheme

The set of equations as defined before is already completed, however there are some equations which need be understood.

3.1 Determination of The Polytrope Index in The Star Core

In general the temperature in the core for stars ($M > M\odot$) is proportional to the pressure variations, and one can write:

$$\frac{dT}{dr} = \Gamma(\beta)\left(\frac{T}{\rho}\right)\frac{dP}{dr} \tag{7}$$

where

$$\Gamma(\beta) = \frac{8-6\beta}{32-24\beta-3\beta^2}$$

or

$$\frac{dT}{dr} = \frac{1}{n}\left(\frac{T}{\rho}\right)\frac{dP}{dr} \tag{8}$$

so,

$$n = \frac{1-\Gamma(\beta)}{\Gamma(\beta)} \tag{9}$$

If $\beta = 1 \rightarrow n = 1.5$ as it should be for a convective index in the absence of radiation pressure. However, in the present application β will vary from 1 to 0.9, which will cause γ vary from $5/3$[3] to $12/10$ for $15M\odot$ star. The ratio specific heats γ is given by $\gamma = \frac{1+n}{n}$ [1].

In terms of practical numerical procedure, first of all one must compute the central gas and radiation pressures from the central density and temperature. However, β will be put constant for the whole core, being considered as average value of the core [3]. By this consideration β is defined as:

$$\beta = 1 - \frac{2}{3}(1-\beta_c) \tag{10}$$

where

$$\beta_c = \frac{P_{g,c}}{P_{g,c}+P_{r,c}}$$

3.2 The Boundary Transition Zone

The boundary happens in the present model at the interface when the two polytropes meets, the convective zone and radiative zone. In this transition zone the polytrope index change from $n = 1.5$ to $n = 3.0$, so this zone marks the boundary of convective core to the radiative shell which involve the convective core. The temperature relation in the radiative zone is given by

[3] For a monoatomic gas, such as helium.

$$\frac{dT}{dr} = -\left(\frac{3\rho k}{4acT^3}\right)\frac{L_r}{4\pi r^2} \tag{11}$$

where

- k = absorption coefficient. In professional models such variable index have a very complicated expression as can be seen in [2], but for now on will be adopted $k = 0.2(1+X)$ as stated in [3], which is independent of density and temperature.
- c = the speed of light = $3\text{x}10^{10}$(cm/s)

At the boundary zone the equations (7) and (11) should be equal, and as the iterative process goes on the relation obtained will converge to a value less than one indicating that the boundary has been passed. Once this has been reached, a interpolation process can be used in order to calculate precisely the transition zone, where the relation is bigger than one and or less than one. Doing such proceeding the mass of the convective core is then determined with enough precision.

3.3 Fitting the Convective - Radiative Polytropes

Since the calculation reached the boundary layer[4], the physical proprieties are known and is need to calculate the initial values and conditions for the radiative zone. However, the polytrope can not be initiated at $x = 0$ since the boundary is not any more the star centre. So, is need choose five initial parameters such as x, F, H, P_c[5] and ρ_c in order to start the calculation in the radiative shell. Is important to say that all these parameters can not be chosen independently from each other and the following numerical procedure is used[6]:

$$P_b = P_c F^{(n+1)} \tag{12}$$

$$\rho_b = \rho_c F^n \tag{13}$$

$$M_{r,b} = -4\pi r_n^3 \rho_c x^2 H \tag{14}$$

$$r_n = \sqrt{\frac{P_c}{\pi G \rho_c^2}} \tag{15}$$

$$r_b = r_n x \tag{16}$$

There are four equations (12) (13), (14) and (16) which one know the variables ($P_b, \rho_b, M_{r,b}$ and r_b) and five parameters (x, F, H, P_c and ρ_c) which need be determined. The adopted scheme allow to choose freely F (Lane–Emden function). The value of F can be choose as indicated below [3]:

Defining

$$w = \log(M)$$

and

$$\log T_c = 7.23937 + 0.274354w - 0.0401771w^2 \tag{17}$$

$$\log \rho_c = 2.27899 - 1.658707w + 0.29329095w^2 \tag{18}$$

[4]Interface between the convective and radiative zone

[5]See [1] for the definition of x, F and H.

[6]G is the gravitational constant.

For

$$\begin{array}{rcl} M < 4 & \rightarrow & F = 9 \\ 4 < M < 10 & \rightarrow & F = 19.58791 - 17.58794w \\ M > 10 & \rightarrow & F = 2 \end{array} \tag{19}$$

4 Numerical Procedure

The full procedure designed to solve the polytrope star has already been given in [1], although the algorithm is basically the same, some considerations are need to be commented towards the total numerical solution of the presented convective–radiative model.

In the convective zone the spatial step size is taken equal to $dx = 0.1$ for all simulations. Is known that, out side of the convective zone to the radiative shell the density decreases towards zero (as the boundary condition specify [1]) at the surface. The spatial step size used to start up the numerical calculation in the radiative shell is taken equal to $dx = 0.3$ and increased to value of 10% in each iteration.

The initial calculation can be started as is done in [3] and is shown as follow:

- compute the initial central temperature by eq(17)
- compute the central density by eq (18)
- compute the central pressure, the first term of eq(1) for the gas pressure and radiative pressure given by the second term of eq(1)
- compute the total pressure by eq(1)
- compute the value of β for the convective core eq(1) and eq(10)
- compute the $\Gamma(\beta)$ as defined by eq(7)
- compute the core polytrope index given by eq(9)
- compute the distance parameter of the convective core using the relation (20) given in [1]
- compute the initial conditions of the Lane-Emden equation for $x = 0; F = 1$ and $\Omega = 0$ [1]
- compute the value of F related to the outer boundary of the polytrope using one relation given by eq(19)
- compute the central nuclear energy production given by relation (5) from T_c (17) e ρ_c (18)

At the centre all values are set as: $M = r = L = 0$. The first step from this initial conditions is calculated by x=dx and should follow the same procedure already described in [1].

5 Results

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-----------------------------------------------------
Initial Values Used in This Calculations
-----------------------------------------------------
 M initial (in solar mass)  =   5.0
 G (cm3/g/s2)               =   6.673000E-008
 a (erg/cm3/K4)             =   7.564640E-015
 R (erg/mol/K)              =   83140000.0000000
 X (hydrogen)               =   7.000000E-001
 Y (helium)                 =   2.700000E-001
 Z (all other elements)     =   3.000000E-002
```

```
mi                        =  6.182380E-001
Solar Mass (g)            =  2.000000E+033
Solar Radius (cm)         =  6.960000E+010
Solar Luminosity(erg/s)   =  3.830000E+033
Solar Density (g/cm3)     =  1.4200000
Speed of light (cm/s)     =  3.000000E+010

Central value of Temperature and Density

Temperature (log(T))  =        7.4101650
Density    (log(Ro)) =        1.2628940
```

i	M	log(P)	log(T)	log(R0)	r/r0	log(E)	log(L)	x	f	h
0	.00000	16.80921	7.41017	1.26289	.00000	3.97381	.00000	.00000	1.00000	.00000
1	.00055	16.80734	7.40946	1.26175	.03494	3.96103	.43348	.10000	.99833	-.03328
2	.00439	16.80172	7.40733	1.25831	.06989	3.92261	1.29862	.20000	.99335	-.06626
3	.01469	16.79235	7.40379	1.25257	.10483	3.85840	1.79127	.30000	.98509	-.09861
4	.03444	16.77923	7.39882	1.24453	.13977	3.76819	2.11257	.40000	.97363	-.13002
5	.06632	16.76235	7.39243	1.23418	.17471	3.65167	2.33449	.50000	.95909	-.16023
6	.11263	16.74170	7.38461	1.22153	.20966	3.50844	2.49031	.60000	.94159	-.18896
7	.17523	16.71727	7.37534	1.20655	.24460	3.33804	2.59896	.70000	.92130	-.21600
8	.25550	16.68903	7.36463	1.18925	.27954	3.14001	2.67295	.80000	.89839	-.24113
9	.35427	16.65697	7.35246	1.16960	.31448	2.91391	2.72152	.90000	.87307	-.26417
10	.47184	16.62105	7.33882	1.14760	.34943	2.65956	2.75189	1.00000	.84556	-.28499
11	.60797	16.58124	7.32369	1.12320	.38437	2.37736	2.76981	1.10000	.81607	-.30348
12	.76190	16.53750	7.30704	1.09640	.41931	2.06908	2.77975	1.20000	.78486	-.31957
13	.93236	16.48977	7.28887	1.06715	.45425	1.73938	2.78492	1.30000	.75216	-.33322

```
Boundary of convective zone is reached

The Fitting Parameters
New Center Press. =   1.090910E+013
New Center Densi. =   3.007211E-002
xfit              =   1.317993E-001
ffit              =          7.2944970
hfit              =        -20.5793100
new rn            =   2.398806E+011

Radiative zone
```

i	M	log(P)	log(T)	log(R0)	r/r0	log(E)	log(L)	x	f	h
14	1.69178	16.26539	7.23277	.89887	.59212	1.73938	2.78492	.17180	6.41064	-21.97721
15	2.56082	15.97900	7.16117	.68407	.74377	1.73938	2.78492	.21580	5.43628	-21.08382
16	3.40519	15.63377	7.07486	.42515	.91058	1.73938	2.78492	.26420	4.45652	-18.70462
17	4.11995	15.23180	6.97437	.12367	1.09407	1.73938	2.78492	.31744	3.53592	-15.67621
18	4.65569	14.77057	6.85907	-.22225	1.29592	1.73938	2.78492	.37600	2.71142	-12.62617
19	5.01345	14.23846	6.72604	-.62134	1.51795	1.73938	2.78492	.44042	1.99603	-9.90984
20	5.22581	13.60586	6.56789	-1.09578	1.76218	1.73938	2.78492	.51129	1.38681	-7.66472
21	5.33833	12.80054	6.36656	-1.69978	2.03083	1.73938	2.78492	.58923	.87234	-5.89521
22	5.39619	11.60361	6.06732	-2.59747	2.32635	1.73938	2.78492	.67498	.43798	-4.54128
23	5.43447	8.39657	5.26557	-5.00275	2.65143	1.73938	2.78492	.76930	.06913	-3.52080

```
Write Surface data
Mass (in Mo)          =        5.4347320
Radius (Ro)           =        2.7191030
Luminosity (log(l/lo)) =        2.7849210
Effect Temp.   (log)  =        4.0000000
```

This results agree very well as compared with Menzel's [4] results.

6 Acknowledgments

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