How to Use *Mathematica* to Solve Typical Problems in Celestial Mechanics

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Abstract

This work demonstrate how to use Mathematica to solve typical problems in celestial mechanics. Since Mathematica is so powerful and the documentation is so extensive, is believed that a good way to the beginner users *is learn by examples. In this work is presented some typical examples which are the right way to begin the exploration which Mathematica can provide. Mathematica* graphics *is the most amazing feature of this software. The unique combination of graphics, symbolic and numerical computing provides both: qualitative and quantitative insight into many problems which by other way would be very difficult to visualize or understand. In the present work three examples are studied. First, the integration of a system involving four differential equations which describe galactic orbits. Second, the zero velocity curves for the* restricted three–body problem *and finally, the Kepler equation solved by a symbolic approach solution.*

Key words: Celestial mechanics, *Mathematica*, didactic support.

1 Introduction

Graphic, intuitive and easy–to–use, the *Mathematica* gives the power need to perform a wide range of technical calculations, from start to finish. It has the familiar *Windows* look and make the user feel already comfortable with.

Mathematica may be one of the most powerful computer program available for personal computers. It may also be one of the most expensive. The full version of *Mathematica* for *Windows* costs about US$1500. The student edition of *Mathematica* for *Windows* is more affordable and still retains all the important features which will be used in this article. There is a drawback for the student edition version it does not support a coprocessor and require an true compatible computer with a 80386 or 80486 CPU. There are versions available for the Macintosh computers with and without coprocessors.

The *Mathematica* software is a powerful blend of graphics, symbolic computing and numeric calculations. The graphics capabilities include two and three dimensional plots, parametric plots, contour and density plots. The user has complete control over how graphics information can be displayed. *Mathematica* also includes an easy to use 3D viewpoint selector which allows the user to interactively specify the viewpoint for graphic display.

Mathematica includes extensive symbolic computing features. In addition to basic operations, symbolic calculations can be performed in the areas of differentiation, integration, sums and products, algebraic equations, differential equations, power series and limits. The basic capability of *Mathematica* can be easily expanded with the use of "packages". These packages are additional special functions written in the *Mathematica* language. A set of "standard" packages are included with *Mathematica* and one can also program their own. The standard packages add additional capability in the areas of algebra, calculus, discrete mathematics, geometry, graphics, linear algebra, number theory, numerical mathematics, statistics and many more are available.

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In this work is used the version 2.2.1 of the *Mathematica* software for *Windows* environment. The examples which are treated are "generic" and should execute on any computer.

2 Theoretical Development

2.1 Projections of Galactic Orbits

The study of periodic orbits which is found in some studies of galactic orbits was carried out by Contopoulos [6] and afterwards by Barbanis [7] in a series of papers. The main study is about the periodic orbits in the field $V = \frac{1}{2}(Ax^2 + By^2) - \varepsilon xy^2$ when ε is considered small.

In the case of the potential field given by

$$V = \frac{1}{2}(Ax^2 + By^2) - \varepsilon xy^2 \tag{1}$$

Is possible to write the energy integral as

$$H = H_0 + \varepsilon H_1 \tag{2}$$

where

$$\begin{aligned} H_0 &= \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + Ax^2 + By^2) & (3) \\ H_1 &= -\varepsilon xy^2 & (4) \end{aligned}$$

where $\mathcal{X}_1 = \dot{x}$, $\mathcal{X}_2 = \dot{y}$ and $\mathcal{Y}_1 = x$, $\mathcal{Y}_2 = y$.

The dynamical system whose equations of motions are written in function of the Hamiltonian H is given by

$$\begin{aligned} \frac{d\mathcal{X}_j}{dt} &= \frac{\partial H}{\partial \mathcal{Y}_j} \\ \frac{d\mathcal{Y}_j}{dt} &= -\frac{\partial H}{\partial \mathcal{X}_j} \qquad \text{(j=1,2)} \end{aligned} \tag{5}$$

It is assumed that H does not depend on the time explicitly and that the system is solved if ε reduces to zero.

In eq(5) x and y are the coordinates in the galactic plane. A, B and ε are positive constants respectively. As stated by Contopoulos when $A = B$ the movement is non–resonant and when $A = 4B$ the movement is resonant. Is possible obtain similar results applying Hori's method [8] to this study.

However, in order to solve by numerical means the system of first order differential equations given by eq(5) is need establish the initial conditions, which are the same as proposed in [7]:

$$\begin{aligned} x &= y{=}0 \\ \dot{x} &= -0.0980\ Kpc/10^7\text{years} \\ \dot{y} &= 0.07480\ Kpc/10^7\text{years} \\ A &= 0.07600\ (10^7)\text{years} \\ B &= 0.55000\ (10^7)\text{years} \\ \varepsilon &= 0.2060\ (10^7)^{-2} Kpc^{-1}\text{years}^{-2} \end{aligned}$$

The *Mathematica* program to solve this system and plot the orbits for a period of $25\,x\,10^7$ years is given as follow:

```
va=0.076
vb=0.550
eps=0.206
sol = NDSolve[
{x1'[t]-va*x3[t]+eps*x4[t]^2==0,
x2'[t] -vb*x4[t]+2*eps*x3[t]*x4[t]==0,
x3'[t]+x1[t]==0,
x4'[t]+x2[t]==0,
x1[0]==-0.0989,x2[0]==0.07480,
x3[0]==0,x4[0]==0},{x1,x2,x3,x4},{t,25},
AccuracyGoal -> Automatic, PrecisionGoal->
Automatic, WorkingPrecision -> 16]
ParametricPlot[Evaluate[{x4[t],x4'[t]} /.
sol], {t,0,25}, PlotRange -> All]
```

$x1 = \dot{x}, x2 = \dot{y}, x3 = x$ and $x4 = y$.

The field represented by the following figures does not represent any real stellar system; in fact it should represent approximately a circular stellar system. The advantage in use this short *Mathematica* code is the fact that with it usage a FORTRAN code of 450 lines approximately, without counting for the graphics routines, was avoided.

2.2 The Surfaces of Zero Relative Velocity

In order to study such problem is need formulate the three–body problem. The general problem of motion of three bodies, assumed to be point masses, subject only to their mutual gravitational attractions has not been solved, although many particular solutions have been found. A particular case is the restricted three–body problem, in this case two bodies of finite mass revolve around one another in circular orbits, and a third body of infinitesimal mass moves in their field. This situation

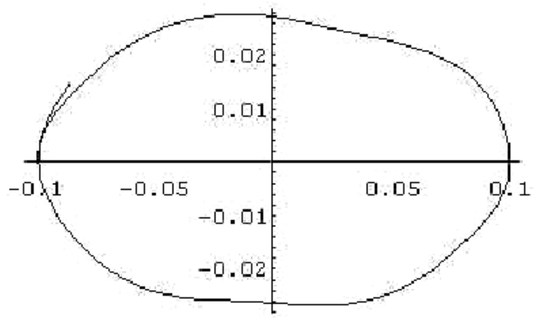

Figure 1: Orbits in phase plane $x1, \dot{x}1$

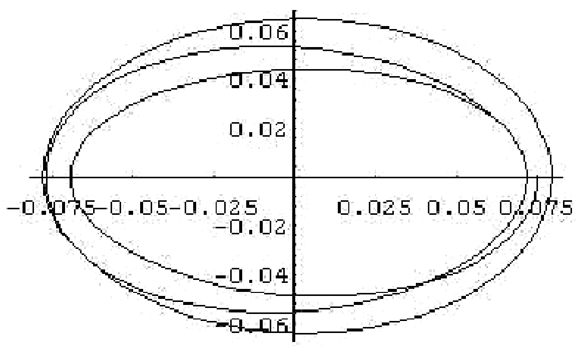

Figure 2: Orbits in phase plane $x2, \dot{x}2$

is approximately realized in many instances in the solar system. See the figure 5 to visualize the geometric situation [1].

As can be observed in figure 5, let the origin be at the center of mass of the two masses and take the axes rotating with the masses, such that they lie along the x–axis. Take the unit of mass to be the sum of their masses, and left separate masses be μ and $(1-\mu)$, where $\mu \leq 1/2$. The axes will be rotating with constant angular velocity, ω, say, and bodies will be fixed at $(x_2, 0, 0)$ and $(x_1, 0, 0)$, where x_1 is negative. Let the unit of distance be $(-x_1 + x_2)$, and let the unit of time be such as to make $k = 1$. Then, in this units is possible to write

$$\omega = k \sqrt{\frac{(1-\mu) + \mu}{(-x_1 + x_2)^3}} = 1 \qquad (6)$$

For positioning the infinitesimal mass at (x, y, z) and let

$$(x - x_1)^2 + y^2 + z^2 = r_1^2 \qquad (7)$$

and

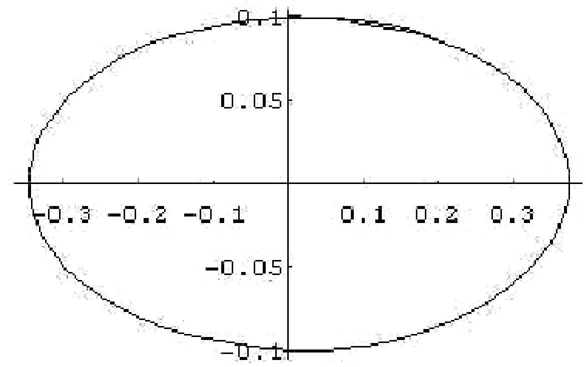

Figure 3: Orbits in phase plane $x3, \dot{x}3$

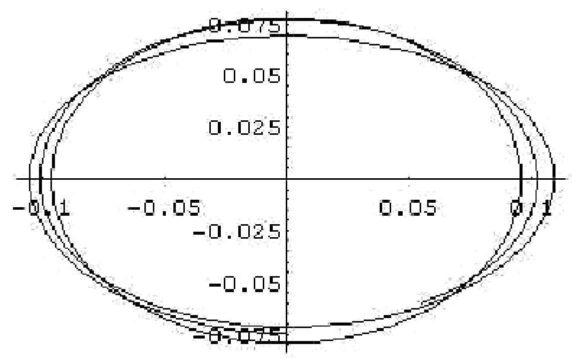

Figure 4: Orbits in phase plane $x4, \dot{x}4$

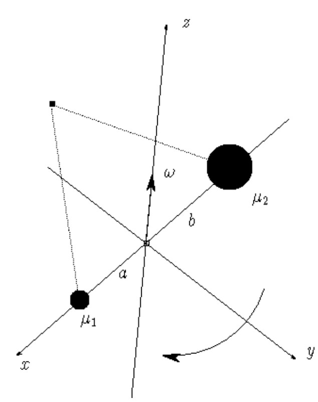

Figure 5: Restricted three–body problem

$$(x - x_2)^2 + y^2 + z^2 = r_2^2 \tag{8}$$

If v is the speed of the infinitesimal mass with respect to the moving axes is possible to write the following equation

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \tag{9}$$

and the modified energy integral

$$V - \frac{1}{2}\omega^2 \rho^2 + \frac{1}{2}\dot{r}^2 = \text{constant} \tag{10}$$

the term $-\frac{1}{2}\omega^2\rho^2$ is the *rotational potential*. Where r have components z along O_z and ρ at right angles to O_z.

The modified energy integral can be written as following

$$v^2 = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C \tag{11}$$

where C is a constant. The eq(11) is the *Jacobi's Integral*.

If $v = 0$ the eq(11) is written as

$$x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} = C \tag{12}$$

where $r_1^2 = (x+\mu)^2 + y^2$ and $r_2^2 = (x-1+\mu)^2 + y^2$.

For some value of C the eq(12) will be the locus of surfaces in space to be described next. Considering the eq(11) as a function of v^2, then is possible to see that v^2 changes sign when a surface is crossed. This is possible when the crossing does not take place at a double point. Hence the motion can take place on one side of the surface but not on other. This is similar to the theorem, in the problem of two bodies, stating that the finite motion is restricted within a circle of radius $2a$. This can also be deduced from the energy integral.

The constant C depends upon the initial position and velocity of the particle. Clearly there will be curves of zero speed given in Cartesian coordinates by

$$-C+x^2+y^2+\frac{2\,\mu}{\sqrt{(-1+\mu+x)^2+y^2}}+\frac{2\,(1-\mu)}{\sqrt{(\mu+x)^2+y^2}} \tag{13}$$

Motion of the particle can occur only in those regions of the $x-y$ plane for which eq(13) > 0. The contour curves given by eq(13), mark the boundaries of the regions within which motion can take place. Lets consider the following cases for possible motions:

- *Case I.* When C is very large, $x^2 + y^2 = C$ (nearly) eq(13) will be positive either if x and y are very large or if r_1 or r_2 are very small. Likewise for small r_1 and r_2, and large C, the x^2 and y^2 terms in eq(13) become insignificant compared with the third and fourth terms. The result is the pair of ovals surrounding $(1-\mu)$ and μ. For this large value of C, motion can not take place in the region between the ovals and the outer contour. Motion can occur if the particle is within the ovals or outside the nearly circular contour located at extern boundary.
- *Case II.* Allowing C to decrease, the ovals around $(1-\mu)$ and μ expand, and the outer contour moves toward the center of the figure. The oval contours in this case have merged into a single closed contour around the two masses.
- *Case III.* Decreasing C further, the regions of stability, that is, the areas of the plane in which motion can occur, become larger. The enlarged oval pattern around the finite masses merges into that outside the exterior oval, leaving only two small region of tadpole–like shape that eventually shrink to points.

In order to have a concrete idea of such contour curves, let to use the following *Mathematica* code to make in a very easy way the work of plotting the contours for $\mu = 0.25$ and $C = 4$;

```
Clear[f]
f[x_,y_]=x^2+y^2+2*(1-mu)/Sqrt[(x+mu)^2+y^2]\
         +(2*mu)/Sqrt[(x-1+mu)^2+y^2] - 4;
mu=.25;
ContourPlot[f[x,y],{x,-2,2},{y,-2,2},
ContourSmoothing -> Automatic,
   ContourShading->False,PlotPoints->50]
```

This situation in the $x-y$ plane is illustrated in figure 6. The basic reference to curves of zero velocity is [3].

By the other hand, if one explicit z as function of x, y is possible rewrite the eq(11) as

$$z = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C \tag{14}$$

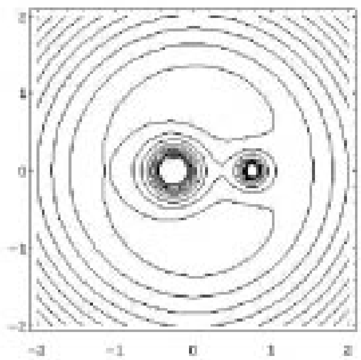

Figure 6: Contour for $C = 4$

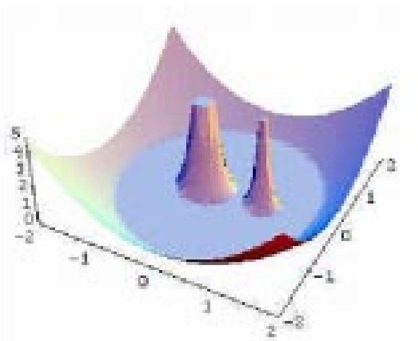

Figure 7: 3D Contour for $C = 4$

The level curves correspond to curves of zero velocity in x, y, z. In order to implement this, the *Mathematica* program is given by

```
Clear[f]
mu=.25;
f[x_,y_]=x^2+y^2+2*(1-mu)/Sqrt[(x+mu)^2+y^2]\
         +(2*mu)/Sqrt[(x-1+mu)^2+y^2] - 4;

Plot3D[f[x,y],{x,-2,2},{y,-2,2},
PlotRange->{0,5},PlotPoints->50,
Boxed->False, Mesh -> False]
```

This situation in the $x-y-z$ plane is illustrated in figure 7.

2.3 The Kepler Equation

This example is concerned with the solution for E, as a function of u and e, of the implicit equation

$$E = u + e\sin(E) \tag{15}$$

where e is to be regarded as a small quantity. Equation (15) is known as the Kepler equation. The problem is capable of formal solution in terms of Bessel function in the form

$$E = u + 2\sum_{n=1}^{\infty} \frac{J_n(ne)\sin(nu)}{n} \tag{16}$$

In order to obtain E as a function of u and e corrected up to the 5th order in e it would be perfectly possible to sum the first 5 terms of this series using a computer. However, consider what is involved in this procedure. Is possible to write a *Mathematica* code to generate the individual Bessel functions and arrange to ignore those terms of order $k > 5$ that arise in e. From the Kepler equation it is clear that $E = u$ to zero order in e. Suppose $E = u + A_k$ is the solution correct to order k in e. Then clearly

$$A_{k+1} = e\sin(u + A_k) \tag{17}$$

Where the right hand member of eq(17) is to be taken only to order $k+1$ in e. Thus an approximation algorithm may be stated as follows

$$E = u + \lim_{n\to\infty} A_n \tag{18}$$

where

$$A_0 = 0 \tag{19}$$

and

$$A_{k+1} = \left[e\sin(u)\left\{1 - \frac{A_k^2}{2!} + \frac{A_k^4}{4!} - \cdots\right\} + e\cos(u)\left\{A_k - \frac{A_k^3}{3!} + \cdots\right\}\right]_{k+1} \tag{20}$$

The *Mathematica* code which implement the eq(18) and eq(20) is written as

```
a[0]=0;
Do [
t1=1-a[k-1]^2/2+a[k-1]^4/24;
t2=a[k-1]-a[k-1]^3/6;
a[k]=Expand[e*Sin[u]*t1+e*Cos[u]*t2,Trig->
True];
Simplify[a[k]];
TeXForm[a[k]] >>"tex.01";
Print [a[k],k], {k,2}]
```

The "output" file "tex.01" is shown below. In this result only terms of order 5 in e are calculated.

$$e\ \sin(u) - \frac{3\,e^3\ \sin(u)}{8} + \frac{5\,e^5\ \sin(u)}{192}$$
$$+\frac{e^2\ \sin(2\,u)}{2} - \frac{e^4\ \sin(2\,u)}{24} + \frac{e^3\ \sin(3\,u)}{8}$$
$$-\frac{5\,e^5\ \sin(3\,u)}{384} + \frac{e^4\ \sin(4\,u)}{48} + \frac{e^5\ \sin(5\,u)}{384}$$

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