On the use of Logarithmic Scales

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1 Introduction

Graphs are the most efficient way for visualizing data. Currently the world
is going through a pandemic and most experts are following the evolution of
the Covid-19 outbreak, by plotting the number of confirmed cases, number
of active cases and other data regarding the issue, since the beginning of the
pandemic. The disease is unfortunately growing in an exponential way and
some useful information, notably the growth rate, can be better inferred for
this type of behavior using logarithmic scales instead of linear ones. It is
the aim of these brief notes to discuss this theme, both in a analytical and
graphical way, providing some justification for the use of different points of
views for the same set of data.

First of all some basic definitions. The most important is of course, the
concept of logarithm. Many of us have memorized the logarithm definition by
the simple formula \( \log_b(x) = y(x) \) or \( b^y(x) = x \). In both cases we can define,
in words, the logarithm as the quantity \( y(x) \) which represents first, a function
(the \( \log_b \)) and second, the exponent to which a base \( b \) should be raised in
order to obtain the quantity \( x \). The logarithmic and the exponentiation
are obviously inverse functions and this leads to the important property
\[ \log_b(a)^c = c \log_b(a). \] The most common values for \( b \) are 2, \( e \approx 2.718 \) and 10. Figure 1 shows the graphs of the logarithmic function for these values of \( b \). Note that when \( y(x) \) acts over a wide range of \( x \) values, the result is comprised into a small range of \( y \) values. The change of the linear scale by its logarithmic counterpart is nothing more than applying the \( y(x) \) function to the independent \( x \) values, then allowing a better visualization of dependent data that otherwise would be hidden.

![Graph of logarithmic functions](image)

Figure 1: The graph of the logarithmic function for most common bases \( b \).

2 An example of exponential growth

Let’s suppose that the number of people infected by the Covid-19, or confirmed cases \( C \), doubles every day \( d \). The exponential function \( C = 2^d \) models adequately this behavior. The first day \( d = 0 \) only one person is infected. The second day \( d = 1 \) there will be 2 people infected and so on. The amount of \( C(d) \) is always twice the amount of the day before \( C(d) = 2 C(d-1) \). This can be conveniently plotted in a graph with linear scale on both axes (see Fig. 2).

But, what about the derivative of this \( C(d) \) function? Well, this calcula-
Figure 2: The amount of confirmed cases over the time $C(d) = 2^d$ and its derivative in a linear scale.

As expected, the curves $C(d)$ and $C'(d)$ differs only by a constant, in this case $\ln 2$ (see Fig. 2). This means that the slopes of the tangent straight lines to the curve $C(d)$ at different values of $d$ are growing continuously over the days since the outbreak. But we know that the amount the cases doubles every day ($C(d) = 2^d$) which clearly indicates a constant growth rate. But how do we see this constancy? Suppose that we decide to plot the equation for confirmed cases using a bi-logarithmic scale of base ten, $b = 10$. This is equivalent to taking the base ten logarithm for both sides of the
equation $C = 2^n$ or $\log C = d \log 2$. After defining the following quantities
$y \equiv \log C, \ x \equiv d$ and $m \equiv \log 2$ we realize that our equation $C(d)$ is now
a straight line $y = xm$ passing through the origin. The slope of this line is
constant and equal to $m = \log 2$. This is the true growth rate of the modeled
event and can only be seen using a logarithmic scale for both axes as shown
in Fig. 3.

![Figure 3: The function $C = 2^d$ in a logarithmic scale.](image)

3 Conclusions

The conclusions of these brief but important notes are the following:

- For phenomena well modeled by exponential curves, and when the
  $y$-values extend by some orders of magnitude, the use of base ten bi-
  logarithmic scales is preferred in order to visualize the true growth rates
  over time.

- The use of linear scales for these phenomena only shows the amount
  of confirmed cases (using the Covid-19 example) over time.