SECTION 11.1 EXERCISES

Answers to odd-numbered problems begin on page A-21.

In Problems 1–6 show that the given functions are orthogonal on the indicated interval.

1. $f_1(x) = x, f_2(x) = x^2; \quad [-2, 2]$
2. $f_1(x) = x^3, f_2(x) = x^2 + 1; \quad [-1, 1]$
3. $f_1(x) = e^x, f_2(x) = xe^{-x} - e^{-x}; \quad [0, 2]$
4. $f_1(x) = \cos x, f_2(x) = \sin^2 x; \quad [0, \pi]$
5. $f_1(x) = x, f_2(x) = \cos 2x; \quad [-\pi/2, \pi/2]$
6. $f_1(x) = e^x, f_2(x) = \sin x; \quad [\pi/4, 5\pi/4]$

In Problems 7–12 show that the given set of functions is orthogonal on the indicated interval. Find the norm of each function in the set.

7. $\{\sin x, \sin 3x, \sin 5x, \ldots\}; \quad [0, \pi/2]$
8. $\{\cos x, \cos 3x, \cos 5x, \ldots\}; \quad [0, \pi/2]$
9. $\{\sin nx\}, n = 1, 2, 3, \ldots; \quad [0, \pi]$
10. $\left\{\sin \frac{n\pi}{p} x\right\}, n = 1, 2, 3, \ldots; \quad [0, p]$
11. $\left\{1, \cos \frac{n\pi}{p} x\right\}, n = 1, 2, 3, \ldots; \quad [0, p]$
12. $\left\{1, \cos \frac{n\pi}{p} x, \sin \frac{m\pi}{p} x\right\}, n = 1, 2, 3, \ldots, m = 1, 2, 3, \ldots; \quad [-p, p]$

In Problems 13 and 14 verify by direct integration that the functions are orthogonal with respect to the indicated weight function on the given interval.

13. $H_0(x) = 1, H_1(x) = 2x, H_2(x) = 4x^2 - 2; \quad w(x) = e^{-x^2}, (-\infty, \infty)$
14. $L_0(x) = 1, L_1(x) = -x + 1, L_2(x) = \frac{1}{2}x^2 - 2x + 1; \quad w(x) = e^{-x}, [0, \infty)$
15. Let $\{\phi_n(x)\}$ be an orthogonal set of functions on $[a, b]$ such that $\phi_0(x) = 1$. Show that $\int_a^b \phi_n(x)\, dx = 0$ for $n = 1, 2, \ldots$.
16. Let $\{\phi_n(x)\}$ be an orthogonal set of functions on $[a, b]$ such that $\phi_0(x) = 1$ and $\phi_1(x) = x$. Show that $\int_a^b (\alpha x + \beta)\phi_n(x)\, dx = 0$ for $n = 2, 3, \ldots$ and any constants α and β.
17. Let $\{\phi_n(x)\}$ be an orthogonal set of functions on $[a, b]$. Show that $\|\phi_m(x) + \phi_n(x)\|^2 = \|\phi_m(x)\|^2 + \|\phi_n(x)\|^2, m \neq n$.
18. From Problem 1 we know that $f_1(x) = x$ and $f_2(x) = x^2$ are orthogonal on $[-2, 2]$. Find constants c_1 and c_2 such that $f_3(x) = x + c_1x^2 + c_2x^3$ is orthogonal to both f_1 and f_2 on the same interval.
19. The set of functions $\{\sin nx\}, n = 1, 2, 3, \ldots$, is orthogonal on the interval $[-\pi, \pi]$. Show that the set is not complete.
20. Suppose f_1, f_2, and f_3 are functions continuous on the interval $[a, b]$. Show that $(f_1 + f_2, f_3) = (f_1, f_3) + (f_2, f_3)$.

EXAMPLE 3 **Convergence to the Periodic Extension**

The Fourier series (13) converges to the periodic extension of (12) on the entire x-axis. The solid dots in Figure 11.2 represent the value

$$\frac{f(0+)+f(0-)}{2}=\frac{\pi}{2}$$

at $0, \pm 2\pi, \pm 4\pi, \ldots$. At $\pm\pi, \pm 3\pi, \pm 5\pi, \ldots$, the series converges to the value

$$\frac{f(\pi-)+f(-\pi+)}{2}=0.$$

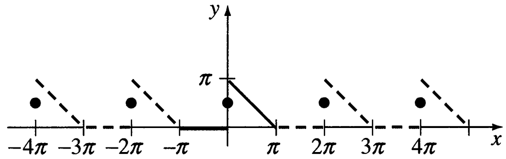


FIGURE 11.2

SECTION 11.2 EXERCISES

Answers to odd-numbered problems begin on page A-21.

In Problems 1–16 find the Fourier series of f on the given interval.

1. $f(x)=\begin{cases}0, & -\pi<x<0\\ 1, & 0\le x<\pi\end{cases}$

2. $f(x)=\begin{cases}-1, & -\pi<x<0\\ 2, & 0\le x<\pi\end{cases}$

3. $f(x)=\begin{cases}1, & -1<x<0\\ x, & 0\le x<1\end{cases}$

4. $f(x)=\begin{cases}0, & -1<x<0\\ x, & 0\le x<1\end{cases}$

5. $f(x)=\begin{cases}0, & -\pi<x<0\\ x^2, & 0\le x<\pi\end{cases}$

6. $f(x)=\begin{cases}\pi^2, & -\pi<x<0\\ \pi^2-x^2, & 0\le x<\pi\end{cases}$

7. $f(x)=x+\pi, \quad -\pi<x<\pi$

8. $f(x)=3-2x, \quad -\pi<x<\pi$

9. $f(x)=\begin{cases}0, & -\pi<x<0\\ \sin x, & 0\le x<\pi\end{cases}$

10. $f(x)=\begin{cases}0, & -\pi/2<x<0\\ \cos x, & 0\le x<\pi/2\end{cases}$

11. $f(x)=\begin{cases}0, & -2<x<-1\\ -2, & -1\le x<0\\ 1, & 0\le x<1\\ 0, & 1\le x<2\end{cases}$

12. $f(x)=\begin{cases}0, & -2<x<0\\ x, & 0\le x<1\\ 1, & 1\le x<2\end{cases}$

13. $f(x)=\begin{cases}1, & -5<x<0\\ 1+x, & 0\le x<5\end{cases}$

14. $f(x)=\begin{cases}2+x, & -2<x<0\\ 2, & 0\le x<2\end{cases}$

15. $f(x)=e^x, \quad -\pi<x<\pi$

16. $f(x)=\begin{cases}0, & -\pi<x<0\\ e^x-1, & 0\le x<\pi\end{cases}$

17. Use the result of Problem 5 to show

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \quad \text{and} \quad \frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots.$$

18. Use Problem 17 to find a series that gives the numerical value of $\pi^2/8$.

19. Use the result of Problem 7 to show

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

20. Use the result of Problem 9 to show

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots.$$

21. **(a)** Use the complex exponential form of the cosine and sine,

$$\cos\frac{n\pi}{p}x = \frac{e^{in\pi x/p} + e^{-in\pi x/p}}{2}, \qquad \sin\frac{n\pi}{p}x = \frac{e^{in\pi x/p} - e^{-in\pi x/p}}{2i},$$

to show that (8) can be written in the **complex form**

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p},$$

where $c_0 = a_0/2$, $c_n = (a_n - ib_n)/2$, and $c_{-n} = (a_n + ib_n)/2$, where $n = 1, 2, 3, \ldots$.

(b) Show that c_0, c_n, and c_{-n} of part (a) can be written as one integral

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}\,dx, \quad n = 0, \pm 1, \pm 2, \ldots.$$

22. Use the results of Problem 21 to find the complex form of the Fourier series of $f(x) = e^{-x}$ on the interval $-\pi < x < \pi$.

11.3 FOURIER COSINE AND SINE SERIES

▪ *Even and odd functions* ▪ *Properties of even and odd functions*
▪ *Fourier cosine and sine series* ▪ *Sequence of partial sums* ▪ *Gibbs phenomenon*
▪ *Half-range expansions*

Even and Odd Functions You may recall that a function f is said to be

$$\textbf{even} \text{ if } f(-x) = f(x) \quad \text{and} \quad \textbf{odd} \text{ if } f(-x) = -f(x).$$

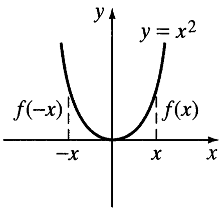


FIGURE 11.3

EXAMPLE 1 **Even/Odd Functions**

(a) $f(x) = x^2$ is even since $f(-x) = (-x)^2 = x^2 = f(x)$. See Figure 11.3.

(b) $f(x) = x^3$ is odd since $f(-x) = (-x)^3 = -x^3 = -f(x)$. See Figure 11.4. ▪

expansion of $f(t) = \pi t$, $0 < t < 1$. With $p = 1$ it follows from (5) and integration by parts that

$$b_n = 2\int_0^1 \pi t \sin n\pi t\,dt = \frac{2(-1)^{n+1}}{n}.$$

From (11) the differential equation of motion is seen to be

$$\frac{1}{16}\frac{d^2x}{dt^2} + 4x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin n\pi t. \qquad \textbf{(13)}$$

To find a particular solution $x_p(t)$ of (13) we substitute (12) into the equation and equate coefficients of $\sin n\pi t$. This yields

$$\left(-\frac{1}{16}n^2\pi^2 + 4\right)B_n = \frac{2(-1)^{n+1}}{n} \quad \text{or} \quad B_n = \frac{32(-1)^{n+1}}{n(64 - n^2\pi^2)}.$$

Thus

$$x_p(t) = \sum_{n=1}^{\infty} \frac{32(-1)^{n+1}}{n(64 - n^2\pi^2)} \sin n\pi t. \qquad \textbf{(14)}$$ ■

Observe in the solution (14) that there is no integer $n \geq 1$ for which the denominator $64 - n^2\pi^2$ of B_n is zero. In general, if there *is* a value of n, say N, for which $N\pi/p = \omega$, where $\omega = \sqrt{k/m}$, then the system described by (11) is in a state of pure resonance. In other words, we have pure resonance if the Fourier series expansion of the driving force $f(t)$ contains a term $\sin(N\pi/L)t$ (or $\cos(N\pi/L)t$) that has the same frequency as the free vibrations.

Of course, if the $2p$-periodic extension of the driving force f onto the negative t-axis yields an even function, then we expand f in a cosine series.

SECTION 11.3 EXERCISES

Answers to odd-numbered problems begin on page A-22.

In Problems 1–10 determine whether the function is even, odd, or neither.

1. $f(x) = \sin 3x$

2. $f(x) = x \cos x$

3. $f(x) = x^2 + x$

4. $f(x) = x^3 - 4x$

5. $f(x) = e^{|x|}$

6. $f(x) = |x^5|$

7. $f(x) = \begin{cases} x^2, & -1 < x < 0 \\ -x^2, & 0 \leq x < 1 \end{cases}$

8. $f(x) = \begin{cases} x + 5, & -2 < x < 0 \\ -x + 5, & 0 \leq x < 2 \end{cases}$

9. $f(x) = x^3$, $0 \leq x \leq 2$

10. $f(x) = 2|x| - 1$

In Problems 11–24 expand the given function in an appropriate cosine or sine series.

11. $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 \leq x < \pi \end{cases}$

12. $f(x) = \begin{cases} 1, & -2 < x < -1 \\ 0, & -1 < x < 1 \\ 1, & 1 < x < 2 \end{cases}$

13. $f(x) = |x|$, $-\pi < x < \pi$

14. $f(x) = x$, $-\pi < x < \pi$

15. $f(x) = x^2$, $-1 < x < 1$

16. $f(x) = x|x|$, $-1 < x < 1$

17. $f(x) = \pi^2 - x^2$, $-\pi < x < \pi$

18. $f(x) = x^3$, $-\pi < x < \pi$

19. $f(x) = \begin{cases} x-1, & -\pi < x < 0 \\ x+1, & 0 \le x < \pi \end{cases}$

20. $f(x) = \begin{cases} x+1, & -1 < x < 0 \\ x-1, & 0 \le x < 1 \end{cases}$

21. $f(x) = \begin{cases} 1, & -2 < x < -1 \\ -x, & -1 \le x < 0 \\ x, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases}$

22. $f(x) = \begin{cases} -\pi, & -2\pi < x < -\pi \\ x, & -\pi \le x < \pi \\ \pi, & \pi \le x < 2\pi \end{cases}$

23. $f(x) = |\sin x|, -\pi < x < \pi$

24. $f(x) = \cos x, -\pi/2 < x < \pi/2$

In Problems 25–34 find the half-range cosine and sine expansions of the given function.

25. $f(x) = \begin{cases} 1, & 0 < x < \frac{1}{2} \\ 0, & \frac{1}{2} \le x < 1 \end{cases}$

26. $f(x) = \begin{cases} 0, & 0 < x < \frac{1}{2} \\ 1, & \frac{1}{2} \le x < 1 \end{cases}$

27. $f(x) = \cos x, 0 < x < \pi/2$

28. $f(x) = \sin x, 0 < x < \pi$

29. $f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 \le x < \pi \end{cases}$

30. $f(x) = \begin{cases} 0, & 0 < x < \pi \\ x - \pi, & \pi \le x < 2\pi \end{cases}$

31. $f(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 \le x < 2 \end{cases}$

32. $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 2 - x, & 1 \le x < 2 \end{cases}$

33. $f(x) = x^2 + x, 0 < x < 1$

34. $f(x) = x(2 - x), 0 < x < 2$

In Problems 35–38 expand the given function in a Fourier series.

35. $f(x) = x^2, \quad 0 < x < 2\pi$

36. $f(x) = x, \quad 0 < x < \pi$

37. $f(x) = x + 1, \quad 0 < x < 1$

38. $f(x) = 2 - x, \quad 0 < x < 2$

In Problems 39 and 40 proceed as in Example 7 to find a particular solution of equation (11) when $m = 1$, $k = 10$, and the driving force $f(t)$ is as given. Assume that when $f(t)$ is extended to the negative t-axis in a periodic manner, the resulting function is odd.

39. $f(t) = \begin{cases} 5, & 0 < t < \pi \\ -5, & \pi < t < 2\pi \end{cases}; \quad f(t + 2\pi) = f(t)$

40. $f(t) = 1 - t, 0 < t < 2; \quad f(t + 2) = f(t)$

In Problems 41 and 42 find a particular solution of equation (11) when $m = \frac{1}{4}$, $k = 12$, and the driving force $f(t)$ is as given. Assume that when $f(t)$ is extended to the negative t-axis in a periodic manner, the resulting function is even.

41. $f(t) = 2\pi t - t^2, 0 < t < 2\pi; \quad f(t + 2\pi) = f(t)$

42. $f(t) = \begin{cases} t, & 0 < t < \frac{1}{2} \\ 1 - t, & \frac{1}{2} \le t < 1 \end{cases}; \quad f(t + 1) = f(t)$

43. Suppose a uniform beam of length L is simply supported at $x = 0$ and $x = L$. If the load per unit length is given by $w(x) = w_0 x/L$, $0 < x < L$, then the differential equation for the deflection $y(x)$ is

$$EI\frac{d^4y}{dx^4} = \frac{w_0 x}{L},$$

where E, I, and w_0 are constants. (See equation (4), Section 5.2.)

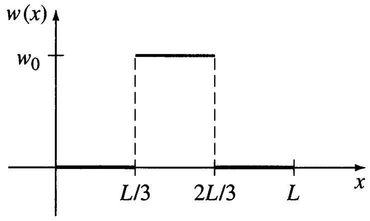


FIGURE 11.15

(a) Expand $w(x)$ in a half-range sine series.

(b) Use the method of Example 7 to find a particular solution $y(x)$ of the differential equation.

44. Proceed as in Problem 43 to find the deflection $y(x)$ when $w(x)$ is as given in Figure 11.15.

45. Prove Property (a) in Theorem 11.2.

46. Prove Property (c).

47. Prove Property (d).

48. Prove Property (f).

49. Prove Property (g).

50. Prove that any function f can be written as a sum of an even and an odd function.

$\left[\textit{Hint:} \text{ Use the identity } f(x) = \dfrac{f(x)+f(-x)}{2} + \dfrac{f(x)-f(-x)}{2}.\right]$

51. Find the Fourier series of

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 \le x < \pi \end{cases}$$

using the identity $f(x) = (|x| + x)/2$, $-\pi < x < \pi$, and the results of Problems 13 and 14. Observe that $|x|/2$ and $x/2$ are even and odd, respectively, on the interval (see Problem 50).

52. The **double sine series** for a function $f(x, y)$ defined over a rectangular region $0 \le x \le b$, $0 \le y \le c$ is given by

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi}{b}x \sin\frac{n\pi}{c}y,$$

where $$A_{mn} = \frac{4}{bc}\int_0^c\int_0^b f(x, y) \sin\frac{m\pi}{b}x \sin\frac{n\pi}{c}y\, dx\, dy.$$

Find the double sine series of $f(x, y) = 1$, $0 \le x \le \pi$, $0 \le y \le \pi$.

53. The **double cosine series** of a function $f(x, y)$ defined over a rectangular region $0 \le x \le b$, $0 \le y \le c$ is given by

$$f(x, y) = A_{00} + \sum_{m=1}^{\infty} A_{m0}\cos\frac{m\pi}{b}x + \sum_{n=1}^{\infty} A_{0n}\cos\frac{n\pi}{c}y + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\cos\frac{m\pi}{b}x\cos\frac{n\pi}{c}y,$$

where

$$A_{00} = \frac{1}{bc}\int_0^c\int_0^b f(x, y)\, dx\, dy$$

$$A_{m0} = \frac{2}{bc}\int_0^c\int_0^b f(x, y)\cos\frac{m\pi}{b}x\, dx\, dy$$

$$A_{0n} = \frac{2}{bc}\int_0^c\int_0^b f(x, y)\cos\frac{n\pi}{c}y\, dx\, dy$$

$$A_{mn} = \frac{4}{bc}\int_0^c\int_0^b f(x, y)\cos\frac{m\pi}{b}x\cos\frac{n\pi}{c}y\, dx\, dy.$$

Find the double cosine series of $f(x, y) = xy$, $0 \le x \le 1$, $0 \le y \le 1$.

1. $y'' + \lambda y = 0, \quad y'(0) = 0, y(1) + y'(1) = 0$
2. $y'' + \lambda y = 0, \quad y(0) + y'(0) = 0, y(1) = 0$
3. Consider $y'' + \lambda y = 0$ subject to $y'(0) = 0, y'(L) = 0$. Show that the eigenfunctions are $\left\{1, \cos\frac{\pi}{L}x, \cos\frac{2\pi}{L}x, \ldots\right\}$. This set, which is orthogonal on $[0, L]$, is the basis for the Fourier cosine series.
4. Consider $y'' + \lambda y = 0$ subject to the periodic boundary conditions $y(-L) = y(L), y'(-L) = y'(L)$. Show that the eigenfunctions are

$$\left\{1, \cos\frac{\pi}{L}x, \cos\frac{2\pi}{L}x, \ldots, \sin\frac{\pi}{L}x, \sin\frac{2\pi}{L}x, \sin\frac{3\pi}{L}x, \ldots\right\}.$$

This set, which is orthogonal on $[-L, L]$, is the basis for the Fourier series.

5. Find the square norm of each eigenfunction in Problem 1.
6. Show that for the eigenfunctions in Example 2

$$\|\sin\sqrt{\lambda_n}x\|^2 = \frac{1}{2}[1 + \cos^2\sqrt{\lambda_n}].$$

7. **(a)** Find the eigenvalues and eigenfunctions of the boundary-value problem

$$x^2y'' + xy' + \lambda y = 0, \qquad y(1) = 0, \quad y(5) = 0.$$

(b) Put the differential equation in self-adjoint form.
(c) Give an orthogonality relation.

8. **(a)** Find the eigenvalues and eigenfunctions of the boundary-value problem

$$y'' + y' + \lambda y = 0, \qquad y(0) = 0, \quad y(2) = 0.$$

(b) Put the differential equation in self-adjoint form.
(c) Give an orthogonality relation.

9. **(a)** Give an orthogonality relation for the Sturm-Liouville problem in Problem 1.
(b) Use a CAS as an aid in verifying the orthogonality relation for the eigenfunctions y_1 and y_2 that correspond to the first two eigenvalues λ_1 and λ_2, respectively.
10. **(a)** Give an orthogonality relation for the Sturm-Liouville problem in Problem 2.
(b) Use a CAS as an aid in verifying the orthogonality relation for the eigenfunctions y_1 and y_2 that correspond to the first two eigenvalues λ_1 and λ_2, respectively.
11. **Laguerre's differential equation** $xy'' + (1 - x)y' + ny = 0, n = 0, 1, 2, \ldots$, has polynomial solutions $L_n(x)$. Put the equation in self-adjoint form, and give an orthogonality relation.
12. **Hermite's differential equation** $y'' - 2xy' + 2ny = 0, n = 0, 1, 2, \ldots$, has polynomial solutions $H_n(x)$. Put the equation in self-adjoint form and give an orthogonality relation.

13. Consider the regular Sturm-Liouville problem

$$\frac{d}{dx}[(1 + x^2)y'] + \frac{\lambda}{1 + x^2}y = 0, \qquad y(0) = 0, \quad y(1) = 0.$$

(a) Find the eigenvalues and eigenfunctions of the boundary-value problem. [*Hint:* Let $x = \tan\theta$ and then use the Chain Rule.]

(b) Give an orthogonality relation.

14. (a) Find the eigenfunctions and the equation that defines the eigenvalues for the boundary-value problem

$$x^2y'' + xy' + (\lambda^2x^2 - 1)y = 0, \qquad y \text{ is bounded at } x = 0, \quad y(3) = 0.$$

(b) Use Table 6.1 to find the approximate values of the first four eigenvalues λ_1, λ_2, λ_3, and λ_4.

Discussion Problem

15. Consider the special case of the regular Sturm-Liouville problem on the interval $[a, b]$:

$$\frac{d}{dx}[r(x)y'] + \lambda p(x)y = 0, \qquad y'(a) = 0, \quad y'(b) = 0.$$

Is $\lambda = 0$ an eigenvalue of the problem? Defend your answer.

11.5

■ *Orthogonal set of Bessel functions* ■ *Fourier-Bessel series* ■ *Differential recurrence relations*
■ *Forms of Fourier-Bessel series* ■ *Convergence of a Fourier-Bessel series*
■ *Orthogonal set of Legendre polynomials* ■ *Fourier-Legendre series*
■ *Convergence of a Fourier-Legendre series*

Fourier series, Fourier cosine series, and Fourier sine series are three ways of expanding a function in terms of an orthogonal set of functions. But such expansions are by no means limited to orthogonal sets of trigonometric functions. We saw in Section 11.1 that a function f defined on an interval (a, b) could be expanded at least formally in terms of any set of functions $\{\phi_n(x)\}$ that is orthogonal with respect to a weight function on $[a, b]$. Many of these so-called generalized Fourier series come from Sturm-Liouville problems arising in physical applications of linear partial differential equations. Fourier series and generalized Fourier series, as well as the two series considered in this section, will appear again in the subsequent consideration of these applications.

11.5.1 Fourier-Bessel Series

In Example 3 of Section 11.4 it was seen that the set of Bessel functions $\{J_n(\lambda_i x)\}$, $i = 1, 2, 3, \ldots$, is orthogonal with respect to the weight function $p(x) = x$ on an interval $[0, b]$ when the eigenvalues λ_i are defined by means of a boundary condition

$$\alpha_2 J_n(\lambda b) + \beta_2 \lambda J_n'(\lambda b) = 0. \qquad \textbf{(1)}$$

SECTION 11.5 EXERCISES

Answers to odd-numbered problems begin on page A-22.

11.5.1

1. Find the first four eigenvalues λ_k defined by $J_1(3\lambda) = 0$. [*Hint:* See Table 6.1 on page 247.]
2. Find the first four eigenvalues λ_k defined by $J_0'(2\lambda) = 0$.

In Problems 3–6 expand $f(x) = 1, 0 < x < 2$ in a Fourier-Bessel series using Bessel functions of order zero that satisfy the given boundary condition.

3. $J_0(2\lambda) = 0$
4. $J_0'(2\lambda) = 0$
5. $J_0(2\lambda) + 2\lambda J_0'(2\lambda) = 0$
6. $J_0(2\lambda) + \lambda J_0'(2\lambda) = 0$

In Problems 7–10 expand the given function in a Fourier-Bessel series using Bessel functions of the same order as in the indicated boundary condition.

7. $f(x) = 5x, \quad 0 < x < 4$
   $3J_1(4\lambda) + 4\lambda J_1'(4\lambda) = 0$
8. $f(x) = x^2, \quad 0 < x < 1$
   $J_2(\lambda) = 0$
9. $f(x) = x^2, \quad 0 < x < 3$
   $J_0'(3\lambda) = 0 \quad$ [*Hint:* $t^3 = t^2 \cdot t$.]
10. $f(x) = 1 - x^2, \quad 0 < x < 1$
    $J_0(\lambda) = 0$

11.5.2

A Fourier-Legendre expansion of a polynomial function defined on the interval $(-1, 1)$ is necessarily a finite series. (Why?) In Problems 11 and 12 find the Fourier-Legendre expansion of the given function.

11. $f(x) = x^2$
12. $f(x) = x^3$

In Problems 13 and 14 write out the first four nonzero terms in the Fourier-Legendre expansion of the given function.

13. $f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$
14. $f(x) = e^x, \quad -1 < x < 1$
15. The first three Legendre polynomials are $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = \frac{1}{2}(3x^2 - 1)$. If $x = \cos\theta$, then $P_0(\cos\theta) = 1$ and $P_1(\cos\theta) = \cos\theta$. Show that $P_2(\cos\theta) = \frac{1}{4}(3\cos 2\theta + 1)$.
16. Use the results given in Problem 15 to find a Fourier-Legendre expansion (23) of $F(\theta) = 1 - \cos 2\theta$.
17. A Legendre polynomial $P_n(x)$ is an even or odd function, depending on whether n is even or odd. Show that if f is an even function on $(-1, 1)$, then (21) and (22) become, respectively,

$$f(x) = \sum_{n=0}^{\infty} c_{2n}P_{2n}(x) \tag{25}$$

$$c_{2n} = (4n + 1)\int_0^1 f(x)P_{2n}(x)\,dx. \tag{26}$$

18. Show that if f is an odd function on the interval $(-1, 1)$, then (21) and (22) become, respectively,

$$f(x) = \sum_{n=0}^{\infty} c_{2n+1} P_{2n+1}(x) \qquad \textbf{(27)}$$

$$c_{2n+1} = (4n + 3) \int_0^1 f(x) P_{2n+1}(x)\, dx. \qquad \textbf{(28)}$$

The series (25) and (27) can also be used when f is defined on only $(0, 1)$. Both series represent f on $(0, 1)$; but on $(-1, 0)$, (25) represents its even extension, whereas (27) represents its odd extension. In Problems 19 and 20 write out the first three nonzero terms in the indicated expansion of the given function. What function does the series represent on $(-1, 1)$?

19. $f(x) = x$, $0 < x < 1$; (25)

20. $f(x) = 1$, $0 < x < 1$; (27)

CHAPTER 11 REVIEW EXERCISES

Answers to odd-numbered problems begin on page A-22.

Answer Problems 1–10 without referring back to the text. Fill in the blank or answer true/false.

1. The functions $f(x) = x^2 - 1$ and $f(x) = x^5$ are orthogonal on $[-\pi, \pi]$. ______

2. The product of an odd function with an odd function is ______.

3. To expand $f(x) = |x| + 1$, $-\pi < x < \pi$ in an appropriate Fourier series we would use a ______ series.

4. Since $f(x) = x^2$, $0 < x < 2$, is not an even function, it cannot be expanded in a Fourier cosine series. ______

5. The Fourier series of $f(x) = \begin{cases} 3, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases}$ will converge to ______ at $x = 0$.

6. $y = 0$ is never an eigenfunction of a Sturm-Liouville problem. ______

7. $\lambda = 0$ is never an eigenvalue of a Sturm-Liouville problem. ______

8. For $\lambda = 25$ the corresponding eigenfunction for the boundary-value problem $y'' + \lambda y = 0$, $y'(0) = 0$, $y(\pi/2) = 0$ is ______.

9. **Chebyshev's differential equation** $(1 - x^2)y'' - xy' + n^2y = 0$ has a polynomial solution $y = T_n(x)$ for $n = 0, 1, 2, \ldots$. The set of Chebyshev polynomials $\{T_n(x)\}$ is orthogonal with respect to the weight function ______ on the interval ______.

10. The set $\{P_n(x)\}$ of Legendre polynomials is orthogonal with respect to the weight function $p(x) = 1$ on $[-1, 1]$ and $P_0(x) = 1$. Hence $\int_{-1}^{1} P_n(x)\, dx =$ ______ for $n > 0$.

11. Show that the set

$$\left\{\sin\frac{\pi}{2L}x, \sin\frac{3\pi}{2L}x, \sin\frac{5\pi}{2L}x, \ldots\right\}$$

is orthogonal on the interval $0 \leq x \leq L$.

12. Find the norm of each function in Problem 11. Construct an orthonormal set.

13. Expand $f(x) = |x| - x$, $-1 < x < 1$, in a Fourier series.

14. Expand $f(x) = 2x^2 - 1$, $-1 < x < 1$, in a Fourier series.

15. Expand $f(x) = e^{-x}$, $0 < x < 1$, in a cosine series.

16. Expand the function given in Problem 15 in a sine series.

17. Find the eigenvalues and eigenfunctions of the boundary-value problem

$$x^2y'' + xy' + 9\lambda y = 0, \qquad y'(1) = 0, \quad y(e) = 0.$$

18. State an orthogonality relation for the eigenfunctions in Problem 17.

19. Expand $f(x) = \begin{cases} 1, & 0 < x < 2 \\ 0, & 2 < x < 4 \end{cases}$ in a Fourier-Bessel series using Bessel functions of order zero that satisfy the boundary condition $J_0(4\lambda) = 0$.

20. Expand $f(x) = x^4$, $-1 < x < 1$, in a Fourier-Legendre series.

1. $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y}$

2. $\dfrac{\partial u}{\partial x} + 3\dfrac{\partial u}{\partial y} = 0$

3. $u_x + u_y = u$

4. $u_x = u_y + u$

5. $x\dfrac{\partial u}{\partial x} = y\dfrac{\partial u}{\partial y}$

6. $y\dfrac{\partial u}{\partial x} + x\dfrac{\partial u}{\partial y} = 0$

7. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$

8. $y\dfrac{\partial^2 u}{\partial x\,\partial y} + u = 0$

9. $k\dfrac{\partial^2 u}{\partial x^2} - u = \dfrac{\partial u}{\partial t}, \quad k > 0$

10. $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad k > 0$

11. $a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

12. $a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2} + 2k\dfrac{\partial u}{\partial t}, \quad k > 0$

13. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

14. $x^2\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

15. $u_{xx} + u_{yy} = u$

16. $a^2 u_{xx} - g = u_{tt}, \quad g$ a constant

In Problems 17–26 classify the given partial differential equation as hyperbolic, parabolic, or elliptic.

17. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$

18. $3\dfrac{\partial^2 u}{\partial x^2} + 5\dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$

19. $\dfrac{\partial^2 u}{\partial x^2} + 6\dfrac{\partial^2 u}{\partial x\,\partial y} + 9\dfrac{\partial^2 u}{\partial y^2} = 0$

20. $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial x\,\partial y} - 3\dfrac{\partial^2 u}{\partial y^2} = 0$

21. $\dfrac{\partial^2 u}{\partial x^2} = 9\dfrac{\partial^2 u}{\partial x\,\partial y}$

22. $\dfrac{\partial^2 u}{\partial x\,\partial y} - \dfrac{\partial^2 u}{\partial y^2} + 2\dfrac{\partial u}{\partial x} = 0$

23. $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial u}{\partial x} - 6\dfrac{\partial u}{\partial y} = 0$

24. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = u$

25. $a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

26. $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad k > 0$

27. Show that the equation

$$k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial u}{\partial t}$$

possesses the product solution

$$u = e^{-k\lambda^2 t}(AJ_0(\lambda r) + BY_0(\lambda r)).$$

28. **(a)** Show that the equation

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

can be put into the form $\partial^2 u/\partial\eta\,\partial\xi = 0$ by means of the substitutions $\xi = x + at$, $\eta = x - at$.

(b) Show that a solution of the equation is

$$u = F(x + at) + G(x - at),$$

where F and G are arbitrary, twice-differentiable functions.

SECTION 12.3 EXERCISES

Answers to odd-numbered problems begin on page A-23.

In Problems 1 and 2 solve the heat equation (1) subject to the given conditions. Assume a rod of length L.

1. $u(0, t) = 0, \quad u(L, t) = 0$

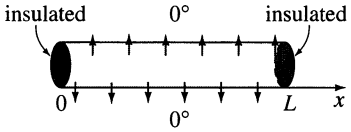
$$u(x,0) = \begin{cases} 1, & 0 < x < L/2 \\ 0, & L/2 < x < L \end{cases}$$

2. $u(0, t) = 0, \quad u(L, t) = 0$

$u(x, 0) = x(L - x)$

3. Find the temperature $u(x, t)$ in a rod of length L if the initial temperature is $f(x)$ throughout and if the ends $x = 0$ and $x = L$ are insulated.

4. Solve Problem 3 if $L = 2$ and $f(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & 1 < x < 2. \end{cases}$



heat transfer from lateral surface of the rod

FIGURE 12.7

5. Suppose heat is lost from the lateral surface of a thin rod of length L into a surrounding medium at temperature zero. If the linear law of heat transfer applies, then the heat equation takes on the form $k\partial^2u/\partial x^2 - hu = \partial u/\partial t$, $0 < x < L$, $t > 0$, h a constant. Find the temperature $u(x, t)$ if the initial temperature is $f(x)$ throughout and the ends $x = 0$ and $x = L$ are insulated. See Figure 12.7.

6. Solve Problem 5 if the ends $x = 0$ and $x = L$ are held at temperature zero.

12.4 WAVE EQUATION

▪ *Solution of a boundary-value problem by separation of variables* ▪ *Standing waves* ▪ *Normal modes* ▪ *First normal mode* ▪ *Fundamental frequency* ▪ *Overtones*

We are now in a position to solve the boundary-value problem (11) of Section 12.2. The vertical displacement of $u(x, t)$ of the vibrating string of length L shown in Figure 12.2(a) is determined from

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < L, \quad t > 0 \tag{1}$$

$$u(0,t) = 0, \qquad u(L,t) = 0, \quad t > 0 \tag{2}$$

$$u(x,0) = f(x), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \quad 0 < x < L. \tag{3}$$

Separating variables in (1) gives

$$\frac{X''}{X} = \frac{T''}{a^2T} = -\lambda^2$$

so that $\qquad X'' + \lambda^2 X = 0 \quad$ and $\quad T'' + \lambda^2 a^2 T = 0$

and therefore

$$X = c_1 \cos \lambda x + c_2 \sin \lambda x$$
$$T = c_3 \cos \lambda at + c_4 \sin \lambda at.$$

SECTION 12.4 EXERCISES

Answers to odd-numbered problems begin on page A-23.

In Problems 1–8 solve the wave equation (1) subject to the given conditions.

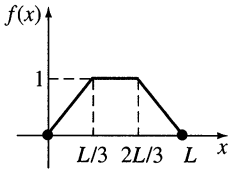


FIGURE 12.9

1. $u(0, t) = 0, \quad u(L, t) = 0$
$u(x, 0) = \frac{1}{4}x(L - x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$

2. $u(0, t) = 0, \quad u(L, t) = 0$
$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = x(L - x)$

3. $u(0, t) = 0, \quad u(L, t) = 0$
$u(x, 0)$, as specified in Figure 12.9, $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$

4. $u(0, t) = 0, \quad u(\pi, t) = 0$
$u(x, 0) = \frac{1}{6}x(\pi^2 - x^2)$,
$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$

5. $u(0, t) = 0, \quad u(\pi, t) = 0$
$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \sin x$

6. $u(0, t) = 0, \quad u(1, t) = 0$
$u(x, 0) = 0.01 \sin 3\pi x$,
$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$

7. $u(0, t) = 0, \quad u(L, t) = 0$

$$u(x, 0) = \begin{cases} \dfrac{2hx}{L}, & 0 < x < \dfrac{L}{2} \\ 2h\left(1 - \dfrac{x}{L}\right), & \dfrac{L}{2} \le x < L \end{cases}, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

The constant h is positive but small compared to L. This is referred to as the "plucked string" problem.

8. $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$
$u(x, 0) = x, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$

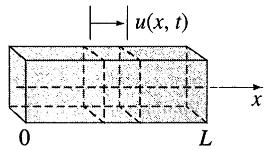


FIGURE 12.10

This problem could describe the longitudinal displacement $u(x, t)$ of a vibrating elastic bar. The boundary conditions at $x = 0$ and $x = L$ are called **free-end conditions.** See Figure 12.10.

9. A string is stretched and secured on the x-axis at $x = 0$ and $x = \pi$ for $t > 0$. If the transverse vibrations take place in a medium that imparts a resistance proportional to the instantaneous velocity, then the wave equation takes on the form

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + 2\beta\frac{\partial u}{\partial t}, \quad 0 < \beta < 1, \quad t > 0.$$

Find the displacement $u(x, t)$ if the string starts from rest from the initial displacement $f(x)$.

10. Show that a solution of the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + u, \quad 0 < x < \pi, \quad t > 0$$
$$u(0, t) = 0, \qquad u(\pi, t) = 0, \quad t > 0$$
$$u(x, 0) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 \le x < \pi \end{cases}$$
$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < \pi$$

is $\quad u(x, t) = \dfrac{4}{\pi}\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^2} \sin(2k-1)x \cos\sqrt{(2k-1)^2 + 1}\, t.$

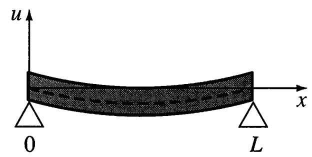


FIGURE 12.11
Simply supported beam

11. The transverse displacement $u(x, t)$ of a vibrating beam of length L is determined from a fourth-order partial differential equation

$$a^2 \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0, \quad 0 < x < L, \quad t > 0.$$

If the beam is **simply supported,** as shown in Figure 12.11, the boundary and initial conditions are

$$u(0, t) = 0, \qquad u(L, t) = 0, \quad t > 0$$
$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0} = 0, \qquad \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=L} = 0, \quad t > 0$$
$$u(x, 0) = f(x), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \quad 0 < x < L.$$

Solve for $u(x, t)$. [*Hint*: For convenience use λ^4 instead of λ^2 when separating variables.]

12. What are the boundary conditions when the ends of the beam in Problem 11 are embedded at $x = 0$ and $x = L$?

13. Consider the boundary-value problem given in (1), (2), and (3) of this section. If $g(x) = 0$ on $0 < x < L$, show that the solution of the problem can be written as

$$u(x, t) = \frac{1}{2}[f(x + at) + f(x - at)].$$

[*Hint:* Use the identity $2 \sin \theta_1 \cos \theta_2 = \sin(\theta_1 + \theta_2) + \sin(\theta_1 - \theta_2)$.]

14. The vertical displacement $u(x, t)$ of an infinitely long string is determined from the initial-value problem

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad -\infty < x < \infty, \quad t > 0$$
$$u(x, 0) = f(x), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x). \tag{9}$$

This problem can be solved without separating variables.

(a) Recall from Problem 28 of Exercises 12.1 that the wave equation can be put into the form $\partial^2 u/\partial\eta\partial\xi = 0$ by means of the substitutions $\xi = x + at$ and $\eta = x - at$. Integrating the last partial differential equation with respect to η and then with respect to ξ shows that $u(x, t) = F(x + at) + G(x - at)$, where F and G are arbitrary twice-differentiable functions, is a solution of the wave equation. Use this solution and the given initial conditions to show that

$$F(x) = \frac{1}{2}f(x) + \frac{1}{2a}\int_{x_0}^{x} g(s)\, ds + c$$

and $$G(x) = \frac{1}{2}f(x) - \frac{1}{2a}\int_{x_0}^{x} g(s)\, ds - c,$$

where x_0 is arbitrary and c is a constant of integration.

(b) Use the results in part (a) to show that

$$u(x, t) = \frac{1}{2}[f(x + at) + f(x - at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\, ds. \qquad \textbf{(10)}$$

Note that when the initial velocity $g(x) = 0$ we obtain

$$u(x, t) = \frac{1}{2}[f(x + at) + f(x - at)], \quad -\infty < x < \infty.$$

The last solution can be interpreted as a superposition of two **traveling waves**, one moving to the right (that is, $\frac{1}{2}f(x - at)$) and one moving to the left ($\frac{1}{2}f(x + at)$). Both waves travel with speed a and have the same basic shape as the initial displacement $f(x)$. The form of $u(x, t)$ given in (10) is called **d'Alembert's solution**.

In Problems 15–17 use d'Alembert's solution (10) to solve the initial-value problem in Problem 14 subject to the given initial conditions.

15. $f(x) = \sin x, \quad g(x) = 1$

16. $f(x) = \sin x, \quad g(x) = \cos x$

17. $f(x) = 0, \quad g(x) = \sin 2x$

18. Suppose $f(x) = 1/(1 + x^2)$, $g(x) = 0$, and $a = 1$ for the initial-value problem stated in Problem 14. Graph d'Alembert's solution in this case at the times $t = 0$, $t = 1$, and $t = 3$.

19. A model for an infinitely long string that is initially held at the three points $(-1, 0)$, $(1, 0)$, and $(0, 1)$ and then simultaneously released at all three points at time $t = 0$ is given by (9) with

$$f(x) = \begin{cases} 1 - |x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases} \quad \text{and} \quad g(x) = 0.$$

(a) Plot the initial position of the string on the interval $[-6, 6]$.

(b) Use a computer algebra system to plot d'Alembert's solution (10) on $[-6, 6]$ for $t = 0.2k$, $k = 0, 1, 2, \ldots, 25$.

(c) Use the animation feature of your computer algebra system to make a movie of the solution. Describe the motion of the string over time.

20. An infinitely long string coinciding with the x-axis is struck at the origin with a hammer whose head is 0.2 inch in diameter. A model where $A_n = \frac{2}{b}\int_0^b F(y) \sin \frac{n\pi}{b} y \, dy$

$$B_n = \frac{1}{\sinh \frac{n\pi}{b} a}\left(\frac{2}{b}\int_0^b G(y) \sin \frac{n\pi}{b} y \, dy - A_n \cosh \frac{n\pi}{b} a\right).$$

SECTION 12.5 EXERCISES

Answers to odd-numbered problems begin on page A-24.

In Problems 1–8 find the steady-state temperature for a rectangular plate with boundary conditions as given.

1. $u(0, y) = 0, \quad u(a, y) = 0$
$u(x, 0) = 0, \quad u(x, b) = f(x)$

2. $u(0, y) = 0, \quad u(a, y) = 0$
$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \quad u(x, b) = f(x)$

3. $u(0, y) = 0, \quad u(a, y) = 0$
$u(x, 0) = f(x), \quad u(x, b) = 0$

4. $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0$
$u(x, 0) = x, \quad u(x, b) = 0$

5. $u(0, y) = 0, \quad u(1, y) = 1 - y$
$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \quad \left.\frac{\partial u}{\partial y}\right|_{y=1} = 0$

6. $u(0, y) = g(y), \quad \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0$
$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \quad \left.\frac{\partial u}{\partial y}\right|_{y=\pi} = 0$

7. $\left.\frac{\partial u}{\partial x}\right|_{x=0} = u(0, y), \quad u(\pi, y) = 1$
$u(x, 0) = 0, \quad u(x, \pi) = 0$

8. $u(0, y) = 0, \quad u(1, y) = 0$
$\left.\frac{\partial u}{\partial y}\right|_{y=0} = u(x, 0), \quad u(x, 1) = f(x)$

In Problems 9 and 10 find the steady-state temperature in the given semi-infinite plate extending in the positive y-direction. In each case assume $u(x, y)$ is bounded as $y \to \infty$.

9.

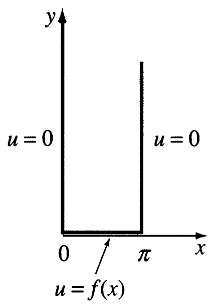


FIGURE 12.14

10.

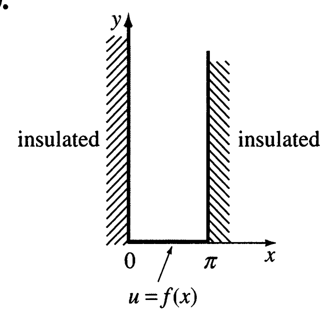


FIGURE 12.15

In Problems 11 and 12 find the steady-state temperature for a rectangular plate with boundary conditions as given.

11. $u(0, y) = 0, \quad u(a, y) = 0$
$u(x, 0) = f(x), \quad u(x, b) = g(x)$

12. $u(0, y) = F(y), \quad u(a, y) = G(y)$
$u(x, 0) = 0, \quad u(x, b) = 0$

In Problems 13 and 14 use the superposition principle to find the steady-state temperature for a square plate with boundary conditions as given.

13. $u(0, y) = 1, \quad u(\pi, y) = 1$
$u(x, 0) = 0, \quad u(x, \pi) = 1$

14. $u(0, y) = 0, \quad u(2, y) = y(2 - y)$
$u(x, 0) = 0, \quad u(x, 2) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \le x < 2 \end{cases}$

12.6 NONHOMOGENEOUS EQUATIONS AND BOUNDARY CONDITIONS

■ *Use of a change of dependent variable* ■ *Steady-state solution* ■ *Transient solution*

The method of separation of variables may not be applicable to a boundary-value problem when the partial differential equation or boundary conditions are nonhomogeneous. For example, when heat is generated at a constant rate r within a rod of finite length, the form of the heat equation is

$$k\frac{\partial^2 u}{\partial x^2} + r = \frac{\partial u}{\partial t}. \tag{1}$$

Equation (1) is nonhomogeneous and is readily shown to be not separable. On the other hand, suppose we wish to solve the usual heat equation $ku_{xx} = u_t$ when the boundaries $x = 0$ and $x = L$ are held at nonzero temperatures k_1 and k_2. Even though the assumption $u(x, t) = X(x)T(t)$ separates the partial differential equation, we quickly find ourselves at an impasse in determining eigenvalues and eigenfunctions since no conclusions can be obtained from $u(0, t) = X(0)T(t) = k_1$ and $u(L, t) = X(L)T(t) = k_2$.

A few problems involving nonhomogeneous equations or nonhomogeneous boundary conditions can be solved by means of a change of dependent variable:

$$u = v + \psi.$$

The basic idea is to determine ψ, a function of *one* variable, in such a manner that v, a function of *two* variables, is made to satisfy a homogeneous partial differential equation and homogeneous boundary conditions. The following example illustrates the procedure.

EXAMPLE 1 Nonhomogeneous Boundary Condition

Solve (1) subject to

$$u(0,t) = 0, \quad u(1,t) = u_0, \quad t > 0$$
$$u(x,0) = f(x), \quad 0 < x < 1.$$

Observe in (5) that $u(x, t) \to \psi(x)$ as $t \to \infty$. In the context of solving forms of the heat equation, ψ is called a **steady-state solution.** Since $v(x, t) \to 0$ as $t \to \infty$, v is called a **transient solution.**

The substitution $u = v + \psi$ can also be used on problems involving forms of the wave equation as well as Laplace's equation.

SECTION 12.6 EXERCISES

Answers to odd-numbered problems begin on page A-24.

In Problems 1 and 2 solve the heat equation $ku_{xx} = u_t$, $0 < x < 1$, $t > 0$ subject to the given conditions.

1. $u(0, t) = 100, \quad u(1, t) = 100$
$u(x, 0) = 0$

2. $u(0, t) = u_0, \quad u(1, t) = 0$
$u(x, 0) = f(x)$

In Problems 3 and 4 solve the partial differential equation (1) subject to the given conditions.

3. $u(0, t) = u_0, \quad u(1, t) = u_0$
$u(x, 0) = 0$

4. $u(0, t) = u_0, \quad u(1, t) = u_1$
$u(x, 0) = f(x)$

5. Solve the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} + Ae^{-\beta x} = \frac{\partial u}{\partial t}, \quad \beta > 0, \quad 0 < x < 1, \quad t > 0$$
$$u(0, t) = 0, \qquad u(1, t) = 0, \quad t > 0$$
$$u(x, 0) = f(x), \quad 0 < x < 1.$$

The partial differential equation is a form of the heat equation when heat is generated within a thin rod from radioactive decay of the material.

6. Solve the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} - hu = \frac{\partial u}{\partial t}, \quad 0 < x < \pi, \quad t > 0$$
$$u(0, t) = 0, \qquad u(\pi, t) = u_0, \quad t > 0$$
$$u(x, 0) = 0, \quad 0 < x < \pi.$$

7. Find a steady-state solution $\psi(x)$ of the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} - h(u - u_0) = \frac{\partial u}{\partial t}, \quad 0 < x < 1, \quad t > 0$$
$$u(0, t) = u_0, \qquad u(1, t) = 0, \quad t > 0$$
$$u(x, 0) = f(x), \quad 0 < x < 1.$$

8. Find a steady-state solution $\psi(x)$ if the rod in Problem 7 is semi-infinite extending in the positive x-direction, radiates from its lateral surface into a medium at temperature zero, and

$$u(0, t) = u_0, \qquad \lim_{x \to \infty} u(x, t) = 0, \quad t > 0$$
$$u(x, 0) = f(x), \quad x > 0.$$

9. When a vibrating string is subjected to an external vertical force that varies with the horizontal distance from the left end, the wave equation takes on the form

$$a^2\frac{\partial^2 u}{\partial x^2} + Ax = \frac{\partial^2 u}{\partial t^2}.$$

Solve this partial differential equation subject to

$$u(0,t) = 0, \qquad u(1,t) = 0 \quad t > 0$$
$$u(x,0) = 0, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \quad 0 < x < 1.$$

10. A string initially at rest on the x-axis is secured on the x-axis at $x = 0$ and $x = 1$. If the string is allowed to fall under its own weight for $t > 0$, the displacement $u(x, t)$ satisfies

$$a^2\frac{\partial^2 u}{\partial x^2} - g = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < 1, \quad t > 0,$$

where g is the acceleration of gravity. Solve for $u(x, t)$.

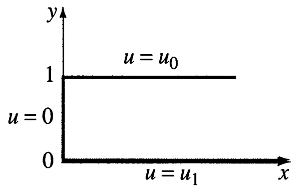


FIGURE 12.16

11. Find the steady-state temperature $u(x, y)$ in the semi-infinite plate shown in Figure 12.16. Assume that the temperature is bounded as $x \to \infty$. [*Hint:* Try $u(x, y) = v(x, y) + \psi(y)$.]

12. Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -h, \quad h > 0$$

occurs in many problems involving electric potential. Solve the above equation subject to the conditions

$$u(0,y) = 0, \qquad u(\pi,y) = 1, \quad y > 0$$
$$u(x,0) = 0, \quad 0 < x < \pi.$$

12.7 USE OF GENERALIZED FOURIER SERIES

▪ *Boundary-value problems that do not lead to Fourier series* ▪ *Using generalized Fourier series*

For certain types of boundary conditions, the method of separation of variables and the superposition principle lead to an expansion of a function in a trigonometric series that is not a Fourier series. To solve the problems in this section we shall utilize the concept of generalized Fourier series developed in Section 11.1.

EXAMPLE 1 **Using Generalized Fourier Series**

The temperature in a rod of unit length in which there is heat transfer from its right boundary into a surrounding medium kept at a constant

As in Example 1, the set of eigenfunctions $\left\{\sin\left(\frac{2n-1}{2}\right)\pi x\right\}$, $n = 1, 2, 3, \ldots$, is orthogonal with respect to the weight function $p(x) = 1$ on the interval $[0, 1]$. The series $\sum_{n=1}^{\infty} A_n \sin\left(\frac{2n-1}{2}\right)\pi x$ is not a Fourier sine series since the argument of the sine is not an integer multiple of $\pi x/L$ ($L = 1$ in this case). The series is again a generalized Fourier series. Hence from (8) of Section 11.1 the coefficients in (7) are

$$A_n = \frac{\int_0^1 x \sin\left(\frac{2n-1}{2}\right)\pi x\, dx}{\int_0^1 \sin^2\left(\frac{2n-1}{2}\right)\pi x\, dx}.$$

Carrying out the two integrations, we arrive at

$$A_n = \frac{8(-1)^{n+1}}{(2n-1)^2\pi^2}.$$

The twist angle is then

$$\theta(x, t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\cos a\left(\frac{2n-1}{2}\right)\pi t \sin\left(\frac{2n-1}{2}\right)\pi x. \quad \blacksquare$$

SECTION 12.7 EXERCISES

Answers to odd-numbered problems begin on page A-24.

1. In Example 1 find the temperature $u(x, t)$ when the left end of the rod is insulated.
2. Solve the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < 1, \quad t > 0$$

$$u(0, t) = 0, \qquad \left.\frac{\partial u}{\partial x}\right|_{x=1} = -h(u(1, t) - u_0), \quad h > 0, \quad t > 0$$

$$u(x, 0) = f(x), \quad 0 < x < 1.$$

3. Find the steady-state temperature for a rectangular plate for which the boundary conditions are

$$u(0, y) = 0, \qquad \left.\frac{\partial u}{\partial x}\right|_{x=a} = -hu(a, y), \qquad 0 < y < b$$

$$u(x, 0) = 0, \qquad u(x, b) = f(x), \qquad 0 < x < a.$$

4. Solve the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < y < 1, \quad x > 0$$

$$u(0, y) = u_0, \qquad \lim_{x\to\infty} u(x, y) = 0, \qquad 0 < y < 1$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \qquad \left.\frac{\partial u}{\partial y}\right|_{y=1} = -hu(x, 1), \quad h > 0, \quad x > 0.$$

5. Find the temperature $u(x, t)$ in a rod of length L if the initial temperature is $f(x)$ throughout and if the end $x = 0$ is kept at temperature zero and the end $x = L$ is insulated.

6. Solve the boundary-value problem

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < L, \qquad t > 0$$

$$u(0, t) = 0, \qquad E\left.\frac{\partial u}{\partial x}\right|_{x=L} = F_0, \quad t > 0$$

$$u(x, 0) = 0, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \qquad 0 < x < L.$$

The solution $u(x, t)$ represents the longitudinal displacement of a vibrating elastic bar that is anchored at its left end and is subjected to a constant force of magnitude F_0 at its right end. See Figure 12.10 on page 492. E is a constant called the modulus of elasticity.

7. Solve the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 1, \quad 0 < y < 1$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \qquad u(1, y) = u_0, \quad 0 < y < 1$$

$$u(x, 0) = 0, \qquad \left.\frac{\partial u}{\partial y}\right|_{y=1} = 0, \qquad 0 < x < 1.$$

8. The initial temperature in a rod of unit length is $f(x)$ throughout. There is heat transfer from both ends, $x = 0$ and $x = 1$, into a surrounding medium kept at a constant temperature zero. Show that

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-k\lambda_n^2 t}(\lambda_n \cos \lambda_n x + h \sin \lambda_n x),$$

where $A_n = \dfrac{2}{(\lambda_n^2 + 2h + h^2)} \displaystyle\int_0^1 f(x)(\lambda_n \cos \lambda_n x + h \sin \lambda_n x)\, dx$

and the λ_n, $n = 1, 2, 3, \ldots$, are the consecutive positive roots of $\tan \lambda = 2\lambda h/(\lambda^2 - h^2)$.

9. A vibrating cantilever beam is embedded at its left end ($x = 0$) and free at its right end ($x = 1$). See Figure 12.18. The transverse displacement $u(x, t)$ of the beam is determined from the boundary-value problem

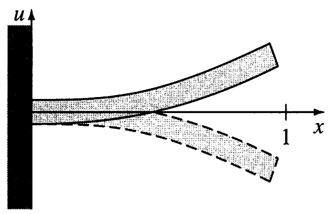


FIGURE 12.18

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0, \quad 0 < x < 1, \quad t > 0$$

$$u(0, t) = 0, \qquad \left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \qquad t > 0$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=1} = 0, \qquad \left.\frac{\partial^3 u}{\partial x^3}\right|_{x=1} = 0, \qquad t > 0$$

$$u(x, 0) = f(x), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \quad x > 0.$$

CHAPTER 12 REVIEW EXERCISES

Answers to odd-numbered problems begin on page A-25.

1. Use separation of variables to find product solutions of

$$\frac{\partial^2 u}{\partial x\,\partial y} = u.$$

2. Use separation of variables to find product solutions of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0.$$

Is it possible to choose a separation constant so that both X and Y are oscillatory functions?

3. Find a steady-state solution $\psi(x)$ of the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < \pi, \quad t > 0$$

$$u(0, t) = u_0, \qquad -\left.\frac{\partial u}{\partial x}\right|_{x=\pi} = u(\pi, t) - u_1, \quad t > 0$$

$$u(x, 0) = 0, \quad 0 < x < \pi.$$

4. Give a physical interpretation for the boundary conditions in Problem 3.

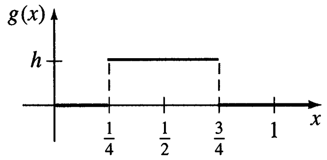


FIGURE 12.22

5. At $t = 0$ a string of unit length is stretched on the positive x-axis. The ends of the string $x = 0$ and $x = 1$ are secured on the x-axis for $t > 0$. Find the displacement $u(x, t)$ if the initial velocity $g(x)$ is as given in Figure 12.22.

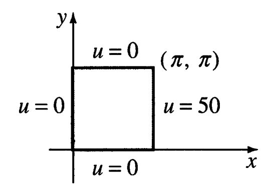


FIGURE 12.23

6. The partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + x^2 = \frac{\partial^2 u}{\partial t^2}$$

is a form of the wave equation when an external vertical force proportional to the square of the horizontal distance from the left end is applied to the string. The string is secured at $x = 0$ one unit above the x-axis and on the x-axis at $x = 1$ for $t > 0$. Find the displacement $u(x, t)$ if the string starts from rest from the initial displacement $f(x)$.

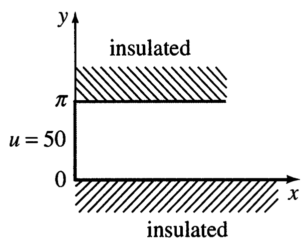


FIGURE 12.24

7. Find the steady-state temperature $u(x, y)$ in the square plate shown in Figure 12.23.

8. Find the steady-state temperature $u(x, y)$ in the semi-infinite plate shown in Figure 12.24.

9. Solve Problem 8 if the boundaries $y = 0$ and $y = \pi$ are held at temperature zero for all time.

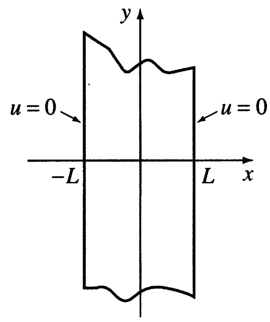


FIGURE 12.25

10. Find the temperature $u(x, t)$ in the infinite plate of width $2L$ shown in Figure 12.25 if the initial temperature is u_0 throughout. [*Hint:* $u(x, 0) = u_0,\ -L < x < L$ is an even function of x.]

11. Solve the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < \pi, \quad t > 0$$
$$u(0, t) = 0, \qquad u(\pi, t) = 0, \quad t > 0$$
$$u(x, 0) = \sin x, \quad 0 < x < \pi.$$

12. Solve the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} + \sin 2\pi x = \frac{\partial u}{\partial t}, \quad 0 < x < 1, \quad t > 0$$
$$u(0, t) = 0, \qquad u(1, t) = 0, \quad t > 0$$
$$u(x, 0) = \sin \pi x, \quad 0 < x < 1.$$

13. Find a formal series solution of the problem

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} + u, \quad 0 < x < \pi, \quad t > 0$$
$$u(0, t) = 0, \qquad u(\pi, t) = 0, \quad t > 0$$
$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < \pi.$$

Do not attempt to evaluate the coefficients in the series.

14. The concentration $c(x, t)$ of a substance that both diffuses in a medium and is convected by currents in the medium satisfies the partial differential equation

$$k\frac{\partial^2 c}{\partial x^2} - h\frac{\partial c}{\partial x} = \frac{\partial c}{\partial t}, \quad h \text{ a constant.}$$

Solve the equation subject to

$$c(0, t) = 0, \qquad c(1, t) = 0, \quad t > 0$$
$$c(x, 0) = c_0, \quad 0 < x < 1,$$

where c_0 is a constant.

Applying the boundary conditions $\Theta(0) = 0$ and $\Theta(\pi) = 0$ to the solution $\Theta = c_1 \cos \lambda\theta + c_2 \sin \lambda\theta$ of (16) gives, in turn, $c_1 = 0$ and $\lambda = n$, $n = 1, 2, 3, \ldots$. Hence $\Theta = c_2 \sin n\theta$. In this problem, unlike the problem in Example 1, $n = 0$ is not an eigenvalue. With $\lambda = n$ the solution of (15) is $R = c_3 r^n + c_4 r^{-n}$. But the assumption that $u(r, \theta)$ is bounded at $r = 0$ prompts us to define $c_4 = 0$. Therefore $u_n = R(r)\Theta(\theta) = A_n r^n \sin n\theta$ and

$$u(r, \theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta.$$

The remaining boundary condition at $r = c$ gives the sine series

$$u_0 = \sum_{n=1}^{\infty} A_n c^n \sin n\theta.$$

Consequently $$A_n c^n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin n\theta \, d\theta,$$

and so $$A_n = \frac{2u_0}{\pi c^n}\frac{1 - (-1)^n}{n}.$$

Hence the solution of the problem is given by

$$u(r, \theta) = \frac{2u_0}{\pi}\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n}\left(\frac{r}{c}\right)^n \sin n\theta. \quad \blacksquare$$

SECTION 13.1 EXERCISES

Answers to odd-numbered problems begin on page A-25.

In Problems 1–4 find the steady-state temperature $u(r, \theta)$ in a circular plate of radius $r = 1$ if the temperature on the circumference is as given.

1. $u(1, \theta) = \begin{cases} u_0, & 0 < \theta < \pi \\ 0, & \pi < \theta < 2\pi \end{cases}$ **2.** $u(1, \theta) = \begin{cases} \theta, & 0 < \theta < \pi \\ \pi - \theta, & \pi < \theta < 2\pi \end{cases}$

3. $u(1, \theta) = 2\pi\theta - \theta^2, \ 0 < \theta < 2\pi$ **4.** $u(1, \theta) = \theta, \ 0 < \theta < 2\pi$

5. Solve the exterior Dirichlet problem for a circular disk of radius c if $u(c, \theta) = f(\theta)$, $0 < \theta < 2\pi$. In other words, find the steady-state temperature $u(r, \theta)$ in a plate that coincides with the entire xy-plane in which a circular hole of radius c has been cut out around the origin and the temperature on the circumference of the hole is $f(\theta)$. [*Hint:* Assume that the temperature is bounded as $r \to \infty$.]

6. Find the steady-state temperature in the quarter-circular plate shown in Figure 13.4.

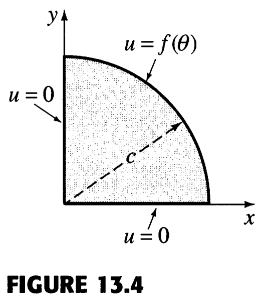


FIGURE 13.4

7. If the boundaries $\theta = 0$ and $\theta = \pi/2$ in Figure 13.4 are insulated, we then have, respectively

$$\left.\frac{\partial u}{\partial \theta}\right|_{\theta=0} = 0, \qquad \left.\frac{\partial u}{\partial \theta}\right|_{\theta=\pi/2} = 0.$$

Find the steady-state temperature if $u(c, \theta) = \begin{cases} 1, & 0 < \theta < \pi/4 \\ 0, & \pi/4 < \theta < \pi/2. \end{cases}$

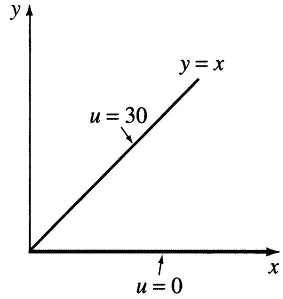


FIGURE 13.5

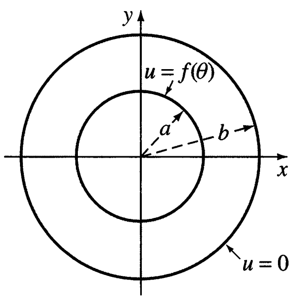


FIGURE 13.6

8. Find the steady-state temperature in the infinite wedge-shaped plate shown in Figure 13.5. [*Hint:* Assume that the temperature is bounded as $r \to 0$ and as $r \to \infty$.]

9. Find the steady-state temperature $u(r, \theta)$ in the circular ring shown in Figure 13.6. [*Hint:* Proceed as in Example 1.]

10. If the boundary conditions for the circular ring in Figure 13.6 are $u(a, \theta) = u_0$, $u(b, \theta) = u_1$, $0 < \theta < 2\pi$, u_0 and u_1 constants, show that the steady-state temperature is given by

$$u(r, \theta) = \frac{u_0 \ln(r/b) - u_1 \ln(r/a)}{\ln(a/b)}.$$

[*Hint:* Try a solution of the form $u(r, \theta) = v(r, \theta) + \psi(r)$.]

11. Find the steady-state temperature $u(r, \theta)$ in a semicircular ring if

$$\begin{aligned} u(a, \theta) &= \theta(\pi - \theta), \quad & u(b, \theta) &= 0, \quad 0 < \theta < \pi \\ u(r, 0) &= 0, & u(r, \pi) &= 0, \quad a < r < b. \end{aligned}$$

12. Find the steady-state temperature $u(r, \theta)$ in a semicircular plate of radius $r = 1$ if

$$\begin{aligned} &u(1, \theta) = u_0, \quad 0 < \theta < \pi \\ &u(r, 0) = 0, \quad u(r, \pi) = u_0, \quad 0 < r < 1, \end{aligned}$$

u_0 a constant.

13. Find the steady-state temperature $u(r, \theta)$ in a semicircular plate of radius $r = 2$ if

$$u(2, \theta) = \begin{cases} u_0, & 0 < \theta < \pi/2 \\ 0, & \pi/2 < \theta < \pi, \end{cases}$$

u_0 a constant, and the edges $\theta = 0$ and $\theta = \pi$ are insulated.

14. Show that the steady-state temperature in Example 1 can be written as the integral

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{c^2 - r^2}{c^2 - 2cr\cos(t - \theta) + r^2} f(t)\, dt.$$

This result is known as **Poisson's integral formula for a circle.** [*Hint:* First show that

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \left[1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{c} \right)^n \cos n(t - \theta) \right] dt.$$

Then use $\cos nv = \frac{1}{2}(e^{inv} + e^{-inv})$ and geometric series to show that

$$1 + 2 \sum_{n=1}^{\infty} u^n \cos nv = \frac{1 - u^2}{1 - 2u \cos v + u^2}, \quad |u| < 1.]$$

Discussion Problem

15. Consider the circular ring shown in Figure 13.6. Discuss how the steady-state temperature $u(r, \theta)$ can be found when the boundary conditions are $u(a, \theta) = f(\theta)$, $u(b, \theta) = g(\theta)$, $0 \le \theta \le 2\pi$.

The remaining boundary condition at $z = 4$ then gives the Fourier-Bessel series

$$u_0 = \sum_{n=1}^{\infty} A_n \sinh 4\lambda_n J_0(\lambda_n r),$$

so that in view of (12) the coefficients are defined by (16) of Section 11.5,

$$A_n \sinh 4\lambda_n = \frac{2u_0}{2^2 J_1^2(2\lambda_n)} \int_0^2 r J_0(\lambda_n r)\, dr.$$

To evaluate the last integral we first use the substitution $t = \lambda_n r$, followed by $\frac{d}{dt}[tJ_1(t)] = tJ_0(t)$:

$$A_n \sinh 4\lambda_n = \frac{u_0}{2\lambda_n^2 J_1^2(2\lambda_n)} \int_0^{2\lambda_n} \frac{d}{dt}[tJ_1(t)]\, dt = \frac{u_0}{\lambda_n J_1(2\lambda_n)}.$$

Finally we arrive at $\quad A_n = \dfrac{u_0}{\lambda_n \sinh 4\lambda_n J_1(2\lambda_n)}.$

Thus the temperature in the cylinder is given by

$$u(r, z) = u_0 \sum_{n=1}^{\infty} \frac{\sinh \lambda_n z J_0(\lambda_n r)}{\lambda_n \sinh 4\lambda_n J_1(2\lambda_n)}.$$ ■

SECTION 13.2 EXERCISES

Answers to odd-numbered problems begin on page A-25.

1. Find the displacement $u(r, t)$ in Example 1 if $f(r) = 0$ and the circular membrane is given an initial unit velocity in the upward direction.
2. A circular membrane of unit radius 1 is clamped along its circumference. Find the displacement $u(r, t)$ if the membrane starts from rest from the initial displacement $f(r) = 1 - r^2$, $0 < r < 1$. [*Hint:* See Problem 10 in Exercises 11.5.]
3. Find steady-state temperature $u(r, z)$ in the cylinder in Example 2 if the boundary conditions are $u(2, z) = 0$, $0 < z < 4$, $u(r, 0) = u_0$, $u(r, 4) = 0$, $0 < r < 2$.
4. If the lateral side of the cylinder in Example 2 is insulated, then

$$\left.\frac{\partial u}{\partial r}\right|_{r=2} = 0, \quad 0 < z < 4.$$

   (a) Find the steady-state temperature $u(r, z)$ when $u(r, 4) = f(r)$, $0 < r < 2$.

   (b) Show that the steady-state temperature in part (a) reduces to $u(r, z) = u_0 z/4$ when $f(r) = u_0$. [*Hint:* Use (11) of Section 11.5.]

5. The temperature in a circular plate of radius c is determined from the boundary-value problem

$$k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial u}{\partial t}, \quad 0 < r < c, \quad t > 0$$
$$u(c, t) = 0, \quad t > 0$$
$$u(r, 0) = f(r), \quad 0 < r < c.$$

Solve for $u(r, t)$.

6. Solve Problem 5 if the edge $r = c$ of the plate is insulated.

7. When there is heat transfer from the lateral side of an infinite circular cylinder of unit radius (see Figure 13.11) into a surrounding medium at temperature zero, the temperature inside the cylinder is determined from

$$k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial u}{\partial t}, \quad 0 < r < 1, \quad t > 0$$
$$\left.\frac{\partial u}{\partial r}\right|_{r=1} = -hu(1, t), \quad h > 0, \quad t > 0$$
$$u(r, 0) = f(r), \quad 0 < r < 1.$$

Solve for $u(r, t)$.

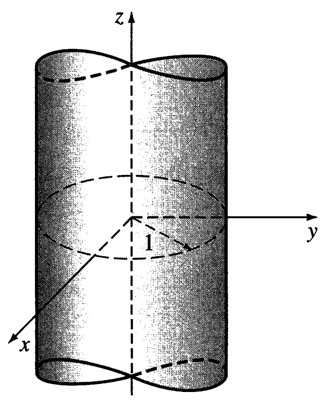


FIGURE 13.11

8. Find the steady-state temperature $u(r, z)$ in a semi-infinite cylinder of unit radius ($z \geq 0$) if there is heat transfer from its lateral side into a surrounding medium at temperature zero and if the temperature of the base $z = 0$ is held at a constant temperature u_0.

9. A circular plate is a composite of two different materials in the form of concentric circles. See Figure 13.12. The temperature in the plate is determined from the boundary-value problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial u}{\partial t}, \quad 0 < r < 2, \quad t > 0$$
$$u(2, t) = 100, \quad t > 0$$
$$u(r, 0) = \begin{cases} 200, & 0 < r < 1 \\ 100, & 1 < r < 2. \end{cases}$$

Solve for $u(r, t)$. [*Hint:* Let $u(r, t) = v(r, t) + \psi(r)$.]

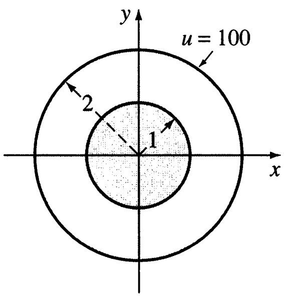


FIGURE 13.12

10. Solve the boundary-value problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \beta = \frac{\partial u}{\partial t}, \quad 0 < r < 1, \quad t > 0, \quad \beta \text{ a constant}$$
$$u(1, t) = 0, \quad t > 0$$
$$u(r, 0) = 0, \quad 0 < r < 1.$$

11. The horizontal displacement $u(x, t)$ of a heavy chain of length L oscillating in a vertical plane satisfies the partial differential equation

$$g\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < L, \quad t > 0.$$

See Figure 13.13.

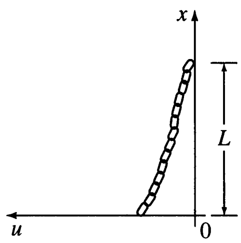


FIGURE 13.13

(a) Using $-\lambda^2$ as a separation constant, show that the ordinary differential equation in the spatial variable x is $xX'' + X' + \lambda^2 X = 0$. Solve this equation by means of the substitution $x = \tau^2/4$.

(b) Use the result of part (a) to solve the given partial differential equation subject to

$$u(L, t) = 0, \quad t > 0$$
$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < L.$$

[*Hint:* Assume the oscillations at the free end $x = 0$ are finite.]

12. In this problem we consider the general case of the vibrating circular membrane of radius c:

$$a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < c, \quad t > 0$$
$$u(c, \theta, t) = 0, \qquad 0 < \theta < 2\pi, \quad t > 0$$
$$u(r, \theta, 0) = f(r, \theta), \quad 0 < r < c, \quad 0 < \theta < 2\pi$$
$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r, \theta), \qquad 0 < r < c, \quad 0 < \theta < 2\pi.$$

(a) Assume that $u = R(r)\Theta(\theta)T(t)$ and that the separation constants are $-\lambda^2$ and $-\nu^2$. Show that the separated differential equations are

$$T'' + a^2\lambda^2 T = 0, \qquad \Theta'' + \nu^2\Theta = 0$$
$$r^2R'' + rR' + (\lambda^2 r^2 - \nu^2)R = 0.$$

(b) Solve the separated equations.

(c) Show that the eigenvalues and eigenfunctions of the problem are as follows:

Eigenvalues: $\nu = n$, $n = 0, 1, 2, \ldots$; eigenfunctions: 1, $\cos n\theta$, $\sin n\theta$.

Eigenvalues: $\lambda_{ni} = x_{ni}/c$, $i = 1, 2, \ldots$, where, for each n, x_{ni} are the positive roots of $J_n(\lambda c) = 0$; eigenfunctions: $J_n(\lambda_{ni} r)$.

(d) Use the superposition principle to determine a multiple series solution. Do not attempt to evaluate the coefficients.

13. (a) Consider Example 1 with $a = 1$, $c = 10$, $g(r) = 0$, and $f(r) = 1 - r/10$, $0 < r < 10$. Find the numerical values of the first three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the boundary-value problem and the first three coefficients A_1, A_2, A_3 of the series solution $u(r, t)$. Write the third partial sum S_3 of the series solution. [*Hint:* Use the table on page 247. Also see Problem 25 in Exercises 6.4.]

(b) Use computer software to graph S_3 for $t = 0, 4, 10, 12, 20$.

Furthermore, when $\lambda^2 = n(n + 1)$, the general solution of the Cauchy-Euler equation (2) is

$$R = c_1 r^n + c_2 r^{-(n+1)}.$$

Since we again expect $u(r, \theta)$ to be bounded at $r = 0$, we define $c_2 = 0$. Hence $u_n = A_n r^n P_n(\cos\theta)$, and

$$u(r, \theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta).$$

At $r = c$,

$$f(\theta) = \sum_{n=0}^{\infty} A_n c^n P_n(\cos\theta).$$

Therefore $A_n c^n$ are the coefficients of the Fourier-Legendre series (23) of Section 11.5:

$$A_n = \frac{2n+1}{2c^n} \int_0^{\pi} f(\theta) P_n(\cos\theta) \sin\theta \, d\theta.$$

It follows that the solution is

$$u(r, \theta) = \sum_{n=0}^{\infty} \left(\frac{2n+1}{2} \int_0^{\pi} f(\theta) P_n(\cos\theta) \sin\theta \, d\theta \right) \left(\frac{r}{c} \right)^n P_n(\cos\theta). \quad \blacksquare$$

SECTION 13.3 EXERCISES

Answers to odd-numbered problems begin on page A-26.

1. Solve the problem in Example 1 if

$$f(\theta) = \begin{cases} 50, & 0 < \theta < \pi/2 \\ 0, & \pi/2 < \theta < \pi. \end{cases}$$

   Write out the first four nonzero terms of the series solution. [*Hint:* See Example 3, Section 11.5.]
2. The solution $u(r, \theta)$ in Example 1 could also be interpreted as the potential inside the sphere due to a charge distribution $f(\theta)$ on its surface. Find the potential outside the sphere.
3. Find the solution of the problem in Example 1 if $f(\theta) = \cos\theta$, $0 < \theta < \pi$. [*Hint:* $P_1(\cos\theta) = \cos\theta$. Use orthogonality.]
4. Find the solution of the problem in Example 1 if $f(\theta) = 1 - \cos 2\theta$, $0 < \theta < \pi$. [*Hint:* See Problem 16, Exercises 11.5.]
5. Find the steady-state temperature $u(r, \theta)$ within a hollow sphere $a < r < b$ if its inner surface $r = a$ is kept at temperature $f(\theta)$ and its outer surface $r = b$ is kept at temperature zero. The sphere in the first octant is shown in Figure 13.16.

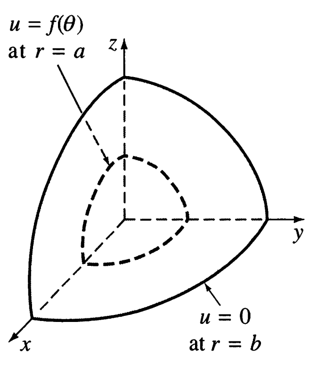


FIGURE 13.16

6. The steady-state temperature in a hemisphere of radius $r = c$ is determined from

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0, \quad 0 < r < c, \quad 0 < \theta < \frac{\pi}{2}$$

$$u\left(r, \frac{\pi}{2}\right) = 0, \qquad 0 < r < c$$

$$u(c, \theta) = f(\theta), \quad 0 < \theta < \frac{\pi}{2}.$$

Solve for $u(r, \theta)$. [*Hint:* $P_n(0) = 0$ only if n is odd. Also see Problem 18, Exercises 11.5.]

7. Solve Problem 6 when the base of the hemisphere is insulated; that is,

$$\left.\frac{\partial u}{\partial \theta}\right|_{\theta=\pi/2} = 0, \quad 0 < r < c.$$

8. Solve Problem 6 for $r > c$.

9. The time-dependent temperature within a sphere of unit radius is determined from

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{\partial u}{\partial t}, \quad 0 < r < 1, \quad t > 0$$

$$u(1, t) = 100, \quad t > 0$$

$$u(r, 0) = 0, \qquad 0 < r < 1.$$

Solve for $u(r, t)$. [*Hint:* Verify that the left side of the partial differential equation can be written as $\frac{1}{r}\frac{\partial^2}{\partial r^2}(ru)$. Let $ru(r, t) = v(r, t) + \psi(r)$. Use only functions that are bounded as $r \to 0$.]

FIGURE 13.17

10. A uniform solid sphere of radius 1 at an initial constant temperature u_0 throughout is dropped into a large container of fluid that is kept at a constant temperature u_1 ($u_1 > u_0$) for all time. See Figure 13.17. Since there is heat transfer across the boundary $r = 1$, the temperature $u(r, t)$ in the sphere is determined from the boundary-value problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{\partial u}{\partial t}, \quad 0 < r < 1, \quad t > 0$$

$$\left.\frac{\partial u}{\partial r}\right|_{r=1} = -h(u(1, t) - u_1), \quad 0 < h < 1$$

$$u(r, 0) = u_0, \qquad 0 < r < 1.$$

Solve for $u(r, t)$. [*Hint:* Proceed as in Problem 9.]

11. Solve the boundary-value problem involving spherical vibrations:

$$a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < c, \quad t > 0$$

$$u(c, t) = 0, \quad t > 0$$

$$u(r, 0) = f(r), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r), \quad 0 < r < c.$$

[*Hint:* Write the left side of the partial differential equation as $a^2 \frac{1}{r} \frac{\partial^2}{\partial r^2}(ru)$. Let $v(r, t) = ru(r, t)$.]

12. A conducting sphere of radius $r = c$ is grounded and placed in a uniform electric field that has intensity E in the z-direction. The potential $u(r, \theta)$ outside the sphere is determined from the boundary-value problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0, \quad r > c, \quad 0 < \theta < \pi$$
$$u(c, \theta) = 0, \quad 0 < \theta < \pi$$
$$\lim_{r\to\infty} u(r, \theta) = -Ez = -Er\cos\theta.$$

Show that $\qquad u(r, \theta) = -Er\cos\theta + E\frac{c^3}{r^2}\cos\theta.$

[*Hint:* Explain why $\int_0^\pi \cos\theta \, P_n(\cos\theta)\sin\theta \, d\theta = 0$ for all nonnegative integers except $n = 1$. See (24) of Section 11.5.]

CHAPTER 13 REVIEW EXERCISES

Answers to odd-numbered problems begin on page A-26.

1. Find the steady-state temperature $u(r, \theta)$ in a circular plate of radius c if the temperature on the circumference is given by

$$u(c, \theta) = \begin{cases} u_0, & 0 < \theta < \pi \\ -u_0, & \pi < \theta < 2\pi. \end{cases}$$

2. Find the steady-state temperature in the circular plate in Problem 1 if

$$u(c, \theta) = \begin{cases} 1, & 0 < \theta < \pi/2 \\ 0, & \pi/2 < \theta < 3\pi/2 \\ 1, & 3\pi/2 < \theta < 2\pi. \end{cases}$$

3. Find the steady-state temperature $u(r, \theta)$ in a semicircular plate of radius 1 if

$$u(1, \theta) = u_0(\pi\theta - \theta^2), \quad 0 < \theta < \pi$$
$$u(r, 0) = 0, \qquad u(r, \pi) = 0, \quad 0 < r < 1.$$

4. Find the steady-state temperature $u(r, \theta)$ in the semicircular plate in Problem 3 if $u(1, \theta) = \sin\theta$, $0 < \theta < \pi$.
5. Find the steady-state temperature $u(r, \theta)$ in the plate shown in Figure 13.18.
6. Find the steady-state temperature $u(r, \theta)$ in the infinite plate shown in Figure 13.19.

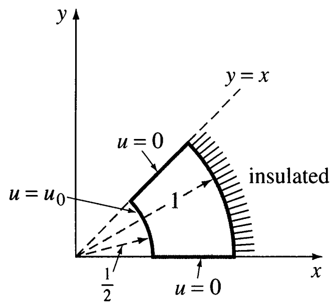


FIGURE 13.18

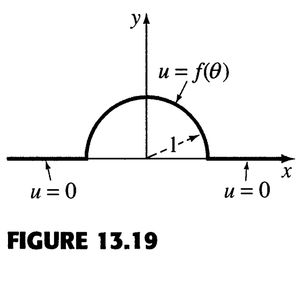


FIGURE 13.19

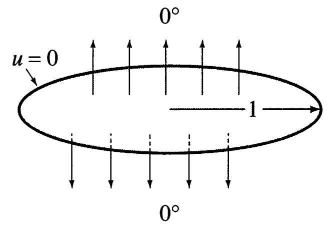


FIGURE 13.20

7. Suppose heat is lost from the flat surfaces of a very thin circular unit disk into a surrounding medium at temperature zero. If the linear law of heat transfer applies, the heat equation assumes the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - hu = \frac{\partial u}{\partial t}, \quad h > 0, \quad 0 < r < 1, \quad t > 0.$$

See Figure 13.20. Find the temperature $u(r, t)$ if the edge $r = 1$ is kept at temperature zero and if initially the temperature of the plate is unity throughout.

8. Suppose x_k is a positive zero of J_0. Show that a solution of the boundary-value problem

$$a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < 1, \quad t > 0$$
$$u(1, t) = 0, \quad t > 0$$
$$u(r, 0) = u_0 J_0(x_k r), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < r < 1$$

is $u(r, t) = u_0 J_0(x_k r) \cos a x_k t$.

9. Find the steady-state temperature $u(r, z)$ in the cylinder in Figure 13.10 if the lateral side is kept at temperature zero, the top $z = 4$ is kept at temperature 50, and the base $z = 0$ is insulated.

10. Solve the boundary-value problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 < r < 1, \quad 0 < z < 1$$
$$\left.\frac{\partial u}{\partial r}\right|_{r=1} = 0, \quad 0 < z < 1$$
$$u(r, 0) = f(r), \quad u(r, 1) = g(r), \quad 0 < r < 1.$$

11. Find the steady-state temperature $u(r, \theta)$ in a sphere of unit radius if the surface is kept at

$$u(1, \theta) = \begin{cases} 100, & 0 < \theta < \pi/2 \\ -100, & \pi/2 < \theta < \pi. \end{cases}$$

[*Hint:* See Problem 20 in Exercises 11.5.]

12. Solve the boundary-value problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < 1, \quad t > 0$$
$$\left.\frac{\partial u}{\partial r}\right|_{r=1} = 0, \quad t > 0$$
$$u(r, 0) = f(r), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r), \quad 0 < r < 1.$$

[*Hint:* Proceed as in Problems 9 and 10, Exercises 13.3, but let $v(r, t) = ru(r, t)$. See Section 12.7.]

13. The function $u(x) = Y_0(\lambda a)J_0(\lambda x) - J_0(\lambda a)Y_0(\lambda x)$, $a > 0$ is a solution of the parametric Bessel equation

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + \lambda^2x^2u = 0$$

on the interval $a \le x \le b$. If the eigenvalues λ_n are defined by the positive roots of the equation $Y_0(\lambda a)J_0(\lambda b) - J_0(\lambda a)Y_0(\lambda b) = 0$, show that the functions

$$u_m(x) = Y_0(\lambda_m a)J_0(\lambda_m x) - J_0(\lambda_m a)Y_0(\lambda_m x)$$
$$u_n(x) = Y_0(\lambda_n a)J_0(\lambda_n x) - J_0(\lambda_n a)Y_0(\lambda_n x)$$

are orthogonal with respect to the weight function $p(x) = x$ on the interval $[a, b]$; that is,

$$\int_a^b xu_m(x)u_n(x)\,dx = 0, \quad m \neq n.$$

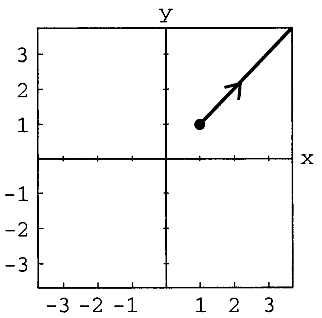
[*Hint:* Follow the procedure on page 460.]

14. Use the results of Problem 13 to solve the following boundary-value problem for the temperature $u(r, t)$ in a circular ring:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial u}{\partial t}, \quad a < r < b, \quad t > 0$$
$$u(a, t) = 0, \quad u(b, t) = 0, \quad t > 0$$
$$u(r, 0) = f(r), \quad a < r < b.$$

7. (a) If $\mathbf{X}(0) = \mathbf{X}_0$ lies on the line $y = 3x$, then $\mathbf{X}(t)$ approaches (0, 0) along this line. For all other initial conditions, $\mathbf{X}(t)$ becomes unbounded and $y = x$ serves as the asymptote.

(b)



9. saddle point **11.** saddle point
13. degenerate stable node **15.** stable spiral **17.** $|\mu| < 1$
19. $\mu < -1$ for a saddle point; $-1 < \mu < 3$ for an unstable spiral point
23. (a) $(-3, 4)$
(b) unstable node or saddle point
(c) (0, 0) is a saddle point.
25. (a) $(\frac{1}{2}, 2)$
(b) unstable spiral point
(c) (0, 0) is an unstable spiral point.

SECTION 10.3 EXERCISES, page 418

1. $r = r_0 e^{\alpha t}$
3. $x = 0$ is unstable; $x = n + 1$ is asymptotically stable.
5. $T = T_0$ is unstable.
7. $x = \alpha$ is unstable; $x = \beta$ is asymptotically stable.
9. $P = a/b$ is asymptotically stable; $P = c$ is unstable.
11. $(\frac{1}{2}, 1)$ is a stable spiral point.
13. $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$ are saddle points; $(\frac{1}{2}, -\frac{7}{4})$ is a stable spiral point.
15. (1, 1) is a stable node; (1, −1) is a saddle point; (2, 2) is a saddle point; (2, −2) is an unstable spiral point.
17. (0, −1) is a saddle point; (0, 0) is unclassified; (0, 1) is stable but we are unable to classify further.
19. (0, 0) is an unstable node; (10, 0) is a saddle point; (0, 16) is a saddle point; (4, 12) is a stable node.
21. $\theta = 0$ is a saddle point. It is not possible to classify either $\theta = \pi/3$ or $\theta = -\pi/3$.
23. It is not possible to classify $x = 0$.
25. It is not possible to classify $x = 0$, but $x = 1/\sqrt{\epsilon}$ and $x = -1/\sqrt{\epsilon}$ are each saddle points.
29. (a) (0, 0) is a stable spiral point.
33. (a) (1, 0), (−1, 0)
35. $|v_0| < \frac{1}{2}\sqrt{2}$
37. If $\beta > 0$, (0, 0) is the only critical point and is stable. If $\beta < 0$, (0, 0), $(\hat{x}, 0)$, and $(-\hat{x}, 0)$, where $\hat{x}^2 = -\alpha/\beta$, are critical points. (0, 0) is stable, while $(\hat{x}, 0)$ and $(-\hat{x}, 0)$ are each saddle points.
39. (b) $(5\pi/6, 0)$ is a saddle point.
(c) $(\pi/6, 0)$ is a center.

SECTION 10.4 EXERCISES, page 430

1. $|\omega_0| < \sqrt{\dfrac{3g}{L}}$
5. (a) First show that $y^2 = v_0^2 + g \ln\left(\dfrac{1 + x^2}{1 + x_0^2}\right)$.
9. (a) The new critical point is $(d/c - \epsilon_2/c,\ a/b + \epsilon_1/b)$.
(b) yes
11. (0, 0) is an unstable node, (0, 100) is a stable node, (50, 0) is a stable node, and (20, 40) is a saddle point.
17. (a) (0, 0) is the only critical point.

CHAPTER 10 REVIEW EXERCISES, page 433

1. true **3.** a center or a saddle point **5.** false
7. false **9.** $\alpha = -1$
11. $r = 1/\sqrt[3]{3t + 1}$, $\theta = t$. The solution curve spirals toward the origin.
13. (a) center **(b)** degenerate stable node
15. (0, 0) is a stable critical point for $\alpha \le 0$.
17. $x = 1$ is unstable; $x = -1$ is asymptotically stable.
19. The system is overdamped when $\beta^2 > 12kms^2$ and underdamped when $\beta^2 < 12kms^2$.

SECTION 11.1 EXERCISES, page 442

7. $\dfrac{\sqrt{\pi}}{2}$ **9.** $\sqrt{\dfrac{\pi}{2}}$ **11.** $\|1\| = \sqrt{p}$; $\left\|\cos\dfrac{n\pi}{p}x\right\| = \sqrt{\dfrac{p}{2}}$

SECTION 11.2 EXERCISES, page 447

1. $f(x) = \dfrac{1}{2} + \dfrac{1}{\pi}\sum_{n=1}^{\infty}\dfrac{1 - (-1)^n}{n}\sin nx$

3. $f(x) = \dfrac{3}{4} + \sum_{n=1}^{\infty}\left\{\dfrac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x - \dfrac{1}{n\pi}\sin n\pi x\right\}$

5. $f(x) = \dfrac{\pi^2}{6} + \sum_{n=1}^{\infty}\left\{\dfrac{2(-1)^n}{n^2}\cos nx + \left(\dfrac{(-1)^{n+1}\pi}{n} + \dfrac{2}{\pi n^3}[(-1)^n - 1]\right)\sin nx\right\}$

7. $f(x) = \pi + 2\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n}\sin nx$

9. $f(x) = \dfrac{1}{\pi} + \dfrac{1}{2}\sin x + \dfrac{1}{\pi}\sum_{n=2}^{\infty}\dfrac{(-1)^n + 1}{1 - n^2}\cos nx$

11. $f(x) = -\dfrac{1}{4} + \dfrac{1}{\pi}\sum_{n=1}^{\infty}\left\{-\dfrac{1}{n}\sin\dfrac{n\pi}{2}\cos\dfrac{n\pi}{2}x + \dfrac{3}{n}\left(1 - \cos\dfrac{n\pi}{2}\right)\sin\dfrac{n\pi}{2}x\right\}$

13. $f(x) = \dfrac{9}{4} + 5\sum_{n=1}^{\infty}\left\{\dfrac{(-1)^n - 1}{n^2\pi^2}\cos\dfrac{n\pi}{5}x + \dfrac{(-1)^{n+1}}{n\pi}\sin\dfrac{n\pi}{5}x\right\}$

15. $f(x) = \dfrac{2\sinh\pi}{\pi}\left[\dfrac{1}{2} + \sum_{n=1}^{\infty}\dfrac{(-1)^n}{1 + n^2}(\cos nx - n\sin nx)\right]$

19. Set $x = \pi/2$.

SECTION 11.3 EXERCISES, page 455

1. odd **3.** neither even nor odd **5.** even **7.** odd
9. neither even nor odd

11. $f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx$

13. $f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nx$

15. $f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x$

17. $f(x) = \frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx$

19. $f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n(1+\pi)}{n}\sin nx$

21. $f(x) = \frac{3}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi}{2} - 1}{n^2}\cos\frac{n\pi}{2}x$

23. $f(x) = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{1+(-1)^n}{1-n^2}\cos nx$

25. $f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n}\cos n\pi x$

$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos\frac{n\pi}{2}}{n}\sin n\pi x$

27. $f(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1-4n^2}\cos 2nx$

$f(x) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2nx$

29. $f(x) = \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{2\cos\frac{n\pi}{2} - (-1)^n - 1}{n^2}\cos nx$

$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}\sin nx$

31. $f(x) = \frac{3}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi}{2} - 1}{n^2}\cos\frac{n\pi}{2}x$

$f(x) = \sum_{n=1}^{\infty}\left\{\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{2}{n\pi}(-1)^n\right\}\sin\frac{n\pi}{2}x$

33. $f(x) = \frac{5}{6} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{3(-1)^n - 1}{n^2}\cos n\pi x$

$f(x) = 4\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}}{n\pi} + \frac{(-1)^n - 1}{n^3\pi^3}\right\}\sin n\pi x$

35. $f(x) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\left\{\frac{1}{n^2}\cos nx - \frac{\pi}{n}\sin nx\right\}$

37. $f(x) = \frac{3}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2n\pi x$

39. $x_p(t) = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n(10-n^2)}\sin nt$

41. $x_p(t) = \frac{\pi^2}{18} + 16\sum_{n=1}^{\infty}\frac{1}{n^2(n^2-48)}\cos nt$

43. **(b)** $y(x) = \frac{2w_0L^4}{EI\pi^5}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}\sin\frac{n\pi}{L}x$

51. $\frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{(-1)^n - 1}{\pi n^2}\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right\}$

53. $f(x, y) = \frac{1}{4} + \frac{1}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^m - 1}{m^2}\cos m\pi x$
$+ \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos n\pi y$
$+ \frac{4}{\pi^4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{[(-1)^m - 1][(-1)^n - 1]}{m^2n^2}\cos m\pi x\cos n\pi y$

SECTION 11.4 EXERCISES, page 463

1. $y = \cos\sqrt{\lambda_n}\,x$; $\cot\sqrt{\lambda} = \sqrt{\lambda}$; 0.7402, 11.7349, 41.4388, 90.8082; $\cos 0.8603x$, $\cos 3.4256x$, $\cos 6.4373x$, $\cos 9.5293x$

5. $\frac{1}{2}(1 + \sin^2\sqrt{\lambda_n})$

7. **(a)** $\lambda = \left(\frac{n\pi}{\ln 5}\right)^2, y = \sin\left(\frac{n\pi}{\ln 5}\ln x\right), n = 1, 2, 3, \ldots$

(b) $\frac{d}{dx}[xy'] + \frac{\lambda}{x}y = 0$

(c) $\int_1^5 \frac{1}{x}\sin\left(\frac{m\pi}{\ln 5}\ln x\right)\sin\left(\frac{n\pi}{\ln 5}\ln x\right)dx = 0, m \neq n$

9. **(a)** $\int_0^1 \cos x_m x \cos x_n x\,dx = 0$, $m \neq n$, where x_m and x_n are positive roots of $\cot x = x$

11. $\frac{d}{dx}[xe^{-x}y'] + ne^{-x}y = 0$; $\int_0^{\infty} e^{-x}L_m(x)L_n(x)\,dx = 0$, $m \neq n$

13. **(a)** $\lambda = 16n^2$, $y = \sin(4n\tan^{-1}x)$, $n = 1, 2, 3, \ldots$

(b) $\int_0^1 \frac{1}{1+x^2}\sin(4m\tan^{-1}x)\sin(4n\tan^{-1}x)\,dx = 0$, $m \neq n$

SECTION 11.5 EXERCISES, page 471

1. 1.277, 2.339, 3.391, 4.441

3. $f(x) = \sum_{i=1}^{\infty}\frac{1}{\lambda_i J_1(2\lambda_i)}J_0(\lambda_i x)$

5. $f(x) = 4\sum_{i=1}^{\infty}\frac{\lambda_i J_1(2\lambda_i)}{(4\lambda_i^2 + 1)J_0^2(2\lambda_i)}J_0(\lambda_i x)$

7. $f(x) = 20\sum_{i=1}^{\infty}\frac{\lambda_i J_2(4\lambda_i)}{(2\lambda_i^2 + 1)J_1^2(4\lambda_i)}J_1(\lambda_i x)$

9. $f(x) = \frac{9}{2} - 4\sum_{i=1}^{\infty}\frac{J_2(3\lambda_i)}{\lambda_i^2 J_0^2(3\lambda_i)}J_0(\lambda_i x)$

11. $f(x) = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)$

13. $f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) - \frac{3}{32}P_4(x) + \cdots$

15. Use $\cos 2\theta = 2\cos^2\theta - 1$.

19. $f(x) = \frac{1}{2}P_0(x) + \frac{5}{8}P_2(x) - \frac{3}{16}P_4(x) + \cdots$
$f(x) = |x|$ on $(-1, 1)$

CHAPTER 11 REVIEW EXERCISES, page 472

1. true **3.** cosine **5.** $\frac{3}{2}$ **7.** false

9. $\frac{1}{\sqrt{1-x^2}}, -1 \leq x \leq 1$

13. $f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{n^2\pi}[(-1)^n - 1]\cos n\pi x + \frac{2}{n}(-1)^n \sin n\pi x\right\}$

15. $f(x) = 1 - e^{-1} + 2\sum_{n=1}^{\infty}\frac{1-(-1)^n e^{-1}}{1+n^2\pi^2}\cos n\pi x$

17. $\lambda = \frac{(2n-1)^2\pi^2}{36}$, $n = 1, 2, 3, \ldots,$

$y = \cos\left(\frac{2n-1}{2}\pi \ln x\right)$

19. $f(x) = \frac{1}{4}\sum_{i=1}^{\infty}\frac{J_1(2\lambda_i)}{\lambda_i J_1{}^2(4\lambda_i)} J_0(\lambda_i x)$

SECTION 12.1 EXERCISES, page 478

1. The possible cases can be summarized in one form $u = c_1 e^{c_2(x+y)}$, where c_1 and c_2 are constants.
3. $u = c_1 e^{y+c_2(x-y)}$ 5. $u = c_1(xy)^{c_2}$
7. not separable
9. $u = e^{-t}(A_1 e^{k\lambda^2 t}\cosh \lambda x + B_1 e^{k\lambda^2 t}\sinh \lambda x)$
   $u = e^{-t}(A_2 e^{-k\lambda^2 t}\cos \lambda x + B_2 e^{-k\lambda^2 t}\sin \lambda x)$
   $u = (c_7 x + c_8)c_9 e^{-t}$
11. $u = (c_1 \cosh \lambda x + c_2 \sinh \lambda x) \times (c_3 \cosh \lambda at + c_4 \sinh \lambda at)$
   $u = (c_5 \cos \lambda x + c_6 \sin \lambda x)(c_7 \cos \lambda at + c_8 \sin \lambda at)$
   $u = (c_9 x + c_{10})(c_{11} t + c_{12})$
13. $u = (c_1 \cosh \lambda x + c_2 \sinh \lambda x)(c_3 \cos \lambda y + c_4 \sin \lambda y)$
   $u = (c_5 \cos \lambda x + c_6 \sin \lambda x)(c_7 \cosh \lambda y + c_8 \sinh \lambda y)$
   $u = (c_9 x + c_{10})(c_{11} y + c_{12})$
15. For $\lambda^2 > 0$ there are three possibilities:

$$u = (c_1 \cosh \lambda x + c_2 \sinh \lambda x) \times (c_3 \cosh \sqrt{1-\lambda^2}y + c_4 \sinh \sqrt{1-\lambda^2}y),\quad \lambda^2 < 1$$
$$u = (c_1 \cosh \lambda x + c_2 \sinh \lambda x) \times (c_3 \cos \sqrt{\lambda^2-1}y + c_4 \sin \sqrt{\lambda^2-1}y),\quad \lambda^2 > 1$$
$$u = (c_1 \cosh x + c_2 \sinh x)(c_3 y + c_4),\quad \lambda^2 = 1$$

The results for the case $-\lambda^2 < 0$ are similar. For $\lambda^2 = 0$ we have

$$u = (c_1 x + c_2)(c_3 \cosh y + c_4 \sinh y).$$

17. elliptic 19. parabolic 21. hyperbolic
23. parabolic 25. hyperbolic
29. $u = e^{n(-3x+y)}$, $u = e^{n(2x+y)}$
31. The equation $x^2 + 4y^2 = 4$ defines an ellipse. The partial differential equation is hyperbolic outside the ellipse, parabolic on the ellipse, and elliptic inside the ellipse.

SECTION 12.2 EXERCISES, page 485

1. $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, 0 < x < L, \quad t > 0$

$u(0,t) = 0, \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0, t > 0$

$u(x,0) = f(x), 0 < x < L$

3. $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, 0 < x < L, \quad t > 0$

$u(0,t) = 100, \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = -hu(L,t), t > 0$

$u(x,0) = f(x), 0 < x < L$

5. $a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, 0 < x < L, \quad t > 0$

$u(0,t) = 0, \quad u(L,t) = 0, t > 0$

$u(x,0) = x(L-x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, 0 < x < L$

7. $a^2\frac{\partial^2 u}{\partial x^2} - 2\beta\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2}, 0 < x < L, \quad t > 0$

$u(0,t) = 0, u(L,t) = \sin \pi t, t > 0$

$u(x,0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, 0 < x < L$

9. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 4, 0 < y < 2$

$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad u(4,y) = f(y), \quad 0 < y < 2$

$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \quad u(x,2) = 0, \quad 0 < x < 4$

SECTION 12.3 EXERCISES, page 489

1. $u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{-\cos\frac{n\pi}{2} + 1}{n}\right)e^{-k(n^2\pi^2/L^2)t}\sin\frac{n\pi}{L}x$

3. $u(x,t) = \frac{1}{L}\int_0^L f(x)\,dx + \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx\right)e^{-k(n^2\pi^2/L^2)t}\cos\frac{n\pi}{L}x$

5. $u(x,t) = e^{-ht}\left[\frac{1}{L}\int_0^L f(x)\,dx + \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx\right)e^{-k(n^2\pi^2/L^2)t}\cos\frac{n\pi}{L}x\right]$

SECTION 12.4 EXERCISES, page 492

1. $u(x,t) = \frac{L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\cos\frac{n\pi a}{L}t\sin\frac{n\pi}{L}x$

3. $u(x,t) = \frac{6\sqrt{3}}{\pi^2}\left(\cos\frac{\pi a}{L}t\sin\frac{\pi}{L}x - \frac{1}{5^2}\cos\frac{5\pi a}{L}t\sin\frac{5\pi}{L}x + \frac{1}{7^2}\cos\frac{7\pi a}{L}t\sin\frac{7\pi}{L}x - \cdots\right)$

5. $u(x,t) = \frac{1}{a}\sin at \sin x$

7. $u(x,t) = \frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}\cos\frac{n\pi a}{L}t\sin\frac{n\pi}{L}x$

9. $u(x,t) = e^{-\beta t}\sum_{n=1}^{\infty} A_n\{\cos q_n t + \frac{\beta}{q_n}\sin q_n t\}\sin nx,$

where $A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$ and $q_n = \sqrt{n^2-\beta^2}$

11. $u(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\frac{n^2\pi^2}{L^2}at + B_n\sin\frac{n^2\pi^2}{L^2}at\right) \times \sin\frac{n\pi}{L}x,$

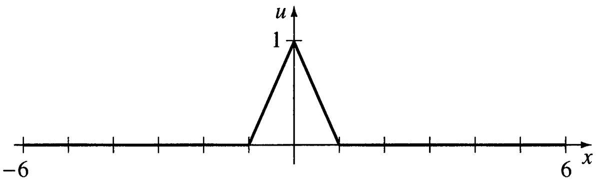
where $A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$

$B_n = \frac{2L}{n^2\pi^2 a}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$

15. $u(x,t) = t + \sin x\cos at$

17. $u(x,t) = \frac{1}{2a}\sin 2x\sin 2at$

19. (a)



SECTION 12.5 EXERCISES, page 498

1. $u(x,y) = \frac{2}{a}\sum_{n=1}^{\infty}\left(\frac{1}{\sinh\frac{n\pi}{a}b}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx\right) \times \sinh\frac{n\pi}{a}y\sin\frac{n\pi}{a}x$

3. $u(x,y) = \frac{2}{a}\sum_{n=1}^{\infty}\left(\frac{1}{\sinh\frac{n\pi}{a}b}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx\right) \times \sinh\frac{n\pi}{a}(b-y)\sin\frac{n\pi}{a}x$

5. $u(x,y) = \frac{1}{2}x + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2\sinh n\pi}\sinh n\pi x\cos n\pi y$

7. $u(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n} \times \frac{n\cosh nx + \sinh nx}{n\cosh n\pi + \sinh n\pi}\sin ny$

9. $u(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\int_0^{\pi} f(x)\sin nx\,dx\right)e^{-ny}\sin nx$

11. $u(x,y) = \sum_{n=1}^{\infty}\left(A_n\cosh\frac{n\pi}{a}y + B_n\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x,$

where $A_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx$

$B_n = \frac{1}{\sinh\frac{n\pi}{a}b}\left(\frac{2}{a}\int_0^a g(x)\sin\frac{n\pi}{a}x\,dx - A_n\cosh\frac{n\pi}{a}b\right)$

13. $u = u_1 + u_2$, where

$u_1(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n\sinh n\pi}\sinh ny\sin nx$

$u_2(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n} \times \frac{\sinh nx + \sinh n(\pi - x)}{\sinh n\pi}\sin ny$

SECTION 12.6 EXERCISES, page 501

1. $u(x,t) = 100 + \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n}e^{-kn^2\pi^2 t}\sin n\pi x$

3. $u(x,t) = u_0 - \frac{r}{2k}x(x-1) + 2\sum_{n=1}^{\infty}\left[\frac{u_0}{n\pi} + \frac{r}{kn^3\pi^3}\right] \times [(-1)^n - 1]e^{-kn^2\pi^2 t}\sin n\pi x$

5. $u(x,t) = \psi(x) + \sum_{n=1}^{\infty}A_n e^{-kn^2\pi^2 t}\sin n\pi x,$

where $\psi(x) = \frac{A}{k\beta^2}[-e^{-\beta x} + (e^{-\beta}-1)x + 1]$

and $A_n = 2\int_0^1 [f(x) - \psi(x)]\sin n\pi x\,dx$

7. $\psi(x) = u_0\left(1 - \frac{\sinh\sqrt{h/k}\,x}{\sinh\sqrt{h/k}}\right)$

9. $u(x,t) = \frac{A}{6a^2}(x - x^3) + \frac{2A}{a^2\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\cos n\pi at\sin n\pi x$

11. $u(x,y) = (u_0 - u_1)y + u_1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{u_0(-1)^n - u_1}{n}e^{-n\pi x}\sin n\pi y$

SECTION 12.7 EXERCISES, page 505

1. $u(x,t) = 2h\sum_{n=1}^{\infty}\frac{\sin\lambda_n}{\lambda_n[h + \sin^2\lambda_n]}e^{-k\lambda_n^2 t}\cos\lambda_n x,$

where the λ_n are the consecutive positive roots of $\cot\lambda = \lambda/h$

3. $u(x,y) = \sum_{n=1}^{\infty}A_n\sinh\lambda_n y\sin\lambda_n x,$

where $A_n = \frac{2h}{\sinh\lambda_n b[ah + \cos^2\lambda_n a]}\int_0^a f(x)\sin\lambda_n x\,dx$ and the λ_n are the consecutive positive roots of $\tan\lambda a = -\lambda/h$

5. $u(x,t) = \sum_{n=1}^{\infty}A_n e^{-k(2n-1)^2\pi^2 t/4L^2}\sin\left(\frac{2n-1}{2L}\right)\pi x,$

where $A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2n-1}{2L}\right)\pi x\,dx$

7. $u(x,y) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)\cosh\left(\frac{2n-1}{2}\right)\pi} \times \cosh\left(\frac{2n-1}{2}\right)\pi x\sin\left(\frac{2n-1}{2}\right)\pi y$

9. (b) 1.8751, 4.6941

SECTION 12.8 EXERCISES, page 509

1. $u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} e^{-k(m^2+n^2)t} \sin mx \sin ny$,

where $A_{mn} = \dfrac{4u_0}{mn\pi^2}[1-(-1)^m][1-(-1)^n]$

3. $u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin mx \sin ny \cos a\sqrt{m^2+n^2}\,t$,

where $A_{mn} = \dfrac{16}{m^3n^3\pi^2}[(-1)^m - 1][(-1)^n - 1]$

5. $u(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sinh \omega_{mn} z \sin \dfrac{m\pi}{a} x \sin \dfrac{n\pi}{b} y$,

where $\omega_{mn} = \sqrt{\left(\dfrac{m\pi}{a}\right)^2 + \left(\dfrac{n\pi}{b}\right)^2}$

$A_{mn} = \dfrac{4}{ab\sinh(c\omega_{mn})}\int_0^b\int_0^a f(x,y)\sin\dfrac{m\pi}{a}x \sin\dfrac{n\pi}{b}y\,dx\,dy$

CHAPTER 12 REVIEW EXERCISES, page 510

1. $u = c_1 e^{(c_2 x + y/c_2)}$ **3.** $\psi(x) = u_0 + \dfrac{(u_1-u_0)}{1+\pi}x$

5. $u(x,t) = \dfrac{2h}{\pi^2 a}\sum_{n=1}^{\infty}\dfrac{\cos\dfrac{n\pi}{4} - \cos\dfrac{3n\pi}{4}}{n^2}\sin n\pi a t \sin n\pi x$

7. $u(x,y) = \dfrac{100}{\pi}\sum_{n=1}^{\infty}\dfrac{1-(-1)^n}{n\sinh n\pi}\sinh nx \sin ny$

9. $u(x,y) = \dfrac{100}{\pi}\sum_{n=1}^{\infty}\dfrac{1-(-1)^n}{n}e^{-nx}\sin ny$

11. $u(x,t) = e^{-t}\sin x$

13. $u(x,t) = e^{-(x+t)}\sum_{n=1}^{\infty} A_n[\sqrt{n^2+1}\cos\sqrt{n^2+1}\,t + \sin\sqrt{n^2+1}\,t]\sin nx$

SECTION 13.1 EXERCISES, page 516

1. $u(r,\theta) = \dfrac{u_0}{2} + \dfrac{u_0}{\pi}\sum_{n=1}^{\infty}\dfrac{1-(-1)^n}{n}r^n \sin n\theta$

3. $u(r,\theta) = \dfrac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\dfrac{r^n}{n^2}\cos n\theta$

5. $u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^{-n}(A_n\cos n\theta + B_n \sin n\theta)$,

where $A_0 = \dfrac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$

$A_n = \dfrac{c^n}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta$

$B_n = \dfrac{c^n}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta$

7. $u(r,\theta) = \dfrac{1}{2} + \dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{\sin\dfrac{n\pi}{2}}{n}\left(\dfrac{r}{c}\right)^{2n}\cos 2n\theta$

9. $u(r,\theta) = A_0 \ln\left(\dfrac{r}{b}\right) + \sum_{n=1}^{\infty}\left[\left(\dfrac{b}{r}\right)^n - \left(\dfrac{r}{b}\right)^n\right] \times [A_n\cos n\theta + B_n \sin n\theta]$,

where $A_0 \ln\left(\dfrac{a}{b}\right) = \dfrac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$

$\left[\left(\dfrac{b}{a}\right)^n - \left(\dfrac{a}{b}\right)^n\right]A_n = \dfrac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta$

$\left[\left(\dfrac{b}{a}\right)^n - \left(\dfrac{a}{b}\right)^n\right]B_n = \dfrac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta$

11. $u(r,\theta) = \dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{1-(-1)^n}{n^3}\dfrac{r^{2n}-b^{2n}}{a^{2n}-b^{2n}}\left(\dfrac{a}{r}\right)^n \sin n\theta$

13. $u(r,\theta) = \dfrac{u_0}{2} + \dfrac{2u_0}{\pi}\sum_{n=1}^{\infty}\dfrac{\sin\dfrac{n\pi}{2}}{n}\left(\dfrac{r}{2}\right)^n\cos n\theta$

SECTION 13.2 EXERCISES, page 522

1. $u(r,t) = \dfrac{2}{ac}\sum_{n=1}^{\infty}\dfrac{\sin\lambda_n at\, J_0(\lambda_n r)}{\lambda_n^2 J_1(\lambda_n c)}$

3. $u(r,z) = u_0\sum_{n=1}^{\infty}\dfrac{\sinh\lambda_n(4-z)\,J_0(\lambda_n r)}{\lambda_n\sinh 4\lambda_n\, J_1(2\lambda_n)}$

5. $u(r,t) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r)e^{-k\lambda_n^2 t}$,

where $A_n = \dfrac{2}{c^2 J_1^2(\lambda_n c)}\int_0^c rJ_0(\lambda_n r)f(r)\,dr$

7. $u(r,t) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r)e^{-k\lambda_n^2 t}$,

where $A_n = \dfrac{2\lambda_n^2}{(\lambda_n^2 + h^2)J_0^2(\lambda_n)}\int_0^1 rJ_0(\lambda_n r)f(r)\,dr$

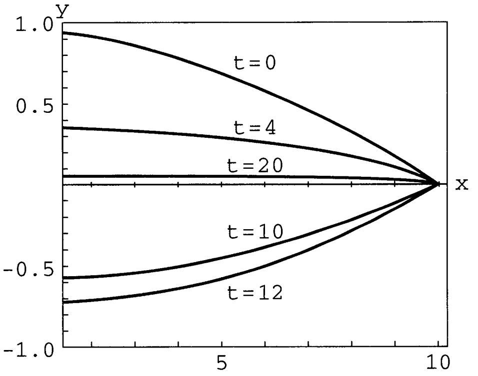
9. $u(r,t) = 100 + 50\sum_{n=1}^{\infty}\dfrac{J_1(\lambda_n)J_0(\lambda_n r)}{\lambda_n J_1^2(2\lambda_n)}e^{-\lambda_n^2 t}$

11. (b) $u(x,t) = \sum_{n=1}^{\infty} A_n\cos(\lambda_n\sqrt{g}t)J_0(2\lambda_n\sqrt{x})$,

where $A_n = \dfrac{2}{LJ_1^2(2\lambda_n\sqrt{L})}\int_0^{\sqrt{L}} vJ_0(2\lambda_n v)f(v^2)\,dv$

13. (a) $\lambda_1 = 0.2405$, $\lambda_2 = 0.5520$, $\lambda_3 = 0.8654$;
$A_1 = 0.7966$, $A_2 = 0.0687$, $A_3 = 0.0535$;
$S_3 = 0.7966\cos(0.2405t)J_0(0.2405r) + 0.0687\cos(0.5520t)J_0(0.5520r) + 0.0535\cos(0.8654t)J_0(0.8654r)$

(b)



SECTION 13.3 EXERCISES, page 526

1. $u(r, \theta) = 50\left[\frac{1}{2}P_0(\cos\theta) + \frac{3}{4}\left(\frac{r}{c}\right)P_1(\cos\theta) - \frac{7}{16}\left(\frac{r}{c}\right)^3 P_3(\cos\theta) + \frac{11}{32}\left(\frac{r}{c}\right)^5 P_5(\cos\theta) + \cdots\right]$

3. $u(r, \theta) = \frac{r}{c}\cos\theta$

5. $u(r, \theta) = \sum_{n=0}^{\infty} A_n \frac{b^{2n+1} - r^{2n+1}}{b^{2n+1}r^{n+1}} P_n(\cos\theta)$, where $\frac{b^{2n+1} - a^{2n+1}}{b^{2n+1}a^{n+1}} A_n = \frac{2n+1}{2}\int_0^{\pi} f(\theta)P_n(\cos\theta)\sin\theta\, d\theta$

7. $u(r, \theta) = \sum_{n=0}^{\infty} A_n r^{2n} P_{2n}(\cos\theta)$, where $A_n = \frac{(4n+1)}{c^{2n}}\int_0^{\pi/2} f(\theta)P_{2n}(\cos\theta)\sin\theta\, d\theta$

9. $u(r, t) = 100 + \frac{200}{\pi r}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} e^{-n^2\pi^2 t}\sin n\pi r$

11. $u(r, t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi a}{c}t + B_n \sin\frac{n\pi a}{c}t\right)\sin\frac{n\pi}{c}r$, where $A_n = \frac{2}{c}\int_0^c rf(r)\sin\frac{n\pi}{c}r\, dr$

$B_n = \frac{2}{n\pi a}\int_0^c rg(r)\sin\frac{n\pi}{c}r\, dr$

CHAPTER 13 REVIEW EXERCISES, page 528

1. $u(r, \theta) = \frac{2u_0}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\left(\frac{r}{c}\right)^n \sin n\theta$

3. $u(r, \theta) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\sin n\theta$

5. $u(r, \theta) = \frac{2u_0}{\pi}\sum_{n=1}^{\infty}\frac{r^{4n} + r^{-4n}}{2^{4n} + 2^{-4n}}\,\frac{1-(-1)^n}{n}\sin 4n\theta$

7. $u(r, t) = 2e^{-ht}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n)}e^{-\lambda_n^2 t}$

9. $u(r, z) = 50\sum_{n=1}^{\infty}\frac{\cosh\lambda_n z\, J_0(\lambda_n r)}{\lambda_n \cosh 4\lambda_n\, J_1(2\lambda_n)}$

11. $u(r, \theta) = 100\left[\frac{3}{2}rP_1(\cos\theta) - \frac{7}{8}r^3P_3(\cos\theta) + \frac{11}{16}r^5P_5(\cos\theta) + \cdots\right]$

SECTION 14.1 EXERCISES, page 533

1. (a) Let $\tau = u^2$ in the integral $\operatorname{erf}(\sqrt{t})$.

7. $y(t) = e^{\pi t}\operatorname{erfc}(\sqrt{\pi t})$

9. Use the property $\int_0^b - \int_0^a = \int_0^b + \int_a^0$.

SECTION 14.2 EXERCISES, page 538

1. $u(x, t) = A\cos\frac{a\pi}{L}t\sin\frac{\pi}{L}x$

3. $u(x, t) = f\left(t - \frac{x}{a}\right)\mathcal{U}\left(t - \frac{x}{a}\right)$

5. $u(x, t) = \left[\frac{1}{2}g\left(t - \frac{x}{a}\right)^2 + A\sin\omega\left(t - \frac{x}{a}\right)\right] \times \mathcal{U}\left(t - \frac{x}{a}\right) - \frac{1}{2}gt^2$

7. $u(x, t) = a\frac{F_0}{E}\sum_{n=0}^{\infty}(-1)^n\left\{\left(t - \frac{2nL + L - x}{a}\right) \times \mathcal{U}\left(t - \frac{2nL + L - x}{a}\right) - \left(t - \frac{2nL + L + x}{a}\right) \times \mathcal{U}\left(t - \frac{2nL + L + x}{a}\right)\right\}$

9. $u(x, t) = (t - x)\sinh(t - x)\mathcal{U}(t - x) + xe^{-x}\cosh t - e^{-x}t\sinh t$

11. $u(x, t) = u_0\operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right)$

13. $u(x, t) = u_1 + (u_0 - u_1)\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)$

15. $u(x, t) = u_0\left[1 - \left\{\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) - e^{x+t}\operatorname{erfc}\left(\sqrt{t} + \frac{x}{2\sqrt{t}}\right)\right\}\right]$

17. $u(x, t) = \frac{x}{2\sqrt{\pi}}\int_0^t \frac{f(t-\tau)}{\tau^{3/2}}e^{-x^2/4\tau}\, d\tau$

19. $u(x, t) = 60 + 40\operatorname{erfc}\left(\frac{x}{2\sqrt{t-2}}\right)\mathcal{U}(t - 2)$

21. $u(x, t) = 100\left[-e^{1-x+t}\operatorname{erfc}\left(\sqrt{t} + \frac{1-x}{2\sqrt{t}}\right) + \operatorname{erfc}\left(\frac{1-x}{2\sqrt{t}}\right)\right]$

23. $u(x, t) = u_0 + u_0 e^{-(\pi^2/L^2)t}\sin\left(\frac{\pi}{L}x\right)$

25. $u(x, t) = u_0 - u_0\sum_{n=0}^{\infty}(-1)^n\left[\operatorname{erfc}\left(\frac{2n+1-x}{2\sqrt{kt}}\right) + \operatorname{erfc}\left(\frac{2n+1+x}{2\sqrt{kt}}\right)\right]$

27. $u(x, t) = u_0 e^{-Gt/C}\operatorname{erf}\left(\frac{x}{2}\sqrt{\frac{RC}{t}}\right)$

29. $u(x, t) = A\sqrt{\frac{k}{\pi t}}\,e^{-x^2/4kt}$; an impulse of heat, or flash burn, takes place at $x = 0$.

SECTION 14.3 EXERCISES, page 548

1. $f(x) = \frac{1}{\pi}\int_0^{\infty}\frac{\sin\alpha\cos\alpha x + 3(1 - \cos\alpha)\sin\alpha x}{\alpha}\, d\alpha$

3. $f(x) = \frac{1}{\pi}\int_0^{\infty}[A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x]\, d\alpha$,

where $A(\alpha) = \frac{3\alpha\sin 3\alpha + \cos 3\alpha - 1}{\alpha^2}$

$B(\alpha) = \frac{\sin 3\alpha - 3\alpha\cos 3\alpha}{\alpha^2}$