AN IMPROVEMENT IN THE GALERKIN PROJECTED RESIDUAL FINITE ELEMENT METHOD (GPR) FOR HELMHOLTZ EQUATION

Eduardo Gomes Dutra do Carmo
carmo@lmn.con.ufrj.br
Programa de Engenharia Nuclear, COPPE - Universidade Federal do Rio de Janeiro
Ilha do Fundão, 21945-970, P.B. 68509, Rio de Janeiro - RJ – Brazil
Gustavo Benitez Alvarez
benitez.gustavo@gmail.com
Departamento de Ciências Exatas, EEIMVR - Universidade Federal Fluminense
Av. dos Trabalhadores, 420, Vila Sta. Cecília, 27255-125, Volta Redonda - RJ – Brazil
Abimael Fernando Dourado Loula
aloc@lncc.br
Departamento de Matemática Aplicada e Computacional, Laboratório Nacional de Computação Científica
Av. Getúlio Vargas 333, 25651-070. P.B. 95113, Petrópolis – RJ - Brazil
Fernando Alves Rochinha
rochinha@adc.coppe.ufrj.br
Programa de Engenharia Mecânica, COPPE – Universidade Federal do Rio de Janeiro
Ilha do Fundão, 21945-970, P.B. 68501, Rio de Janeiro - RJ – Brazil

Abstract. *In XXVIII CILAMCE edition a new finite element method for Helmholtz equation was introduced: The Galerkin Plus Multiplies Projection of Residual Method (GMPR). This method was obtained adding to the Galerkin formulation an appropriate numbers of projections of the residual of PDE within each element. This allows that the element matrix has a maximum number of free parameters. Also, for rectangular domain, uniform mesh and bilinear elements, a methodology to choose these free parameters was presented. The criterion adopted to determine the free parameters consists of minimizing the phase error of the approximate solution. The GMPR method is a "variationally" consistent finite element formulation and convergent for the homogeneous Helmholtz equation. In spite of everything, this initial version of the GMPR method is not necessarily convergent for all non homogeneous problems. Therefore, in this work we introduced a necessary modification to obtain a FEM of the GPR class (initially denoted by GMPR and now by GPR) with uniform convergence properties. The modification is introduced through a new projection term in the weak formulation: the residual gradient projection term. With this modification the GPR formulation remains "variationally" consistent and uniform convergence properties are recovered in all cases: homogeneous and non homogeneous Helmholtz problems. We presented some numerical tests with source term to show the good rates of convergence of the GPR formulation.*
Keywords: *Finite element method, Stabilization, GMPR, GPR, GLS, Helmholtz equation*

1. INTRODUCTION

It is known that numerical approximation of time-harmonic acoustic, elastic and electromagnetic wave problems governed by the Helmholtz equation is particularly challenging. The oscillatory behavior of the exact solution and the quality of the numerical approximation depend on the wave number *k*. To approximate Helmholtz equation with acceptable accuracy the resolution of the mesh should be adjusted to the wave number according to a rule of thumb (Ihlenburg et al, 1995), which prescribes a minimum number of elements per wavelength. Despite of this rule, the performance of the Galerkin finite element method deteriorates as *k* increases. This misbehavior, known as pollution of the finite element solution, can only be avoided after a drastic refinement of the mesh, which normally entails significant barriers for the numerical analysis of Helmholtz equation at mid and high frequencies.

A great effort has been devoted to alleviate the pollution effect. There exist several attempts to minimize the phase error of finite element approximations to Helmholtz equation. In one-dimension a Galerkin Least Square (GLS) stabilization, as proposed in (Harari et al, 1992), can completely eliminate the phase error, but not in two (Thompson et al, 1995) or three dimensions (Thompson et al, 2004). For two dimensions, stencils with minimal pollution error are constructed in (Babuška et al, 1995) through the Quasi Stabilized Finite Element Method (QS). As the Quasi-Stabilized Finite Element Method is not based on a variational formulation it is not clear how this formulation can be applied to non uniform meshes, high-order polynomials and non homogeneous problems. Finite element methods based on variational formulations, such as Residual-Based Finite Element Method (RBFEM) (Oberai et al, 2000) and Discontinuous Finite Element Method at Element Level (DGB) (Loula et al, 2007; Rochinha et al, 2007), have also been developed to minimize the phase error in two dimensions.

These two methods present some disadvantages. The RBFEM method is obtained from the Galerkin approximation by appending terms that are proportional to residuals on element interiors and inter-element boundaries. These terms implicate in an extra computational effort when the RBFEM formulation is compared with a classical continuous finite element formulation. The DGB method is a discontinuous finite element formulation, where discontinuities are introduced locally, inside each element. This method needs the condensation technique to eliminate degrees of freedom introduced by the discontinuities and implicate in an extra computational effort. Besides, these two methods are not able to minimize the phase error in three dimensions.

In XXVIII CILAMCE edition our first ideas about a new finite element method with multiplies projection of residual were developed to address the Helmholtz equation (Dutra do Carmo et al, 2006). The Galerkin Plus Multiplies Projection of Residual Method (GMPR) was obtained adding to the Galerkin formulation an appropriate numbers of projections of the residual of PDE within each element. This allows that the element matrix has a maximum number of free parameters. Also, for rectangular domain, uniform mesh and bilinear elements, a methodology to choose these free parameters was presented. The criterion adopted to determine the free parameters consists of minimizing the phase error of the approximate solution. The GMPR method is a "variationally" consistent finite element formulation and convergent for the homogeneous Helmholtz equation. In spite of everything, this initial version of the GMPR method is not necessarily convergent for all non homogeneous problems. Therefore, in this work we introduced a necessary modification to obtain a FEM of the GPR class (initially denoted by GMPR and now by GPR) with uniform convergence properties. The modification is introduced through a new projection term in the weak formulation: the residual gradient projection term. With this modification the GPR formulation remains "variationally" consistent and uniform convergence properties are recovered in all cases: homogeneous and non homogeneous Helmholtz problems. We presented some numerical tests with source term to show the good rates of convergence of the GPR formulation.

2. THE HELMHOLTZ EQUATION

2.1 The boundary value problem

Let $\Omega \subset R^n$ $(n \geq 1)$ be an open bounded domain with a Lipschitz continuous smooth piecewise boundary. Let Γ_g, Γ_q and Γ_r subsets of Γ satisfying $\Gamma_g \cap \Gamma_q = \Gamma_g \cap \Gamma_r = \Gamma_q \cap \Gamma_r = \emptyset$ and $\Gamma_g \cup \Gamma_q \cup \Gamma_r = \Gamma$. We shall consider the interior Helmholtz problem:

$$L(u) \equiv -\nabla \cdot (\nabla u) - k^2 u = f \quad \text{in } \Omega, \tag{1}$$

$$u = g \quad \text{on } \Gamma_g, \tag{2}$$

$$\nabla u \cdot \hat{n} = q \quad \text{on } \Gamma_q, \tag{3}$$

$$\nabla u \cdot \hat{n} + \alpha u = r \quad \text{on } \Gamma_r, \tag{4}$$

where u denotes a scalar field that describes time-harmonic acoustic, elastic or electromagnetic steady state waves. The coefficient $k \in R$ is the wave number, $f \in L^2(\Omega)$ is the source term, $g \in H^{\frac{1}{2}}(\Gamma_g) \cap C^0(\Gamma_g)$, $q \in L^2(\Gamma_q)$ and $r \in L^2(\Gamma_r)$ are the prescribed boundary conditions. The coefficient $\alpha \in L^\infty(\Gamma_r)$ is positive on Γ_r and $\hat{n}$ denotes the outward normal unit vector defined almost everywhere on Γ.

2.2 The associated variational problem

Let S and V defined as $S = \{u \in H^1(\Omega): u = g \text{ on } \Gamma_g\}$, $V = \{v \in H^1(\Omega): v = 0 \text{ on } \Gamma_g\}$ The variational problem associated to the boundary value problem defined by Eqs. (1-4) consist of finding $u \in S$ satisfying the following variational equation:

$$A(u,v) \equiv \int_\Omega [(\nabla u) \cdot \nabla v - k^2 uv] d\Omega + \int_{\Gamma_r} \alpha uv\, d\Gamma = \int_\Omega f\, v\, d\Omega + \int_{\Gamma_q} qv\, d\Gamma + \int_{\Gamma_r} rv\, d\Gamma \equiv F(v) \quad \forall\, v \in V, \tag{5}$$

The major challenges, in term of FEM, is to find a consistent formulation in continuous or discontinuous finite dimensional spaces, such that, its approximate solution is stable and the closest possible of the correspondent solution in infinite dimensional space given by Eq. (5). Here, we will just treat with continuous finite dimensional spaces.

2.3 The associated Galerkin finite element formulation

Let $M^h = \{\Omega_1, \ldots, \Omega_{NE}\}$ be a partition of Ω in no degenerated finite element Ω_e, such that Ω_e can be mapped in standard elements by isoparametric mapping and that satisfy $\Omega_e \cap \Omega_{e'} = \emptyset$ if $e \neq e'$ and $\Omega \cup \Gamma = \bigcup_{e=1}^{ne} (\Omega_e \cup \Gamma_e)$, where Γ_e denotes the boundary of Ω_e.

Let $p \geq 1$ an integer and consider $P^p(\Omega_e)$ defined as the space of polynomials of degree less than or equal to p. Let $H^h(\Omega) = \{\varphi \in H^1(\Omega); \varphi_e \in P^p(\Omega_e)\}$ and $H^{\frac{1}{2},h}(\Gamma_g) = \{\varphi \in H^{\frac{1}{2}}(\Gamma_g); \exists \phi \in H^1(\Omega) \text{ and } \phi = \varphi \text{ on } \Gamma_g\}$ are the finite dimension spaces and let g^h be the interpolate of g on $H^{\frac{1}{2},h}(\Gamma_g)$. The Galerkin formulation consists of finding $u^h \in S^h = \{\varphi \in H^h(\Omega); \varphi = g^h \text{ on } \Gamma_g\}$ that satisfies $\forall v^h \in V^h = \{\varphi \in H^h(\Omega); \varphi = 0 \text{ on } \Gamma_g\}$:

$$A(u^h, v^h) = F(v^h), \tag{6}$$

For purely diffusive problems the solution of Galerkin FEM is the best approximation in the energy norm. It is well know that the Galerkin FEM is shown unstable and little accuracy for Helmholtz equation. Its numerical solution presents spurious oscillations that do not corresponding with the physical solution of problem.

3. THE GALERKIN PROJECTED RESIDUAL METHOD

In XXVIII CILAMCE edition our first ideas about a new finite element method with multiplies projection of residual were developed to address the Helmholtz equation (Dutra do Carmo et al, 2006). This method, initially denoted by GMPR and now by GPR, was obtained adding to the Galerkin formulation an appropriate numbers of projections of the residual of PDE within each element. This allows that the element matrix has a maximum number of free parameters. Formally it is represented by the following equation

$$A(u^h, v^h) + \sum_{e=1}^{ne} \left(\sum_{l=1}^{\mathrm{N}} \tau_l^e \left(L(u_e^h), \frac{L(v_e^h)\psi_{l,e}}{k^2} \right)_{L^2(\Omega_e)} \right) = F_{GPR}(v^h) \quad \forall v^h \in V^h, \tag{7}$$

where N is the dimension of a local real linear space $E_{GPR}(\Omega_e)$ defined as $E_{GPR}(\Omega_e) = \{\psi : \Omega_e \to R; \psi = \sum_{i=1}^{npel}\sum_{j=1}^{npel} C_{i,j} L(\eta_i) L(\eta_j), \quad C_{i,j} \in R\}$ with basis denoted by $\psi_{l,e}$ and $npel$ denotes the number of nodal points of the element Ω_e and $\eta_i \; (i = 1, \ldots, npel)$ denotes the usual local shape functions associated to nodal point *i*. More details on $E_{GPR}(\Omega_e)$ and $\psi_{l,e}$ can be found in Carmo et al (2008). The free stabilization parameters are denoted by τ_l^e and

$$F_{GPR}(v^h) = F(v^h) + \sum_{e=1}^{ne} \left(\sum_{l=1}^{\mathrm{N}} \tau_i^e \left(f_e, \frac{L(v_e^h)\psi_{l,e}}{k^2} \right)_{L^2(\Omega_e)} \right). \tag{8}$$

This method is a “variationally” consistent finite element formulation and convergent for the homogeneous Helmholtz equation. However, this initial version is not necessarily convergent for all non homogeneous problems. Numerical experiments indicate that the method, built this way, it is not convergent for some source term, for example, when $\nabla^2 u_{exact} \neq 0$. Therefore, in order to obtain a FEM of the GPR class with uniform convergence properties a new projection term was included to the initial formulation. With this modification the GPR formulation remains “variationally” consistent and uniform convergence properties are recovered in all cases: homogeneous and non homogeneous

Helmholtz problems. The new term is introduced through a projection of residual gradient of the PDE, which can be formally represented by finding $u^h \in S^h$ that satisfies $\forall v^h \in V^h$ the variational equation

$$A(u^h, v^h) + \sum_{e=1}^{ne}\left(\sum_{l=1}^{\mathrm{N}} \tau_l^e \left(\left(L(u_e^h), \frac{L(v_e^h)\psi_{l,e}}{k^2} \right)_{L^2(\Omega_e)} + \underbrace{2\int_{\Omega_e} \frac{(\nabla L(u_e^h)\cdot \nabla L(v_e^h))\psi_{l,e}}{k^4} d\Omega}_{\text{new projection term}} \right)\right) = F_{GPR}(v^h),$$

$$F_{GPR}(v^h) = F(v^h) + \sum_{e=1}^{ne}\left(\sum_{l=1}^{\mathrm{N}} \tau_l^e \left(\left(f_e, \frac{L(v_e^h)\psi_{l,e}}{k^2} \right)_{L^2(\Omega_e)} + 2\int_{\Omega_e} \frac{(\nabla f_e)\cdot \nabla L(v_e^h))\psi_{l,e}}{k^4} d\Omega \right)\right). \tag{9}$$

Note that, a new GPR formulation is consistent, in sense that the exact solution of Eq. (5) is also solution of Eq. (9).

3.1 The element matrix

Let u_e^h be the restriction of u^h to Ω_e given by:

$$u_e^h = \sum_{m=1}^{npel} \hat{u}_e^h(m)\eta_m, \tag{10}$$

where $\hat{u}_e^h(m)$ denote the value of u_e^h in local node m of Ω_e element. Therefore, we have:

$$\left(L(u_e^h), \frac{L(\eta_i)\psi_{l,e}}{k^2} \right)_{L^2(\Omega_e)} + 2\int_{\Omega_e} \frac{(\nabla L(u_e^h)\cdot \nabla L(\eta_i))\psi_{l,e}}{k^4} d\Omega =$$

$$\sum_{m=1}^{npel} \hat{u}_e^h(m)\left[\left(L(\eta_m), \frac{L(\eta_i)\psi_{l,e}}{k^2} \right)_{L^2(\Omega_e)} + 2\int_{\Omega_e} \frac{(\nabla L(\eta_m)\cdot \nabla L(\eta_i))\psi_{l,e}}{k^4} d\Omega \right], \tag{11}$$

$$i = 1,\ldots,npel \text{ and } l = 1,\ldots,\mathrm{N}. \tag{12}$$

Consider M^l $(l = 1,\ldots,\mathrm{N})$ as being a set of *npel*×*npel* matrices defined as:

$$M_{ij}^l = \left(L(\eta_j), \frac{L(\eta_i)\psi_{l,e}}{k^2} \right)_{L^2(\Omega_e)} + 2\int_{\Omega_e} \frac{(\nabla L(\eta_j)\cdot \nabla L(\eta_i))\psi_{l,e}}{k^4} d\Omega. \tag{13}$$

Therefore,

$$\left(L(u_e^h), \frac{L(\eta_i)\psi_{l,e}}{k^2} \right)_{L^2(\Omega_e)} + 2\int_{\Omega_e} \frac{(\nabla L(u_e^h)\cdot \nabla L(\eta_i))\psi_{l,e}}{k^4} d\Omega = \sum_{m=1}^{npel} M_{im}^l \hat{u}_e^h(m), \tag{14}$$

and $\left[A^e_{GPR}\right]_{im}$ denoting entries of the element matrix detailed through

$$\left[A^e_{GPR}\right]_{im} = A^e(\eta_m, \eta_i) + \sum_{l=1}^{\mathrm{N}} \tau_l^e M^l_{im} \ . \tag{15}$$

We can notice that the element matrix is formed by the usual part of Galerkin plus a projected residual with the correspondent projected residual gradient of the differential equation at element level. A possible criterion to determine the free parameters $\tau_1^e, \ldots, \tau_{\mathrm{N}}^e$, corresponding to each projection of residual, consists of fitting the element matrix of GPR method to given matrix determined through some stability and/or convergence criteria. This matrix will be denominated GPR-generating matrix and denoted by M^{gen}. For Helmholtz equation with uniform mesh and bilinear quadrilateral elements we have the basis $\psi_1, \ldots, \psi_{\mathrm{N}}$ for $E_{GPR}(\Omega_e)$:

$$\begin{aligned}
&\psi_1 = +k^4 \eta_1 \eta_1, \\
&\psi_2 = +k^4 \eta_1 \eta_2, \\
&\psi_3 = +k^4 \eta_1 \eta_3, \\
&\psi_4 = +k^4 \eta_1 \eta_4, \\
&\psi_5 = +k^4 \eta_2 \eta_2, \\
&\psi_6 = +k^4 \eta_2 \eta_3, \\
&\psi_7 = +k^4 \eta_3 \eta_3, \\
&\psi_8 = +k^4 \eta_3 \eta_4, \\
&\psi_9 = +k^4 \eta_4 \eta_4,
\end{aligned} \tag{16}$$

$$\mathrm{N} = 9 \text{ and } \tau_l^e = \tau_l \ \ \forall \Omega_e . \tag{17}$$

With above conditions and Dirichlet boundary condition the element matrix M^{QS} that minimizes the phase error is associated to the stencil given in Babuška (1995). In this case, is interesting to choose the GPR-generating matrix as:

$$M^{gen} = \lambda_3 M^{QS}, \tag{18}$$

where λ_3 is a parameter that should be determined and the matrix M^{QS} is

$$M^{QS} = \begin{bmatrix} \tau_1^* & \tau_2^* & \tau_3^* & \tau_4^* \\ \tau_2^* & \tau_5^* & \tau_6^* & \tau_3^* \\ \tau_3^* & \tau_6^* & \tau_7^* & \tau_8^* \\ \tau_4^* & \tau_3^* & \tau_8^* & \tau_9^* \end{bmatrix}, \tag{19}$$

being that the τ_i^* can be determined though the standard dispersion analysis following the steps:

1) From typical dispersion analysis, a plane wave solution $e^{i\tilde{k}(x\cos\theta+y\sin\theta)}$ $(i=\sqrt{-1}, 0\le\theta\le\pi)$ propagating in the θ with wave number $\tilde{k}$ is imposed to the interior stencil of GPR, yielding

$$\bar{\tau}_0+\bar{\tau}_1\cos(\tilde{k}h\sin\theta)+\bar{\tau}_2\cos(\tilde{k}h\cos\theta)+\bar{\tau}_3\cos(\tilde{k}h\sin\theta)\cos(\tilde{k}h\cos\theta)=0\,, \tag{20}$$

$$\bar{\tau}_0=\tau_1^*+\tau_5^*+\tau_7^*+\tau_9^*\,, \tag{21}$$

$$\bar{\tau}_1=2(\tau_4^*+\tau_6^*)\,, \tag{22}$$

$$\bar{\tau}_2=2(\tau_2^*+\tau_8^*)\,, \tag{23}$$

$$\bar{\tau}_3=4\tau_3^*\,. \tag{24}$$

Notice that the parameters $\bar{\tau}_0,\bar{\tau}_1,\bar{\tau}_2$ and $\bar{\tau}_3$ depend on k but not on $\tilde{k}$. The stencil Eq. (20) is a linear algebric equation with four unknowns $\bar{\tau}_0,\bar{\tau}_1,\bar{\tau}_2$ and $\bar{\tau}_3$. Choosing two different directions θ_1 and θ_2 for the plane wave the interior stencil generates two linearly independent equations. Thus, two unknowns are still undetermined within the dispersion analysis.

2) Due to the mesh symmetric, the following restrictions for the free parameters can be imposed

$$\tau_1^*=\tau_5^*=\tau_7^*=\tau_9^*=\tfrac{1}{4}\Rightarrow\bar{\tau}_0=1\,, \tag{25}$$

$$\tau_4^*=\tau_6^*=\tfrac{1}{2}\bar{\tau}_1 \text{ and } \tau_2^*=\tau_8^*=\tfrac{1}{2}\bar{\tau}_2=\tfrac{1}{2}\bar{\tau}_1\Rightarrow\bar{\tau}_1=\bar{\tau}_2\,. \tag{26}$$

It should be emphasized that for uniform meshes only two free parameters are necessary to retrieve the optimal stencil obtained in Babuška (1995). For non uniform meshes these restrictions can not be imposed, since the mesh is not symmetrical. With the imposed restrictions, the interior stencil leads to

$$1+\bar{\tau}_1(\cos(\tilde{k}h\sin\theta)+\cos(\tilde{k}h\cos\theta))+\bar{\tau}_3\cos(\tilde{k}h\sin\theta)\cos(\tilde{k}h\cos\theta)=0\,. \tag{27}$$

3) Minimizing the phase error of the approximate solution, following the work (Babuška et al, 1995), yields

$$\bar{\tau}_1=\frac{(r_1-r_2)}{(r_2w_1-r_1w_2)}\,, \tag{28}$$

$$\bar{\tau}_3=\frac{(w_2-w_1)}{(r_2w_1-r_1w_2)}\,, \tag{29}$$

with

$$r_1=\cos(kh\cos\tfrac{\pi}{16})\cos(kh\sin\tfrac{\pi}{16})\,,\qquad r_2=\cos(kh\cos\tfrac{3\pi}{16})\cos(kh\sin\tfrac{3\pi}{16})\,, \tag{30}$$

$$w_1=\cos(kh\cos\tfrac{\pi}{16})+\cos(kh\sin\tfrac{\pi}{16})\,,\quad w_2=\cos(kh\cos\tfrac{3\pi}{16})+\cos(kh\sin\tfrac{3\pi}{16})\,, \tag{31}$$

Therefore, the GPR-generating matrix M^{gen} corresponds to the matrix given in Babuška (1995) is

$$M^{gen} = \lambda_3 \begin{bmatrix} \frac{1}{4} & \frac{\bar{\tau}_1}{2} & \frac{\bar{\tau}_3}{4} & \frac{\bar{\tau}_1}{2} \\ \frac{\bar{\tau}_1}{2} & \frac{1}{4} & \frac{\bar{\tau}_1}{2} & \frac{\bar{\tau}_3}{4} \\ \frac{\bar{\tau}_3}{4} & \frac{\bar{\tau}_1}{2} & \frac{1}{4} & \frac{\bar{\tau}_1}{2} \\ \frac{\bar{\tau}_1}{2} & \frac{\bar{\tau}_3}{4} & \frac{\bar{\tau}_1}{2} & \frac{1}{4} \end{bmatrix}. \tag{32}$$

To determine the nine free parameters τ_l^e in Eq. (15) by fitting the element matrix of GPR method to M^{gen} we considered the *J* functional defined as:

$$J = \sum_{m=1}^{npel} \sum_{i=1}^{npel} \left[A^e(\eta_m, \eta_i) + (\sum_{l=1}^{Denl} \tau_l M_{im}^l) - M_{im}^{gen} \right]^2 . \tag{33}$$

Due to the mesh symmetric, the following restrictions for the free parameters can be imposed again

$$\tau_1 = \tau_5 = \tau_7 = \tau_9 = 0, \tag{34}$$

$$\tau_2 = \tau_4 = \tau_6 = \tau_8 = \lambda_1, \tag{35}$$

$$\tau_3 = \lambda_2 . \tag{36}$$

Therefore, the *J* functional can be writing as

$$J = \sum_{m=1}^{npel} \sum_{i=1}^{npel} \left[A^e(\eta_m, \eta_i) + \lambda_1 (M_{im}^2 + M_{im}^4 + M_{im}^6 + M_{im}^8) + \lambda_2 M_{im}^3 - \lambda_3 M_{im}^{QS} \right]^2 . \tag{37}$$

Finally, for each Ω_e the parameters λ_1, λ_2 and λ_3 are determined solving the following system of algebraic equations

$$\frac{\partial J}{\partial \lambda_m} = 0, \; m = 1,2,3 . \tag{38}$$

It should be observed that for the parameters λ_1, λ_2, λ_3 and τ_l determined by Eqs. (34-36) and Eq. (38), the element matrix of GPR method coincides with the GPR generating matrix.

We should emphasize that the GPR formulation possesses a general methodology, that it is valid for any geometry and dimension of the domain, as well as, for any local approach space. A GPR method is derived for each particular choice of the set of free parameters τ_l^e. Usually, these parameters are determined through a dispersion analysis of the finite element approximation restricted to uniform meshes. In this sense, when a dispersion analysis is applied to the GPR method the stencil of the Quasi Stabilized Finite Element Method (Babuška et al, 1995) can be retrieved for a certain choice of the stabilization parameters. It's well known that the element matrix associated to this stencil minimizes the phase error in relation to uniform meshes. Also, it's well known that a FEM with two free parameters can retrieve this stencil. Since there is no notice about an optimal stencil for non-uniform meshes, choosing free parameters of stabilized finite element formulation applied to Helmholtz equation is an open question for unstructured meshes, in general. And the optimal values of the stabilization parameters, determined for uniform mesh, are surely not optimal for non-uniform meshes. If an element matrix associated to optimal stencil for non-uniform meshes were known (GPR-genetating matrix), the free parameters could be chosen to retrieve this stencil. Since non-uniform meshes are not symmetrical, in order to retrieve this kind of stencil one should expect a FEM formulation with more than two free parameters. In this case, a FEM with a greater number of free parameters has a greater capability to retrieve this optimal stencil.

4. NUMERICAL RESULTS

In the present section a number of examples to illustrate the main features and potential of GPR method applied to Helmholtz equation are presented. The first group the numerical test deals with homogenous Helmholtz equation and a second group the numerical test deals with inhomogeneous Helmholtz equation. These numerical tests show that uniform convergence properties are recovered in all cases: homogeneous and non homogeneous Helmholtz problems. In all examples a unity square domain, bilinear shape functions, 3x3 Gaussian integration, uniform mesh (160x160) and the same wave number (k = 100) are adopted.

The first group the numerical test deals with plane-waves propagating in 2-D domains. As the propagation direction is not known *a priori*, the free parameters are the ones computed in the previous section. Three 2-D examples are presented to show the importance of having a finite element formulation capable to minimize the phase error for homogenous Helmholtz equation. These examples illustrate as the accuracy and stability of some FEM with large phase errors (such as, Galerkin and GLS methods) deteriorate and compare them with stabilized formulations able to minimizing the phase error (such as, QS, DGB, RBFEM and GPR methods). It should be highlight that for uniform meshes and homogenous equation the solution of QS, DGB and GPR methods coincide.

The first example this group the numerical test have Dirichlet boundary conditions such that the exact solution is a plane-wave propagating in θ-direction: $u(x,y)=\cos(k(x\cos\theta+ysin\theta))$. In all examples of this group the numerical test the stabilization parameter of GLS method is determined by eliminating the phase error in the direction $\theta=\frac{\pi}{8}$, as proposed in (Thompson et al, 1995).

Figures 1 and 2 present a comparison between the relative errors in L^2-norm and H^1-seminorm of the GPR, continuous interpolant (CI) and QS solutions. In this case, the solution of the QS and GPR methods coincide. Fig. 3 shows the nodal interpolant, GPR and GLS solutions in sections $x=0.5$ along the y direction for $\theta=(\pi/4)$ for this example.

The next example is similar to previous example, but now the exact solution is given by a superposition of n mono-energetic plane-waves propagating in n different θ-directions: $u(x,y)=\sum_{i=1}^{n}\cos(k(x\cos\theta_i+ysin\theta_i))$. Firstly, three plane waves propagating in the directions $\theta_1=0$, $\theta_2=\frac{\pi}{8}$ and $\theta_3=\frac{\pi}{4}$ are considered. The relative errors in L^2-norm, H^1-seminorm and H^1-norm are present in Table 1. Figure 4 shows the nodal interpolant, GPR and GLS solutions in sections $x=0.5$ along the y direction. Figure 5 shows the same FEM solutions in section $y=0.5$ along the x direction. Again, the results show the good performance of the GPR formulation and how this formulation reduces the phase error over all wave vector orientations θ.

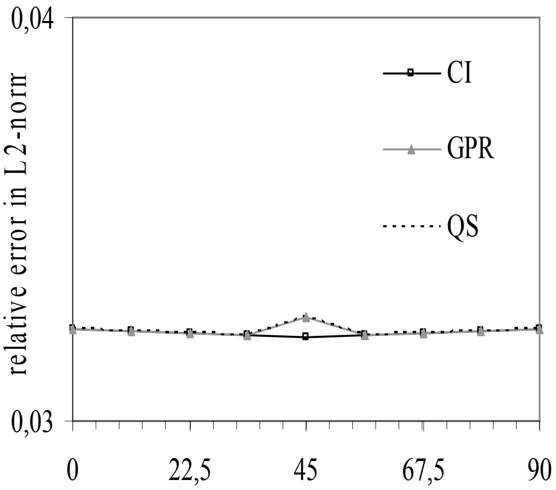

Fig. 1 Relative error of the CI, GPR and QS solutions in the L^2-norm as a function of θ-direction.

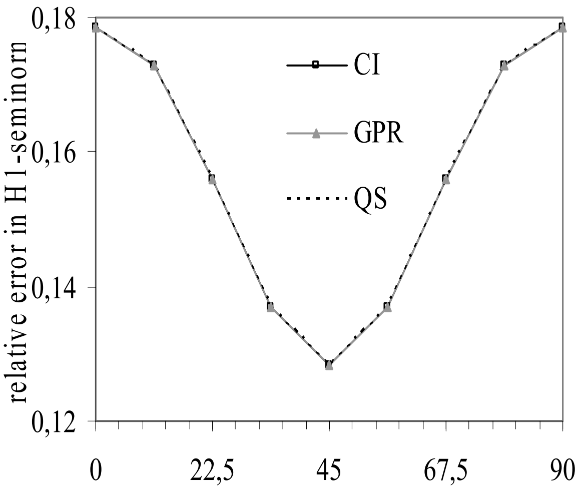

Fig. 2 Relative error of the CI, GPR and QS solutions in the H^1-seminorm as a function of θ-direction.

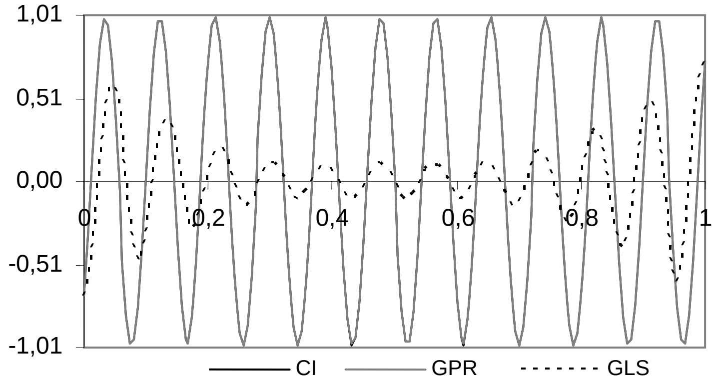

Fig. 3 Solution of homogeneous problem in two dimension at sections *x=0.5* for *θ=(π/4)*.

Table 1. Relative errors of FEMs for three and six plane waves

	Relative Errors of three finite element methods			
Three plane waves	CI	GMPR	GLS	Galerkin
L^2-norm	3.22E-02	3.23E-02	5.40E-01	1.71E+00
H^1-seminorm	1.56E-01	1.56E-01	5.59E-01	1.72E+00
H^1-norm	1.56E-01	1.56E-01	5.59E-01	1.72E+00
Six plane waves	CI	GMPR	GLS	Galerkin
L^2-norm	3.22E-02	3.23E-02	5.45E-01	3.24E+00
H^1-seminorm	1.56E-01	1.56E-01	5.69E-01	3.24E+00
H^1-norm	1.56E-01	1.56E-01	5.69E-01	3.24E+00

Secondly, six plane waves propagating in the directions $\theta_1 = 0$, $\theta_2 = \frac{\pi}{20}$, $\theta_3 = \frac{\pi}{10}$, $\theta_4 = \frac{3\pi}{20}$, $\theta_5 = \frac{\pi}{5}$ and $\theta_6 = \frac{\pi}{4}$ are considered. Figures 6 and 7 show the nodal interpolant, GPR and GLS solutions in sections *x=0.5* and *y=0.5* respectively. Very similar conclusions to the previous example can be drawn. We should observe that, in these two examples the directions of plane waves propagations are always different to $\theta_1 = \frac{\pi}{16}$ and $\theta_2 = \frac{3\pi}{16}$, which are the directions for asymptotically optimal interior stencil.

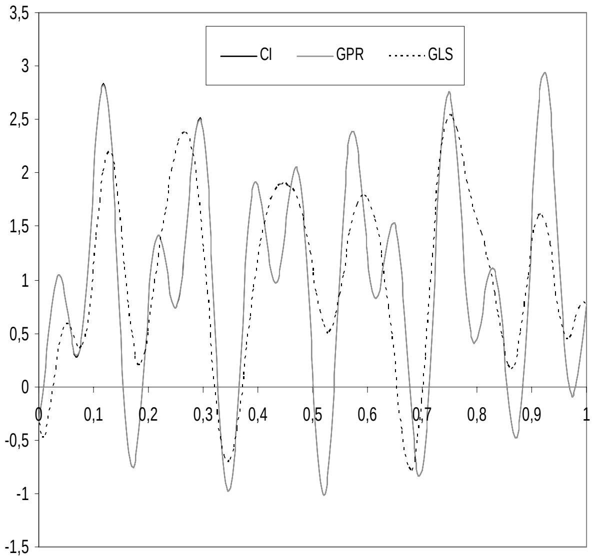

Fig. 4 GPR and GLS solutions of homogeneous problem in two dimension at sections *x=0.5*, three plane-waves.

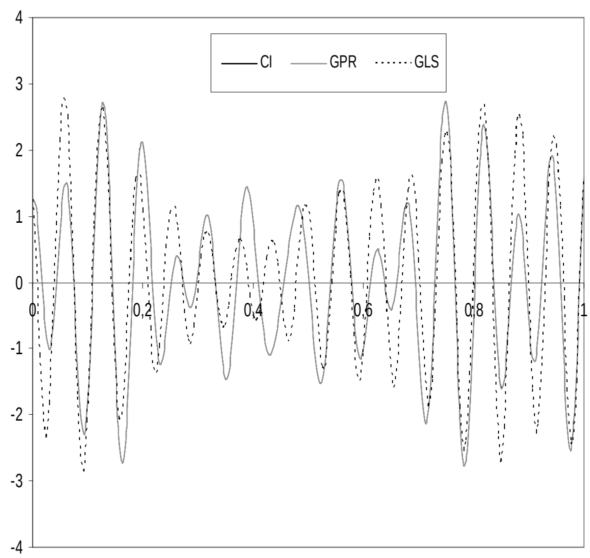

Fig. 5 GMPR and GLS solutions of homogeneous problem in two dimension at sections *y=0.5*, three plane-waves.

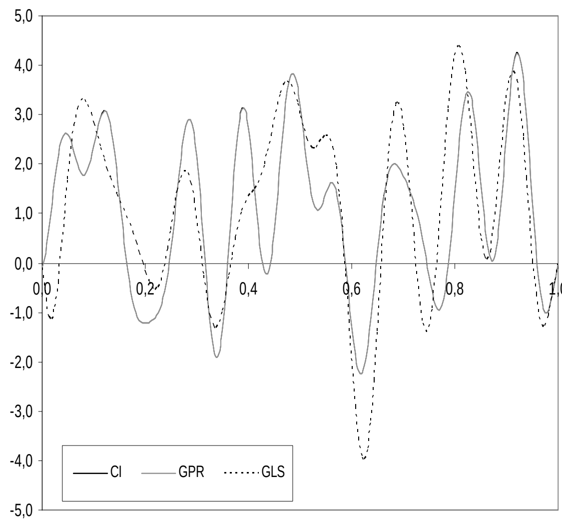

Fig. 6 GMPR and GLS solutions of homogeneous problem in two dimension at sections *x=0.5*, six plane-waves.

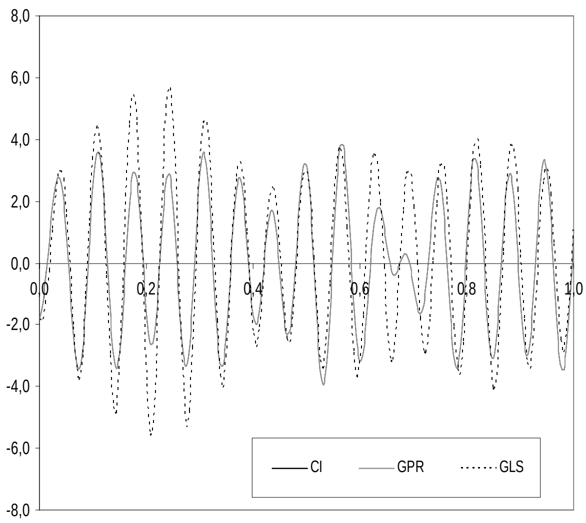

Fig. 7 GMPR and GLS solutions of homogeneous problem in two dimension at sections *y=0.5*, six plane-waves.

The second group the numerical test have source term (inhomogeneous Helmholtz equation) and Dirichlet boundary conditions such that the exact solution is a plane-wave propagating in θ-direction plus a polynomial function, $u(x,y) = p(x,y) + \sin(k(x\cos\theta + y\sin\theta))$. In the first example $p(x,y) = x + y$ (case 1), the second example $p(x,y) = x^2 + y^2$ (case 2) and the third example $p(x,y) = (1 + x + y)^3$ (case 3). That is, $f(x,y) = -k^2(x+y)$, $f(x,y) = -4 - k^2(x^2 + y^2)$ and $f(x,y) = -12(1+x+y) - k^2(1+x+y)^3$ respectively. In Fig. 8 and Fig. 9 the errors of the GPR method in L^2-norm and H^1-seminorm relative to the continuous bilinear interpolant are presented respectively. The GPR approximation is very close to the continuous interpolant for any θ-direction of plane-wave. Notice that for the case 1 the errors of the GPR method just has the part corresponding to the error of the plane wave, since the bilinear shape functions approximate the linear polynomial function exactly. Figure 10 shows the GPR solutions in sections $x = 0.5$ along the y direction obtained with $\theta = \frac{\pi}{4}$ for cases 1 and 2 of the source term. These results show clearly that the GPR solution is very close to the exact solution for this θ-direction of plane-wave which corresponds to the direction of largest phase lag for GPR approximation.

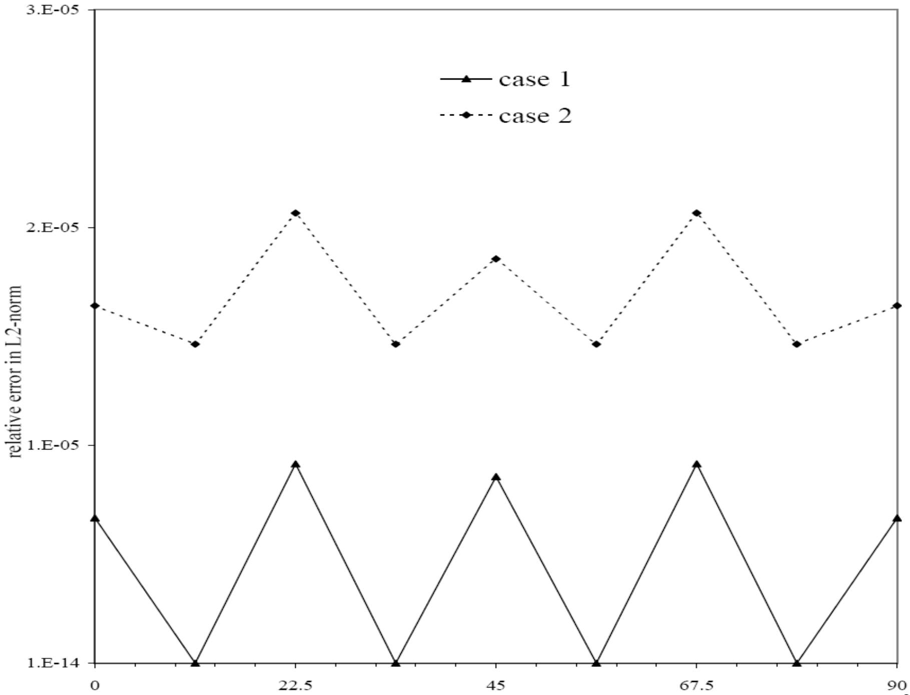

Fig. 8 Non homogeneous Helmholtz equation. Error of the GPR solutions in the L^2-norm as a function of θ-direction relative to continuous interpolant.

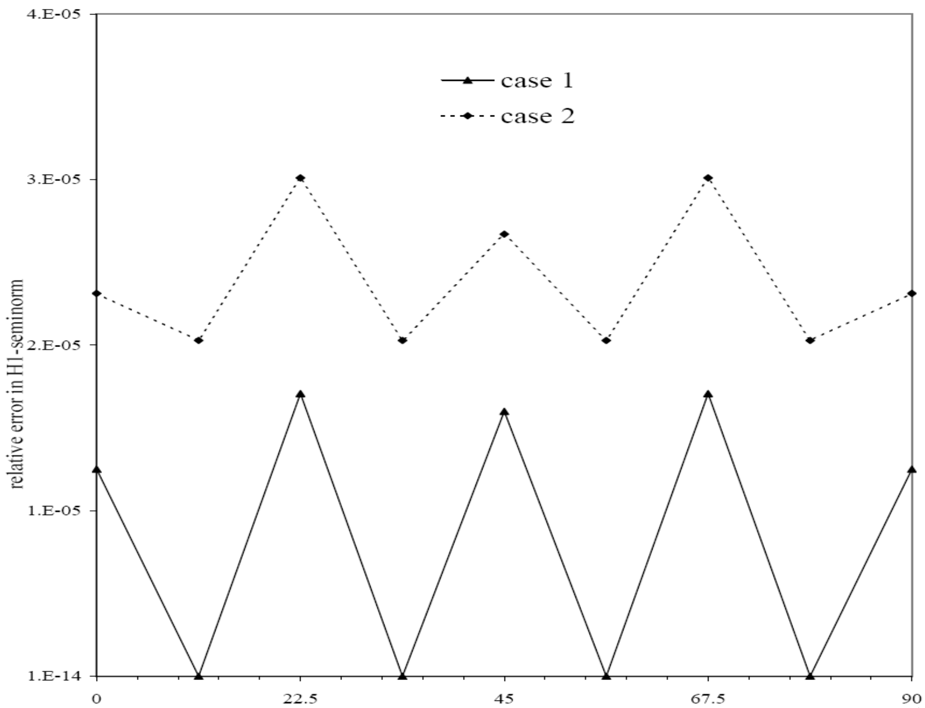

Fig. 9 Non homogeneous Helmholtz equation. Error of the GPR solutions in the H^1-norm as a function of θ-direction relative to continuous interpolant.

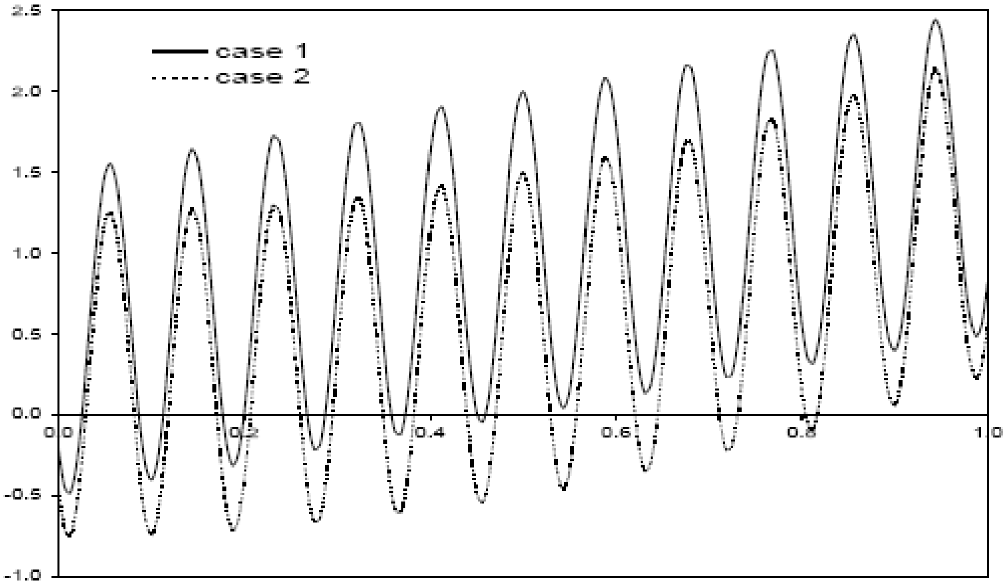

Fig. 10 Non homogeneous Helmholtz equation. GPR solutions in two dimensions at sections $x = 0.5$ for $\theta = \frac{\pi}{4}$.

A convergence study is carried out for non homogeneous Helmholtz equation and we observe uniform convergence, independently of the value of the wave number k. Fig. 11 present, for cases 2 and 3 with $k = 10$, the errors of the GPR solutions in the L^2-norm and H^1-seminorm as a function of h relative to continuous interpolant. A uniform refinement is employed starting with a (10x10) mesh until a (100x100) mesh. The results show the good rates of convergence for the GPR approximation.

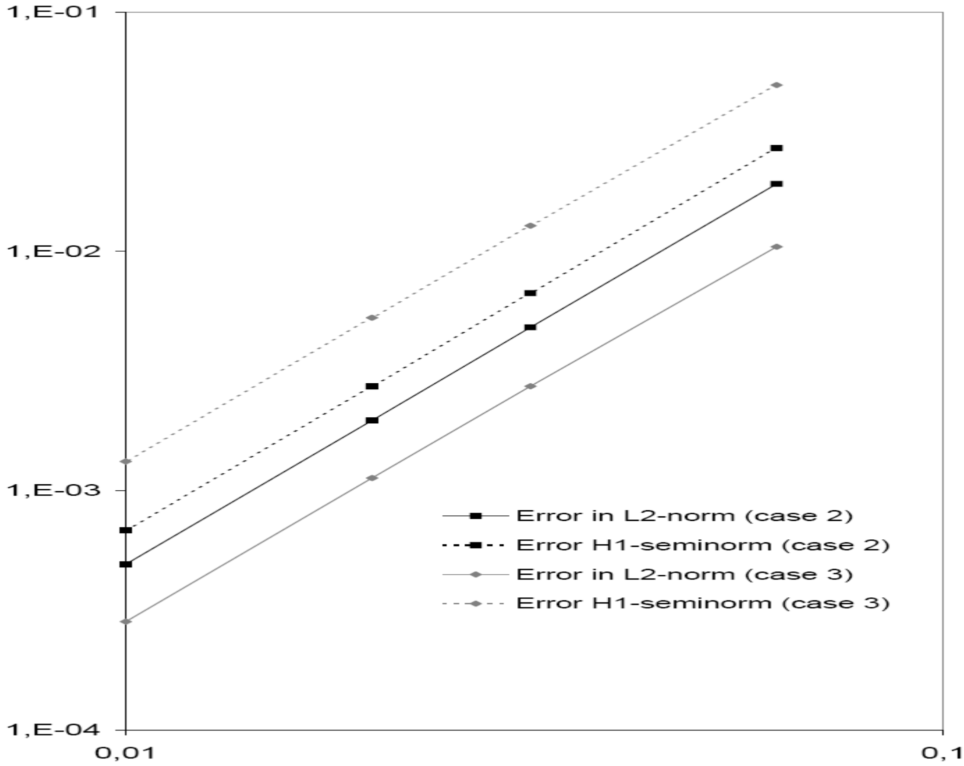

Fig. 11 Convergence study for non homogeneous Helmholtz equation. Error of the GPR solutions in the L^2-norm and H^1-seminorm as a function of h relative to continuous interpolant for cases 2 and 3 with $k = 10$.

5. CONCLUSIONS

We present a modified version of the GPR finite element formulation initially developed in Dutra do Carmo (2006) for Helmholtz equation. The initial GPR version was obtained adding to the Galerkin formulation an appropriate numbers of projections of the residual of PDE within each element. This allows that the element matrix has a maximum number of free parameters. Also, for rectangular domain, uniform mesh and bilinear elements, a methodology to choose these free parameters was developed. The criterion adopted to determine the free parameters consists of minimizing the phase error of the approximate solution. The initial version is a "variationally" consistent finite element formulation and convergent for the homogeneous Helmholtz equation. However, the initial version of the GPR method is not necessarily convergent for all non homogeneous problems.

Herein, we introduced a necessary modification to obtain a FEM of the GPR class with uniform convergence properties. The modification is introduced through a new projection term in the weak formulation: the residual gradient projection term. With this modification the GPR formulation remains "variationally" consistent and uniform convergence properties are recovered in all cases: homogeneous and non homogeneous Helmholtz problems.

The numerical simulations presented here emphasize the importance of having a FEM that minimizes the phase error consistently. Also, the numerical tests with source term show the good rates of convergence of the GPR formulation. The good performance of the modified formulation obtained for Helmholtz equation, stimulates to apply the GMPR method to other problems in future works.

Acknowledgements

The authors wish to thank the Brazilian research-funding agencies CNPq and FAPERJ for their support to this work.

REFERENCES

Alvarez, G. B., Loula, A. F. D., Dutra do Carmo, E. G. and Rochinha, F. A., 2006. A discontinuous finite element formulation for the Helmholtz equation. *Comput. Methods Appl. Mech. Engrg.,* vol. 195, pp. 4018-4035.

Babuška, I., Ihlenburg, F., Paik, E.T. and Sauter, S.A., 1995. A generalized finite element method for solving the Helmholtz equation in two dimensions with minimal pollution. *Comput. Methods Appl. Mech. Engrg.*, vol. 128, pp. 325-359.

Dutra do Carmo, E. G. and Duarte, A. V. C., 2002. New formulations and numerical analysis of discontinuous Galerkin methods. *Computational and Applied Mathematics*, vol. 21, No. 3, pp. 661-715.

Dutra do Carmo, E.G., Alvarez, G. B., Loula, A. F. D. and Rochinha, F. A., 2006. The Galerkin plus multiplies projection of residual method (GMPR) applied to Helmholtz equation. *Proceedings of the XXVII Iberian Latin-American Congress on Computational Methods in Engineering – CILAMCE 2006*.

Dutra do Carmo, E. G., Alvarez, G. B., Loula, A. F. D. and Rochinha, F. A., 2008. A nearly optimal Galerkin projected residual finite element method for Helmholtz problem. *Comput. Methods Appl. Mech. Engrg.*, vol. 197, pp. 1362-1375.

Farhat, C., Macedo, A.P. and Lesoinne, M., 2000. A two-level domain decomposition method for the iterative solution of high frequency exterior Helmholtz problems. *Numer. Math.*, vol. 85, pp. 283-308.

Farhat, C., Harari, I. And Franca, L., 2001. The discontinuous enrichment method. *Comput. Methods Appl. Mech. Engrg.*, vol. 190, pp. 6455-6479.

Farhat, C., Harari, I. and Hetmaniuk, U., 2003. A discontinuous Galerkin method with Lagrange multipliers for the solution of Helmholtz problems in the mid-frequency regime. *Comput. Methods Appl. Mech. Engrg.*, vol. 192, pp. 1389-1419.

Franca, L.P., Farhat, C., Macedo, A.P. and Lesoinne, M., 1997. Residual-free bubbles for the Helmholtz equation. *Int. J. Numer. Methods Engrg.*, vol. 40, pp. 4003-4009.

Franca, L. P. and Macedo, A. P., 1998. A two-level finite element method and its application to the Helmholtz equation. *Int. J. Numer. Methods Engrg.*, vol. 43, No. 1, pp. 23-32.

Harari, I. and Hughes, T.J.R., 1991. Finite element method for the Helmholtz equation in an exterior domain: Model problems. *Comp. Meth. Appl. Mech. Eng.*, vol. 87, pp. 59-96.

Harari, I. and Hughes, T.J.R., 1992. Galerkin/least squares finite element methods for the reduced wave equation with non-reflecting boundary conditions in unbounded domains. *Comp. Meth. Appl. Mech. Eng.*, vol. 98, pp. 411-454.

Hughes, T. J. R., 1995. Multiscale phenomena: Green's functions, the Dirichlet-to-Neumann formulation, subgrid scale models, bubbles and the origins of stabilized methods. *Comput. Methods Appl. Mech. Engrg.*, vol. 127, No. 1/4, pp. 387-401.

Ihlenburg, F. and Babuška, I., 1995. Finite element solution of the Helmholtz equation with high wave number Part I: The h-version of the FEM. *Comput. Math. Appli.*, vol. 30, No. 9, pp. 9-37.

Loula, A. F. D., Alvarez, G. B., Dutra do Carmo, E. G. and Rochinha, F. A., 2007. A discontinuous finite element method at element level for Helmholtz equation. *Comput. Methods Appl. Mech. Engrg.*, vol. 196, pp. 867-878.

Oberai, A. A. and Pinsky, P. M., 2000. A residual-based finite element method for the Helmholtz equation. *Int. J. Numer. Methods Engrg.*, vol. 49, pp. 399-419.

Thompson, L.L. and Pinsky, P.M., 1995. A Galerkin least squares finite element method for the two-dimensional Helmholtz equation. *Int. J. Numer. Methods Eng.*, vol. 38, No. 3, pp. 371-397.

Thompson, L.L. and Pinsky, P.M., 2004. Acoustics: Encyclopedia of Computational Mechanics. *Edited by E. Stein, R. de Borst and T. J. R. Hughes. Jonh Wiley & Sons, Ltda.*

Rochinha, F. A., Alvarez, G. B., Dutra do Carmo, E. G. and Loula, A. F. D., 2007. A locally discontinuous enriched finite element formulation for acoustics. *Commun. Numer. Meth. Eng.*, vol. 23, No. 6, pp. 623-637.