A STABILIZED FINITE ELEMENT FORMULATION FOR DIFFUSIVE-REACTIVE SINGULARLY PERTURBED EQUATION: THE GALERKIN PROJECTED RESIDUAL METHOD

Eduardo Gomes Dutra do Carmo
carmo@lmn.con.ufrj.br
Programa de Engenharia Nuclear, COPPE - Universidade Federal do Rio de Janeiro
Ilha do Fundão, 21945-970, P.B. 68509, Rio de Janeiro - RJ – Brazil
Gustavo Benitez Alvarez
benitez.gustavo@gmail.com
Departamento de Ciências Exatas, EEIMVR - Universidade Federal Fluminense
Av. dos Trabalhadores, 420, Vila Sta. Cecília, 27255-125, Volta Redonda - RJ – Brazil
Abimael Fernando Dourado Loula
aloc@lncc.br
Departamento de Matemática Aplicada e Computacional, Laboratório Nacional de Computação Científica
Av. Getúlio Vargas 333, 25651-070. P.B. 95113, Petrópolis – RJ - Brazil
Fernando Alves Rochinha
rochinha@adc.coppe.ufrj.br
Programa de Engenharia Mecânica, COPPE – Universidade Federal do Rio de Janeiro
Ilha do Fundão, 21945-970, P.B. 68501, Rio de Janeiro - RJ – Brazil

Abstract. *The Galerkin Projected Residual Method (GPR) is a finite element formulation developed to scalar and linear second-order boundary value problems. The method is obtained adding to the Galerkin formulation an appropriate numbers of projections of the residual of PDE within each element. These multiple projections allow the generation of appropriate number of free stabilization parameters in the element matrix depending on the local space of approximation and on the differential operator. The free parameters can be determined imposing some convergence and/or stability criteria or by postulating the element matrix with the desired stability properties. The element matrix of most stabilized methods (such as, GLS and GGLS methods) can be obtained from this new method with appropriate choices of the stabilization parameters. The GPR formulation has been applied with success to Helmholtz problem. In this work we applied the GPR method to diffusion-reaction singularly perturbed problem. In this case, the methodology to choose the free parameters consists in to postulate an element matrix with the desired stability properties (GPR-generating matrix) and the free parameters are determinate solving a least square problem at element level. The methodology is applicable to the both: uniforms and non uniform meshes with bilinear rectangular elements or linear triangular elements. Some numerical tests show the optimal rates of convergence for regular solutions and good stability of the GPR formulation in problems with sharp layer.*
Keywords: *Finite element method, Stabilization, GPR, GLS, diffusive-reactive equation*

1. INTRODUCTION

Boundary-value problems governed by second-order linear partial differential equations model several physical phenomena. Usually, the Galerkin Finite Element Method (FEM) is used to solve numerically these boundary value problems. However, only for purely diffusive problems the Galerkin method provides the optimal solution. In many other problems the Galerkin FEM is unstable and inaccurate, presenting spurious oscillations that do not correspond to the actual solution of the problem. Stable and accuracy numerical solution via FEM for these problems has been the greatest challenge. The reaction-diffusion equation is a representative example of the great effort has been devoted to obtain stable and accurate FEM. In the references we cite some representative works.

Here we will consider only continuous stabilized FEM for reaction-diffusion equation. Recently, a new continuous stable FEM was developed to scalar and linear second-order boundary value problems: the Galerkin Projected Residual Method. The method is obtained adding to the Galerkin formulation an appropriate numbers of projections of the residual of PDE within each element. These multiple projections allow the generation of appropriate number of free stabilization parameters in the element matrix depending on the local space of approximation and on the differential operator. The free parameters can be determined imposing some convergence and/or stability criteria or by postulating the element matrix with the desired stability properties. The element matrix of most stabilized methods (such as, GLS and GGLS methods) can be obtained from this new method with appropriate choices of the stabilization parameters.

The GPR formulation has been applied with success to Helmholtz problem (Dutra do Carmo et al, 2008). In this work we applied the GPR method to diffusion-reaction singularly perturbed problem (Dutra do Carmo, submitted). In this case, the methodology to choose the free parameters consists in to postulate an element matrix with the desired stability properties (GPR-generating matrix) and the free parameters are determinate solving a least square problem at element level. The methodology is applicable to the both: uniforms and non uniform meshes with bilinear rectangular elements or linear triangular elements. Some numerical tests show the optimal rates of convergence for regular solutions and good stability of the GPR formulation in problems with sharp layer.

2. THE REACTIVE-DIFFUSIVE EQUATION

2.1 The boundary value problem

Let $\Omega \subset R^n \quad (n \geq 1)$ be an open bounded domain with a Lipschitz continuous smooth piecewise boundary. Let Γ_g, Γ_q and Γ_r subsets of Γ satisfying $\Gamma_g \cap \Gamma_q = \Gamma_g \cap \Gamma_r = \Gamma_q \cap \Gamma_r = \varnothing$ and $\Gamma_g \cup \Gamma_q \cup \Gamma_r = \Gamma$. We shall consider the problem:

$$L(u) \equiv -\nabla \cdot (D\nabla u) + \sigma u = f \quad \text{in } \Omega, \tag{1}$$

$$u = g \quad \text{on } \Gamma_g, \tag{2}$$

$$D\nabla u \cdot \hat{n} = q \quad \text{on } \Gamma_q, \tag{3}$$

$$D\nabla u \cdot \hat{n} + \alpha u = r \quad \text{on } \Gamma_r. \tag{4}$$

where the functions D (diffusive coefficient) and σ (reactive coefficient) are assumed satisfy: $0 < D \leq \overline{D}$ and $0 < \sigma \leq \overline{\sigma}$ with $\overline{D}$ and $\overline{\sigma}$ being positive real constants. $f \in L^2(\Omega)$

is the source term, $g \in H^{\frac{1}{2}}(\Gamma_g) \cap C^0(\Gamma_g)$, $q \in L^2(\Gamma_q)$ and $r \in L^2(\Gamma_r)$ are the prescribed boundary conditions. The coefficient $\alpha \in L^{\infty}(\Gamma_r)$ is positive on Γ_r and $\hat{n}$ denotes the outward normal unit vector defined almost everywhere on Γ.

2.2 The associated variational problem

Let S and V defined as $S = \{u \in H^1(\Omega): u = g \text{ on } \Gamma_g\}$, $V = \{v \in H^1(\Omega): v = 0 \text{ on } \Gamma_g\}$ The variational problem associated to the boundary value problem defined by Eqs. (1-4) consist of finding $u \in S$ satisfying the following variational equation:

$$A(u,v) \equiv \int_{\Omega} [D\nabla u \cdot \nabla v + \sigma uv]\, d\Omega + \int_{\Gamma_r} \alpha uv\, d\Gamma = \int_{\Omega} f\, v\, d\Omega + \int_{\Gamma_q} qv\, d\Gamma + \int_{\Gamma_r} rv\, d\Gamma \equiv F(v) \quad \forall\, v \in V, \qquad (5)$$

The major challenges, in term of FEM, is to find a consistent formulation in continuous or discontinuous finite dimensional spaces, such that, its approximate solution is stable and the closest possible of the correspondent solution in infinite dimensional space given by Eq. (5). In the present work we pursue this goal for the diffusive reactive problem. A similar effort was developed in Dutra do Carmo (2008) for the Helmholtz equation.

2.3 The associated Galerkin finite element formulation

Let $M^h = \{\Omega_1, \ldots, \Omega_{NE}\}$ be a partition of Ω in no degenerated finite element Ω_e, such that Ω_e can be mapped in standard elements by isoparametric mapping and that satisfy $\Omega_e \cap \Omega_{e'} = \emptyset$ if $e \neq e'$ and $\Omega \cup \Gamma = \bigcup_{e=1}^{ne} (\Omega_e \cup \Gamma_e)$, where Γ_e denotes the boundary of Ω_e.

Let $p \geq 1$ an integer and consider $P^p(\Omega_e)$ defined as the space of polynomials of degree less than or equal to p. Let $H^h(\Omega) = \{\varphi \in H^1(\Omega); \varphi_e \in P^p(\Omega_e)\}$ and $H^{\frac{1}{2},h}(\Gamma_g) = \{\varphi \in H^{\frac{1}{2}}(\Gamma_g); \exists \phi \in H^1(\Omega) \text{ and } \phi = \varphi \text{ on } \Gamma_g\}$ are the finite dimension spaces and let g^h be the interpolate of g on $H^{\frac{1}{2},h}(\Gamma_g)$. The Galerkin formulation consists of finding $u^h \in S^h = \{\varphi \in H^h(\Omega); \varphi = g^h \text{ on } \Gamma_g\}$ that satisfies $\forall v^h \in V^h = \{\varphi \in H^h(\Omega); \varphi = 0 \text{ on } \Gamma_g\}$:

$$A(u^h, v^h) = F(v^h), \qquad (6)$$

For purely diffusive problems the solution of Galerkin FEM is the best approximation in the energy norm. It is well know that the Galerkin FEM is shown unstable and little accuracy for diffusion-reaction singularly perturbed equation. Its numerical solution presents spurious oscillations that do not corresponding with the physical solution of problem.

3. THE GALERKIN PROJECTED RESIDUAL METHOD

The GPR method was previously introduced for Helmholtz equation in Dutra do Carmo (2006 and 2008). This method was obtained adding to the Galerkin formulation an appropriate numbers of projections of the residual of PDE within each element. This allows that the element matrix has a maximum number of free parameters. Other theoretical details on the method can be found in Dutra do Carmo (submitted). The GPR method applied to reactive-diffusive equation can be formally statement as: find $u^h \in S^h$ satisfying $\forall v^h \in V^h$

$$A(u^h, v^h) + \sum_{e=1}^{ne}\left(\sum_{l=1}^{\mathrm{N}} \tau_l^e \left(L(u_e^h), L(v_e^h)\psi_{l,e}\right)_{L^2(\Omega_e)}\right) = F(v^h) + \sum_{e=1}^{ne}\left(\sum_{l=1}^{\mathrm{N}} \tau_l^e \left(f_e, L(v_e^h)\psi_{l,e}\right)_{L^2(\Omega_e)}\right), \qquad (7)$$

where N is the dimension of a local real linear space $E_{GPR}(\Omega_e)$ defined as $E_{GPR}(\Omega_e) = \{\psi : \Omega_e \to R; \psi = \sum_{i=1}^{npel}\sum_{j=1}^{npel} C_{i,j} L(\eta_i)L(\eta_j), \quad C_{i,j} \in R\}$ with basis denoted by $\psi_{l,e}$ and $npel$ denotes the number of nodal points of the element Ω_e and $\eta_i \ (i = 1, \ldots, npel)$ denotes the usual local shape functions associated to nodal point *i*. More details on $E_{GPR}(\Omega_e)$ and $\psi_{l,e}$ can be found in Carmo et al (2008 and submitted). The free stabilization parameters are denoted by τ_l^e. Note that, a new GPR formulation is consistent, in sense that the exact solution of Eq. (5) is also solution of Eq. (7).

3.1 The element matrix

Let u_e^h be the restriction of u^h to Ω_e given by:

$$u_e^h = \sum_{m=1}^{npel} \hat{u}_e^h(m)\eta_m , \qquad (8)$$

where $\hat{u}_e^h(m)$ denote the value of u_e^h in local node m of Ω_e element. Therefore, we have:

$$\left(L(u_e^h), L(\eta_i)\psi_{l,e}\right)_{L^2(\Omega_e)} = \sum_{m=1}^{npel} \hat{u}_e^h(m)\left(L(\eta_m), L(\eta_i)\psi_{l,e}^{enl}\right)_{L^2(\Omega_e)} , \qquad (9)$$

$$i = 1, \ldots, npel \text{ and } l = 1, \ldots, \mathrm{N} . \qquad (10)$$

Consider $M^l \ (l = 1, \ldots, \mathrm{N})$ as being a set of *npel*×*npel* matrices defined as:

$$M_{ij}^l = \left(L(\eta_j), L(\eta_i)\psi_{l,e}\right)_{L^2(\Omega_e)} . \qquad (11)$$

Therefore,

$$\left(L(u_e^h), L(\eta_i)\psi_{l,e}^{enl}\right)_{L^2(\Omega_e)} = \sum_{m=1}^{npel} \hat{u}_e^h(m) M_{im}^l , \qquad (12)$$

and $\left[A_{GPR}^e\right]_{im}$ denoting entries of the element matrix detailed through

$$\left[A_{GPR}^e\right]_{im} = A^e(\eta_m, \eta_i) + \sum_{l=1}^{\mathrm{N}} \tau_l^e M_{im}^l . \qquad (13)$$

We can notice that the element matrix is formed by the usual part of Galerkin plus a projected residual of the differential equation at element level. In Dutra do Carmo (submitted) is proof that the functions $\psi_{l,e}$ are linearly independent if and only if the N matrices M^l are linearly independent. This allows choosing an appropriate base for the space of matrices generated by the GPR method. A particular GPR method is derived for each specific choice of the set of free parameters $\tau_1^e,\ldots,\tau_{\mathrm{N}}^e$, corresponding to each projection of residual. A possible criterion to determine the free parameters consists of fitting the element matrix of GPR method to given matrix determined through some stability and/or convergence criteria. This matrix will be denominated GPR-generating matrix and denoted by M^{gen}. Then the components of the vector $\tau_1^e,\ldots,\tau_{\mathrm{N}}^e$ can be determined, for example, by solving the following minimization problem at element level:

$$\frac{\partial \mathrm{F}}{\partial \tau_m^e}=0,\ \ m=1,\ldots,\mathrm{N}\,;\quad \text{with } \mathrm{F}(M_{im}^{gen})=\sum_{i=1}^{npel}\sum_{j=1}^{npel}\left[(\sum_{l=1}^{\mathrm{N}}\tau_l^e M_{im}^{l})-M_{im}^{gen}\right]^2 . \tag{14}$$

Considering that successful stabilized finite element methods have been already applied to reaction diffusion problems, such as the Gradient Galerkin Least Squares (GGLS) and the Unusual Stabilization (USFEM), we design our method departing from a nontrivial combination of both. Indeed, the possibility of directly combining two stabilizing formulations was explored in Dutra do Carmo et al (submitted), Valentin et al (1995) and Hauke et al (2001and 2002) aiming at obtaining their best features. Here we built the GPR generating matrix through the weighted stiffness matrix inspired on the GGLS method

$$K_{ij}^e=\int_{\Omega_e}\chi^{e,2}\sigma(\mathbf{J}\nabla\eta_j)\cdot(\mathbf{J}\nabla\eta_i)d\Omega\,, \tag{15}$$

and also using the stabilization matrix of the USFEM method to introduce the weighted mass matrix

$$B_{ij}^e=-\int_{\Omega_e}\chi^{e,1}\sigma\eta_j\eta_i d\Omega\,, \tag{16}$$

where $\chi^{e,1}$ and $\chi^{e,2}$ are dimensionless functions, understood as the weights of the nontrivial combination mentioned above, and $\mathbf{J}$ is the Jacobian matrix corresponding to the mapping between reference and actual elements. The resulting GPR-generating matrix was adopted to reproduce the ability of the GGLS method in capturing thin sharp layers along with the stable behavior obtained by the USFEM when applied to problems where those layers are no longer confined to thin regions. Indeed, as will be confirmed by the numerical experiments reported in the next section, by exploring this combination the GPR method developed here achieves optimal convergence even in the presence of sharp gradients. Figure 1 presents numerical results illustrating typical instability of Galerkin approximations for a predominantly reactive reaction diffusion problem. We clearly observe the spurious oscillations close to the boundary layers compared to the nodally exact solution presented in Fig. 2. The well known GGLS stabilization is capable to reduce these oscillations as shown in Figures 3 and 4. We also observe the improved performance of GGLS method with bilinear elements (Fig. 3) compared to linear elements (Fig. 4).

It is also worth mentioning that the expression inspired on the GGLS method was used above as, indeed, Eq. (15) differs fundamentally from the original form of the stabilizing GGLS term due to the presence of the Jacobian $\mathbf{J}$ replacing the h^2 in order to handle distorted elements. This is also confirmed by the numerical tests. Considering the definition of the Jacobian matrix $\mathbf{J}$, we observe that K_{ij}^{e} can be equivalently given by

$$K_{ij}^{e} = \int_{\Omega_e} \chi^{e,2} \sigma \left(\mathbf{J}\mathbf{J}^{-1} \nabla_{loc} \eta_j\right) \cdot \left(\mathbf{J}\mathbf{J}^{-1} \nabla_{loc} \eta_i\right) d\Omega = \int_{\Omega_e} \chi^{e,2} \sigma \nabla_{loc} \eta_j \cdot \nabla_{loc} \eta_i d\Omega \tag{17}$$

with

$$\nabla_{loc} \eta = \left(\frac{\partial \eta}{\partial \xi_1}, \ldots, \frac{\partial \eta}{\partial \xi_1} \right), \tag{18}$$

for quadrilateral elements and hexahedron elements, and

$$\nabla_{loc} \eta = \left(\frac{\partial \eta}{\partial \mathrm{L}_1} - \frac{\partial \eta}{\partial \mathrm{L}_{nt}}, \ldots, \frac{\partial \eta}{\partial \mathrm{L}_n} - \frac{\partial \eta}{\partial \mathrm{L}_{nt}} \right), \quad (nt = n+1), \tag{19}$$

for triangular elements and tetrahedral elements.

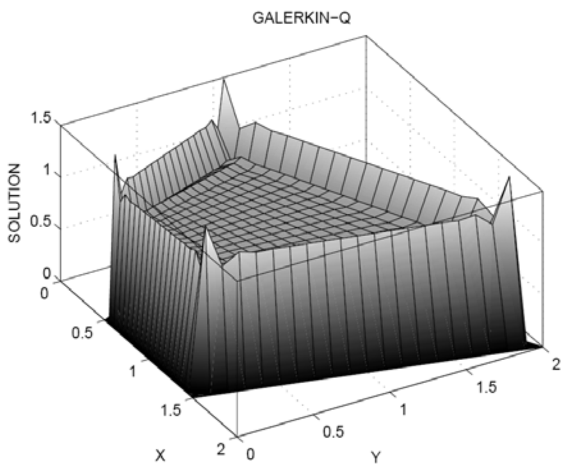

Fig. 1 Galerkin - Bilinear Elements

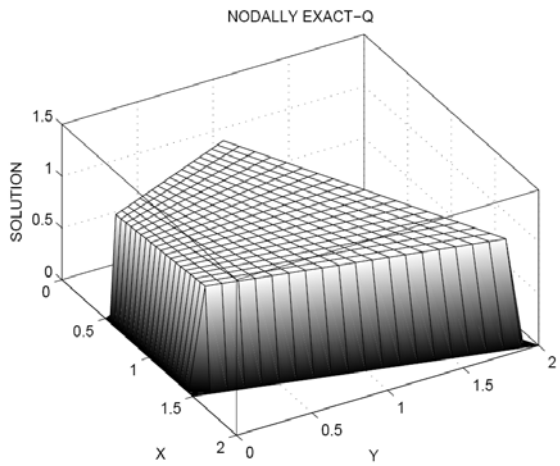

Fig. 2 Nodally Exact

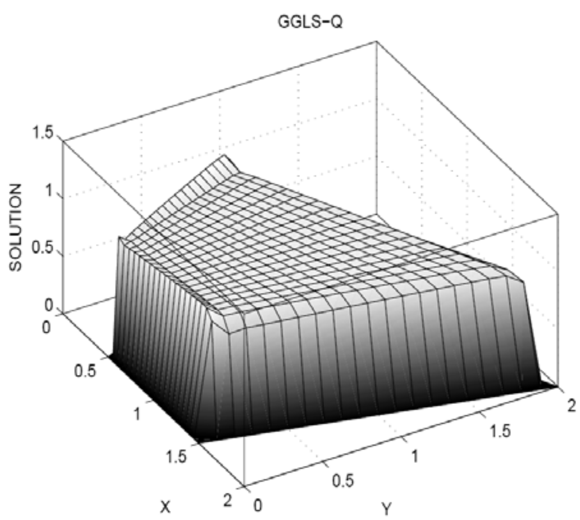

Fig. 3 GGLS - Bilinear Elements

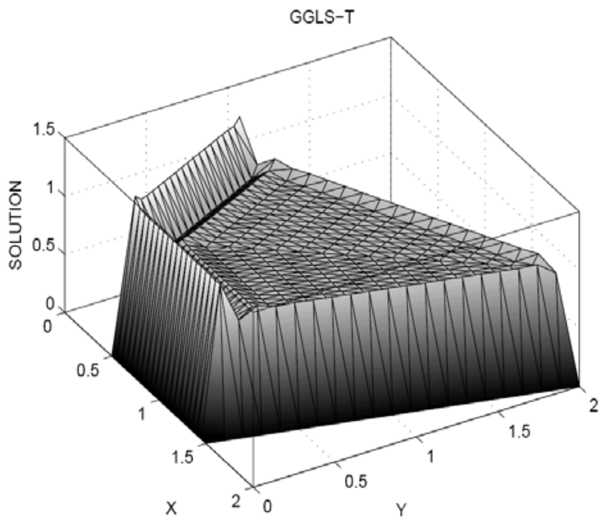

Fig. 4 GGLS - Bilinear Elements

For the elements Ω_e such that $M^{gen,e} \neq \mathbf{0}_{matrix}$ the components of the vector $\tau^e = (\tau_1^e, \ldots, \tau_{\mathrm{N}}^e)$ are determined as being the solution of the minimization problem given by Eq. (14) considering the matrix $M^{gen,e}$ given by

$$M^{gen,e} = K^e + B^e \tag{20}$$

with

$$\chi^{e,1} = \varsigma^{e,0} \chi^e \left|1 - \chi^e\right|^{\left(\frac{1}{1-\chi^e}\right)}, \tag{21}$$

$$\chi^{e,2} = (\chi^e)^{\left(\frac{1}{1-\chi^e}\right)} \varsigma^{e,2} \varsigma^{e,0}, \tag{22}$$

$$\chi^e = \frac{1}{\chi^{e,0}(P_{reat}^e) + P_{reat}^e}, \tag{23}$$

$$P_{reat}^e = \frac{6D}{\sigma (h_e)^2}, \tag{24}$$

$$\chi^{e,0}(P_{reat}^e) = \begin{cases} 1 & \text{if } P_{reat}^e \leq 1 \\ P_{reat}^e & \text{if } P_{reat}^e > 1 \end{cases}, \tag{25}$$

$$h_e = \left(\int_{\Omega_e} d\Omega \right)^{\frac{1}{n}}. \tag{26}$$

We still need to determine the real constant $\varsigma^{e,0}$ and the dimensionless function $\varsigma^{e,2}$. To this end, for each Ω_e and for each $\Omega_{e'}$ we consider $[\varphi]_{ee'}$ being defined as follows:

$$[\varphi]_{ee'} = \int_{\Gamma_e \cap \Gamma_{e'}} \left|\varphi_e - \varphi_{e'}\right| d\Gamma, \quad \varphi \in L^2(\Omega) \text{ and } \varphi_e \in H^1(\Omega_e) \tag{27}$$

which keeps track of possible discontinuities across element edges. We also introduce Γ_{int} defined as

$$\Gamma_{\text{int}} = \bigcup_{e=1}^{ne} \left(\bigcup_{e'=1}^{ne} \Gamma_{ee'}^* \right), \tag{28}$$

$$\Gamma_{ee'}^* = \Gamma_{e'e}^* = \begin{cases} \Gamma_e \cap \Gamma_{e'} & \text{if } ([f]_{ee'} \neq 0 \text{ or } [\sigma]_{ee'} \neq 0 \text{ or } [D]_{ee'} \neq 0) \\ \varnothing & \text{if } ([f]_{ee'} = 0 \text{ and } [\sigma]_{ee'} = 0 \text{ and } [D]_{ee'} = 0) \end{cases}, \tag{29}$$

which is the union of the external boundary with the internal edges between two elements presenting discontinuous properties or sources. It should be observed that for diffusive reactive problems, sharp layers will only occur inside an element Ω_e if $\Gamma_e \cap (\Gamma \cup \Gamma_{\text{int}}) \neq \varnothing$.

Based on this observation and inspired on references Franca et al (1989) and Franca et al (2005), we accomplished a large number of computational experiments with bilinear rectangular elements and linear triangular elements and conclude that the following expressions for the real constant $\varsigma^{e,0}$ and the dimensionless function $\varsigma^{e,2}$ present very good stability and accuracy properties:

$$\varsigma^{e,0} = \begin{cases} 0 \ \text{ if } \Gamma_e \cap (\Gamma \cup \Gamma_{\text{int}}) = \varnothing \\ 1 \ \text{ if } \Gamma_e \cap (\Gamma \cup \Gamma_{\text{int}}) \neq \varnothing \end{cases}, \tag{30}$$

$$\varsigma^{e,2} = \begin{cases} \varsigma^{e,q} & \text{if } \Omega_e \text{ is a quadrilateral element or is a hexahedron element} \\ \varsigma^{e,q} & \text{if } \Omega_e \text{ is a triangular element or is a tetrahedral element} \end{cases}, \tag{31}$$

where $\varsigma^{e,q}$ for bilinear quadrilateral element and trilinear hexahedron element and $\varsigma^{e,t}$ for linear triangular element and linear tetrahedral element are respectively data as follows:

$$\varsigma^{e,q} = \begin{cases} 2/3 + (N^e_{face} - 1)/6 & \text{if } \ meas(\Gamma \cap \Gamma_e) > 0 \\ 2/3 & \text{if } \ meas(\Gamma \cap \Gamma_e) = 0 \end{cases}, \tag{32}$$

$$\varsigma^{e,t} = \begin{cases} \varsigma^{e,q} & \text{if } \ \Gamma_{\mathrm{e}}(\Gamma \cup \Gamma_{\text{int}}) = \varnothing \\ (P^e_{reat})^{-(\chi^e+1)} & \text{if } \ \Gamma_{\mathrm{e}}(\Gamma \cup \Gamma_{\text{int}}) \neq \varnothing \end{cases}, \tag{33}$$

where N^e_{face} is the number of faces of Ω_e contained in Γ.

From Eq. (30) we observe that the extra computational effort demanded by the proposed GPR formulation compared to the Galerkin method is not significant as it corresponds basically to the calculations of the stabilization matrices and forcing vectors of the elements Ω_e such that $\Gamma_e \cap (\Gamma \cup \Gamma_{\text{int}}) \neq \varnothing$. It must be emphasized that those elements are mapped beforehand, which significantly reduces the computational burden. Also, for GPR finite element approximations with polynomials of degree bigger than 1, additional numeric experiments need to be accomplished to validate the proposed expressions for $\varsigma^{e,q}$ and $\varsigma^{e,t}$.

4. NUMERICAL RESULTS

To assess the overall performance of the proposed GPR method, a comprehensive number of numerical tests were carried out. Special emphasis was placed in sharp layers and distorted meshes, which often are present on real applications. We will only describe, along their corresponding results, the most significant numeric experiments.

The assessment of our method was accomplished through the use of examples with exact solutions, and comparisons with well known stabilized formulations were also considered here. We will denote by "EMM" (Enriched Multiscale Method) the method presented in reference Franca et al (2005), "USFEM" (Unusual Stabilization) the method presented in Franca et al (2000) and "ASGS" (Algebraic Subgrid Scales) the method introduced in Codina (2000) and Hauke (2002).

4.1 Quadrilateral domain using non uniform meshes

This experiment demonstrates the performance of GPR method when applied to a reactive dominant problem defined over a quadrilateral domain of vertexes (0.5, 0.0), (1.5, 0.0), (2.0, 2.0) and (0.0, 1.0) with $D = 10^{-6}$, $\sigma = 1$, $f = 1$ and homogeneous Dirichlet boundary conditions.

Special emphasis is placed on the use of non uniform meshes. Results were obtained for the mesh of quadrilateral elements shown in the Fig. 5 and for the mesh of triangular elements shown in the Fig. 6. Results in 3D plots for “USFEM”, “ASGS”, “GPR” and nodally exact solutions are presented in Figures 7, 8, 9 and 10, respectively, for the mesh of 20x20 quadrilateral elements. As reference Franca et al (2005) doesn’t present the corresponding formulation for distorted quadrilateral elements, we didn’t present results for the method denominated “EMM”. Clearly, we can observe the great performance of the “GPR” method, showing that the effects of the distortion on the elements do not cause loss of accuracy and stability, maintaining the accuracy and the stability observed with uniform meshes. However, the effect of the distortion on the elements is clearly observed for the methods “USFEM” and “ASGS”, with evident losses of accuracy and stability. Figures 11 and 12 present 2D plots comparing these solutions in two sections.

Similar results in 3D plots are presented in Figures 13, 14, 15 and 16 for the mesh of 800 triangular elements. Once more, we can observe the great accuracy and stability of the methods “GPR” and presenting equivalent performance, and indicating again that the effects of the distortion on the elements do not cause loss of accuracy and stability. Again, it is clear the effect of the distortion on the elements for the methods “USFEM” and “ASGS”, with remarkable losses of accuracy and stability. In this case the “EMM” method, not shown here, presents performance to the “GPR” method. Figures 17 and 18 present 2D plots comparing these solutions in two sections.

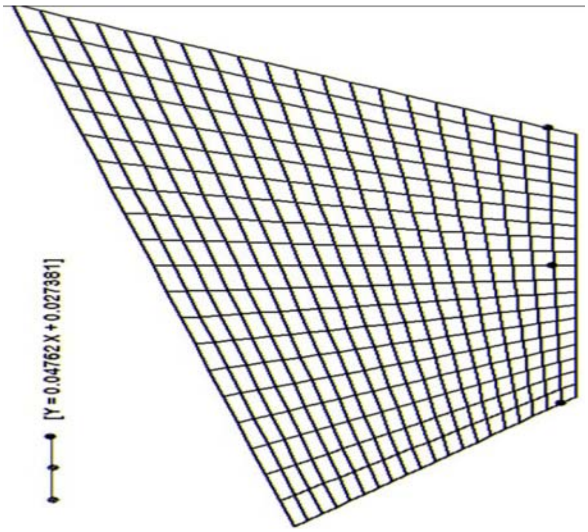

Fig. 5 Nonuniform mesh – Quadrilaterals

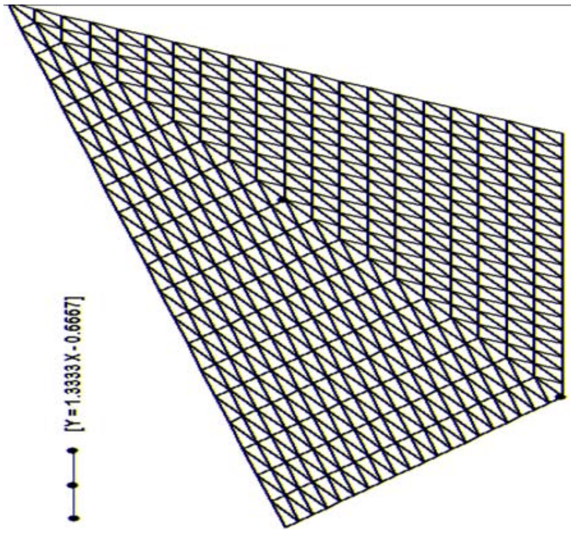

Fig. 6 Nonuniform mesh - Triangles

4.2 Convergence Study

The second numerical experiment consists of obtaining the convergence rates expressed in terms of $L^2(\Omega)$ and $H^1(\Omega)$ norms for the problem defined over the quadrilateral domain given in Section 4.1 above, with $\sigma = 1$, $f = (2\pi^2 D + 1)\sin(\pi x)\sin(\pi y)$ and boundary conditions $u = \sin(\pi x)\sin(\pi y)$ on Γ. The results were obtained for quadrilateral meshes with partitions 10x10, 20x20, 40x40, 80x80 and 120x120 and for triangular meshes with 400, 1600, 3600, 6400 and 10000 elements. The convergence results are presented in graphs $-\log(Hmesh) \times \log\|Error\|$ in the natural basis, where $Hmesh = (ne)^{1/2}$, $\|Error\| = \|u - u^h\|_{L^2(\Omega)}$ or $\|Error\| = |u - u^h|_{H^1(\Omega)}$ with *ne* denoting the number of elements, *n* the dimension of the domain and $\left|\circ\right|_{H^1(\Omega)}$ the H^1–seminorm. The quadrilateral mesh 20x20 is shown in the Fig. 5 and the triangular mesh with 800 elements in the Fig. 6. The convergence study was performed for different values of the diffusion coefficient, e.g.: $D=1$, $D=10^{-3}$ and $D=10^{-6}$. Results for quadrilateral meshes are presented in Figures 19, 21 and 23 and for triangular meshes are presented in the Figures 20, 22 and 24, for different values of the diffusion coefficient D. Clearly, for all values of D tested, the GPR method presents optimum rates of convergence.

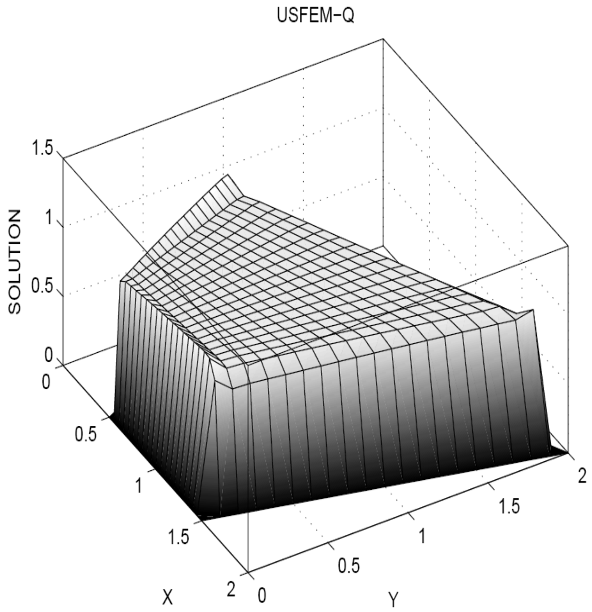

Fig. 7 USFEM - Bilinear Elements

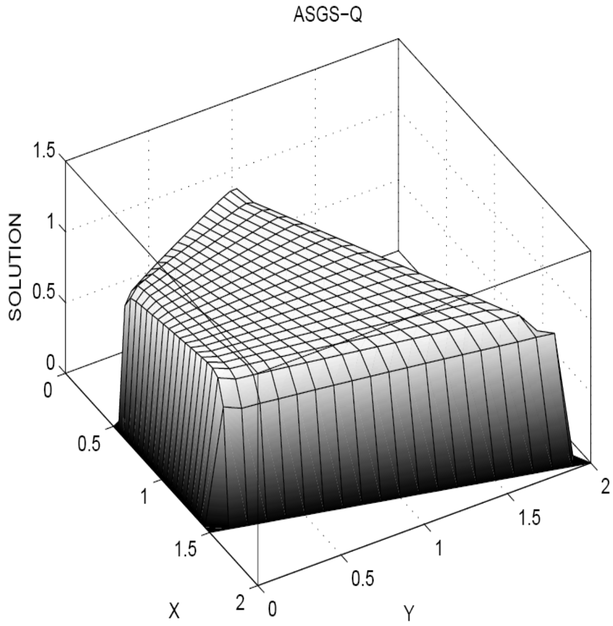

Fig. 8 ASGS - Bilinear Elements

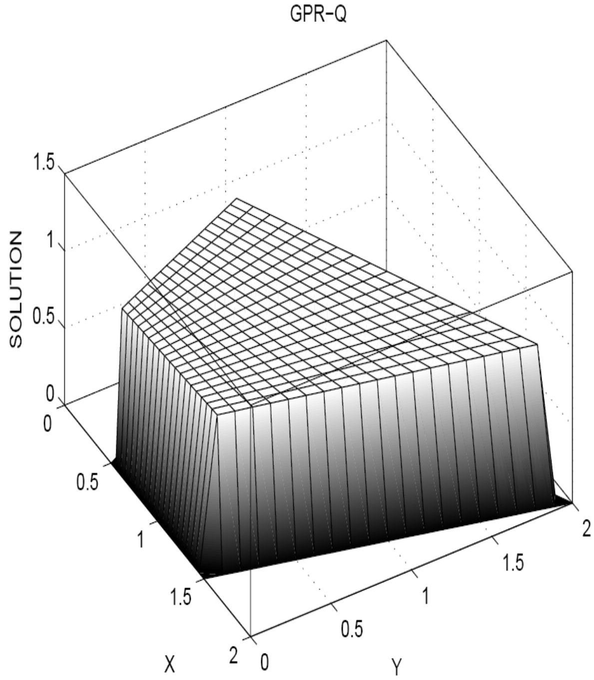

Fig. 9 GPR - Bilinear Elements

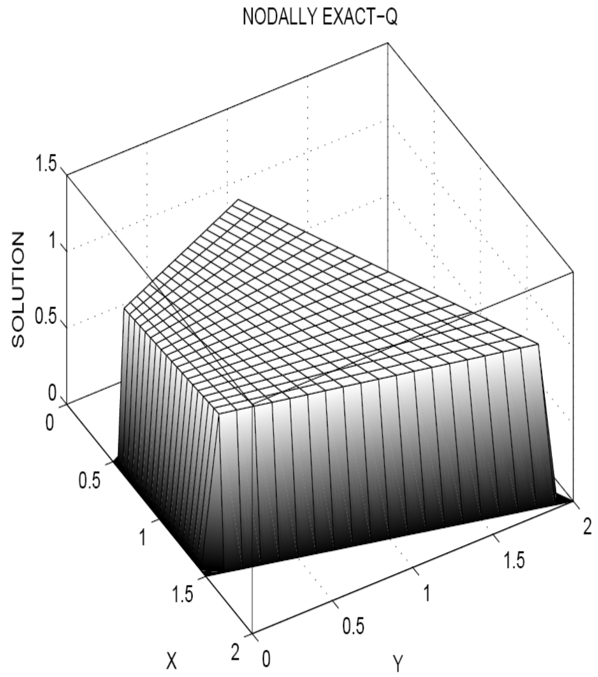

Fig. 10 Nodally Exact

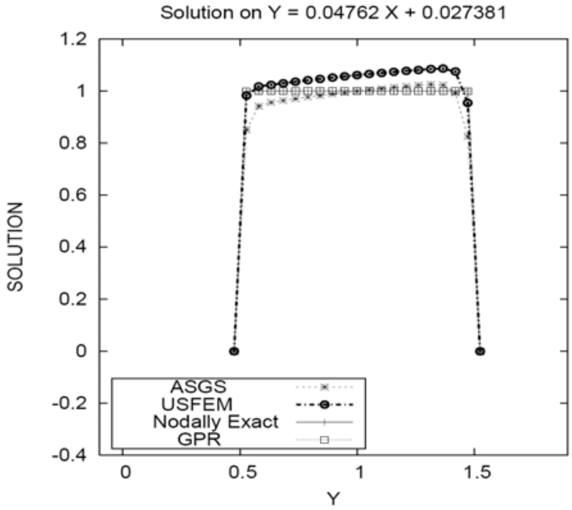

Fig. 11 Bilinear - inclined section

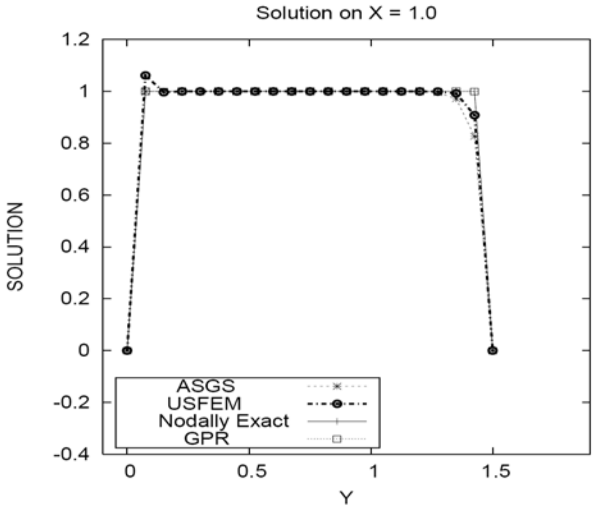

Fig. 12 Bilinear - vertical section

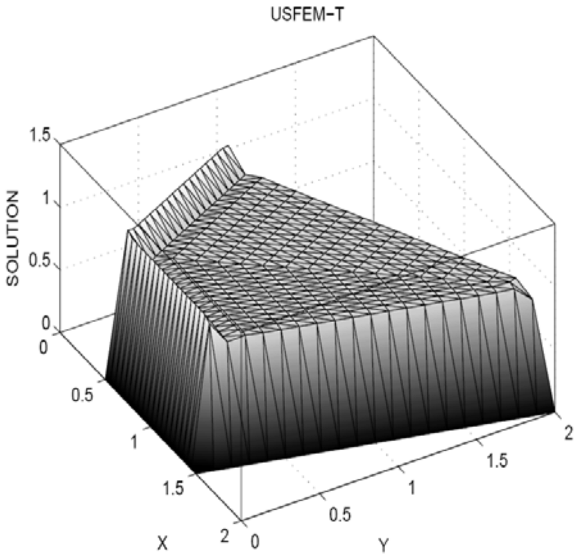

Fig. 13 USFEM - Linear Elements

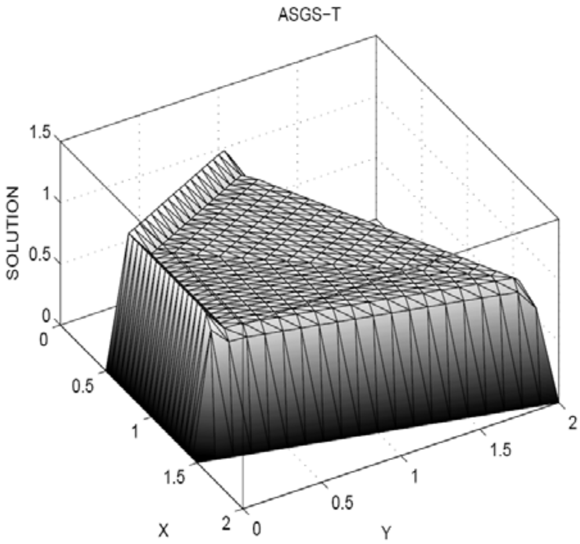

Fig. 14 ASGS - Linear Elements

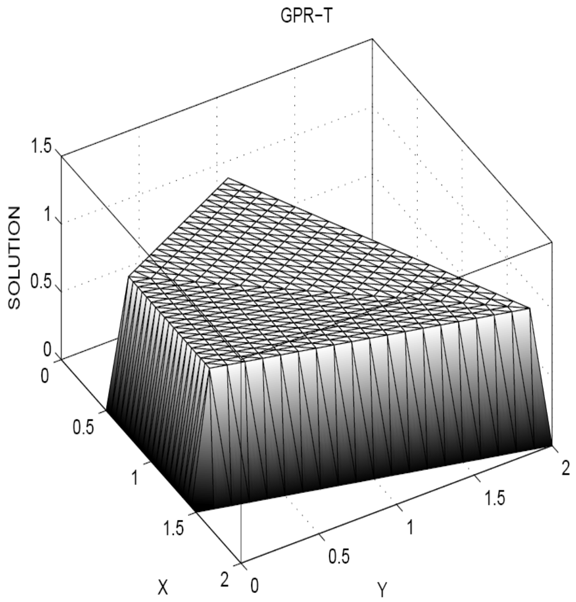

Fig. 15 GPR - Linear Elements

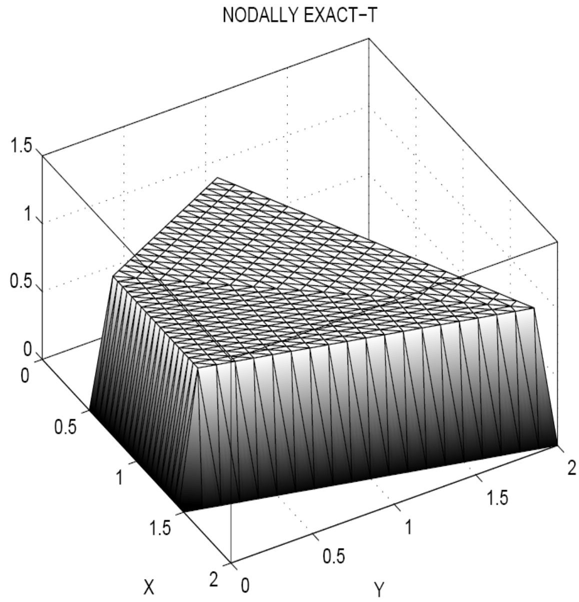

Fig. 16 Nodally Exact

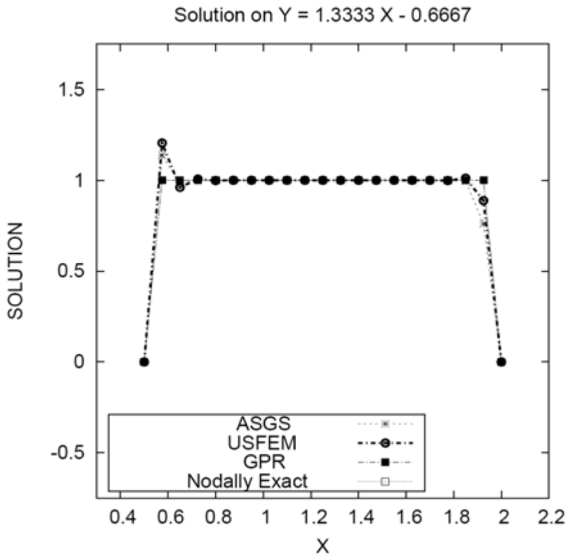

Fig. 17 Linear - inclined section

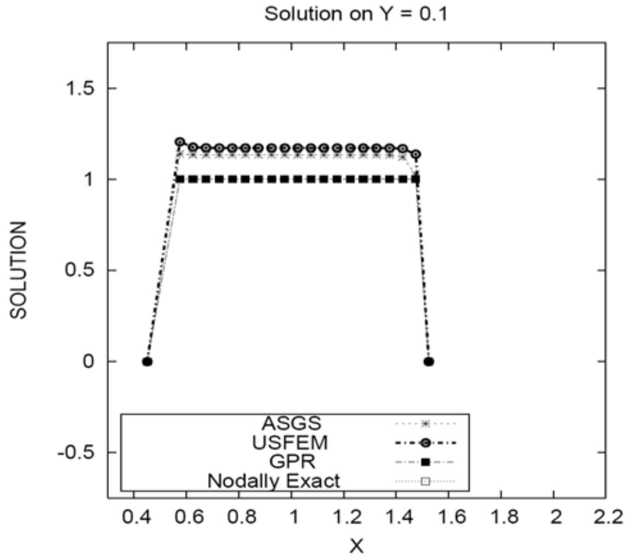

Fig. 18 Linear - horizontal section

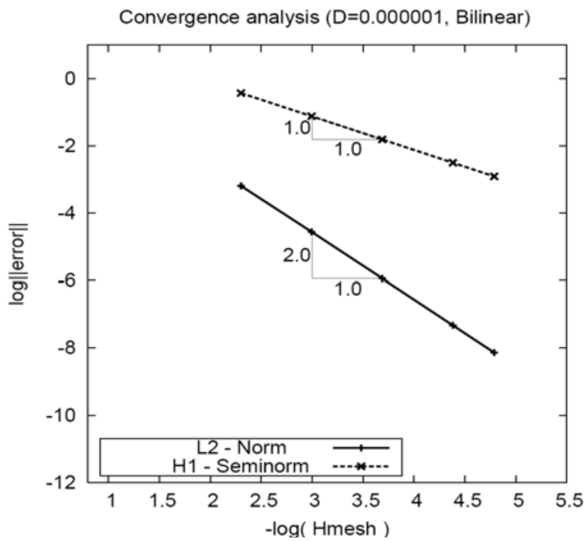

Fig. 19 Bilinear Elements

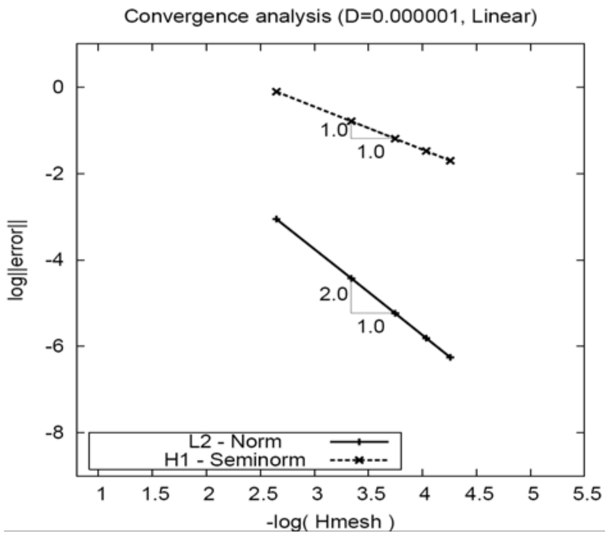

Fig. 20 Linear Elements

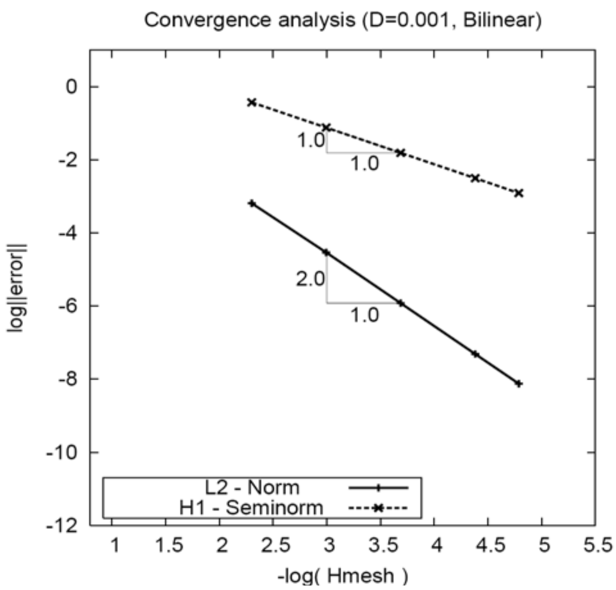

Fig. 21 Bilinear Elements

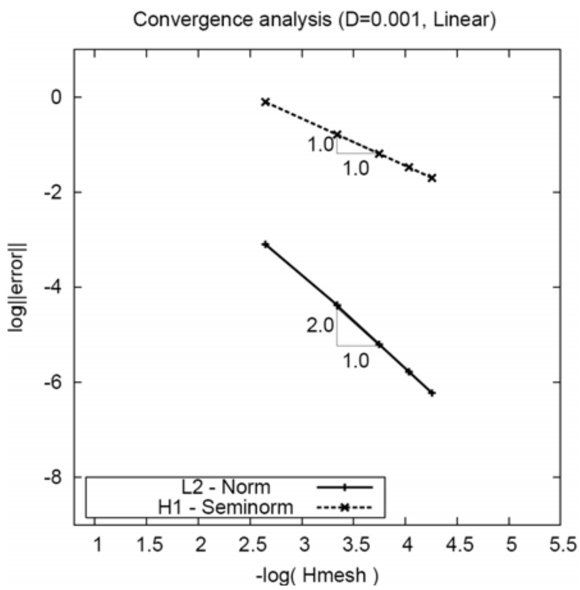

Fig. 22 Linear Elements

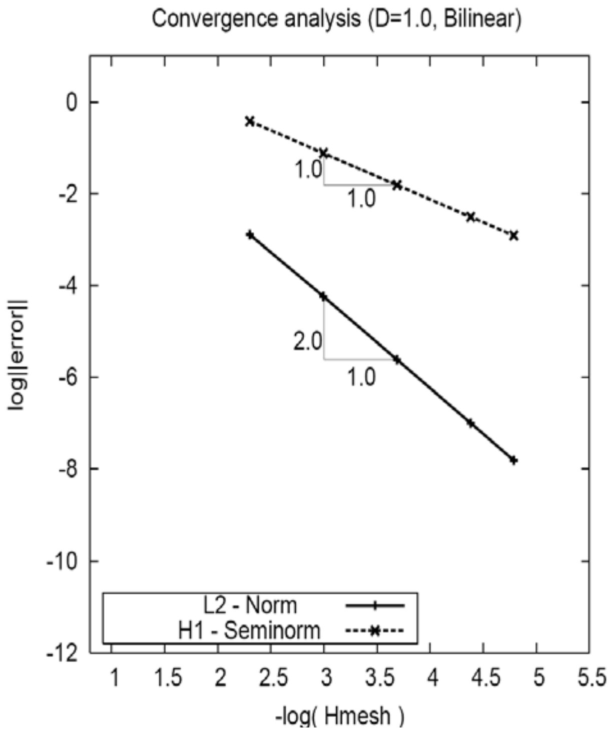

Fig. 23 Bilinear Elements

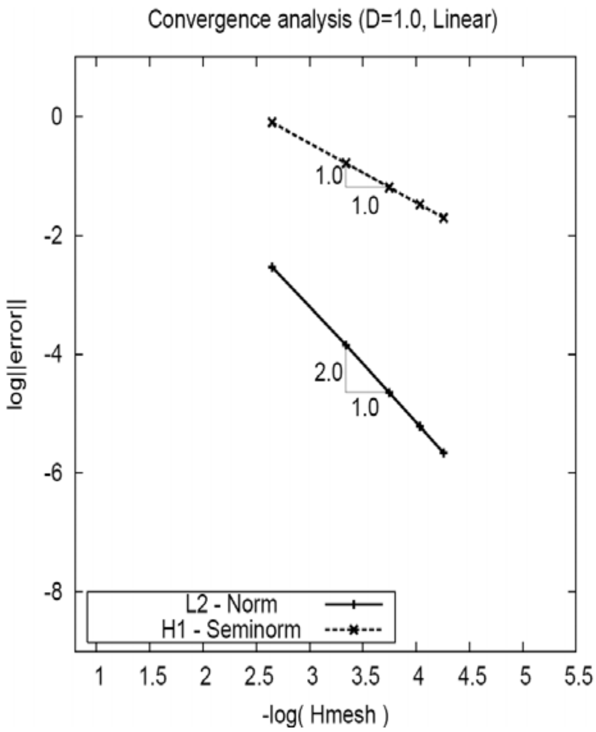

Fig. 24 Linear Elements

5. CONCLUSIONS

We present a new consistent FEM to be applied to diffusive-reactive boundary value problems. The method is obtained adding to the Galerkin formulation an appropriate numbers of projections of the residual of PDE within each element. These multiple projections allow the generation of appropriate number of free stabilization parameters in the element matrix depending on the local space of approximation and on the differential operator. The free parameters can be determined imposing some convergence and/or stability criteria or by postulating the element matrix with the desired stability properties. In this work, the methodology to choose the free parameters consists in to postulate an element matrix with the desired stability properties (GPR-generating matrix) and the free parameters are determinate solving a least square problem at element level. The methodology is applicable to the both: uniforms and non uniform meshes with bilinear rectangular elements or linear triangular elements.

A comprehensive number of numerical experiments was undertaken in order to assess and analyze the proposed method. They clearly indicate that the new method possesses a great performance in terms of accuracy and of stability which compensates the extra computational effort. Moreover, it was possible to embed in generating matrix the mesh distortion what we are convinced helped on reducing the sensitivity of the solution to mesh distortions frequently found in real applications.

The presented results are representative as they deal with both regular situations and some presenting sharp layers. A study with a typical regular problem indicates that the proposed method presents optimum convergence rates. Boundary layers were captured with high precision as well. The good performance of the proposed formulation obtained for diffusive-reactive problem and Helmholtz equation, stimulates to apply the GPR methodology to other problems in future works.

Acknowledgements

The authors wish to thank the Brazilian research-funding agencies CNPq and FAPERJ for their support to this work.

REFERENCES

Burman, E. and Hansbo, P., 2004. Edge stabilization for Galerkin approximations of convection–diffusion–reaction problems. *Comput. Methods Appl. Mech. Engrg.*, vol. 193, pp. 1437-1453.

Codina, R., 1998. Comparison of some finite element methods for solving the diffusion-convection-reaction equation. *Computer Methods in Applied Mechanics and Engineering*, vol. 156, pp. 185-210.

Codina, R., 2000. On stabilized finite element methods for linear systems of convection–diffusion-reaction equations. *Comput. Methods Appl. Mech. Engrg.*, vol. 188, pp. 61-82.

Codina, R., 2001. A stabilized finite element method for generalized stationary incompressible flows. *Comput. Methods Appl. Mech. Engrg.*, vol. 190, pp. 2681-2706.

Dutra do Carmo, E.G., Alvarez, G. B., Loula, A. F. D. and Rochinha, F. A., 2006. The Galerkin plus multiplies projection of residual method (GMPR) applied to Helmholtz equation. *Proceedings of the XXVII Iberian Latin-American Congress on Computational Methods in Engineering – CILAMCE 2006*.

Dutra do Carmo, E. G., Alvarez, G. B., Loula, A. F. D. and Rochinha, F. A., 2008. A nearly optimal Galerkin projected residual finite element method for Helmholtz problem. *Comput. Methods Appl. Mech. Engrg.*, vol. 197, pp. 1362-1375.

Dutra do Carmo, E. G., Alvarez, G. B., Rochinha, F. A. and Loula, A. F. D., In Press. Galerkin projected residual method applied to diffusive-reactive problems. *Comput. Methods Appl. Mech. Engrg.*.

Franca, L. P. and Do Carmo, E. G. D., 1989. The Galerkin gradient least squares method. *Comput. Methods Appl. Mech. Engrg.*, vol. 74, pp. 41-54.

Franca, L. P. and Farhat, C., 1995. Bubble functions prompt unusual stabilized finite element methods. *Comput. Methods Appl. Mech. Engrg.*, vol. 123, pp. 299-308.

Franca, L. P. and Valentin, F., 2000. On an improved unusual stabilized finite element method for the advective-reative-diffusive equation. *Comput. Methods Appl. Mech. Engrg.*, vol. 190, pp. 1785-1800.

Franca, L. P., Madureira, A. L. and Valentin, F., 2005. Towards multiscale functions: enriching finite element spaces with local but not bubble like functions. *Comput. Methods Appl. Mech. Engrg.*, vol. 194, pp. 3006-3021.

Hauke, G. and Olivares, A. G., 2001. Variational subgrid scale formulations for the advection-diffusion-reaction equation. *Comput. Methods Appl. Mech. Engrg.*, vol. 190, pp. 6847-6865.

Hauke, G., 2002. A simple subgrid scale stabilized method for the advection-diffusion reaction equation. *Comput. Methods Appl. Mech. Engrg.*, vol. 191, pp. 2925-2947.

Idelsohn, S., Nigro, N., Storti, M. and Buscaglia, G., 1996. A Petrov-Galerkin formulation for advection-reaction-diffusion problems. *Computer Methods in Applied Mechanics and Engineering*, vol. 136, pp. 27-46.

Lacasse, D., Garon, A. and Pelletier, D., 2007. Development of an adaptive Discontinuous-Galerkin finite element method for advection–reaction equations. *Comput. Methods Appl. Mech. Engrg.*, vol. 196, pp. 2071-2083.

Li, J. and Navon, I. M., 1998. Uniformly convergent finite element methods for singularly perturbed elliptic boundary value problems I: Reaction-diffusion Type. *Computers & Mathematics with Applications*, vol. 35, pp. 57-70.

Li, J., 2002. Uniform convergence of discontinuous finite element methods for singularly perturbed reaction-diffusion problems. *Computers & Mathematics with Applications*, vol. 44, pp. 231-240.

Moore, P. K., 2003. Implicit interpolation error-based error estimation for reaction-diffusion equations in two space dimensions. *Comput. Methods Appl. Mech. Engrg.*, vol. 192, pp. 4379-4401.

Parvazinia, M. and Nassehi, V., 2007. Multiscale finite element modeling of diffusion-reaction equation using bubble functions with bilinear and triangular elements. *Comput. Methods Appl. Mech. Engrg.*, vol. 196, pp. 1095-1107.

Romkes, A., Prudhomme, S. and Oden, J. T., 2006. Convergence analysis of a discontinuous finite element formulation based on second order derivatives. *Comput. Methods Appl. Mech. Engrg.*, vol. 195, pp. 3461-3482.

Sandboge, R., 1998. Adaptive finite element methods for systems of reaction-diffusion equations. *Comput. Methods Appl. Mech. Engrg.*, vol. 166, pp. 309-328.