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A NEW STABILIZED FINITE ELEMENT FORMULATION FOR SCALAR CONVECTION-DIFFUSION PROBLEMS

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Abstract. *In general, the solution of the diffusion-convection problem possesses boundary layers. The approximate solution of the classic finite element method possesses spurious oscillations in the presence of boundary layers. In this work a new stabilized and accurate finite element formulation for convection-dominated problems is presented. The basis of the new formulation is a new upwind function. The upwind function chosen for the new method degenerates into the SUPG or CAU methods, depending on the approximate solution's regularity. The accuracy and stability of the new formulation for the linear and scalar advection-diffusion equation is demonstrated in several numerical examples.*

Keywords: *Stabilized FEM, Diffusion-Convection, Stabilization, Petrov-Galerkin*

1. INTRODUCTION

In general, the solution of diffusive-convective problems possesses boundary layers, that are small subregions where derivatives of the solution are very large. In this case, the classic finite element method or Galerkin method is inappropriate, because its numerical solution presents spurious oscillations. This difficulty of the Galerkin method has been motivating, during the last two decades, to obtain new methods that eliminate such spurious oscillations. Today, these methods are known as stabilized finite element methods.

It is well known that the stability of the Galerkin method can be improved introducing a small "artificial" diffusion. Hughes & Brooks (1982) proposed the SUPG method (Streamline Upwind/Petrov-Galerkin), which consists of modifying the weighting functions to produce a small artificial diffusion in the streamline direction. Later on, Hughes, Franca and Hulbert (1989) developed the GLS method (Galerkin-Least-Squares) that adds a least squares term to the Galerkin method.

The SUPG and GLS methods are obtained by adding stabilization terms to the Galerkin formulation, and which do not introduce an excessive diffusion. When applied to the convection-dominated problems, both methods produce similar numerical results. They also present properties of good stability and accuracy, if the exact solution is smooth or the gradient of the solution is in the streamline direction. For problems whose solution is not smooth, spurious oscillations can remain in sub regions where boundary layers exist.

A great variety of stabilized finite element formulations have been developed to solve this problem (Carmo & Alvarez, 2003, Hauke, 2002, Sampaio & Coutinho, 2001, Ilinca, Hétu & Pelletier,2000, Papastavrou & Verfürth, 2000, Codina, 1998, Almeida & Silva, 1997 and Hughes, 1995). Many of these attempts have used SUPG or GLS methods as a starting point. In particular, Galeão & Carmo (1988) used the idea of "approximate upwind direction" to develop the CAU method (Consistent Approximate Upwind), which preserves the term of SUPG or GLS and adds a nonlinear term. The latter provides an extra control concerning the function's derivative in the direction of the approximate gradient.

The CAU's approximate solution is much more stable than the solution obtained by SUPG or GLS methods when the problem presents boundary layers. However, when the exact solution of the problem is regular, the approximate solution of the CAU method presents an undesirable crosswind diffusion. This led Carmo & Galeão (1991) to develop the CCAU method (Controlled Consistent Approximate Upwind), which keeps the basic characteristics of methods such as the CAU class ones, to build the approximate upwind direction and to incorporate a parameter of feedback control. This parameter modifies the weighting functions Petrov-Galerkin depending on the regularity of the approximate solution.

The CCAU method presents properties of good stability and accuracy, when it is applied to convection-dominated problems, both for smooth solutions and not smooth solutions. In spite of this, CCAU is not a method that presents a simple computational algorithm. Up to date, it has not been generalized for the diffusion-convection problems where the magnitude that describes the transport is a vector.

In this work we propose a new Petrov-Galerkin method, which belongs to the CAU's class and introduces just the "right amount" of artificial diffusion. The method shows excellent properties of stability and accuracy, both for problems with boundary layers and for smooth problems. It's worth mentioning that, throughout the paper one can verify how simple the method is, as well as how easy implementing its computational algorithm becomes.

2. THE SCALAR DIFFUSION-ADVECTION STATIONARY EQUATION

2.1 The boundary value problem

Let $\Omega \subset R^n$ be an open bounded domain, whose boundary Γ is a piecewise smooth boundary. The unit outward normal vector $\hat{n}$ to Γ is defined almost everywhere. We shall consider the problem:

$$-\nabla \cdot (K\nabla\phi) + u \cdot \nabla\phi = f \quad \text{in } \Omega \,, \tag{1}$$

$$\phi = g \quad \text{on } \Gamma \,, \tag{2}$$

where ϕ denotes the unknown quantity of the problem, K is the diffusivity tensor, $u = (u_1, \ldots, u_n)$ is the transport advective field, f is the volume source term and g is the boundary value prescribed for ϕ on Γ.

2.2 Variational Formulation

Consider the set of all the kinematically admissible functions S and the space of the admissible variations V defined as: $S = \{\psi \in H^1(\Omega) : \psi = g \text{ on } \Gamma\}$, $V = \{\eta \in H^1(\Omega) : \eta = 0 \text{ on } \Gamma\}$, where $H^1(\Omega)$ is the standard Sobolev space. The variational problem associated to the boundary value problem defined as Eq. (1) and Eq. (2), involves finding $\phi \in S$ that satisfies the variational equation:

$$\int_\Omega [(K\nabla\phi) \cdot \nabla\eta + (u \cdot \nabla\phi)\eta]\, d\Omega = \int_\Omega f\, \eta\, d\Omega \quad \forall\, \eta \in V \,. \tag{3}$$

2.3 Finite element formulations

Consider $M^h = \{\Omega_1, \ldots, \Omega_{ne}\}$ a finite element partition of Ω, such that: $\overline{\Omega} = \bigcup_{e=1}^{ne} \overline{\Omega}_e$, $\overline{\Omega} = \Omega \bigcup \Gamma$, $\overline{\Omega}_e = \Omega_e \bigcup \Gamma_e$ and $\Omega_e \bigcap \Omega_{e'} = \emptyset$ se $e \neq e'$. The respective finite element set and space of S and V are defined as: $S^{h,k} = \{\phi^h \in H^1(\Omega) : \phi_e^h \in P^k(\Omega_e),\ \phi^h = g^h \text{ on } \Gamma\}$ and $V^{h,k} = \{\eta^h \in H^1(\Omega) : \eta_e^h \in P^k(\Omega_e),\ \eta^h = 0 \text{ on } \Gamma\}$, where $P^k(\Omega_e)$ being the space of polynomials of degree k, g^h denotes the interpolation of g and ϕ_e^h denotes the restriction of ϕ^h to Ω_e.

The variational problem Eq. (3) can be approached by way of the following problems: to find $\phi^h \in S^{h,k}$ that satisfies $\forall\, \eta^h \in V^{h,k}$,

$$A_G(\phi^h, \eta^h) = F_G(\eta^h) \quad \text{Galerkin method,} \tag{4}$$

$$A_G(\phi^h, \eta^h) + A_{GLS}(\phi^h, \eta^h) = F_G(\eta^h) + F_{GLS}(\eta^h) \quad \text{GLS method,} \tag{5}$$

$$A_G(\phi^h, \eta^h) + A_{SUPG}(\phi^h, \eta^h) + A_{CAU}(\phi^h, p^h(\phi^h)) = F_G(\eta^h) + F_{SUPG}(\eta^h) \quad \text{CAU method,} \tag{6}$$

$$A_G = \sum_{e=1}^{ne} \int_{\Omega_e} [(K\nabla\phi_e^h) \cdot \nabla\eta_e^h + (u \cdot \nabla\phi_e^h)\eta_e^h]\, d\Omega_e, \quad F_G = \sum_{e=1}^{ne} \int_{\Omega_e} f\, \eta_e^h\, d\Omega_e \,, \tag{7}$$

$$A_{GLS} = \sum_{e=1}^{ne} \int_{\Omega_e} [-\nabla \cdot (K_e \nabla\phi_e^h) + (u_e \cdot \nabla\phi_e^h)]\, p_e^h\, d\Omega_e, \quad F_{GLS} = \sum_{e=1}^{ne} \int_{\Omega_e} f_e\, p_e^h\, d\Omega_e \,, \tag{8}$$

$$p_e^h = \frac{\tau_e h_e}{2|u_e|}\left[-\nabla \cdot (K_e \nabla \eta_e^h) + u_e \cdot \nabla \eta_e^h\right] \quad \forall \Omega_e \in M^h, \;\; h_e = 2\frac{|u_e|}{|b_e|}, \tag{9}$$

$$b_{e,i} = \sum_{j=1}^{n} \frac{\partial \xi_i}{\partial x_j} u_{e,j} \quad P_e = \frac{h_e |u_e|}{2K}, \;\; \tau_e = \max\left\{0, \; 1 - \frac{C_1}{C_2 P_e}\right\}, \tag{10}$$

$$A_{SUPG} = \sum_{e=1}^{ne} \int_{\Omega_e} [-\nabla \cdot (K_e \nabla \phi_e^h) + (u_e \cdot \nabla \phi_e^h)] \frac{h_e \tau_e}{2|u_e|} (u_e \cdot \nabla \eta_e^h) \, d\Omega_e, \tag{11}$$

$$F_{SUPG} = \sum_{e=1}^{ne} \int_{\Omega_e} f_e \frac{h_e \tau_e}{2|u_e|} (u_e \cdot \nabla \eta_e^h) \, d\Omega_e, \tag{12}$$

$$A_{CAU} = \sum_{e=1}^{ne} \int_{\Omega_e} C_e(\phi_e^h) \nabla \phi_e^h \cdot \nabla \eta_e^h \, d\Omega_e, \tag{13}$$

$$C_e(\phi_e^h) = \begin{cases} 0, & if \;\; \dfrac{|\mathrm{Re}(\phi_e^h)|}{|u_e| |\nabla \phi_e^h|} \ge \dfrac{\tau_e^c h_e^c}{\tau_e h_e} \\[2ex] \dfrac{\tau_e h_e}{2}\left[\dfrac{\tau_e^c h_e^c}{\tau_e h_e} - \dfrac{|\mathrm{Re}(\phi_e^h)|}{|u_e| |\nabla \phi_e^h|}\right] \dfrac{|\mathrm{Re}(\phi_e^h)|}{|\nabla \phi_e^h|}, & if \;\; \dfrac{|\mathrm{Re}(\phi_e^h)|}{|u_e| |\nabla \phi_e^h|} < \dfrac{\tau_e^c h_e^c}{\tau_e h_e} \end{cases}, \tag{14}$$

$$\mathrm{Re}(\phi_e^h) = u_e \cdot \nabla \phi_e^h - \nabla \cdot (K_e \nabla \phi_e^h) - f_e \quad \forall \Omega_e \in M^h, \tag{15}$$

$$h_e^c = 2\frac{|u_e - v_e^h|}{|b_e^c|}, \;\; b_{e,i}^c = \sum_{j=1}^{n} \frac{\partial \xi_i}{\partial x_j} (u_{e,j} - v_{e,j}^h), \tag{16}$$

$$P_e^c = \frac{h_e^c |u_e - v_e^h|}{2|K_e|}, \;\; \tau_e^c = \max\left\{0, \; 1 - \frac{C_1}{C_2 P_e^c}\right\}, \tag{17}$$

$$v_e^h = \begin{cases} u_e, & \text{if } |\nabla \phi_e^h| = 0 \\[2ex] u_e - \dfrac{\mathrm{Re}(\phi_e^h)}{|\nabla \phi_e^h|^2} \nabla \phi_e^h, & \text{if } |\nabla \phi_e^h| \ne 0 \end{cases}, \tag{18}$$

where $\xi_i(\Omega_e)$ $(i = 1, \ldots, n)$ is the local coordinates system that map Ω_e in the usual standard elements $\hat{\Omega}_e$; $u_{e,j}$ is the "*j*" component of u_e; and C_1 and C_2 are constants that depend of the element type, Ω_e and k ($C_1 = C_2 = 1$ for bilinear quadrilaterals elements).

If the diffusive term in the advection diffusion equation is dominant, the Galerkin method produces a good numerical approach of the exact solution. When the advective term is dominant, the Galerkin method can present spurious oscillations. The term Eq. (8) supplies a control of derivatives in the direction of the advective field. However, the GLS method doesn't supply control of derivatives in another directions different from the direction of the advective field. This allows the appearance of local spurious oscillations in the presence of internal and/or external boundary layers, when $P_e >> 1$. The non-linear term Eq. (13) uses the vectorial field v_e^h to determine another upwind direction $U_e^h = u_e - v_e^h$, different from the direction u_e. Notice that from Eq. (18), U_e^h is in the direction of $\nabla \phi_e^h$. This allows the CAU method to possess greater stability than the GLS or SUPG methods in regions near to internal and/or external boundary layers. It is known (Carmo & Galeão, 1991) that in the case of problems with smooth exact solution, the CAU method presents loss of accuracy in its solution, when compared to the SUPG or the GLS solutions. We will develop a new method in the next section, which does not carry the CAU method's difficulties.

3. THE STREAMLINE AND APPROXIMATE UPWIND/PETROV-GALERKIN METHOD

The fundamental idea that supports the development of SAUPG method (Carmo & Alvarez, 2003) is the divergence in the performance of both SUPG and CAU methods. The SUPG method lacks stability when applied to problems with boundary layers, while the CAU method loses accuracy in the solution for smooth problems. We are interested in building a new Petrov-Galerkin method that degenerates into SUPG and CAU methods, as extreme cases. Then, the new perturbation for the weighting function should have the form:

$$\psi_e^h = \eta_e^h + {}_{SAUPG}p_e^h\,, \quad {}_{SAUPG}\,p_e^h = {}_{SUPG}p_e^h + \tau_e^{SAUPG}(u_e - v_e^h)\cdot\nabla\eta_e^h\,, \tag{19}$$

where v_e^h is the auxiliary vectorial field used in CAU formulation (Galeão & Carmo, 1988) to determine the upwind approximate direction and it is given by Eq. (18). Therefore, the new formulation involves finding $\phi^h \in S^{h,k}$ that satisfies:

$$A_G(\phi^h,\eta^h) + A_{SUPG}(\phi^h,\eta^h) + A_{SAUPG}(\phi^h,\eta^h) = F_G(\eta^h) + F_{SUPG}(\eta^h)\,, \tag{20}$$

$$A_{SAUPG}(\phi^h, p^h(\phi^h)) = \sum_{e=1}^{ne}\int_{\Omega_e} B_e(\phi_e^h)\nabla\phi_e^h\cdot\nabla\eta_e^h\,d\Omega_e\,. \tag{21}$$

The new method degenerate into the SUPG and CAU methods, as extreme cases, if the function $B_e(\phi_e^h)$ is searched as:

$$B_e(\phi_e^h) = B_1[B_2]^{\gamma-1}[C_e(\phi_e^h)]^{\gamma} = B_1[B_2C_e(\phi_e^h)]^{\gamma-1}C_e(\phi_e^h)\,, \tag{22}$$

where B_1, B_2 and γ are functions that will depend on the regularity of the approximate solution. Also notice that the function γ and the weighting function should have the following asymptotic behavior:

$$\gamma \approx \begin{cases} 1 & \text{for non-regular solution} \\ 2 & \text{for very regular solution} \end{cases}, \quad {}_{SAUPG}\,p_e^h \approx \begin{cases} {}_{CAU}\,p_e^h & \text{for non-regular solution} \\ {}_{SUPG}\,p_e^h & \text{for very regular solution} \end{cases}. \tag{23}$$

In Eq. (19), the smaller the difference $(u_e - v_e^h)$, if compared to u_e, the smaller the contribution of the new term to the weighting function perturbation. In the limit, when that difference tends to zero, the term of SUPG only contributes to the weighting function. For this reason, we define the following dimensionless variable in order to evaluate the regularity of the approximate solution,

$$\alpha_e = \frac{|u_e - v_e^h|}{|u_e|} = \frac{|\mathrm{Re}(\phi_e^h)|}{|u_e||\nabla\phi_e^h|}\,. \tag{24}$$

We chose B_2 as:

$$B_2 = \frac{2}{|u_e|h_e\tau_e}, \tag{25}$$

where h_e and τ_e are respectively determined by Eq. (9) and Eq. (10). Now, consider the following parameter $\bar{h}_e$ defined as:

$$\bar{h}_e = \min\{1,\ \mathrm{adim}(h_e)\} \quad \forall \Omega_e \in M^h, \tag{26}$$

where $\mathrm{adim}(h_e)$ is a function that turns its argument dimensionless. Define the following function,

$$\beta_1 = \begin{cases} 1 & \text{if } \alpha_e \geq 1 \\ \beta_2 = \begin{cases} \beta_3 & \text{if } \beta_3 < \alpha_e \\ \frac{1}{2}[\alpha_e + \beta_3] & \text{if } \beta_3 \geq \alpha_e \end{cases} & \text{if } \alpha_e < 1 \end{cases}, \tag{27}$$

where $\beta_3 = [\bar{h}_e]^{q_1(\alpha_e)}$. We choose B_1 as follows,

$$B_1(\alpha_e) = [\beta_1]^{(\gamma-1)(\gamma_1+1)}, \quad \gamma_1(\alpha_e) = \begin{cases} q_2(\alpha_e) & \text{se } \alpha_e < 1 \\ 0 & \text{se } \alpha_e \geq 1 \end{cases}, \tag{28}$$

$q_1(\alpha_e)$ and $q_2(\alpha_e)$ are functions that depends on the regularity of the approximate solution.

The function γ determines the degree of diffusion of the new method, and in order to find the dependency of γ with α_e, we define the following functions:

$$\gamma = \begin{cases} 1 & \text{if } \alpha_e \geq 1 \\ \begin{bmatrix} 1 & \text{if } \gamma_2 \geq 1 \\ [2-\gamma_2]^{\left(\frac{1-\gamma_2}{1+\gamma_2}\right)} & \text{if } \gamma_2 < 1 \end{bmatrix} & \text{if } \alpha_e < 1 \end{cases}, \qquad \gamma_2 = \frac{\gamma_4}{[\beta_2]^{\gamma_3}}, \tag{29}$$

$$\gamma_3 = \begin{cases} \frac{1}{2} & \text{if } \alpha_e < \frac{1}{4} \\ [\frac{1}{4} + \alpha_e] & \text{if } \alpha_e \geq \frac{1}{4} \end{cases}, \qquad \gamma_4 = \begin{cases} [\alpha_e]^{q_3(\alpha_e)} & \text{if } [\overline{\mathrm{Re}}]^{q_3(\alpha_e)} \leq 1 \\ [\overline{\mathrm{Re}}\ \alpha_e]^{q_3(\alpha_e)} & \text{if } [\overline{\mathrm{Re}}]^{q_3(\alpha_e)} > 1 \end{cases}, \tag{30}$$

where $\overline{\mathrm{Re}} = \mathrm{adim}(|u_e|\|\nabla\phi_e^h\|)$. In (Carmo & Alvarez, 2003) the procedures is described to obtain the dependence of the functions $q_1(\alpha_e)$, $q_2(\alpha_e)$ and $q_3(\alpha_e)$ with the parameter α_e by numerical experiments.

$$q_1(\alpha_e) = q_2(\alpha_e) = 1 - (\alpha_e)^2, \qquad q_3(\alpha_e) = 3 + \frac{1}{2}\alpha_e + (\alpha_e)^2. \tag{31}$$

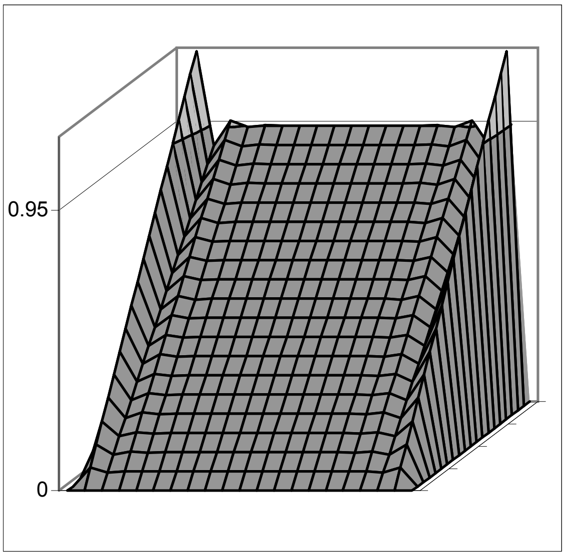
Finally, we have been building up a Petrov-Galerkin class finite element method to solve diffusive-convective transport problems. The new method consists of finding $\phi^h \in S^{h,k}$ satisfying Eq. (20), with the new upwind function successively determined by Eq. (22), Eq. (25), Eq. (28) and Eq. (29).

4. NUMERICAL RESULTS

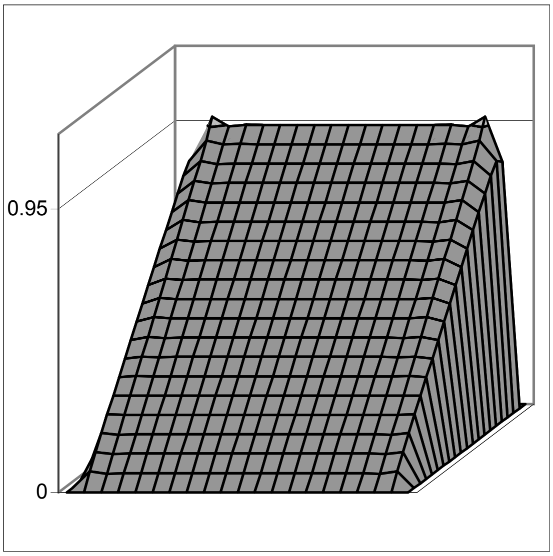
We present in this section the numerical results obtained from several standard numerical tests, which are divided into two types of problems: with internal and/or external boundary layers (examples1 and 2) and with smooth solutions (example 3). In all cases, the medium is assumed homogeneous and isotropic with $K = 10^{-10}$. The considered domain is a square of unitary sides (0,1)×(0,1). Bilinear isoparametric quadrilaterals elements were used and three iterations were accomplished for the CAU and SAUPG methods. A regular mesh (20×20) was used for all problems.

Example 1: an inclined plane with a 45^o slope

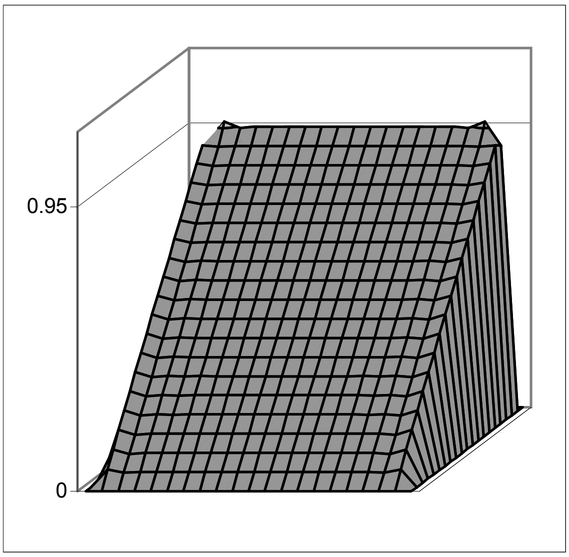
The source term $f(x, y) = 1$ and homogeneous Dirichlet boundary conditions are assumed. The advective field is $u = (1,0)$. The SAUPG method's good performance, when compared to the GLS or CAU method's solutions is shown in Fig. 1.



(GLS)



(CAU)



(SAUPG)

Figure 1- An inclined plane with a 45^o slope.

Example 2: advection skew to the mesh

In this problem we have $f(x, y) = 0$ and the following boundary conditions:

$\phi(x,0) = 0$, $\phi(x,1) = 1$

$\phi(1, y) = 0$, $\forall y \in [0, 1]$

$\phi(0, y) = 0$, $\forall y \in [0, 0.6]$

$\phi(0, y) = y - 0.6$, $\forall y \in [0.6, 0.65]$

$\phi(0, y) = 18(y - 0.65) + 0.05$, $\forall y \in [0.65, 0.70]$

$\phi(0, y) = (y - 0.70) + 0.95$, $\forall y \in [0.70, 0.75]$

$\phi(0, y) = 1$, $\forall y \in [0.75, 1]$.

and three differents advective fields skew to the mesh were considered. Observing Fig. 2, 3 and 4 we can verify that the SAUPG and the CAU solutions are in agreement.

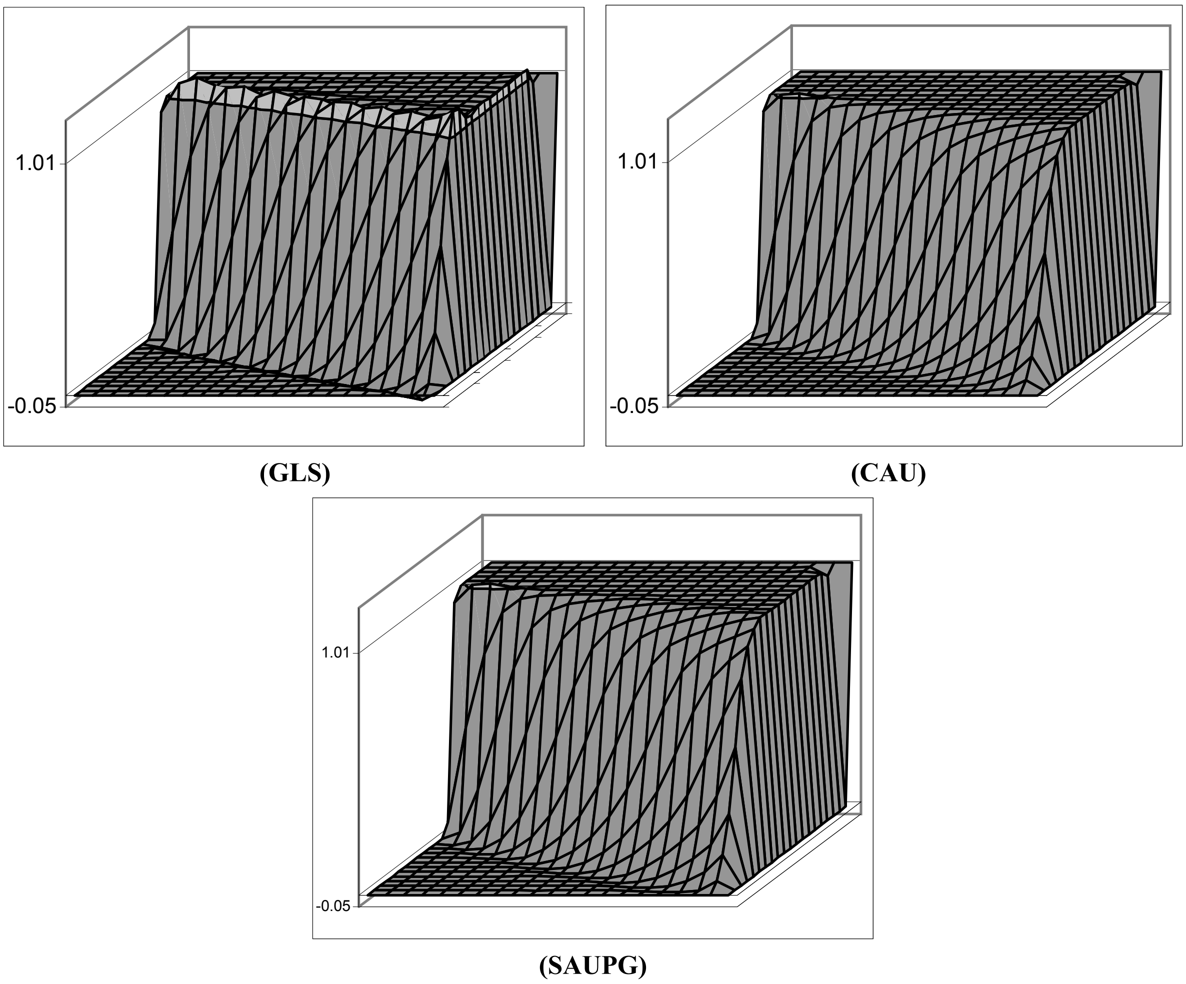
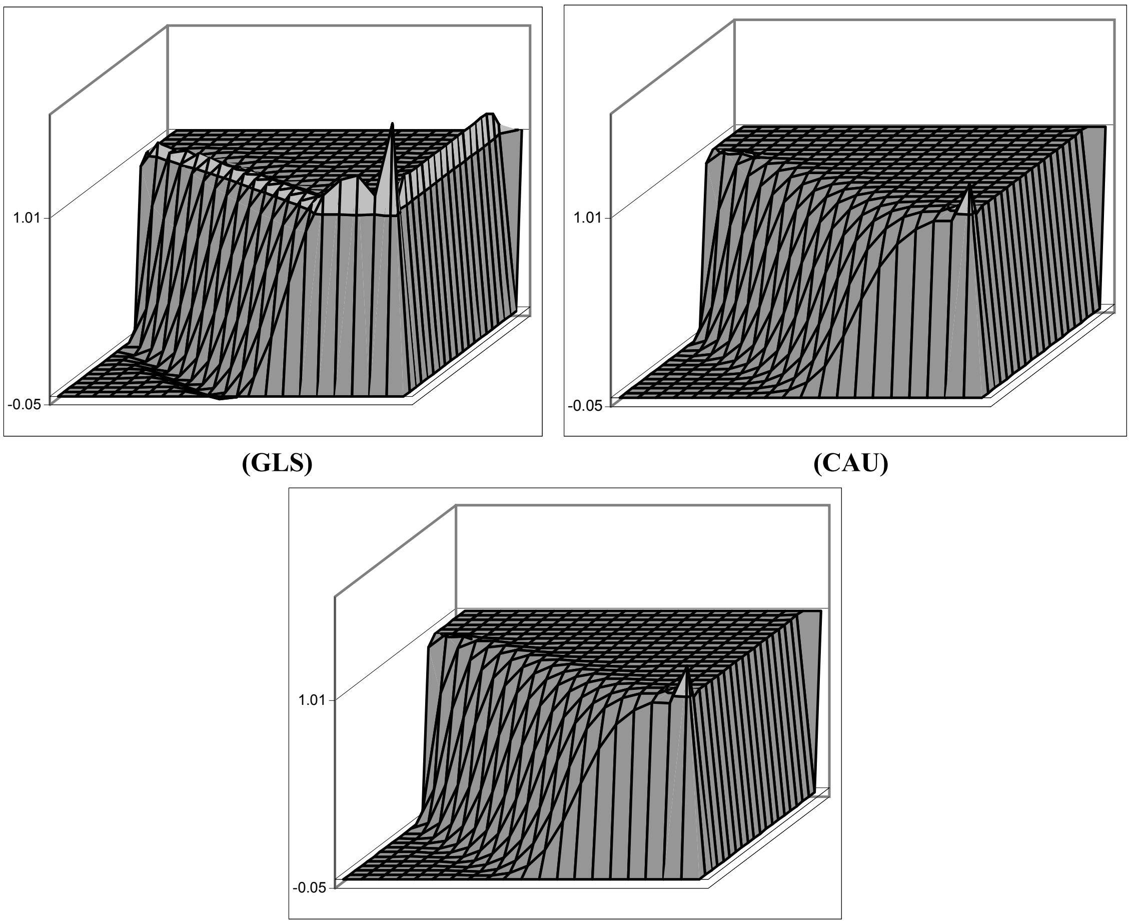


Figure 2- Advection skew to the mesh: case $u = (2,-1)$.



(GLS) **(CAU)**

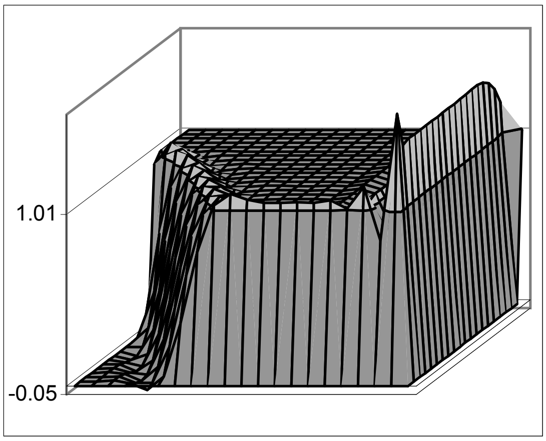
(SAUPG)

Figure 3- Advection skew to the mesh: case $u = (1,-1)$.

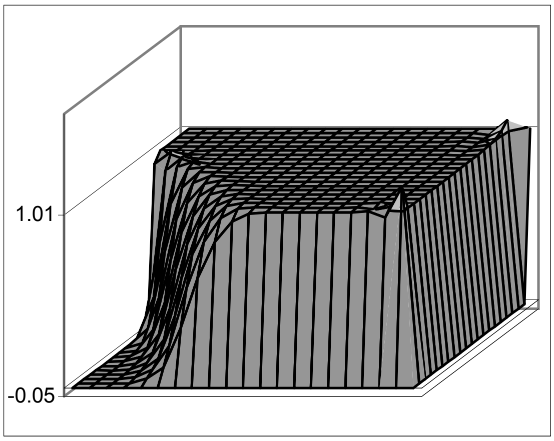
The numerical examples shown in Fig. 1, 2, 3 and Fig. 4 confirm the stability of the proposed method, when applied to problems with boundary layers. Observing Fig. 1-4 we can verify that the SAUPG stability is similar to that of the CAU method. In order to confirm the accuracy of the proposed method, when applied to smooth problems, we shown the example 3.

Example 3: advection of a sine hill in a rotating flow field

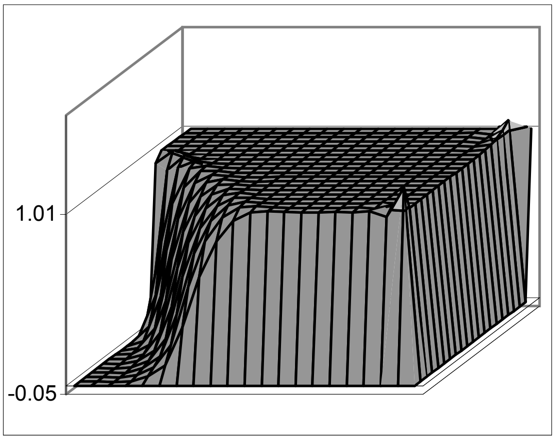
The problems statement is shown in Fig. 5. For this classical test problem, the flow's rotation is determined by the advective field $u = (-y, x)$. Along the external boundary $\phi = 0$, and on the internal 'boundary' OA a sine hill function $\phi(0, y) = sin(2\pi y)\ \forall y \in [-0.5,0]$ is prescribed. For this problem, GLS and SAUPG methods yield to very good results as shown in Fig. 6.



(GLS)



(CAU)



(SAUPG)

Figure 4- Advection skew to the mesh: case $u = (1,-2)$.

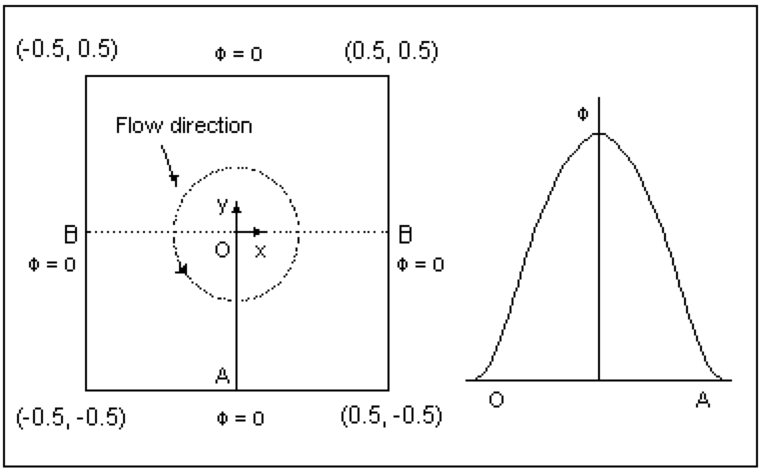
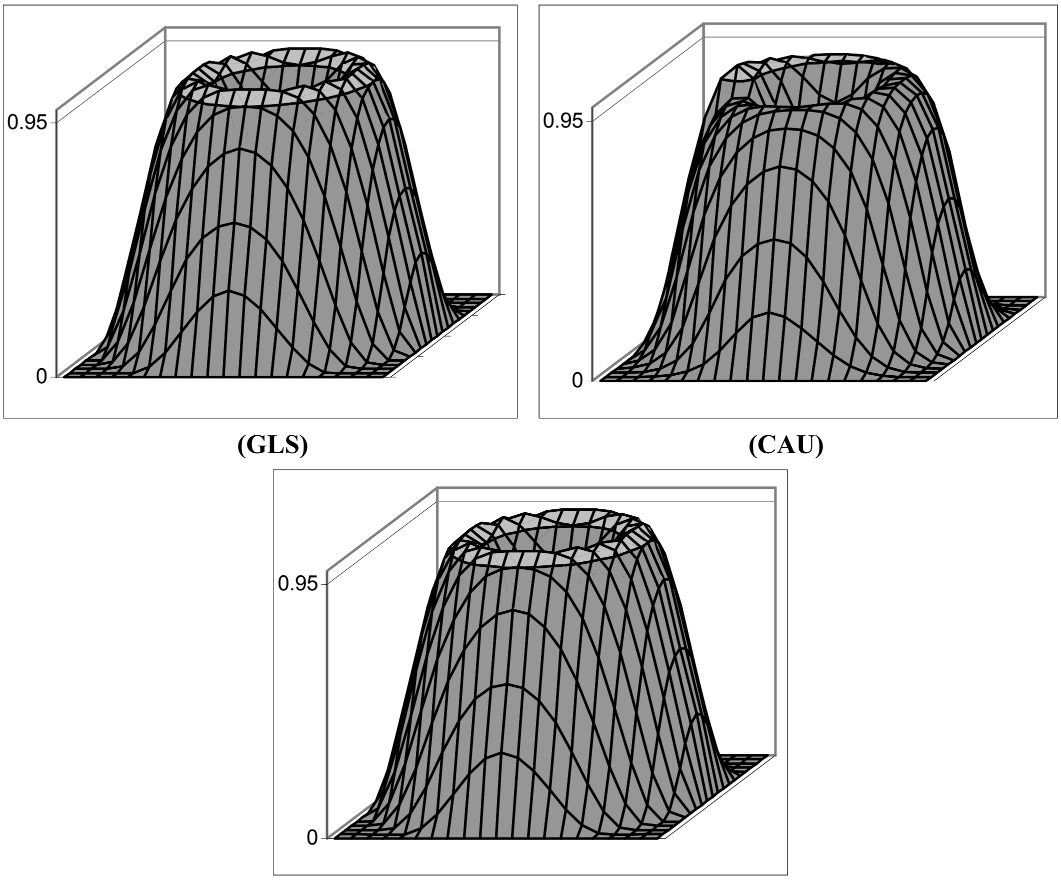


Figure 5- Advection in a rotating flow field: problem statement.



(GLS) **(CAU)**

(SAUPG)

Figure 6- Advection in a rotating flow field.

4. CONCLUSIONS

The numerical results presented in the previous section allow us to conclude that we have developed a new stabilized finite element formulation to solve scalar diffusion-convection problems. The newly built Petrov-Galerkin method based on the CAU method is a non-linear method, and its computational algorithm can be easily implemented. Also, the stability of the method near of external and/or internal boundary layers is similar to that of the CAU method, and its accuracy when applied to smooth solution is similar to that of the SUPG and GLS methods. The new method can be easily combined with adaptive refinement techniques.

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