A NEW STABILIZED FINITE ELEMENT FORMULATION FOR SCALAR CONVECTION-DIFFUSION PROBLEMS AND ITS UP-WIND FUNCTION USING LINEAR AND QUADRATIC ELEMENTS

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Abstract. *A new stabilized and accurate finite element formulation for convection dominated problems is presented. The new method uses the GLS formulation as starting point. The paper also presents a study on the function of the Péclet number that appears in the upwind function of the GLS, CAU and SAUPG methods, using linear and quadratic triangular elements, as well as bilinear and biquadratic quadrilateral elements. The numerical experiments indicate that in the case of elements with faces on the "outflow boundary", the upwind function is strongly dependent on the geometry and on the degree of polynomial interpolation. A new function of the Péclet number is then proposed, to take these effects into consideration, creating a new method of stabilization for higher order elements.*

Keywords: *Stabilized FEM, Diffusion-Convection, Stabilization*

1. INTRODUCTION

It is well known that Galerkin finite element method is unstable for convection dominated convection-diffusion problems for presenting spurious oscillations. During the last two decades, innumerous attempts to obtain new methods have been carried out to avoid such spurious oscillations. These new methods are known as stabilized finite element methods. References about some of them can be found in Carmo et all (2003). Among them, the SUPG - Streamline Upwind/Petrov-Galerkin method (Brooks et all, 1982) and later the GLS - Galerkin-Least-Squares method (Huhes et all, 1989) are two methods that stand out. Both methods add stabilization terms to the Galerkin formulation, providing good stability and accuracy to the numerical solution, when the exact solution of the problem is smooth (Johnson, 1984). However, the spurious oscillations remain when the exact solution possesses boundary layers.

Many attempts have been used the SUPG or GLS methods as starting point for new stabilized formulations. In particular, the CAU - Consistent Approximate Upwind method (Galeão et all, 1988) maintain the term of SUPG and adds a nonlinear term providing an extra control concerning the function's derivative in the direction of the approximate gradient, and eliminates the spurious oscillations. Nevertheless, when the exact solution of the problem is smooth, the CAU's approximate solution presents undesirable crosswind diffusion (Carmo et all, 1991). Building up a stable method that captures discontinuities and also preserves the accuracy of the SUPG or GLS methods for problems with smooth solution has been a great challenge. Recently, the authors develop the SAUPG method (Carmo et all, 2003), which takes advantage of the best qualities of the SUPG and CAU methods.

In this work, based on the idea if the SAUPG method, we introduces a new stabilized method using the GLS method as a starting point. Also, we studied the upwind function dependent on the geometry of the element (triangular and quadrilateral elements) and on the degree of the polynomial in the shape functions.

2. THE STATIONARY SCALAR DIFFUSION-ADVECTION EQUATION

2.1 The boundary value problem

Let $\Omega \subset R^n$ be an open bounded domain, whose boundary Γ is a piecewise smooth boundary. The unit outward normal vector $\hat{n}$ to Γ is defined almost everywhere. We shall consider the problem:

$$-\nabla \cdot (D \nabla \phi) + u \cdot \nabla \phi = f \quad \text{in } \Omega, \tag{1}$$

$$\phi = g \quad \text{on } \Gamma, \tag{2}$$

where ϕ denotes the unknown quantity of the problem, D is the diffusivity tensor, $u = (u_1, \ldots, u_n)$ is the transport advective field, f is the volume source term and g is the boundary value prescribed for ϕ on Γ.

2.2 Variational Formulation

Consider the set of all the admissible functions S and the space of the admissible variations V defined as: $S = \{\psi \in H^1(\Omega) : \psi = g \text{ on } \Gamma\}$, $V = \{\eta \in H^1(\Omega) : \eta = 0 \text{ on } \Gamma\}$, where $H^1(\Omega)$ is the standard Sobolev space. The variational formulation of the boundary value problem defined as Eq. (1) and Eq. (2), involves finding $\phi \in S$ that satisfies the variational equation:

$$\int_{\Omega} [(D\nabla\phi)\cdot\nabla\eta + (u\cdot\nabla\phi)\eta]\, d\Omega = \int_{\Omega} f\,\eta\, d\Omega \quad \forall\, \eta \in V\,. \tag{3}$$

2.3 Finite element formulations

Consider $M^h = \{\Omega_1, \ldots, \Omega_{ne}\}$ a finite element partition of Ω, such that: $\overline{\Omega} = \bigcup_{e=1}^{ne} \overline{\Omega}_e$, $\overline{\Omega} = \Omega \bigcup \Gamma$, $\overline{\Omega}_e = \Omega_e \bigcup \Gamma_e$ and $\Omega_e \bigcap \Omega_{e'} = \emptyset$ se $e \neq e'$. The respective finite element set and space of S and V are defined as: $S^{h,k} = \{\phi^h \in H^1(\Omega) : \phi_e^h \in P^k(\Omega_e),\ \phi^h = g^h \text{ on } \Gamma\}$ and $V^{h,k} = \{\eta^h \in H^1(\Omega) : \eta_e^h \in P^k(\Omega_e),\ \eta^h = 0 \text{ on } \Gamma\}$, where $P^k(\Omega_e)$ being the space of polynomials of degree k, g^h denotes the interpolation of g and ϕ_e^h denotes the restriction of ϕ^h to Ω_e.

Galerkin, Streamline Upwind/Petrov-Galerkin (SUPG), Galerkin Least-Squares (GLS) and Consistent Approximate Upwind (CAU) finite element formulations of problem Eq. (3) consist in: finding $\phi^h \in S^{h,k}$ that satisfies $\forall \eta^h \in V^{h,k}$,

Galerkin method

$$A_G(\phi^h, \eta^h) = F_G(\eta^h)\,, \tag{4}$$

$$A_G = \sum_{e=1}^{ne} \int_{\Omega_e} [(D\nabla\phi_e^h)\cdot\nabla\eta_e^h + (u\cdot\nabla\phi_e^h)\eta_e^h]\, d\Omega_e, \quad F_G = \sum_{e=1}^{ne} \int_{\Omega_e} f\,\eta_e^h\; d\Omega_e\,, \tag{5}$$

SUPG method

$$A_G(\phi^h, \eta^h) + A_{SUPG}(\phi^h, \eta^h) = F_G(\eta^h) + F_{SUPG}(\eta^h)\,, \tag{6}$$

$$A_{SUPG} = \sum_{e=1}^{ne} \int_{\Omega_e} [-\nabla\cdot(D_e\nabla\phi_e^h) + (u_e\cdot\nabla\phi_e^h)] \frac{h_e\tau_e}{2|u_e|}(u_e\cdot\nabla\eta_e^h)\; d\Omega_e\,, \tag{7}$$

$$F_{SUPG} = \sum_{e=1}^{ne} \int_{\Omega_e} f_e \frac{h_e\tau_e}{2|u_e|}(u_e\cdot\nabla\eta_e^h)\; d\Omega_e\,, \tag{8}$$

$$h_e = 2\frac{|u_e|}{|b_e|},\ \ b_{e,i} = \sum_{j=1}^{n} \frac{\partial\xi_i}{\partial x_j} u_{e,j},\ \ \tau_e = \max\left\{0,\ 1 - \frac{C_1}{C_2 P_e}\right\},\ \ P_e = \frac{h_e|u_e|}{2D}\,, \tag{9}$$

GLS method

$$A_G(\phi^h, \eta^h) + A_{LS}(\phi^h, \eta^h) = F_G(\eta^h) + F_{LS}(\eta^h)\,, \tag{10}$$

$$A_{LS} = \sum_{e=1}^{ne} \int_{\Omega_e} [-\nabla\cdot(D_e\nabla\phi_e^h) + (u_e\cdot\nabla\phi_e^h)]\, p_e^h\; d\Omega_e, \quad F_{LS} = \sum_{e=1}^{ne} \int_{\Omega_e} f_e\, p_e^h\; d\Omega_e\,, \tag{11}$$

$$p_e^h = \frac{\tau_e h_e}{2|u_e|}\left[-\nabla\cdot(D_e\nabla\eta_e^h) + u_e\cdot\nabla\eta_e^h\right]\ \ \forall\Omega_e \in M^h\,, \tag{12}$$

CAU method

$$A_G(\phi^h, \eta^h) + A_{SUPG}(\phi^h, \eta^h) + A_{CAU}(\phi^h, \eta^h)) = F_G(\eta^h) + F_{SUPG}(\eta^h)\,, \tag{13}$$

$$A_{CAU} = \sum_{e=1}^{ne} \int_{\Omega_e} C_e(\phi_e^h)\nabla\phi_e^h\cdot\nabla\eta_e^h\; d\Omega_e\,, \tag{14}$$

$$C_e(\phi_e^h)=\begin{cases}0, & if\ \dfrac{\left|\mathrm{Re}(\phi_e^h)\right|}{\left|u_e\right|\left|\nabla\phi_e^h\right|}\geq\dfrac{\tau_e^c h_e^c}{\tau_e h_e}\\ \dfrac{\tau_e h_e}{2}\left[\dfrac{\tau_e^c h_e^c}{\tau_e h_e}-\dfrac{\left|\mathrm{Re}(\phi_e^h)\right|}{\left|u_e\right|\left|\nabla\phi_e^h\right|}\right]\dfrac{\left|\mathrm{Re}(\phi_e^h)\right|}{\left|\nabla\phi_e^h\right|}, & if\ \dfrac{\left|\mathrm{Re}(\phi_e^h)\right|}{\left|u_e\right|\left|\nabla\phi_e^h\right|}<\dfrac{\tau_e^c h_e^c}{\tau_e h_e}\end{cases}, \tag{15}$$

$$\mathrm{Re}(\phi_e^h)=u_e\cdot\nabla\phi_e^h-\nabla\cdot(D_e\nabla\phi_e^h)-f_e \quad \forall\Omega_e\in M^h, \tag{16}$$

$$h_e^c=2\frac{\left|u_e-v_e^h\right|}{\left|b_e^c\right|},\quad b_{e,i}^c=\sum_{j=1}^{n}\frac{\partial\xi_i}{\partial x_j}(u_{e,j}-v_{e,j}^h),\quad \tau_e^c=\max\left\{0,\ 1-\frac{C_1}{C_2P_e^c}\right\}, \tag{17}$$

$$P_e^c=\frac{h_e^c\left|u_e-v_e^h\right|}{2D_e},\quad v_e^h=\begin{cases}u_e, & \text{if } \left|\nabla\phi_e^h\right|=0\\ u_e-\dfrac{\mathrm{Re}(\phi_e^h)}{\left|\nabla\phi_e^h\right|^2}\nabla\phi_e^h, & \text{if } \left|\nabla\phi_e^h\right|\neq 0\end{cases}, \tag{18}$$

where $\xi_i(\Omega_e)\ (i=1,\ldots,n)$ is the local coordinate system that maps Ω_e in the usual standard elements $\hat{\Omega}_e$; $u_{e,j}$ is the "*j*" component of u_e; and C_1 and C_2 are constants that depend of the element type, Ω_e and k ($C_1=C_2=1$ for bilinear quadrilaterals elements).

When the advective term is dominant, the Galerkin method is totally unstable, unless the mesh is highly refined such that $\hat{h}_e\leq\frac{2K_e}{|u_e|}\ \forall\Omega_e\in M^h$, where $\hat{h}_e$ denote the diameter of Ω_e. The instability of the Galerkin method is due to the weak control on the gradient of the solution. The term $A_{LS}(\phi^h,\eta^h)$ introduces control in the streamline direction. However, when the problem possesses boundary layers the GLS method presents spurious oscillations. The approximate upwind direction $U_e^h=u_e-v_e^h$ is in the direction of $\nabla\phi_e^h$. The CAU method possess a additional stability, in regions near internal and/or external boundary layers, compared to GLS or SUPG methods for adding another residual to control the gradient of the solution in any direction.

3. THE GALERKIN LEAST SQUARES AND APPROXIMATE UPWIND METHOD

The main idea that supports the development of SAUPG method (Carmo & Alvarez, 2003) is the performance of both SUPG and CAU methods. The SUPG method lacks stability when applied to problems with boundary layers, while the CAU method loses accuracy in smooth problems. In Carmo et all (2004) the stabilization term of SUPG was replaced by the GLS term to develop the GLSAU formulation. This formulation consists in finding $\phi^h\in S^{h,k}$ that satisfies $\forall\eta^h\in V^{h,k}$:

$$A_G(\phi^h,\eta^h)+A_{LS}(\phi^h,\eta^h)+A_{AU}(\phi^h,\eta^h)=F_G(\eta^h)+F_{LS}(\eta^h), \tag{19}$$

$$A_{AU}(\phi^h,\eta^h)=\sum_{e=1}^{ne}\int_{\Omega_e}B_e(\phi_e^h)\nabla\phi_e^h\cdot\nabla\eta_e^h\,d\Omega_e, \tag{20}$$

$$B_e(\phi_e^h)=B_1[B_2]^{\gamma-1}[C_e(\phi_e^h)]^{\gamma}=B_1[B_2C_e(\phi_e^h)]^{\gamma-1}C_e(\phi_e^h), \tag{21}$$

where B_1, B_2 and γ are functions that depend on the regularity of the approximate solution and were determined for the SAUPG method. In Carmo et all (2003) the non-dimensional variable $\alpha_e = \frac{|u_e - v_e^h|}{|u_e|} = \frac{|\mathrm{Re}(\phi_e^h)|}{|u_e||\nabla\phi_e^h|}$ was define to evaluate the regularity of the approximate solution, and the functions B_1 and B_2 were defined as:

$$B_1(\alpha_e) = [\beta_1]^{(\gamma-1)(\gamma_1+1)}, \ B_2 = \frac{2}{|u_e| h_e \tau_e}, \ \gamma_1(\alpha_e) = \begin{cases} q_2(\alpha_e) & \text{se } \alpha_e < 1 \\ 0 & \text{se } \alpha_e \geq 1 \end{cases} \tag{22}$$

$$\beta_1 = \begin{cases} 1 & \text{if } \alpha_e \geq 1 \\ \beta_2 = \begin{cases} \beta_3 & \text{if } \beta_3 < \alpha_e \\ \frac{1}{2}[\alpha_e + \beta_3] & \text{if } \beta_3 \geq \alpha_e \end{cases} & \text{if } \alpha_e < 1 \end{cases}, \tag{23}$$

$$\beta_3 = [\bar{h}_e]^{q_1(\alpha_e)}, \ \bar{h}_e = \min\{1, \operatorname{adim}(h_e)\} \quad \forall \Omega_e \in M^h, \tag{24}$$

where $\operatorname{adim}(\cdot)$ is a function that turns its argument dimensionless. Following the procedure described in Carmo et all (2003) the function γ is given by:

$$\gamma = \begin{cases} 1 & \text{if } \alpha_e \geq 1 \\ \begin{bmatrix} 1 & \text{if } \gamma_2 \geq 1 \\ [2-\gamma_2]^{\left(\frac{1-\gamma_2}{1+\gamma_2}\right)} & \text{if } \gamma_2 < 1 \end{bmatrix} & \text{if } \alpha_e < 1 \end{cases}, \qquad \gamma_2 = \frac{\gamma_4}{[\beta_2]^{\gamma_3}}, \tag{25}$$

$$\gamma_2 = \frac{\gamma_4}{[\beta_2]^{\gamma_3}}, \ \gamma_3 = \begin{cases} \frac{1}{2} & \text{if } \alpha_e < \frac{1}{4} \\ [\frac{1}{4} + \alpha_e] & \text{if } \alpha_e \geq \frac{1}{4} \end{cases}, \quad \gamma_4 = \begin{cases} [\alpha_e]^{q_3(\alpha_e)} & \text{if } [\overline{\mathrm{Re}}]^{q_3(\alpha_e)} \leq 1 \\ [\overline{\mathrm{Re}}\,\alpha_e]^{q_3(\alpha_e)} & \text{if } [\overline{\mathrm{Re}}]^{q_3(\alpha_e)} > 1 \end{cases}, \tag{26}$$

where $\overline{\mathrm{Re}} = \operatorname{adim}(|u_e||\nabla\phi_e^h|)$. Also, the functions q_1, q_2 and q_3 dependent on the parameter α_e are obtained with the procedure described in Carmo et all (2003),

$$q_1(\alpha_e) = q_2(\alpha_e) = 1 - (\alpha_e)^2, \qquad q_3(\alpha_e) = 3 + \frac{1}{2}\alpha_e + (\alpha_e)^2. \tag{27}$$

When the solution is very smooth, γ tends to 2 and the $A_{LS}(\phi^h, \eta^h)$ prevails on $A_{AU}(\phi^h, \eta^h)$ in Eq. (19). In the case where the solution is not smooth, the contribution of the $A_{AU}(\phi^h, \eta^h)$ term is then significant. For the new method, the parameter τ_e defined by Eq. (9), as well as, the τ_e^c give by Eq. (17) are redefined as: $\tau_e^c = \tau_o \max\{0, 1 - \frac{1}{P_e^c}\}$ and $\tau_e = \tau_o \max\{0, 1 - \frac{1}{P_e}\}$, where P_e and P_e^c are respectively given by Eq. (9) and Eq. (18). The parameter τ_o was determined through the numerical experiments for bilinear quadrilateral element in Carmo et all (2003) as $\tau_o = 1$.

4. NUMERICAL RESULTS

In this section we present the numerical results obtained in several standard numerical tests with two types of problems presenting internal and/or external boundary layers and with smooth solutions. In all cases, the medium is assumed homogeneous and isotropic with $D = 10^{-10}$. The considered domain is a square of unitary sides (0,1)×(0,1). Five iterations were accomplished for the CAU and GLSAU methods.

Example 1: advection skew to the mesh

In this problem we have $f(x,y)=0$ and the following boundary conditions:

$\phi(x,0)=0$, $\phi(x,1)=1$

$\phi(1,y)=0$, $\forall y\in[0,1]$

$\phi(0,y)=0$, $\forall y\in[0,0.6]$

$\phi(0,y)=y-0.6$, $\forall y\in[0.6,0.65]$

$\phi(0,y)=18(y-0.65)+0.05$, $\forall y\in[0.65,0.70]$

$\phi(0,y)=(y-0.70)+0.95$, $\forall y\in[0.70,0.75]$

$\phi(0,y)=1$, $\forall y\in[0.75,1]$,

and advectives field skew to the mesh are $u=(1,-2)$ and $u=(1,-1)$.

Example 2: external boundary layers and semicircular inernal layer

In this example we have $f(x,y)=0$, where the Dirichlet boundary conditions are the same as in the latter example and the advective field is $u=(y,-x)$.

Example 3: advection of a sine hill in a rotating flow field

The problems statement is shown in Fig. 1. The flow's rotation is determined by the advective field $u=(-y,x)$. Along the external boundary $\phi=0$, and on the internal 'boundary' OA a sine hill function $\phi(0,y)=sin(2\pi y)$ $\forall y\in[-0.5,0]$ is prescribed.

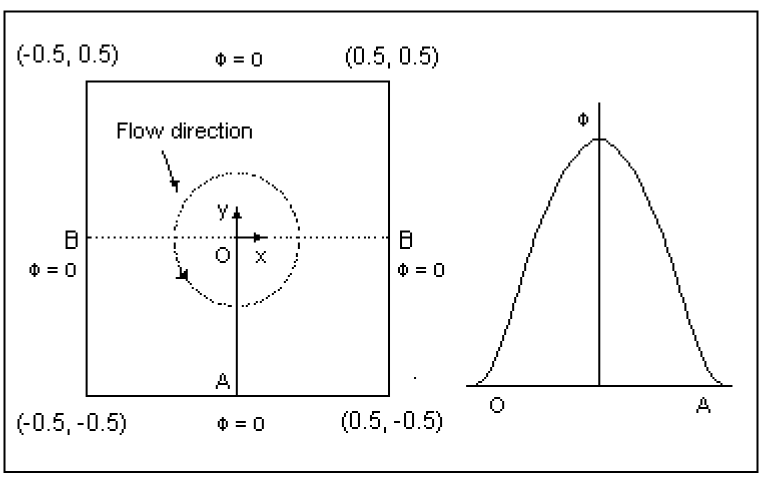

Figure 1. Advection in a rotating flow field: problem statement.

In Carmo et all (2004) was verified, numerically, that the functions $q_i(\alpha_e)$ (*i=1,2,3*) Eq. (27), determined through the numerical experiments for bilinear quadrilateral elements in Carmo et all (2003), are also valid for linear or quadratic triangular elements, as well as, for biquadratic quadrilateral elements. Also, it was verified in all the examples that the three methods introduce diffusivity in excess for the external boundary layers, when the τ_o defined for bilinear quadrilateral elements is used. At present, not a perfect theoretical support has been obtained and not much has been studied about the parameter τ_e dependence on the element Péclet number for the higher order quadratic rectangular or triangular elements. Some few attempts can be found in Heinrich (1980), Mizukami (1985) and Codina et all (1992). In Carmo et all (2004) was propose Eq. (28), based on a numerical analyses of the stability and on the fact that the internal and external boundary layers are of different physical nature (Johnson et all, 1984). The purpose was to avoid that the upwind function is not overestimated or underestimated.

$$\tau_o = \begin{cases} 1 & \text{if } \Gamma_e \cap \Gamma^+ = \varnothing \\ \tau_o^e & \text{if } \Gamma_e \cap \Gamma^+ \neq \varnothing \end{cases}, \quad \tau_o^e = \begin{cases} \left(\frac{1}{2}\right)^{(k-1)} & \text{for quadrilateral elements} \\ \left(\frac{1}{2}\right)^{k} & \text{for triangular elements} \end{cases} . \tag{28}$$
$$\Gamma^+ = \{x \in \Gamma : \hat{n} \cdot u > 0\}$$

All figures coming afterward show the numerical results for each example. In each figure, the cases (a) correspond to each one of the three methods, using τ_o given by Eq. (28). Cases (b) refer to the solutions of the methods obtained with $\tau_o = 1$, that is to say, the upwind functions corresponding to the value determined for the bilinear quadrilateral elements.

4.1 Approximate solutions using quadratic quadrilateral elements (9-noded)

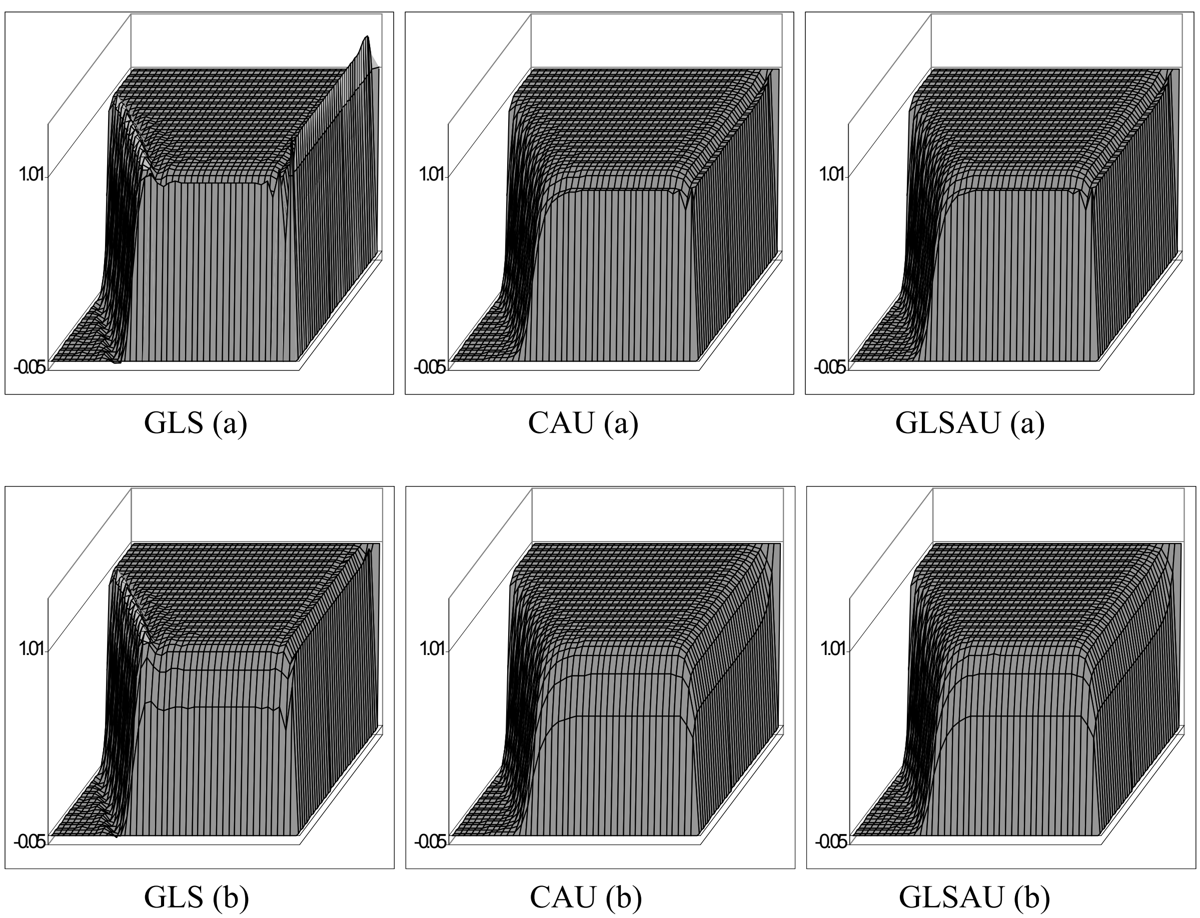

Figure 2- Example 1, $u = (1,-2)$, (a) $\tau_o^e = \frac{1}{2}$, (b) $\tau_o^e = 1$.

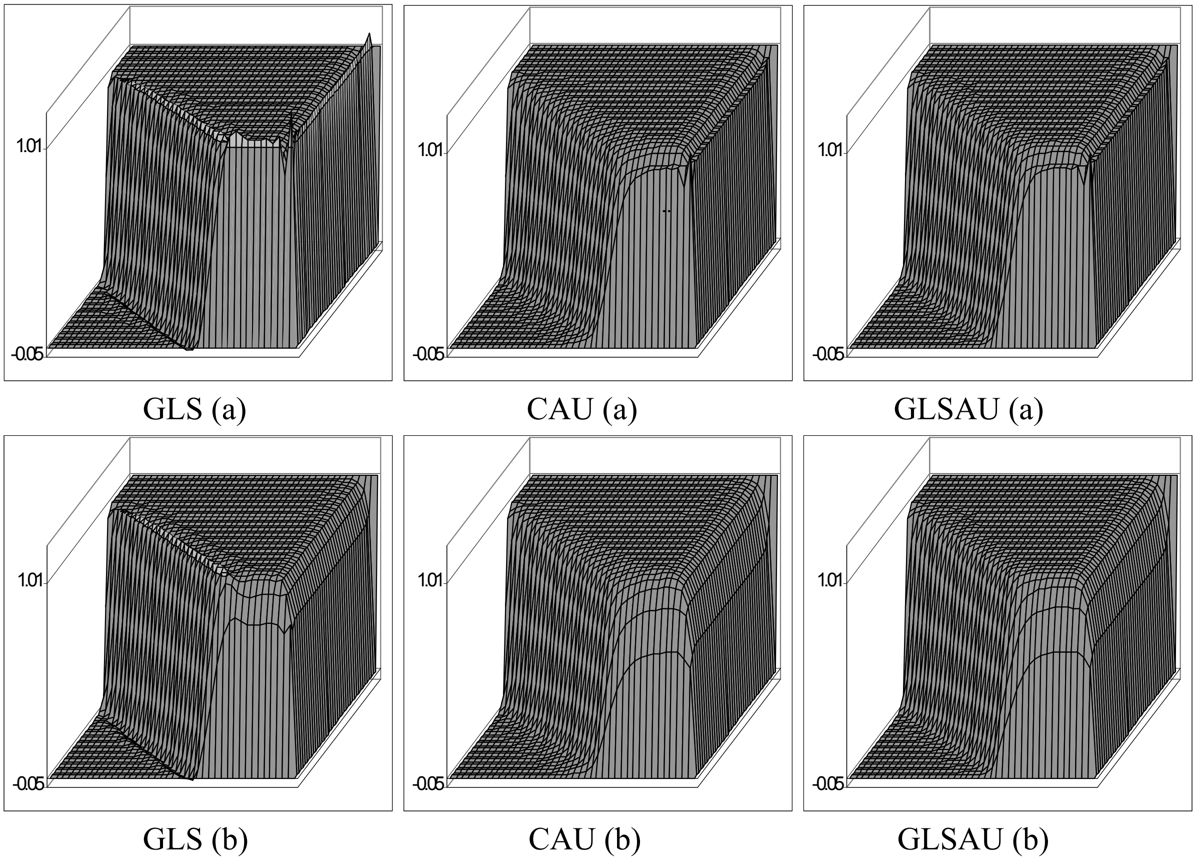

Figure 3- Example 1, $u = (1,-1)$, (a) $\tau_o^e = \frac{1}{2}$, (b) $\tau_o^e = 1$.

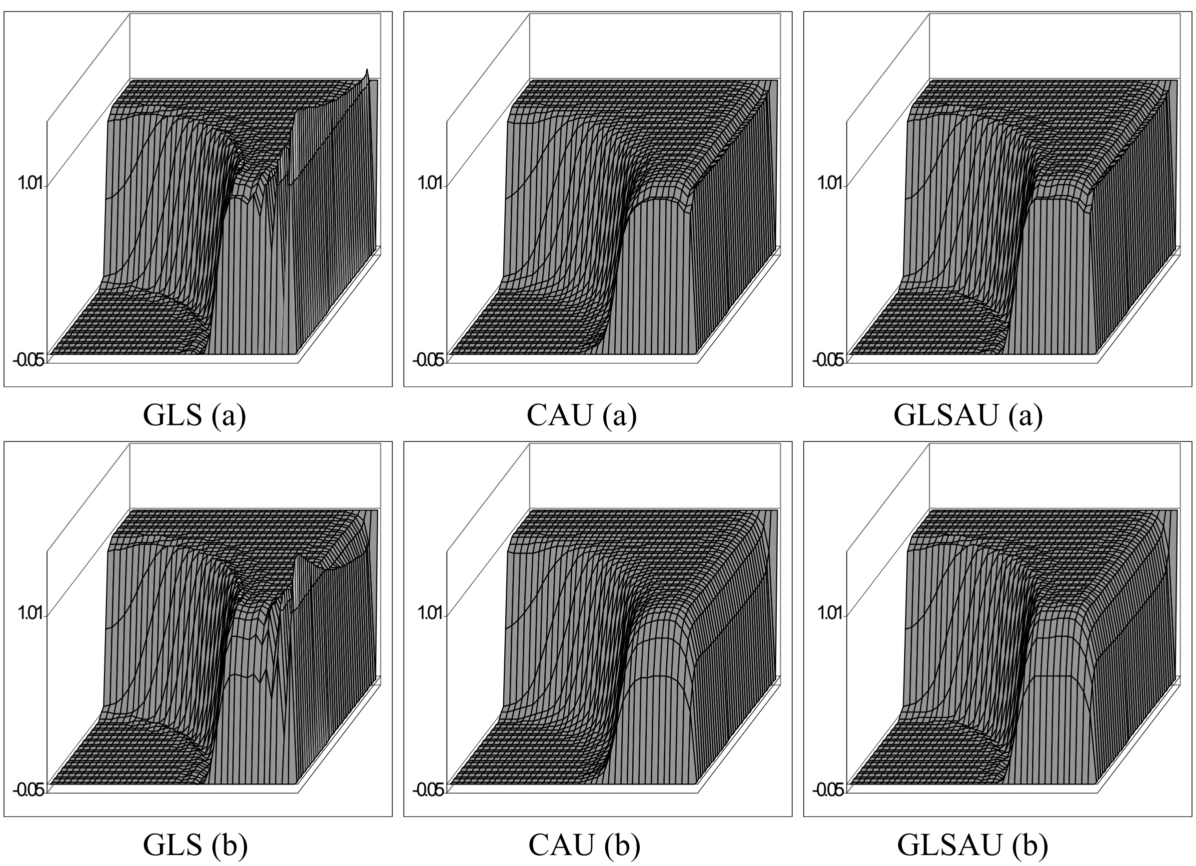

Figure 4 - Example 2, (a) $\tau_o^e = \frac{1}{2}$, (b) $\tau_o^e = 1$.

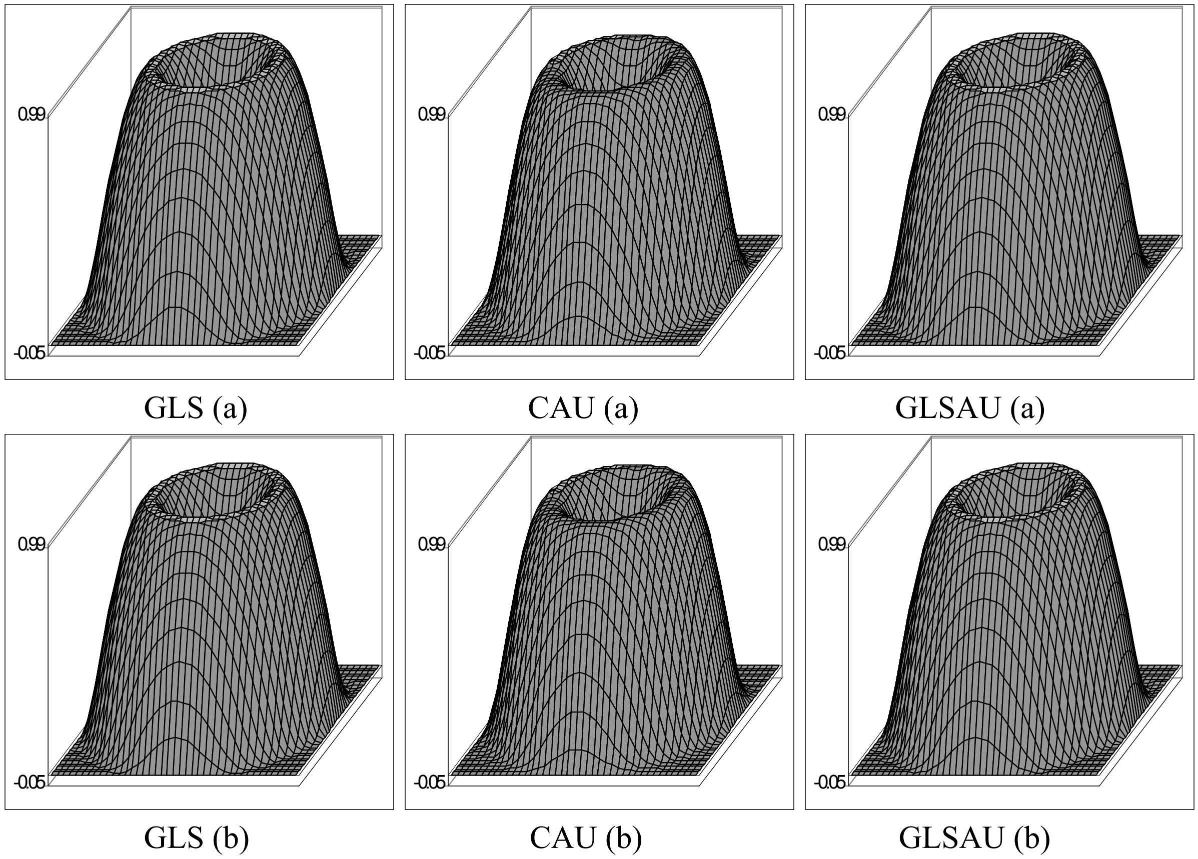

Figure 5 - Example 3, (a) $\tau_o^e = \frac{1}{2}$, (b) $\tau_o^e = 1$.

For the examples 1 and 2, we can observe in Figs. (2), (3) and (4) that the approximate solutions of the GLSAU and CAU methods are very close. In the cases (b) undershoots are obtained for the external boundary layers. For the example 3, the GLS and GLSAU methods yield to very good results (see Figure 5).

4.2 Approximate solutions using linear triangular elements

Here the observations done in the previous subsection continue being valid. Similar results are obtained for quadratic triangular elements, so they are not shown here. Again, for the examples 1 and 2, we can observe in Figs. (6), (7) and (8) the similarity between the approximate solutions of the GLSAU and CAU methods. In the cases (b) undershoots are obtained for the external boundary layers. It should be highlighted, that the internal sharp layer of the example 1 (see Figure 7) is very well approached by the GLS, CAU and GLSAU solutions. This is something expected, since in this case the advective field u is aligned to the mesh. For the example 3, the GLS and GLSAU methods yield to very good results as showed in Figure 5.

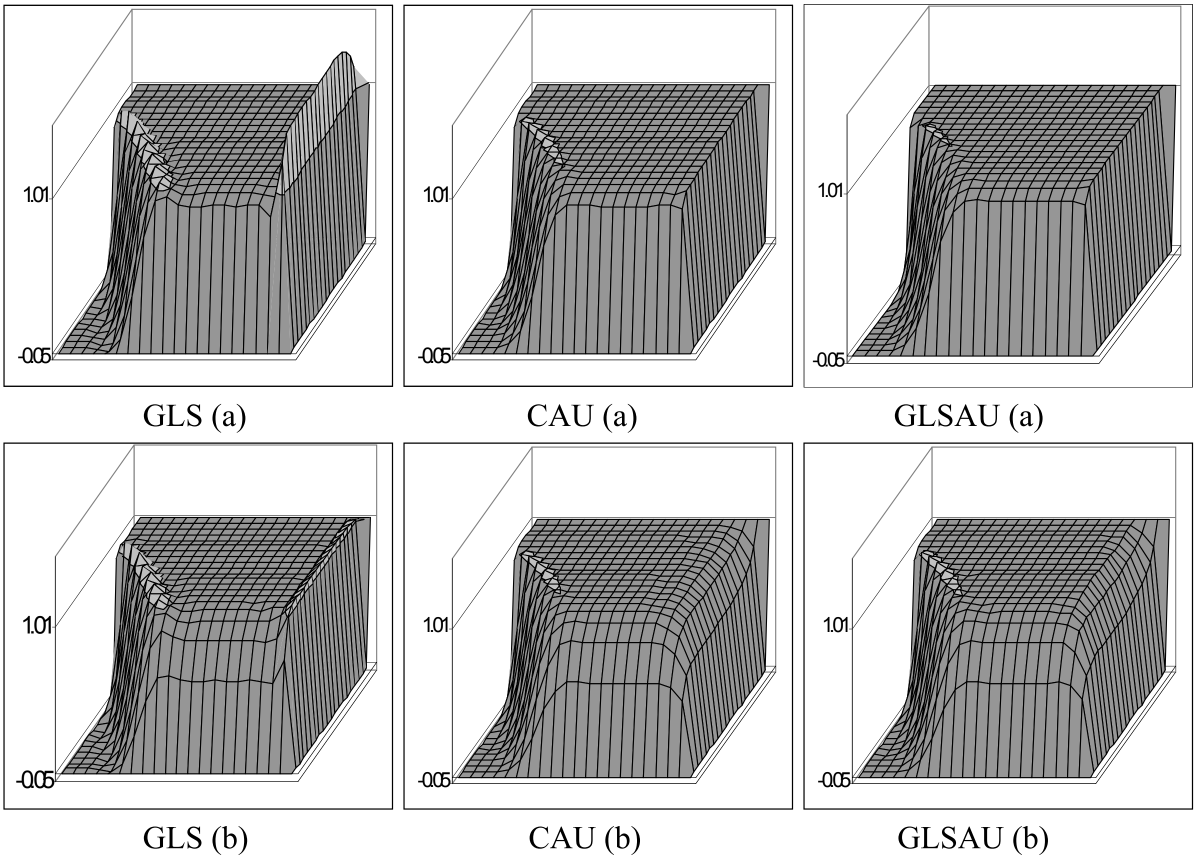

Figure 6- Example 1, $u = (1,-2)$, (a) $\tau_o^e = \frac{1}{2}$, (b) $\tau_o^e = 1$.

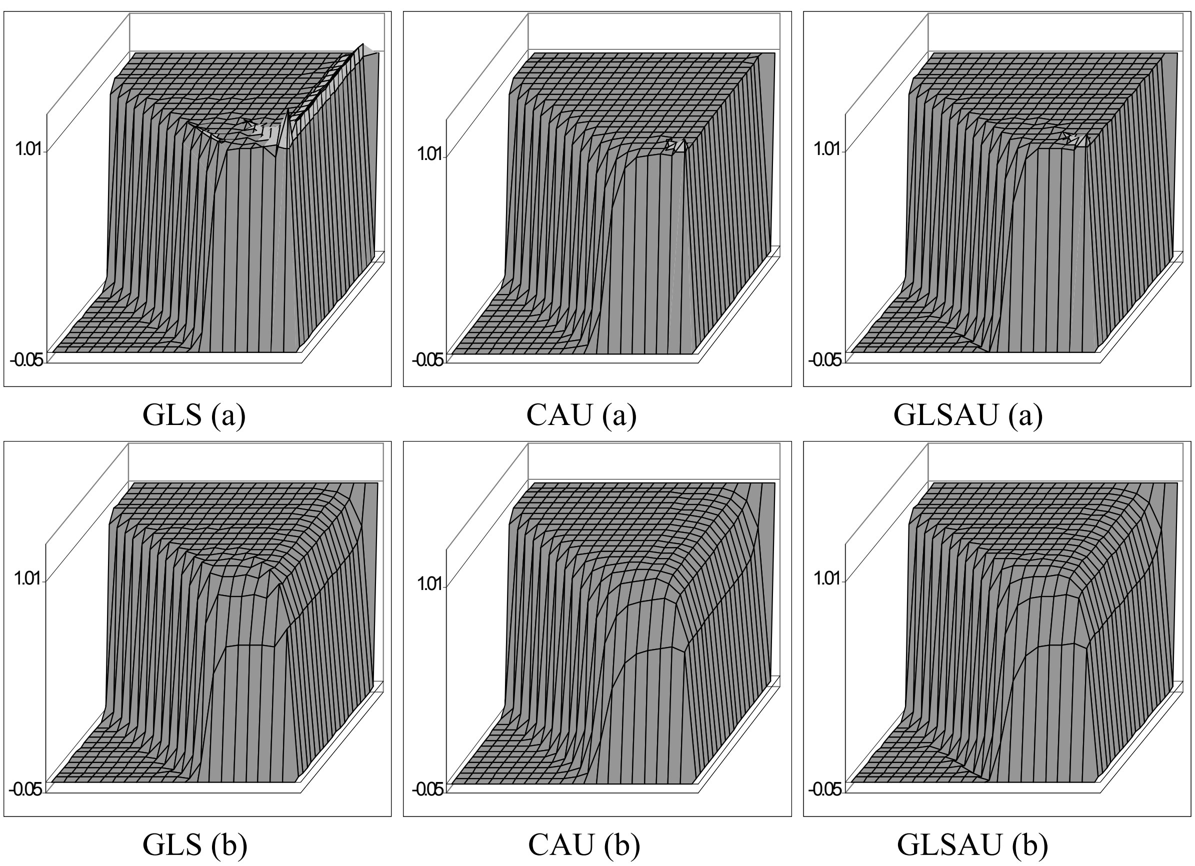

Figure 7 - Example 1, $u = (1,-1)$, (a) $\tau_o^e = \frac{1}{2}$, (b) $\tau_o^e = 1$.

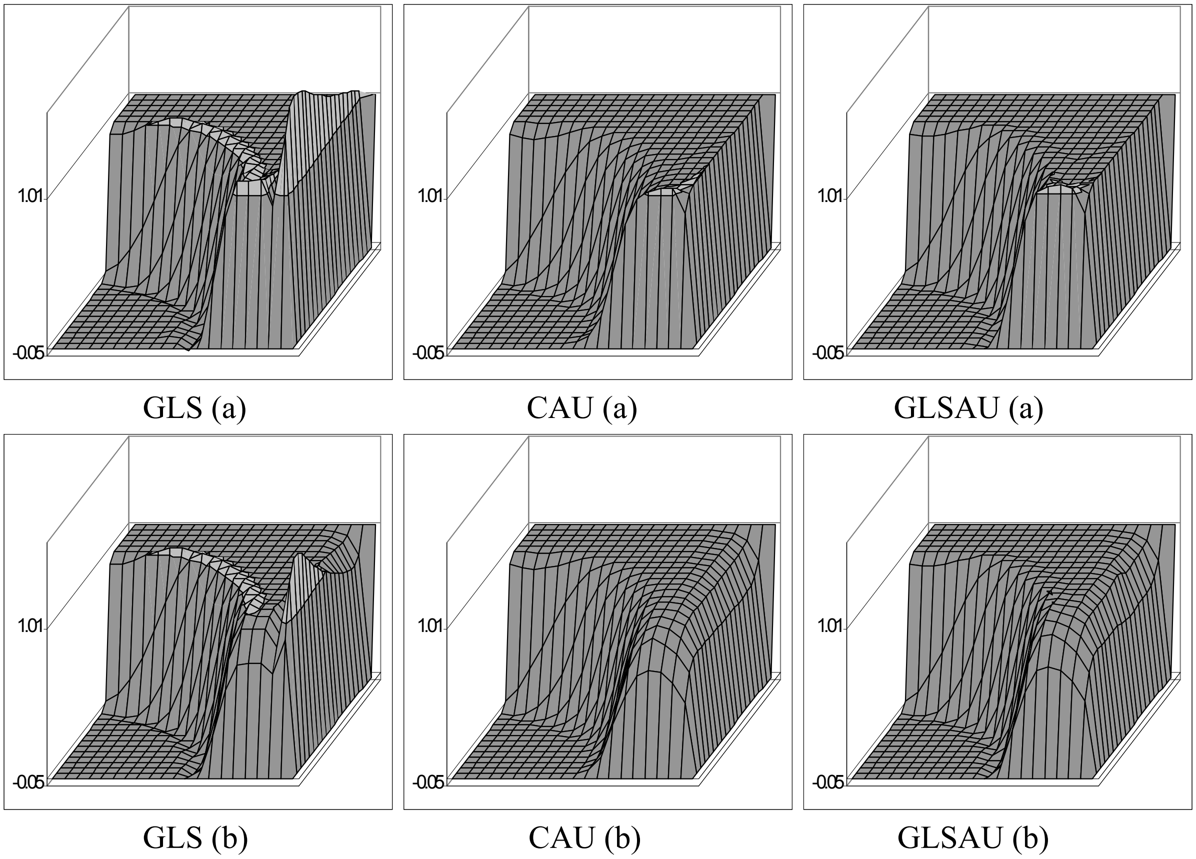

Figure 8 - Example 2, (a) $\tau_o^e = \frac{1}{2}$, (b) $\tau_o^e = 1$.

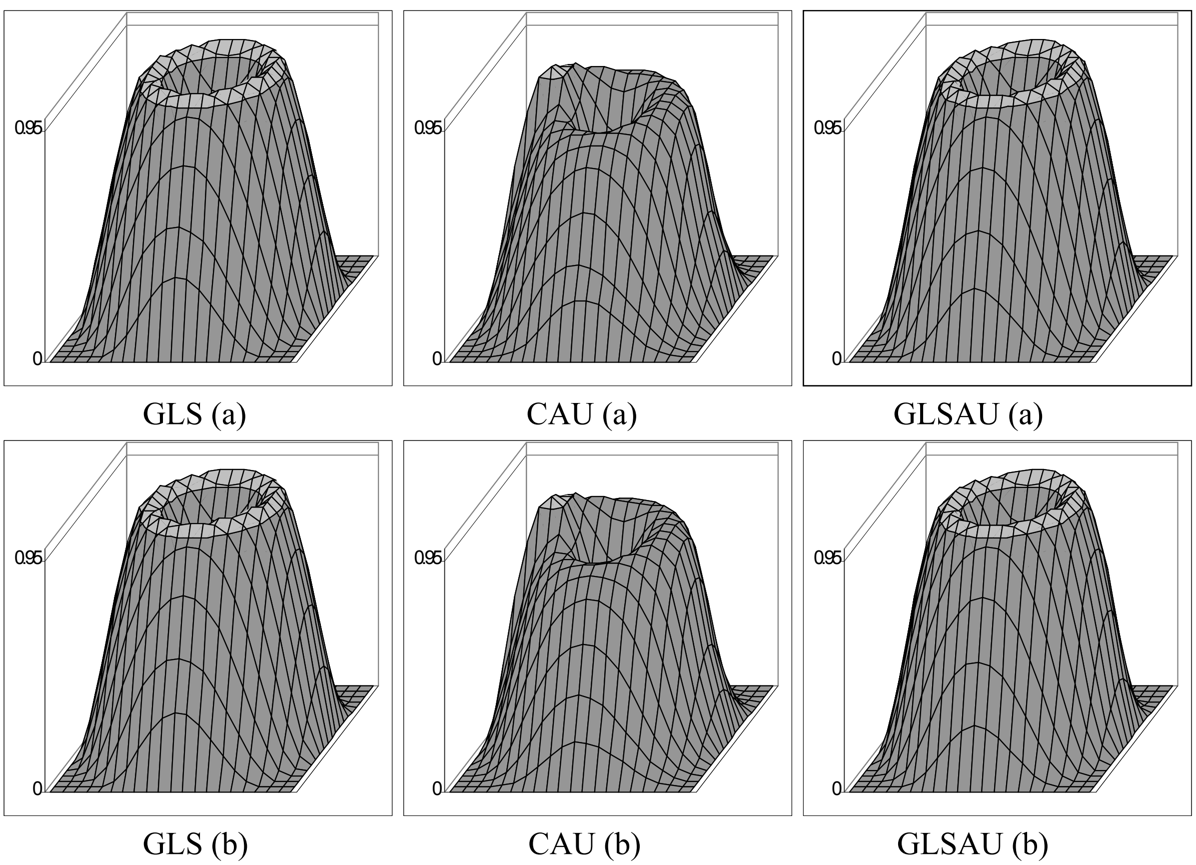

Figure 9 - Example 3, (a) $\tau_o^e = \frac{1}{2}$, (b) $\tau_o^e = 1$.

5. CONCLUSIONS

In this work, we have presented a new stabilized finite element formulation to solve scalar convection diffusion problems. The formulation preserves the best qualities of the GLS and CAU methods, in terms of stability and accuracy. The numerical experiments showed that the functions $q_1(\alpha_e)$, $q_2(\alpha_e)$ and $q_3(\alpha_e)$ do not depend on the geometry of the element nor on the degree of the polynomial in the shape functions.

We also studied the upwind function dependence on the geometry of the element and the degree of the polynomial in the shape functions. The numerical results led us to define τ_o as in Eq. (28), that is to say, value 1 is necessary for the internal sharp layer, while τ_o^e is the optimum value to avoid overshoots and undershoots for the external boundary layer. This modification on stabilizing parameter near the outflow boundaries increases the quality of the numerical solution for the GLS, CAU and GLSAU methods. Although we lack a theoretical error estimate, the simplicity of the new scheme, as well as the parameter τ must be considered for practical purposes.

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