THE GALERKIN PLUS MULTIPLIES PROJECTION OF RESIDUAL METHOD (GMPR) APPLIED TO HELMHOLTZ EQUATION

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Abstract. *A new finite element method, developed to second-order generic scalar linear boundary-value problems, is presented for Helmholtz equation. The method is obtained adding to the Galerkin formulation appropriate multiple projections of residual of the differential equation at element level. These multiple projections of the residual allow that the element matrix has a maximum number of free parameters. The number of these parameters will depend on the local space of approximation and the differential operator. These free parameters can be determined seeking to satisfy some convergence and/or stability criterion. Several stabilized methods (such like, GLS and GGLS methods, etc.) can be obtained starting from new method for an appropriate choice of the free parameters. Also, we present a methodology to choose of free parameters for rectangular domain, uniform mesh and bilinear elements. In this case, the criterion adopted to determine the free parameters consists of minimizing the phase error of the approximate solution.*

Keywords: *Finite element method, Stabilization, GMPR, GLS, Helmholtz equation*

1. INTRODUCTION

Boundary-value problems governed by second-order linear partial differential equations model several physical phenomena. Stable and accuracy numerical solution via finite element method (FEM) for this problem has been the greatest challenge. To improve the stability and accuracy of finite element methods for these boundary-value problems it is necessary to have some criterion that allows obtaining the stability and/or accuracy wanted. Given a certain criterion, the stabilized method should be capable to simulate the wanted criterion. A form of obtaining such stabilized methods consists of adding residual terms at element level to the classical Galerkin formulation. This is the base of the GLS, GGLS methods and other stabilized methods. Therefore, it is desirable that a stabilized method based on these ideas with a maximum number of free parameters can be developed. The new method should be built in a way that, if the criterion to determine the free parameters is the same as adopted, for example, in the GLS, GGLS methods or another linear stabilized method, then these methods can be obtained starting from the new formulation.

A method with this capacity can be applied to Helmholtz equation. In this case, a possible criterion to be adopted, to determine the free parameters of the formulation, can consist of minimizing the phase error of the approximate solution. We should emphasize that several methods that minimize the phase error of the approximate solution to solve the Helmholtz equation exist. The Quasi-Stabilized Finite Element Method (QS) (Babuška et al, 1995) is one of these methods. Though, this method is not based on a variational formulation. This hinders the extension of the QS method for no uniforms meshes, high-order polynomials and problems with source term. In one-dimension the approximate solution of GLS method (Harari et al, 1992) eliminates the phase error for Helmholtz problem. In two-dimensions, two methods, originating from of variational formulations, that minimize the phase error exist: Residual-Based Finite Element Method (RBFEM) (Oberai et al, 2000) and Discontinuous Finite Element Method at Element Level (DGB) (Loula et al, submitted).

The RBFEM method is obtained from the Galerkin approximation by appending terms that are proportional to residuals on element interiors and inter-element boundaries. The terms that are proportional to the residuals on elements interiors can be understood as an extension of GLS method, considering the stabilization parameter dependent on the position. The free parameters are completed through residual terms on boundaries of the element. The residual on inter-element boundaries terms are motivated by the variational multiscale formulation. These terms implicate in an extra computational effort when the RBFEM formulation is compared with a classical continuous finite element formulation. Besides, does not exist a systematic for the choice of the functions at element level that allows obtaining these residual terms.

The DGB method is a discontinuous finite element formulation, where discontinuities are introduced locally, inside each element. These discontinuities can be viewed as discontinuous bubbles and the corresponding degrees of freedom can be eliminated at element level by static condensation yielding a global matrix topologically equivalent to those of classical C^0 finite element approximations. However, the DGB method needs the condensation technique to eliminate degrees of freedom introduced by the discontinuities. The free parameters, related to the weak enforcement of continuity inside each element, need to be determined. For uniform meshes we present a methodology to determine explicitly the stabilization parameters minimizing the pollution effect. But, for no uniforms meshes this methodology can transform the DGB formulation in a non linear method, in the sense that will be necessary to solve a non linear problem at element level to determine the free parameters and to accomplish the condensation technique. Besides, the local discontinuities introduced on inter-bubbles boundaries implicate in an extra computational effort.

In this paper, a new formulation, developed for second-order generic scalar linear boundary-value problems, is applied to Helmholtz equation. This formulation is obtained adding appropriate multiple projections of residual of the differential equation, only at element level, to the Galerkin formulation. These projections of residual at element level are obtained through a methodology, in such way that the formulation carries the dependence of differential equation operator and of local approach space explicitly. Besides, this methodology allows introducing in the formulation a maximum number of free parameters, which can be determined following some criterion to improve the stability and accuracy of the approximate solution. Explicit values of the free parameters minimizing the phase error are presented via a dispersion analysis. The formulation is general for any dimension of the domain. Some numerical experiments to evaluate the performance of the new formulation are presented. Finally, section 5 contains some conclusions and final remarks.

2. THE HELMHOLTZ EQUATION

2.1 The boundary value problem

Let $\Omega \subset R^n \quad (n \geq 1)$ be an open bounded domain with a Lipschitz continuous smooth piecewise boundary. Let Γ_g, Γ_q and Γ_r subsets of Γ satisfying $\Gamma_g \cap \Gamma_q = \Gamma_g \cap \Gamma_r = \Gamma_q \cap \Gamma_r = \varnothing$ and $\Gamma_g \cup \Gamma_q \cup \Gamma_r = \Gamma$. We shall consider the interior Helmholtz problem:

$$L(u) \equiv -\nabla \cdot (\nabla u) - k^2 u = f \quad \text{in } \Omega, \tag{1}$$

$$u = g \quad \text{on } \Gamma_g, \tag{2}$$

$$\nabla u \cdot \hat{n} = q \quad \text{on } \Gamma_q, \tag{3}$$

$$\nabla u \cdot \hat{n} + \alpha u = r \quad \text{on } \Gamma_r, \tag{4}$$

where u denotes a scalar field that describes time-harmonic acoustic, elastic or electromagnetic steady state waves. The coefficient $k \in R$ is the wave number, $f \in L^2(\Omega)$ is the source term, $g \in H^{\frac{1}{2}}(\Gamma_g) \cap C^0(\Gamma_g)$, $q \in L^2(\Gamma_q)$ and $r \in L^2(\Gamma_r)$ are the prescribed boundary conditions. The coefficient $\alpha \in L^\infty(\Gamma_r)$ is positive on Γ_r and $\hat{n}$ denotes the outward normal unit vector defined almost everywhere on Γ.

2.2 The associated variational problem

Let S and V defined as $S = \left\{u \in H^1(\Omega) : u = g \text{ on } \Gamma_g\right\}$, $V = \left\{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_g\right\}$ The variational problem associated to the boundary value problem defined by Eqs. (1-4) consist of finding $u \in S$ satisfying the following variational equation:

$$A(u,v) \equiv \int_\Omega [(\nabla u) \cdot \nabla v - k^2 uv] d\Omega + \int_{\Gamma_r} \alpha uv \, d\Gamma = \int_\Omega f v \, d\Omega + \int_{\Gamma_q} qv \, d\Gamma + \int_{\Gamma_r} rv \, d\Gamma \quad \forall \, v \in V, \tag{5}$$

The major challenges, in term of FEM, is to find a consistent formulation in continuous or discontinuous finite dimensional spaces, such that, its approximate solution is stable and the closest possible of the correspondent solution in infinite dimensional space given by Eq. (5).

In this paper we will just treat with continuous finite dimensional spaces. The continuous Galerkin FEM is the most used approximation.

2.3 The associated Galerkin finite element formulation

Let $M^h = \{\Omega_1, \ldots, \Omega_{NE}\}$ be a partition of Ω in no degenerated finite element Ω_e, such that Ω_e can be mapped in standard elements by isoparametric mapping and that satisfy $\Omega_e \cap \Omega_{e'} = \emptyset$ if $e \neq e'$ and $\Omega \cup \Gamma = \bigcup_{e=1}^{ne} (\Omega_e \cup \Gamma_e)$, where Γ_e denotes the boundary of Ω_e.

Let $p \geq 1$ an integer and consider $P^p(\Omega_e)$ defined as the space of polynomials of degree less than or equal to p. Let $H^h(\Omega) = \{\varphi \in H^1(\Omega); \varphi_e \in P^p(\Omega_e)\}$ and $H^{\frac{1}{2},h}(\Gamma_g) = \{\varphi \in H^{\frac{1}{2}}(\Gamma_g); \exists \phi \in H^1(\Omega) \text{ and } \phi = \varphi \text{ on } \Gamma_g\}$ are the finite dimension spaces and let g^h be the interpolate of g on $H^{\frac{1}{2},h}(\Gamma_g)$. The Galerkin formulation consists of finding $u^h \in S^h = \{\varphi \in H^h(\Omega); \varphi = g^h \text{ on } \Gamma_g\}$ that satisfies $\forall v^h \in V^h = \{\varphi \in H^h(\Omega); \varphi = 0 \text{ on } \Gamma_g\}$:

$$A_G(u^h, v^h) = F(v^h), \tag{6}$$

$$A_G(u^h, v^h) = \sum_{e=1}^{ne} \int_{\Omega_e} [\nabla u_e^h \cdot \nabla v_e^h - k^2 u_e^h v_e^h] \, d\Omega + \int_{\Gamma_r} \alpha u^h v^h \, d\Gamma, \tag{7}$$

$$F_G(v^h) = \sum_{e=1}^{ne} \int_{\Omega_e} f v_e^h \, d\Omega + \int_{\Gamma_q} q v^h \, d\Gamma + \int_{\Gamma_r} r v^h \, d\Gamma, \tag{8}$$

It is well know that the Galerkin FEM is shown unstable and little accuracy for Helmholtz equation. Its numerical solution presents spurious oscillations that do not corresponding with the physical solution of problem.

3. THE GALERKIN PLUS MULTIPIES PROJECTION OF RESIDUAL METHOD

In this section we presented the Galerkin plus multiply projection of residual method (GMPR). As it was said previously, the idea consists of adding to the Galerkin FEM multiple projections of the residual with one free parameter associated to each projection. The maximum number of free parameters depends on the differential operator and the local approach space. It is to say, the maximum number of linearly independent projections of residual will depend on properties of operator (such as symmetry, etc) and on order of interpolant polynomials. These free parameters would be then determined by appropriate criteria for each specific problem, seeking to obtain more accuracy and stable approximate solutions. For this, consider for each fixed element Ω_e the space $E_{gmpr}(\Omega_e)$ defined as proceeds:

$$E_{gmpr}(\Omega_e) = \{\psi : \Omega_e \to R; \psi = \sum_{i=1}^{npel} \sum_{j=1}^{npel} C_{i,j} L(\eta_i) L(\eta_j), \quad C_{i,j} \in R\} \tag{9}$$

where $L(\cdot)$ is defined by Eq. (1), $npel$ denotes the number of nodal points of the element Ω_e and $\eta_i\,(i=1,\ldots,npel)$ denotes the usual local shape functions associated to nodal point i. From Eq. (9) follows that, $E_{gmpr}(\Omega_e)$ is the space generated by $\{L(\eta_i)L(\eta_j);i,j=1,\ldots,npel\}$.

Let d_{gmpr} be a dimension of $E_{gmpr}(\Omega_e)$ and $\psi_1,\ldots,\psi_{d_{gmpr}}$ a local base for $E_{gmpr}(\Omega_e)$ built as following:

$$\psi_i = L(\eta_{Lig(i,1)})L(\eta_{Lig(i,2)}), i=1,\ldots,d_{gmpr} \tag{10}$$

where $Lig(\cdot,\cdot)$ is a function of $\{1,\ldots,d_{gmpr}\}\times\{1,2\}$ in $\{1,\ldots,npel\}\times\{1,\ldots,npel\}$ such that, the set $\{L(\eta_{Lig(1,1)})L(\eta_{Lig(1,2)}),\ldots,L(\eta_{Lig(i,1)})L(\eta_{Lig(i,2)}),\ldots,L(\eta_{Lig(d_{gmpr},1)})L(\eta_{Lig(d_{gmpr},2)})\}$ is a base for $E_{gmpr}(\Omega_e)$. There is vary ways to build a function $Lig(\cdot,\cdot)$. A simple way consists of defining the functions $\phi_1,\ldots,\phi_l,\ldots\phi_{npel\times npel}$ as:

$$\phi_l = L(\eta_i)L(\eta_j) \Leftrightarrow l=(i-1)npel+j\,, \tag{11}$$

$$l=1,\ldots,\overline{npel}\,, \tag{12}$$

$$i=1,\ldots npel,\ j=1,\ldots npel\,, \tag{13}$$

$$\overline{npel}=\frac{nepl(npel+1)}{2} \tag{14}$$

and to choose:

$$Lig(1,1)=1,\ Lig(1,2)=1\,. \tag{15}$$

For $m=2,\ldots,d_{gmpr}$ fixed we defined $Lig(m,1)=i$ and $Lig(m,2)=j$, if and only if, $l=(i-1)npel+j$ is the smallest integer satisfying:

$$npel[Lig(m-1,1)-1]+Lig(m-1,2)<l\le\overline{npel}\,, \tag{16}$$

$$\{\phi_1,\ldots,\phi_{l_{m-1}},\phi_{l_m}\}\ \text{is a linearly independent set,} \tag{17}$$

$$l_\mu = npel[Lig(\mu,1)-1]+Lig(\mu,2)\,, \tag{18}$$

$$l_m = l = npel[Lig(m,1)-1]+Lig(m,2)\,. \tag{19}$$

Then, let $Lig(\cdot,\cdot)$ be a function defined by the calculation algorithm for the base of $E_{gmpr}(\Omega_e)$ space. Therefore, $\{L(\eta_{Lig(i,1)})L(\eta_{Lig(i,2)}),\ i=1,\ldots,d_{gmpr}\}$ is a local base for $E_{gmpr}(\Omega_e)$ and we defined ψ_i by Eq. (10) for this $Lig(\cdot,\cdot)$ function.

Let $\psi_i^\perp\ (i=1,\ldots,d_{gmpr})$ be the functions defined as:

$$\psi_1^\perp=\psi_1\,, \tag{20}$$

$$\psi_i^\perp=\psi_i-\sum_{l=1}^{i-1}\frac{\left(\psi_i,\psi_l^\perp\right)_{L^2(\Omega_e)}}{\left\|\psi_l^\perp\right\|^2_{L^2(\Omega_e)}}\psi_l^\perp,\ i=2,\ldots,d_{gmpr}\,, \tag{21}$$

in other words, $\psi_1^{\perp},\ldots,\psi_{d_{gmpr}}^{\perp}$ is a orthogonal base in $L^2(\Omega_e)$ for $E_{gmpr}(\Omega_e)$, where $(\cdot,\cdot)_{L^2(\Omega_e)}$ and $\|\cdot\|_{L^2(\Omega_e)}$ denote the inner product and norm in $L^2(\Omega_e)$ respectively, as defined in (Adams, 1975) . Therefore, $\hat{\psi}_1^{\perp}$ $(i=1,\ldots,d_{gmpr})$ defined as:

$$\hat{\psi}_i^{\perp}=\frac{\hat{\psi}_i^{\perp}}{\left\|\hat{\psi}_i^{\perp}\right\|_{L^2(\Omega_e)}}, \tag{22}$$

is a orthonormal base in $L^2(\Omega_e)$ for $E_{gmpr}(\Omega_e)$.

Let $C_{l,i}$ $(l=1,\ldots,\overline{n}pel$ and $i=1,\ldots,d_{gmpr})$ be coefficients, such that:

$$\phi_l=\sum_{i=1}^{d_{gmpr}}C_{l,i}\hat{\psi}_i^{\perp}. \tag{23}$$

Let $\tau_1,\ldots,\tau_{d_{gmpr}}$ real parameters. The Galerkin plus multiply projection of residual method consists in finding $u^h\in S^h$ that satisfies $\forall v^h\in V^h$ the variational equation:

$$A_G(u^h,v^h)+\sum_{e=1}^{ne}\left(\sum_{i=1}^{d_{gmpr}}\tau_i\left(L(u^h),L(v^h)\hat{\psi}_i^{\perp}\right)_{L^2(\Omega_e)}\right)=F_G(v^h)+\sum_{e=1}^{ne}\left(\sum_{i=1}^{d_{gmpr}}\tau_i\left(f_e,L(v^h)\hat{\psi}_i^{\perp}\right)_{L^2(\Omega_e)}\right), \tag{24}$$

where f_e denote the restriction of f to Ω_e. Note that, a new finite element formulation is consistent, in sense that the exact solution of Eq. (5) is also solution of Eq. (24).

3.1 The element matrix

Let u_e^h be the restriction of u^h to Ω_e given by:

$$u_e^h=\sum_{i=1}^{npel}\hat{u}_e^h(i)\eta_i, \tag{25}$$

where $\hat{u}_e^h(i)$ denote the value of u_e^h in local node i of Ω_e element. Therefore, we have:

$$\left(L(u_e^h),L(\eta_i)\hat{\psi}_l^{\perp}\right)_{L^2(\Omega_e)}=\sum_{m=1}^{npel}\hat{u}_e^h(m)\left(L(\eta_m),L(\eta_i)\hat{\psi}_l^{\perp}\right)_{L^2(\Omega_e)}, \tag{26}$$

$$i=1,\ldots,npel \text{ and } l=1,\ldots,d_{gmpr}. \tag{27}$$

Remind that $\hat{\psi}_1^{\perp},\ldots,\psi_{d_{gmpr}}^{\perp}$ is a orthonormal base in $L^2(\Omega_e)$ for $E_{gmpr}(\Omega_e)$ and by Eqs. (11-14) and Eq. (23) can be obtained,

$$\left(L(\eta_m),L(\eta_i)\hat{\psi}_l^{\perp}\right)_{L^2(\Omega_e)}=\left(\phi_j,\hat{\psi}_l^{\perp}\right)_{L^2(\Omega_e)}=C_{j,l}, \tag{28}$$

$$j=(m-1)npel+i \text{ if } i\geq m, \text{ or } j=(i-1)npel+m \text{ if } i\leq m. \tag{29}$$

Therefore,

$$\left(L(u_e^h), L(\eta_i)\hat{\psi}_l^{\perp}\right)_{L^2(\Omega_e)} = \sum_{m=1}^{npel} C_{i,l}\hat{u}_e^h(m), \tag{30}$$

and if M_{im}^e denote the elements of the element matrix we obtain:

$$M_{im}^e = A_G^e(\eta_m, \eta_i) + \sum_{l=1}^{d_{gmpr}} \tau_l C_{i,l}\ , \tag{31}$$

$$A_G^e(\eta_m, \eta_i) = \int_{\Omega_e} [\nabla \eta_m \cdot \nabla \eta_i + k^2 \eta_m \eta_i]\, d\Omega + \int_{\Gamma_r} \alpha \eta_m \eta_i d\Gamma\ , \tag{32}$$

where $C_{i,l}$ is given by Eq. (28) and *j* is defined by Eq. (29). We can notice that the element matrix is formed by the usual part of Galerkin more d_{gmpr} projections of residual of the differential equation at element level. The free parameters $\tau_1, \ldots, \tau_{d_{gmpr}}$, corresponding to each projection of residual, can be determined through some criterion adopted to improve the accuracy and/or stability of the approximate solution for each specific problem. For Helmholtz equation with uniform mesh and bilinear quadrilateral elements we have:

$$\begin{aligned}
\phi_1 &= +k^4 \eta_1 \eta_1, \\
\phi_2 &= +k^4 \eta_1 \eta_2, \\
\phi_3 &= +k^4 \eta_1 \eta_3, \\
\phi_4 &= +k^4 \eta_1 \eta_4, \\
\phi_5 &= +k^4 \eta_2 \eta_2, \\
\phi_6 &= +k^4 \eta_2 \eta_3, \\
\phi_7 &= +k^4 \eta_2 \eta_4 = \phi_3, \\
\phi_8 &= +k^4 \eta_3 \eta_3, \\
\phi_9 &= +k^4 \eta_3 \eta_4, \\
\phi_{10} &= +k^4 \eta_4 \eta_4,
\end{aligned} \tag{33}$$

and the base $\psi_1, \ldots, \psi_{d_{gmpr}}$ for $E_{gmpr}(\Omega_e)$ is:

$$\begin{aligned}
\psi_1 &= \phi_1, \\
\psi_2 &= \phi_2, \\
\psi_3 &= \phi_3, \\
\psi_4 &= \phi_4, \\
\psi_5 &= \phi_5, \\
\psi_6 &= \phi_6, \\
\psi_7 &= \phi_8, \\
\psi_8 &= \phi_9, \\
\psi_9 &= \phi_{10},
\end{aligned} \tag{34}$$

therefore,

$$d_{gmpr} = 9 . \tag{35}$$

Let a_{im}^{l} be defined by:

$$a_{im}^{l} = C_{j,l} \Leftrightarrow \begin{cases} j = (m-1)npel + i \text{ if } i \geq m \\ j = (i-1)npel + m \text{ if } m \geq i \end{cases}, \tag{36}$$

$$i = 1,\ldots,npel,\ m = 1,\ldots,npel \text{ and } l = 1,\ldots,d_{gmpr} . \tag{37}$$

Therefore, the elements of element matrix are:

$$M_{im}^{e} = A_{G}^{e}(\eta_m,\eta_i) + \sum_{l=1}^{d_{gmpr}} \tau_l a_{im}^{l} . \tag{38}$$

By Eq. (33), Eqs. (36-37) and Eq. (28) we obtain $a_{13}^{l} = a_{24}^{l}\ \forall l \in \{1,\ldots,d_{gmpr}\}$. Also, by Eq. (32) proceed $A_{G}^{e}(\eta_3,\eta_1) = A_{G}^{e}(\eta_4,\eta_2)$.

Let τ_l^* defined as:

$$\tau_l^* = A_{G}^{e}(\eta_m,\eta_i) + \sum_{\mu=1}^{d_{gmpr}} \tau_\mu a_{im}^{\mu} , \tag{39}$$

$$l = (i-1)npel + m,\ i = 1,\ldots,npel,\ m = 1,\ldots,npel , \tag{40}$$

and therefore the element matrix becomes:

$$M^{e} = \begin{bmatrix} \tau_1^* & \tau_2^* & \tau_3^* & \tau_4^* \\ \tau_2^* & \tau_5^* & \tau_6^* & \tau_3^* \\ \tau_3^* & \tau_6^* & \tau_7^* & \tau_8^* \\ \tau_4^* & \tau_3^* & \tau_8^* & \tau_9^* \end{bmatrix} . \tag{41}$$

Notice that due to symmetry of the Helmholtz operator and that is using first-order interpolant polynomials has nine projections, and therefore nine free parameters. In this paper, for a uniform mesh we adopted the dispersion analysis criteria to determine the free parameters $\tau_1^*,\ldots,\tau_{d_{gmpr}}^*$ that compose the element matrix M^e given by Eq. (41), through the following steps:

1) Using an uniform mesh with bilinear quadrilateral elements, the interior stencil is obtained for a plane wave $e^{ik(x\cos\theta + y\sin\theta)}$ $(i = \sqrt{-1}, 0 \leq \theta \leq \pi)$ as a function of the components of the matrix M^e, given by:

$$\bar{\tau}_0 + \bar{\tau}_1 \cos(\tilde{k}h\sin\theta) + \bar{\tau}_2 \cos(\tilde{k}h\cos\theta) + \bar{\tau}_3 \cos(\tilde{k}h\sin\theta)\cos(\tilde{k}h\cos\theta) = 0 , \tag{42}$$

$$\bar{\tau}_0 = \tau_1^* + \tau_5^* + \tau_7^* + \tau_9^* , \tag{43}$$

$$\bar{\tau}_1 = 2(\tau_4^* + \tau_6^*) , \tag{44}$$

$$\bar{\tau}_2 = 2(\tau_2^* + \tau_8^*)\,, \tag{45}$$

$$\bar{\tau}_3 = 4\tau_3^*\,. \tag{46}$$

Notice that the parameters $\bar{\tau}_0, \bar{\tau}_1, \bar{\tau}_2$ and $\bar{\tau}_3$ depend on k but not on $\widetilde{k}$. The stencil Eq. (42) is an equation with four unknowns $\bar{\tau}_0, \bar{\tau}_1, \bar{\tau}_2$ and $\bar{\tau}_3$. Choosing two different directions θ_1 and θ_2 for the plane wave follows that the interior stencil becomes two linearly independent equations. Therefore, two unknowns will be undetermined if we used the dispersion analysis criteria.

2) Let us assume the following restrictions for the free parameters:

$$\tau_1^* = \tau_5^* = \tau_7^* = \tau_9^* = \tfrac{1}{4} \Rightarrow \bar{\tau}_0 = 1\,, \tag{47}$$

$$\tau_4^* = \tau_6^* = \tfrac{1}{2}\bar{\tau}_1 \text{ and } \tau_2^* = \tau_8^* = \tfrac{1}{2}\bar{\tau}_2 = \tfrac{1}{2}\bar{\tau}_1 \Rightarrow \bar{\tau}_1 = \bar{\tau}_2\,. \tag{48}$$

We should emphasize that for a non-uniform meshes these restrictions should not necessarily be assumed. Then the interior stencil is:

$$1 + \bar{\tau}_1(\cos(\widetilde{k}h\sin\theta) + \cos(\widetilde{k}h\cos\theta)) + \bar{\tau}_3\cos(\widetilde{k}h\sin\theta)\cos(\widetilde{k}h\cos\theta) = 0\,. \tag{49}$$

3) To minimize the phase error of the approximate solution, following the work (Babuška et al, 1995) has:

$$r_1 = \cos(kh\cos\theta_1)\cos(khsin\theta_1)\,,\ r_2 = \cos(kh\cos\theta_2)\cos(khsin\theta_2)\,, \tag{50}$$

$$w_1 = \cos(kh\cos\theta_1) + \cos(khsin\theta_1)\,,\ w_2 = \cos(kh\cos\theta_2) + \cos(khsin\theta_2)\,, \tag{51}$$

$$\bar{\tau}_1 = \frac{(r_1 - r_2)}{(r_2 w_1 - r_1 w_2)},\quad \bar{\tau}_3 = \frac{(w_2 - w_1)}{(r_2 w_1 - r_1 w_2)},\quad \theta_1 = \tfrac{\pi}{16} \text{ and } \theta_2 = \tfrac{3\pi}{16}\,. \tag{52}$$

Therefore, the matrix M^e corresponds with the matrix given by (Babuška et al, 1995) and it has the following form:

$$M^e = \begin{bmatrix} \frac{1}{4} & \frac{\bar{\tau}_1}{2} & \frac{\bar{\tau}_3}{4} & \frac{\bar{\tau}_1}{2} \\ \frac{\bar{\tau}_1}{2} & \frac{1}{4} & \frac{\bar{\tau}_1}{2} & \frac{\bar{\tau}_3}{4} \\ \frac{\bar{\tau}_3}{4} & \frac{\bar{\tau}_1}{2} & \frac{1}{4} & \frac{\bar{\tau}_1}{2} \\ \frac{\bar{\tau}_1}{2} & \frac{\bar{\tau}_3}{4} & \frac{\bar{\tau}_1}{2} & \frac{1}{4} \end{bmatrix}. \tag{53}$$

This way, τ_l^* are determined by Eqs. (47-48). Finally, τ_μ are determined by Eqs. (39-40). The determination of τ_μ becomes fundamental for problems that possess source term. We should highlight that it was not necessary any consideration as made in the reference (Loula et al, submitted), besides not needing of any term in boundary of element (Oberai et al, 2000). Beyond, we should emphasize that the GMPR formulation possesses a general methodology, that it is valid for any geometry and dimension of the domain, as well as, for any local approach space.

4. NUMERICAL RESULTS

The plane-wave is solution of Helmholtz equation. In general, the direction of wave propagation is not known. Here, we present three 2-D examples that show the importance of having a finite element formulation to minimize the phase error. These examples illustrate as the accuracy and stability of any FEM that none minimize the phase error (such as, Galerkin and GLS methods) is deteriorates when we compared with a formulation, that possesses the property of minimizing the phase error (such as, QS, DGB, RBFEM and GMPR methods). For uniform meshes the solution of QS, DGB and GMPR methods coincides.

For this consider the homogenous Helmholtz equation in a unity square domain. The Dirichlet boundary conditions are such that the exact solution is a plane-wave propagating in θ-direction: $u(x,y) = \cos(k(x\cos\theta + ysin\theta))$. In all examples bilinear shape functions, 3x3 Gaussian integration, uniform mesh (160x160) and the same wave number (k=100) are adopted. Also, for all examples, the stabilization parameter of GLS method is choice at direction $\theta = \frac{\pi}{8}$, as proposed in (Thompson et al, 1995).

Figures 1 and 2 present a comparison between the relative errors in L²-norm and H¹-seminorm of the GMPR, continuous interpolant (CI) and QS solutions. In this case, the solution of the QS and GMPR methods coincide. Fig. 3 shows the nodal interpolant, GMPR and GLS solutions in sections *x=0.5* along the y direction for *θ=(π/4)* for this example.

The next example is similar to previous example, but now the exact solution is given by a superposition of *n* mono-energetic plane-waves propagating in *n* different θ-directions: $u(x,y) = \sum_{i=1}^{n} \cos(k(x\cos\theta_i + ysin\theta_i))$. Firstly, three plane waves propagating in the directions $\theta_1 = 0$, $\theta_2 = \frac{\pi}{8}$ and $\theta_3 = \frac{\pi}{4}$ are considered. The relative errors in L²-norm, H¹-seminorm and H¹-norm are present in Table 1. Figure 4 shows the nodal interpolant, GMPR and GLS solutions in sections *x=0.5* along the *y* direction. Figure 5 shows the same FEM solutions in section *y=0.5* along the *x* direction. Again, the results show the good performance of the GMPR formulation and how this formulation reduces the phase error over all wave vector orientations *θ*.

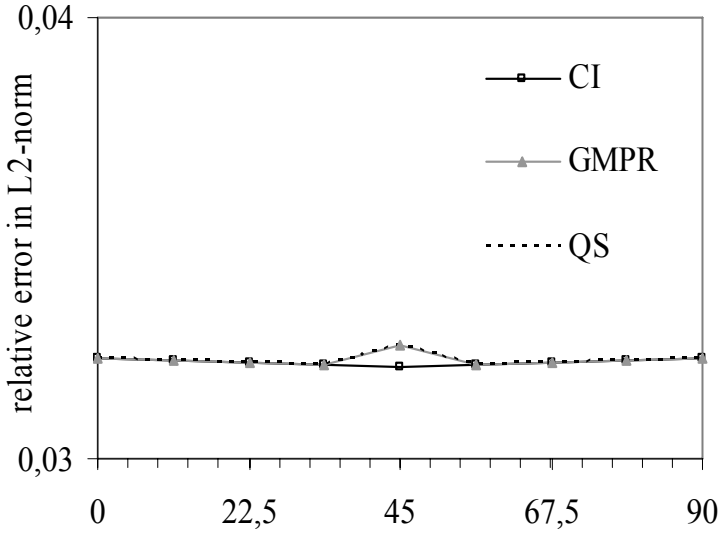

Fig. 1 Relative error of the CI, GMPR and QS solutions in the L²-norm as a function of *θ*-direction.

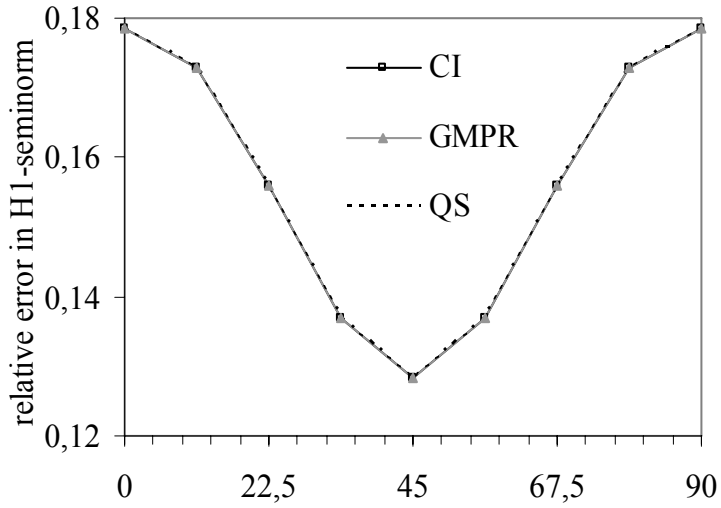

Fig. 2 Relative error of the CI, GMPR and QS solutions in the H¹-seminorm as a function of θ-direction.

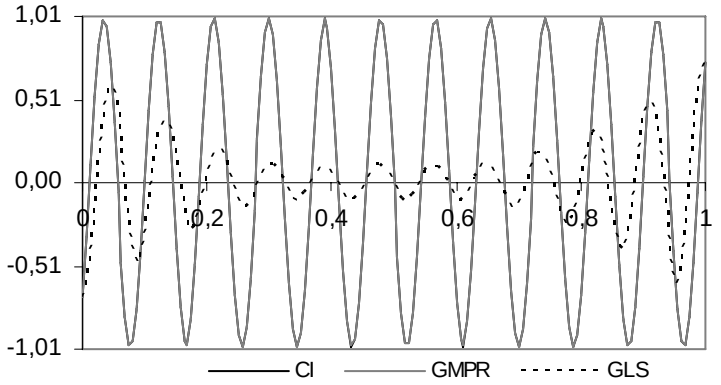

Fig. 3 Solution of homogeneous problem in two dimension at sections *x=0.5* for *θ=(π/4)*.

Table 1. Relative errors of FEMs for three and six plane waves

	Relative Errors of three finite element methods			
Three plane waves	CI	GMPR	GLS	Galerkin
L²-norm	3.22E-02	3.23E-02	5.40E-01	1.71E+00
H¹-seminorm	1.56E-01	1.56E-01	5.59E-01	1.72E+00
H¹-norm	1.56E-01	1.56E-01	5.59E-01	1.72E+00
Six plane waves	CI	GMPR	GLS	Galerkin
L²-norm	3.22E-02	3.23E-02	5.45E-01	3.24E+00
H¹-seminorm	1.56E-01	1.56E-01	5.69E-01	3.24E+00
H¹-norm	1.56E-01	1.56E-01	5.69E-01	3.24E+00

Secondly, six plane waves propagating in the directions $\theta_1 = 0$, $\theta_2 = \frac{\pi}{20}$, $\theta_3 = \frac{\pi}{10}$, $\theta_4 = \frac{3\pi}{20}$, $\theta_5 = \frac{\pi}{5}$ and $\theta_6 = \frac{\pi}{4}$ are considered. Figures 6 and 7 show the nodal interpolant, GMPR and GLS solutions in sections *x=0.5* and *y=0.5* respectively. Very similar conclusions to the previous example can be drawn. We should observe that, in these

two examples the directions of plane waves propagations are always different to $\theta_1 = \frac{\pi}{16}$ and $\theta_2 = \frac{3\pi}{16}$, which are the directions for asymptotically optimal interior stencil.

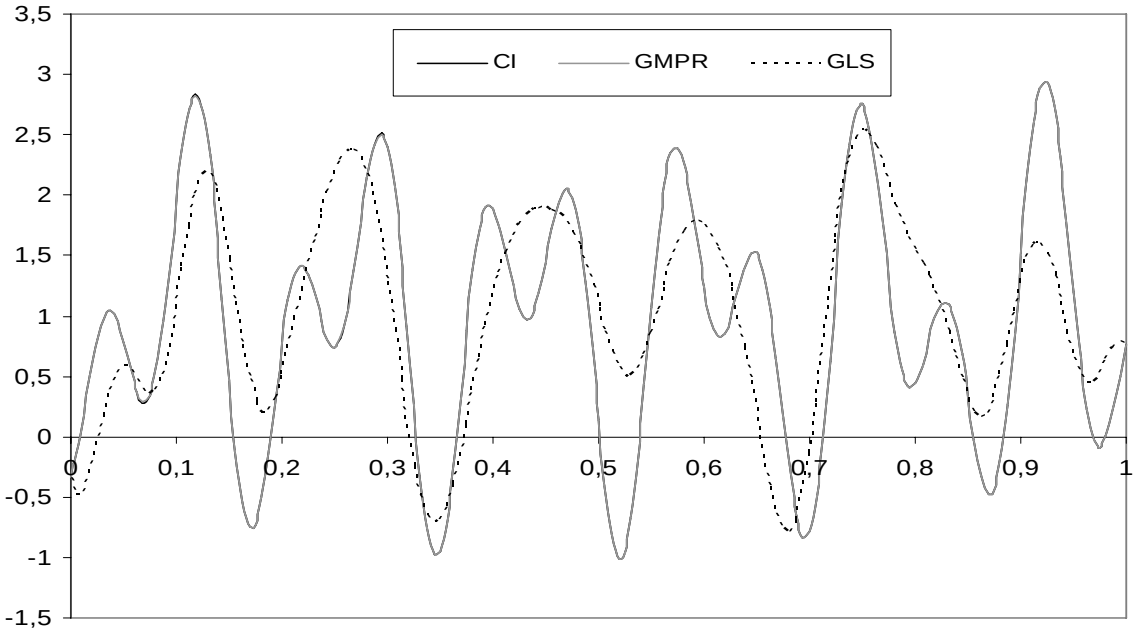

Fig. 4 GMPR and GLS solutions of homogeneous problem in two dimension at sections *x=0.5*, three plane-waves.

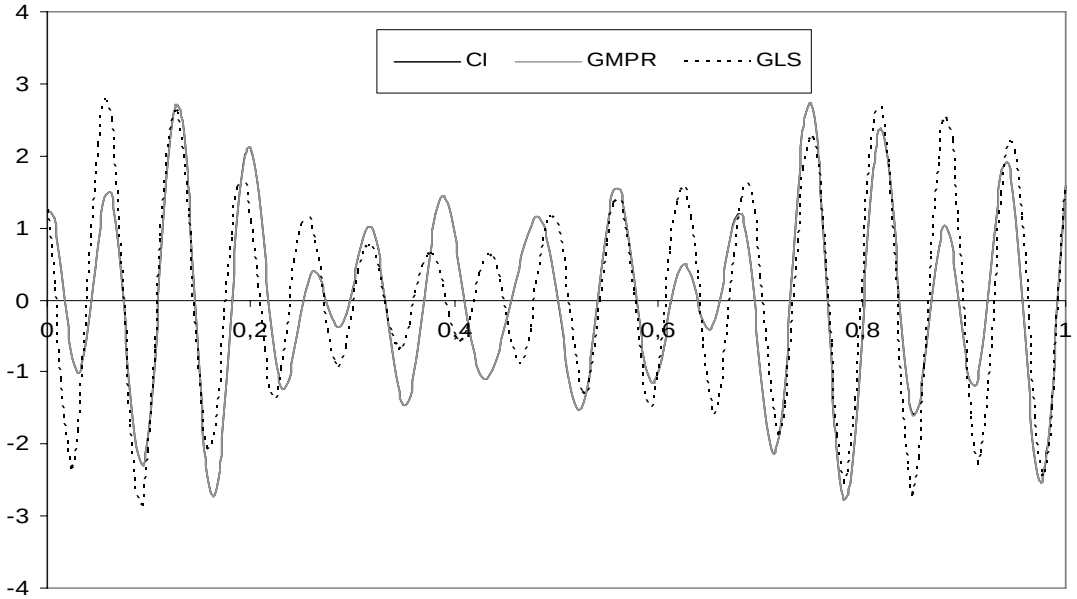

Fig. 5 GMPR and GLS solutions of homogeneous problem in two dimension at sections *y=0.5*, three plane-waves.

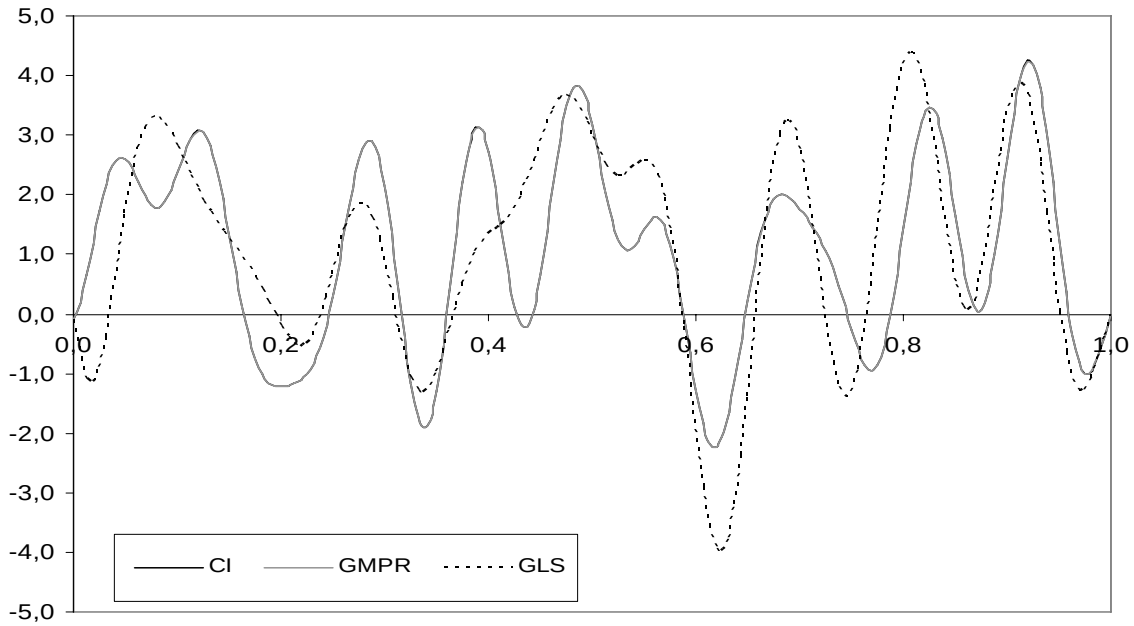

Fig. 6 GMPR and GLS solutions of homogeneous problem in two dimension at sections *x=0.5*, six plane-waves.

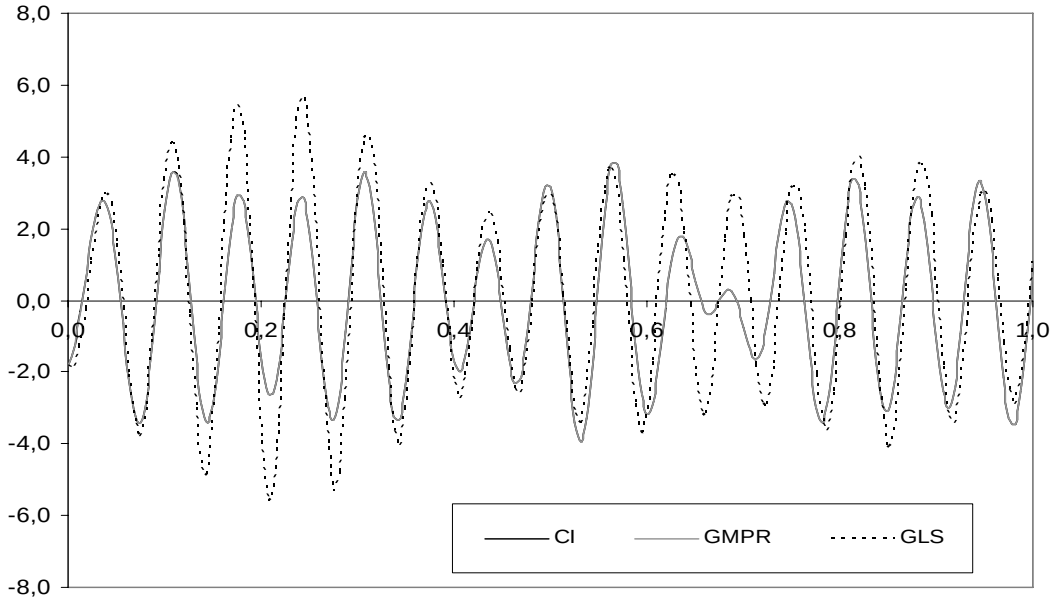

Fig. 7 GMPR and GLS solutions of homogeneous problem in two dimension at sections *y=0.5*, six plane-waves.

5. CONCLUSIONS

Herein, we present a new consistent finite element formulation for Helmholtz equation. The formulation is valid for any dimension of the domain and any order of local basis functions.

The GMPR method introduces a maximum number of free parameters via multiple projections of the residual for each local approach space. These free parameters can be determined by appropriate criteria allowing to find the continuous finite dimensional space

S^h that best approximates the infinite dimensional space S. Explicit values of the free parameters minimizing the phase error are presented via a dispersion analysis.

For bilinear shape functions the GMPR method need 3x3 Gaussian integration, since the local base of $E_{gmpr}(\Omega_e)$ are quadratic polynomial functions. Even so, the GMPR method only has volume terms. The RBFEM formulation and DGB method have surface terms besides the volume terms. Therefore, the extra computational effort of GMPR method is minor that the extra computational effort of RBFEM and DGB methods.

The numerical simulations presented prove the importance of having a FEM that minimizes the phase error. The good performance of the proposed formulation obtained for Helmholtz equation, stimulates to apply the GMPR method to other problems in future works.

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