THE GALERKIN SYMMETRICAL PROJECTED RESIDUAL METHOD (GSPR) FOR DIFFUSION-ADVECTION EQUATION

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Abstract. *In previous works the method GPR (Galerkin Projected Residual Method) was introduced. The method is obtained adding to the Galerkin formulation an appropriate numbers of projections of the residual of PDE within each element. These multiple projections allow the generation of appropriate number of free stabilization parameters in the element matrix depending on the local space of approximation and on the differential operator. The free parameters can be determined imposing some convergence and/or stability criteria or by postulating the element matrix with the desired stability properties. The element matrix of most stabilized methods (such as, GLS and GGLS methods) can be obtained from this new method with appropriate choices of the stabilization parameters. The GPR formulation has been applied with success to Helmholtz problem and to diffusion-reaction singularly perturbed problem.*

In this work, based on the initial ideas of the GPR method, we developed the Galerkin Symmetrical Projected Residual Method (GSPR). The GSPR method introduces new ideas about the space of matrices associated with the multiple projections of the residual. In this case, the space of matrices is splitting in symmetrical and skew-symmetrical matrices. We observed that only the symmetrical matrices are decisive to stabilize the numerical method and therefore to determine the free parameters. The methodology to choose the free parameters is similar to used for GPR method and consists in to postulate an element matrix with the desired stability properties (generating matrix) and the free parameters are determinate solving a least square problem at element level. We presented some numerical tests for problems with sharp layer and two different postulated generating matrices are proposed to show the importance of this matrix in the stabilization properties of the method.

Keywords: *FEM, Stabilization, GPR, GLS, Diffusion-advection equation*

1 INTRODUCTION

The advection-diffusion equation model several physical phenomena. The Galerkin finite element method is often used to obtain numerical solutions for this boundary value problem. In general, only for purely diffusive problem the Galerkin approximate solution is the optimal solution. It is well known that the Galerkin finite element method is unstable and inaccuracy for this equation with dominate advection (Brooks et al., 1982), (Johnson et al., 1984). Its numerical solution presents spurious oscillations that do not corresponding with the physical solution of problem.

Stable and accuracy numerical solution via finite element method (FEM) for this problem has been the greatest challenge. A great variety of FEM have been developed to obtain stable and accuracy solution, but it is impossible to list all of the works in this direction. Comparisons between different methods and a recent bibliographical review can be found in (Codina, 1998), (John et al., 2007) and (John et al., 2008). Many of these attempts have used continuous (Hughes et al., 1989), (Franca et al., 1989), (Hughes, 1995), (Oñate, 1998), (Ilinca et al., 2000), (Franca et al., 2000), (Nesliturk et al., 2003), (Burman et al., 2004), (Franca et al., 2005), (Lube et al., 2006) and discontinuous finite element spaces (Hughes et al., 2006) to development de new FEM. The main challenge, in term of FEM, is to find a consistent formulation in continuous or discontinuous finite dimensional spaces, such that, its approximate solution is stable and the closest possible of the correspondent solution in infinite dimensional space.

Here we will just treat with continuous finite dimensional spaces. In previous works the GPR method (Galerkin Projected Residual Method) was introduced. This method is obtained adding to the Galerkin formulation an appropriate numbers of projections of the residual of PDE within each element. These multiple projections allow the generation of appropriate number of free stabilization parameters in the element matrix depending on the local space of approximation and on the differential operator. The free parameters can be determined imposing some convergence and/or stability criteria or by postulating the element matrix with the desired stability properties. The element matrix of most stabilized methods (such as, GLS and GGLS methods) can be obtained from this new method with appropriate choices of the stabilization parameters.

In this paper, the fundamental ideas of the GPR method are maintained to develop the Galerkin Symmetrical Projected Residual Method (GSPR). The GSPR method introduces new ideas about the space of matrices associated with the multiple projections of the residual. The space of matrices is splitting in symmetrical and skew-symmetrical matrices and only the symmetrical matrices are decisive to determine the free parameters. The methodology to choose the free parameters is similar to used for GPR method and consists in to postulate an element matrix with the desired stability properties (generating matrix) and the free parameters are determinate solving a least square problem at element level. Two different generating matrices are postulated to determine the free parameters and the stabilization properties of the method are studied by numerical tests.

2 THE ADVECTION-DIFFUSION EQUATION

2.1 The model boundary value problem

Let $\Omega \subset R^n$ $(n \geq 1)$ be an open bounded domain with a Lipschitz continuous smooth piecewise boundary Γ. Let Γ_g, Γ_q and Γ_r are subsets of Γ satisfying $\Gamma_g \cap \Gamma_q = \Gamma_g \cap \Gamma_r = \Gamma_q \cap \Gamma_r = \varnothing$ and $\Gamma_g \cup \Gamma_q \cup \Gamma_r = \Gamma$. We shall consider the problem:

$$L(\phi) \equiv -\nabla \cdot (D\nabla\phi) + u \cdot \nabla\phi = f \quad \text{in } \Omega\,, \tag{1}$$
$$\phi = g \quad \text{on } \Gamma_g\,, \tag{2}$$
$$D\nabla\phi \cdot \hat{n} = q \quad \text{on } \Gamma_q\,, \tag{3}$$
$$D\nabla\phi \cdot \hat{n} + \alpha\phi = r \quad \text{on } \Gamma_r\,. \tag{4}$$

where the functions D (diffusive coefficient) and u (advection field) are assumed satisfy: $0 < D \le \overline{D}$ and $0 \le -\frac{1}{2}\nabla \cdot u$ with $\overline{D}$ being positive real constant. $f \in L^2(\Omega)$ is the source term, $g \in H^{\frac{1}{2}}(\Gamma_g) \cap C^0(\Gamma_g)$, $q \in L^2(\Gamma_q)$ and $r \in L^2(\Gamma_r)$ are the prescribed boundary conditions. The coefficient $\alpha \in L^\infty(\Gamma_r)$ and $\hat{n}$ denotes the outward normal unit vector defined almost everywhere on Γ.

2.2 The associated variational problem

Let S and V defined as $S = \{\phi \in H^1(\Omega) : \phi = g \text{ on } \Gamma_g\}$, $V = \{\eta \in H^1(\Omega) : \eta = 0 \text{ on } \Gamma_g\}$. The variational problem associated to the boundary value problem defined by Eqs. (1-4) consist of finding $\phi \in S$ satisfying the following variational equation:

$$A(\phi,\eta) \equiv \int_\Omega [D\nabla\phi \cdot \nabla\eta + u \cdot \nabla\phi\eta]\, d\Omega + \int_{\Gamma_r} \alpha\phi\eta\, d\Gamma = \int_\Omega f\eta d\Omega + \int_{\Gamma_q} q\eta d\Gamma + \int_{\Gamma_r} r\eta d\Gamma \equiv F(\eta)\, \forall\ \eta \in V\,. \tag{5}$$

The major challenges, in term of FEM, is to find a consistent formulation in continuous or discontinuous finite dimensional spaces, such that, its approximate solution is stable and the closest possible of the correspondent solution in infinite dimensional space given by (5). In this paper we will just treat with continuous finite dimensional spaces. The continuous Galerkin FEM is the most used approximation.

2.3 The associated Galerkin finite element formulation

Let $M^h = \{\Omega_1, \ldots, \Omega_{ne}\}$ be a partition of Ω in no-degenerated finite element Ω_e, such that Ω_e can be mapped in standard elements by isoparametric mapping and that satisfy $\Omega_e \cap \Omega_{e'} = \emptyset$ if $e \ne e'$ and $\Omega \cup \Gamma = \bigcup_{e=1}^{ne} (\Omega_e \cup \Gamma_e)$, where Γ_e denotes the boundary of Ω_e.

Let $k \ge 1$ an integer and consider $P^k(\Omega_e)$ defined as the space of polynomials of degree less than or equal to k. Let $H^{h,k}(\Omega) = \{\eta \in H^1(\Omega); \eta_e \in P^k(\Omega_e)\}$, $S^{h,k} = \{\eta \in H^{h,k}(\Omega); \eta = g^h \text{ on } \Gamma_g\}$ and $V^{h,k} = \{\eta \in H^{h,k}(\Omega); \eta = 0 \text{ on } \Gamma_g\}$ are the finite dimension spaces, where η_e denotes the restriction of η to Ω_e. Let g^h be the interpolate of g. The Galerkin formulation consists of finding $\phi^h \in S^{h,k}$ that satisfies:

$$A(\phi^h, \eta^h) = F(\eta^h) \quad \forall \eta^h \in V^{h,k}\,. \tag{6}$$

Only for purely diffusive problems the solution of Galerkin FEM is an accurate approximation. It is well known that the Galerkin finite element method is unstable and inaccuracy for several problems described by scalar and linear second-order partial differential equations. Its numerical solution presents spurious oscillations that do not corresponding with the physical solution of problem.

3 THE GALERKIN SYMMETRICAL PROJECTED RESIDUAL METHOD (GSPR)

The GPR method was previously introduced for Helmholtz equation and diffusion-reaction equation in (Dutra do Carmo et al., 2008). This method was obtained adding to the Galerkin formulation an appropriate numbers of projections of the residual of PDE within each element. This allows that the element matrix has a maximum number of free parameters. Other theoretical details on the method can be found in Dutra do Carmo (2008).

Here, the space of matrices associated with the multiple projections of the residual is splitting in symmetrical and skew-symmetrical matrices. We observed that, when a GPR-generating matrix is symmetrical, only the symmetrical matrices are decisive to stabilize the numerical method and therefore to determine the free parameters. The GSPR method applied to advection-diffusion equation consist of finding $\phi^h \in S^{h,k}$ satisfying $\forall \eta^h \in V^{h,k}$ the variational equation:

$$A(\phi^h,\eta^h)+\sum_{e=1}^{ne}\left(\underbrace{\left(L(\phi_e^h),\tau_0^e L(\eta_e^h)\right)_{L^2(\Omega_e)}}_{A^{GLS,e}(\tau_0^e,\phi_e^h,\eta_e^h)}+\underbrace{\sum_{l=1}^{\mathrm{N}}\tau_l^e\left(L(\phi_e^h),L^*(\eta_e^h)\psi_{l,e}\right)_{L^2(\Omega_e)}}_{A^{GSPR,e}(\tau^e,\phi_e^h,\eta_e^h)}\right)=$$

$$F(\eta^h)+\sum_{e=1}^{ne}\left(\underbrace{\left(f_e,\tau_0^e L(\eta_e^h)\right)_{L^2(\Omega_e)}}_{F^{GLS,e}(\tau_0^e,\eta_e^h)}+\underbrace{\sum_{l=1}^{\mathrm{N}}\tau_l^e\left(f_e,L^*(\eta_e^h)\psi_{l,e}\right)_{L^2(\Omega_e)}}_{F^{GSPR,e}(\tau^e,\eta_e^h)}\right), \tag{7}$$

where τ_0^e is the GLS parameter. Works as (Hughes et al., 1986), (Tezduyar et al., 2000), (Dutra do Carmo et al., 2004), (Knobloch, 2006), (Knobloch, 2008) study an optimal choice for the free parameters of some stabilized methods. The τ_l^e are the free parameters of the GSPR formulation, N is the dimension of a real linear space $E_{GPR}(\Omega_e)$ defined as $E_{GPR}(\Omega_e)=\{\psi:\Omega_e\to\mathbb{R};\psi=\sum_{i=1}^{npel}\sum_{j=1}^{npel}C_{i,j}\hat{L}(\eta_i,\eta_j),\quad C_{i,j}\in\mathbb{R}\}$ with basis denoted by $\psi_{l,e}$, $npel$ denotes the number of nodal points of the element Ω_e and $\eta_i\,(i=1,\ldots,npel)$ denotes the usual local shape functions associated to nodal point *i*. $L^*(\circ)$ is a linear operator. $\hat{L}(\circ,\circ)$ denotes an arbitrary symmetrical operator and do not necessarily have to be linear. The choice of this operator will depend on each specific problem and it is crucial to generate an adequate space of matrices for the GSPR method. Note that, a new GSPR formulation is consistent, in sense that the exact solution of Eq. (5) is also solution of Eq. (7).

In order to determine the N parameters τ_l^e we shall determine the element matrix of the GSPR method. Let ϕ_e^h be the restriction of ϕ^h to Ω_e given by:

$$\phi_e^h=\sum_{j=1}^{npel}\hat{\phi}_e^h(j)\eta_j, \tag{8}$$

where $\hat{\phi}_e^h(j)$ denote the value of $\phi_e^h(j)$ in local node j of element Ω_e. Let $M_{ij}^{e,l,S}$ and $M_{ij}^{e,l,AS}$ two matrices of $npel \times npel$ order with components given by:

$$M_{ij}^{e,l,S} = \frac{1}{2}\Big(\big(L(\eta_j), L^*(\eta_i)\psi_{l,e}\big)_{L^2(\Omega_e)} + \big(L(\eta_i), L^*(\eta_j)\psi_{l,e}\big)_{L^2(\Omega_e)}\Big), \tag{9}$$

$$M_{ij}^{e,l,AS} = \frac{1}{2}\Big(\big(L(\eta_j), L^*(\eta_i)\psi_{l,e}\big)_{L^2(\Omega_e)} - \big(L(\eta_i), L^*(\eta_j)\psi_{l,e}\big)_{L^2(\Omega_e)}\Big). \tag{10}$$

Therefore, we have:

$$\big(L(\phi_e^h), L^*(\eta_i)\psi_{l,e}\big)_{L^2(\Omega_e)} = \sum_{j=1}^{npel} \hat{\phi}_e^h(j)\big(M_{ij}^{e,l,S} + M_{ij}^{e,l,AS}\big), \tag{11}$$

$$\sum_{l=1}^{\mathrm{N}} \tau_l^e \big(L(\phi_e^h), L^*(\eta_i)\psi_{l,e}\big)_{L^2(\Omega_e)} = \sum_{j=1}^{npel}\sum_{l=1}^{\mathrm{N}} \hat{\phi}_e^h(j)\tau_l^e\big(M_{ij}^{e,l,S} + M_{ij}^{e,l,AS}\big). \tag{12}$$

If we chose $\hat{L}(\eta_i, \eta_j)$ as being the symmetrical linear operator defined as:

$$\hat{L}(\eta_i, \eta_j) = \frac{1}{2}\big(L(\eta_j), L^*(\eta_i) + L(\eta_i), L^*(\eta_j)\big), \tag{13}$$

then front (7) and (9-13) follows that

$$M_{ij}^{GSPR,e} \equiv A^{GLS,e}(\tau_0^e, \eta_i, \eta_j) + A^{GSPR,e}(\tau^e, \eta_i, \eta_j) = \sum_{l=1}^{\mathrm{N}} \tau_l^e \big(M_{ij}^{e,l,S} + M_{ij}^{e,l,AS}\big), \tag{14}$$

where $M_{ij}^{GSPR,e}$ denotes the element matrix of GSPR method.

The free parameters τ_l^e can be determined through some criterion adopted to improve the accuracy and/or stability of the approximate solution. In general, if through some criterion adopted we find that the adequate element matrix is $A^{GLS,e} + M^{gen,e}$, where $M^{gen,e}$ is a symmetrical matrix. As all skew-symmetrical matrix it verifies

$$\sum_{i=1}^{npel}\sum_{j=1}^{npel} M_{ij}^{e,l,AS}\varphi_j\varphi_i = 0 \quad \forall(\varphi_1,\ldots,\varphi_{npel}) \in \mathbb{R}^{npel} \text{ and } \forall l, \tag{15}$$

then only the symmetrical matrices are decisive to determine the free parameters. Therefore, the τ_l^e can be determined by solving the following minimization problem at element level. Find $\tau_1^e,\ldots,\tau_{\mathrm{N}}^e$ that minimize the least square functional

$$\frac{\partial \mathrm{F}(M^{gen,e})}{\partial \tau_m^e} = 0, \quad m = 1,\ldots,\mathrm{N}, \tag{16}$$

$$\mathrm{F}(M^{gen}) = \sum_{i=1}^{npel}\sum_{j=1}^{npel}\left[\Big(\sum_{l=1}^{\mathrm{N}} \tau_l^e M_{ij}^{GSPR,e,l}\Big) - M_{ij}^{gen,e}\right]^2, \tag{17}$$

where $M_{ij}^{gen,e}$ denote the (*i,j*) - entry of $M^{gen,e}$.

Remark 1 A particular GSPR method is derived for each specific choice of the set of free parameters τ_l^e. This set of parameters can be determined, as illustrated above, by knowing or postulating an element matrix with the stability properties coherent with the differential operator. With this strategy in mind, a consistent variational formulation can be derived associated with any postulated element matrix.

Remark 2 The set of parameters can also be determined using information on the solution of the homogeneous or non homogeneous problem like in optimal or nearly optimal Petrov-Galerkin formulations, Multiscale or Residual Free Bubble stabilizations.

Remark 3 In (Dutra do Carmo et al., 2008) a stabilization matrix obtained via standard dispersion analysis for the homogeneous Helmholtz equation is adopted to develop a variationally consistent GPR formulation capable to deal with the non homogeneous equation.

Note that, the strategy to determine the vector τ_l^e depends on differential operator of the problem and the chosen matrix $M^{gen,e}$.

For advection-diffusion equation we design our method with L^* as identity operator. It is well known that to generate stability in advection-diffusion problems is necessary to add to the GLS formulation some type of capture operator. Unfortunately, the stabilized formulations based on capture operators are no linear even when the problem is linear (Hughes et al., 1986), (Galeão et al., 1988), (Dutra do Carmo et al., 2003), (Dutra do Carmo et al., 2004). The GSPR formulation can be capable to supply a linear stabilized formulation for this problem. The GLS formulation controls the gradient in the direction of the field *u*. In order to get the control of derivatives in another directions, we consider for each element Ω_e the vectors $u=(u_1,u_2)$, $u^{\perp}=(-u_2,u_1)$, the mesh parameter $h_e=\left(\int_{\Omega_e} d\Omega\right)^{1/2}$ and the following parameters $P_e=\dfrac{h_e|u_e|}{D_e}$, $\tau_e^*=\dfrac{1}{2}\dfrac{h_e}{|u_e|}Sup\left\{0,1-\dfrac{C_0^k}{P_e}\right\}$ and $\tau_0^e=\tau_e^*$, where C_0^k depends on the degree of the polynomial and $C_0^1=1$. We postulated two generating matrix to obtain the control of derivatives:

$$M1^{gen,e}=\tau_e^*\beta_e\left[\alpha_e\int_{\Omega_e}(u^{\perp}\cdot\nabla\eta_j)(u^{\perp}\cdot\nabla\eta_i)d\Omega+(1-\alpha_e)\int_{\Omega_e}(u\cdot\nabla\eta_j)(u\cdot\nabla\eta_i)d\Omega\right], \tag{18}$$

$$M2^{gen,e}=2|u_e|^2\tau_e^*\beta_e\int_{\Omega_e}\nabla\eta_j\cdot\nabla\eta_i d\Omega\,, \tag{19}$$

where $0<\alpha_e<1$ and β_e are two parameters that should be determined. In all cases the numerical experiments suggest $\alpha_e=0.5$, $\beta_e=1$ for $M1^{gen,e}$ and $\beta_e=0.175$ for $M2^{gen,e}$.

4 NUMERICAL RESULTS

In the present section two 2-D examples to illustrate the main features and potential of GSPR method applied to advection-diffusion equation are presented. For these cases the solution of the GPR method is similar to GSPR formulation. Both examples deal with homogenous equation and Dirichlet boundary conditions. In all examples a unity square domain, bilinear shape functions, 3x3 Gaussian integration, uniform mesh (20x20), $D = 10^{-10}$ and three advection fields u ($(1,-1), (1,-2), (2,-1)$) skew to the mesh are adopted.

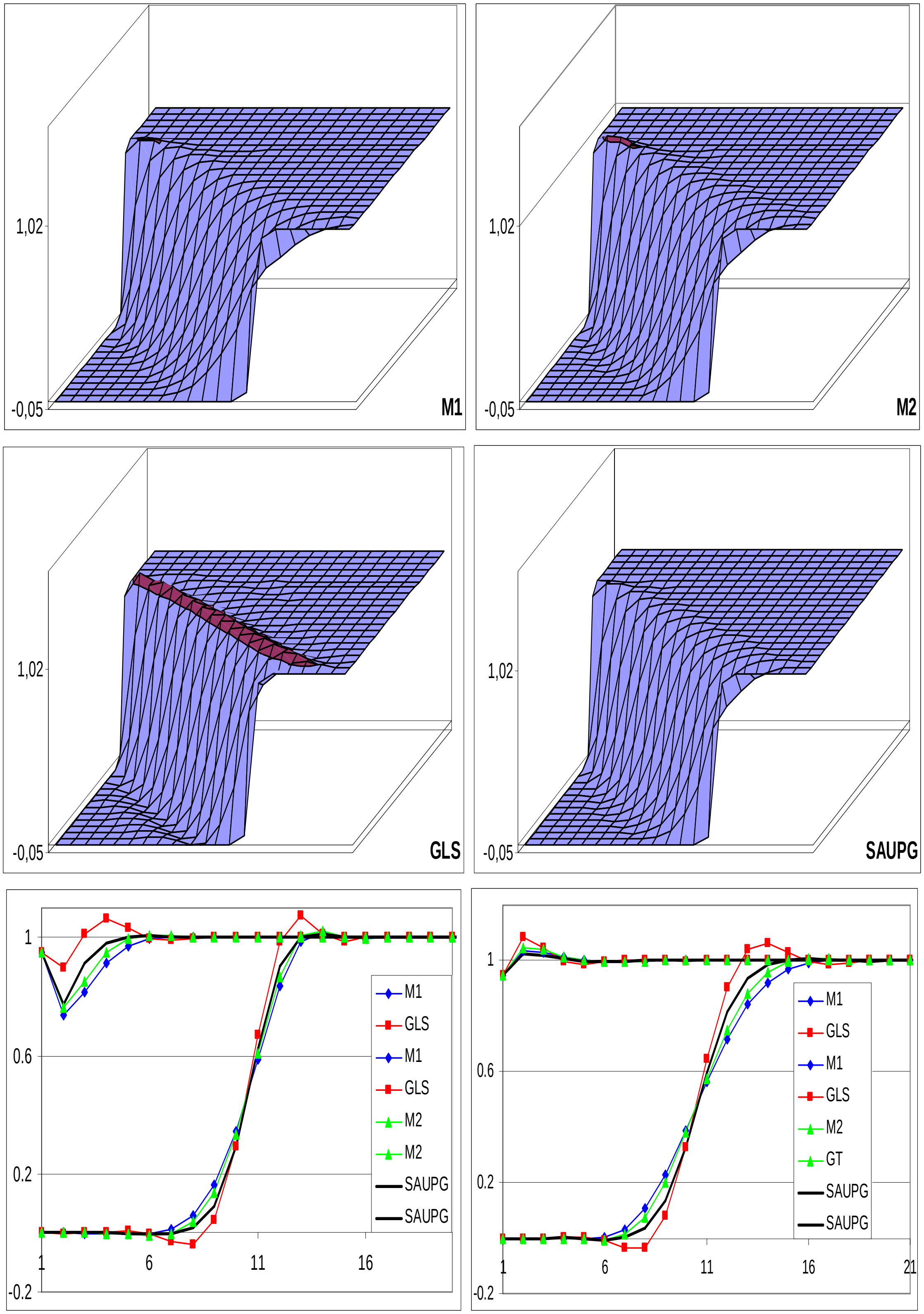

Figure 1 – Solutions for Example 1 with advection skew to mesh: case *u=(1,-1)*

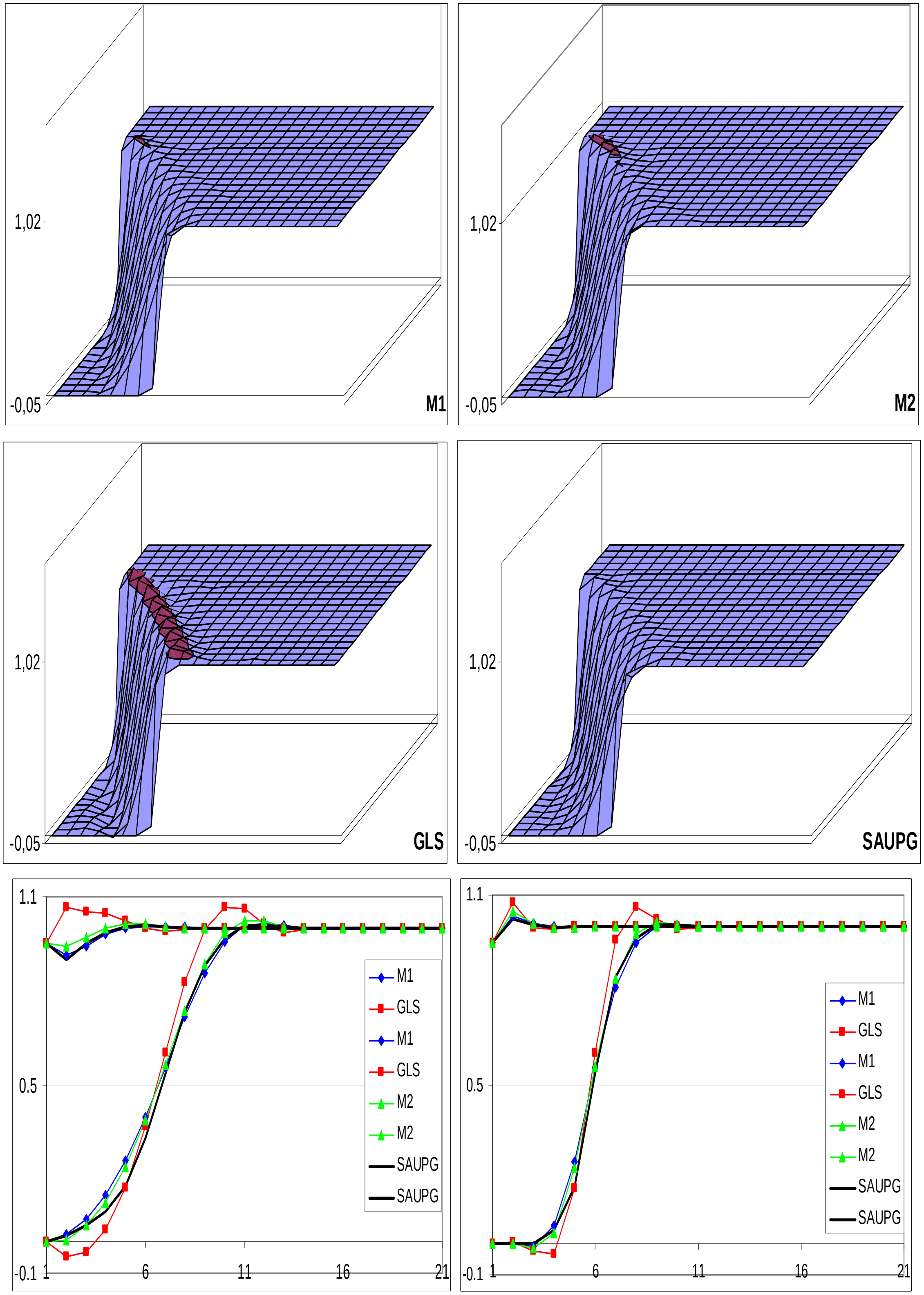

Figure 2 - Solutions for Example 1 with advection skew to mesh: case *u=(1,-2)*

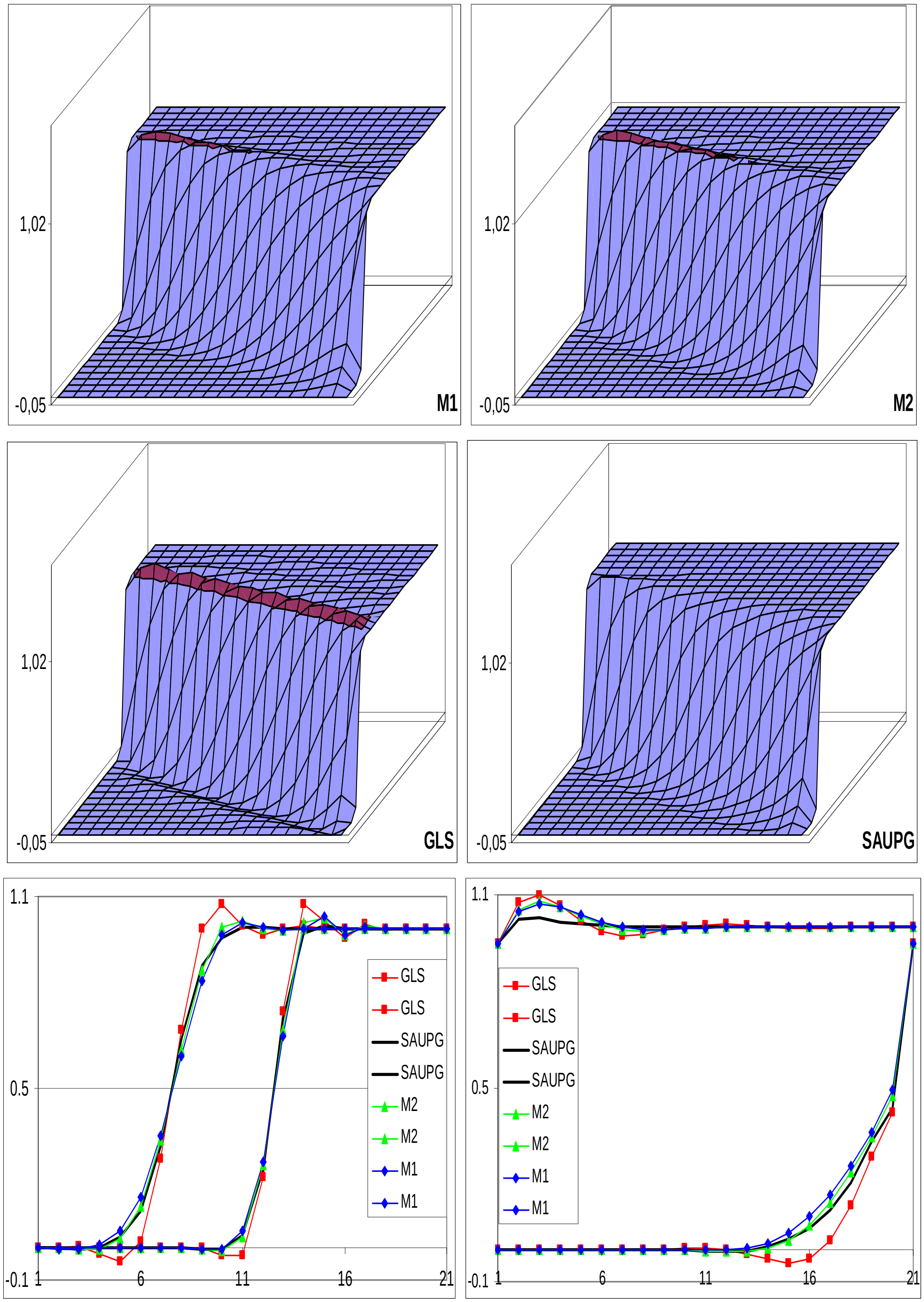

Figure 3 - Solutions for Example 1 with advection skew to mesh: case *u=(2,-1)*

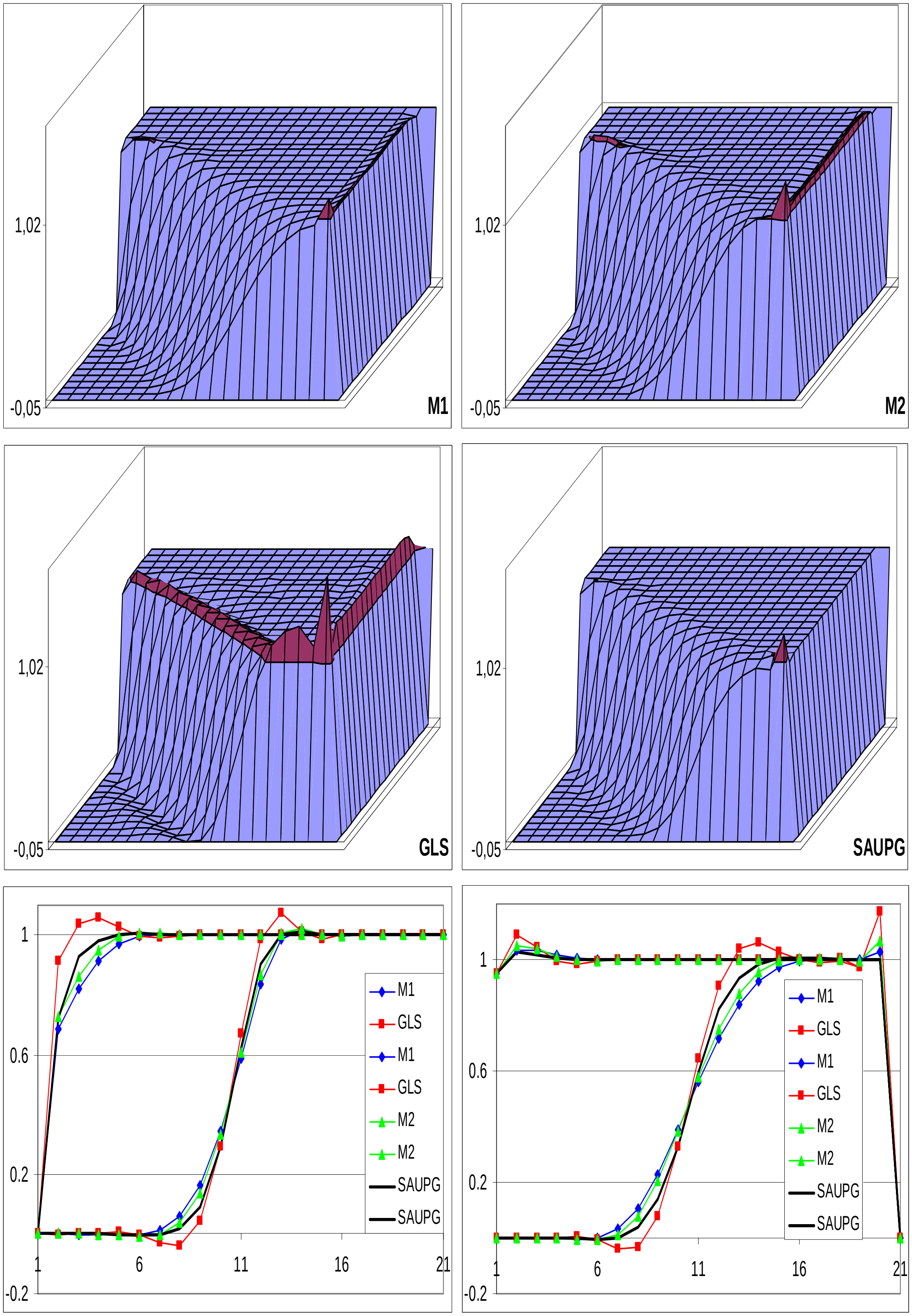

Figure 4 - Solutions for Example 2 with advection skew to mesh: case *u=(1,-1)*

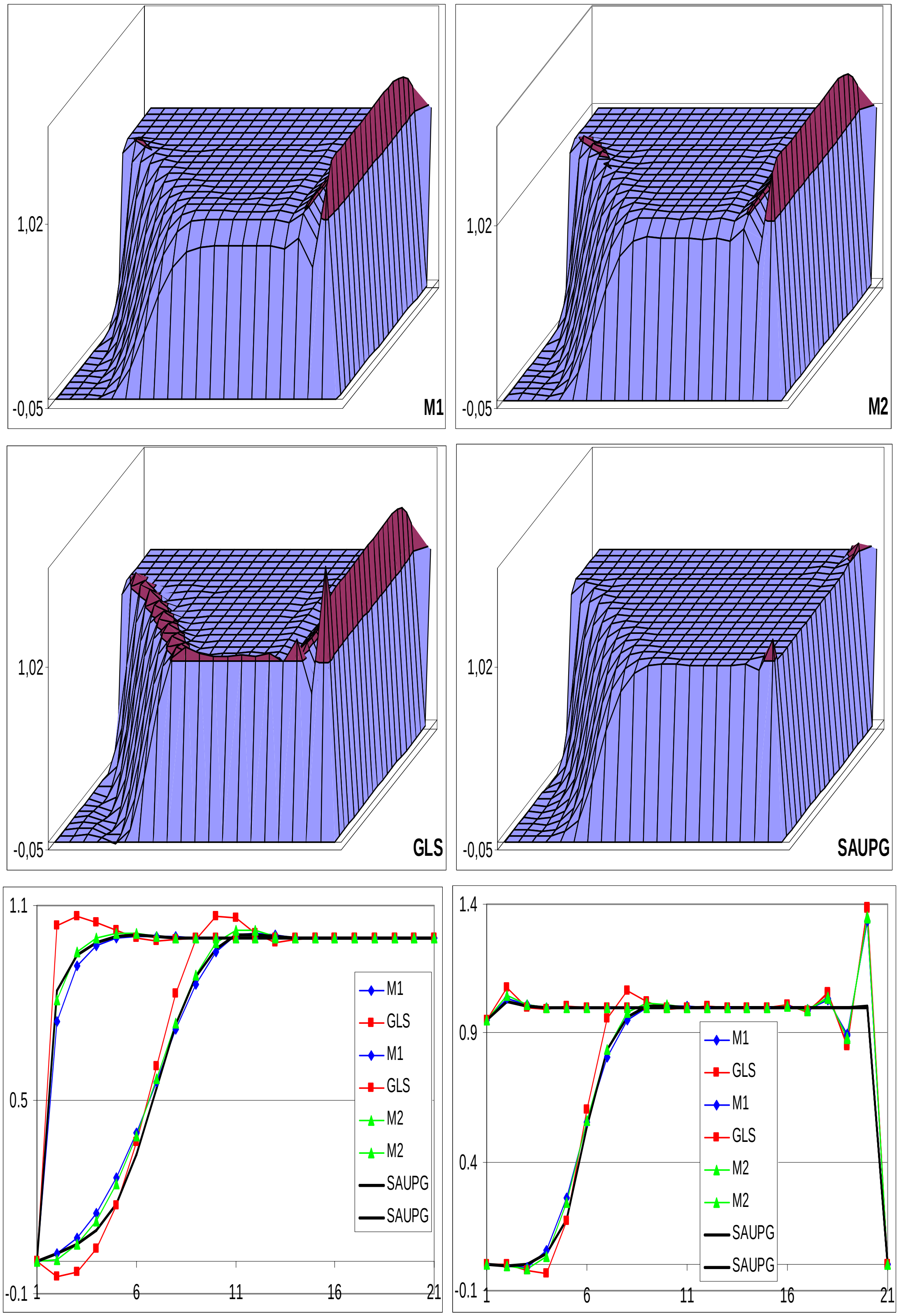

Figure 5 - Solutions for Example 2 with advection skew to mesh: case *u=(1,-2)*

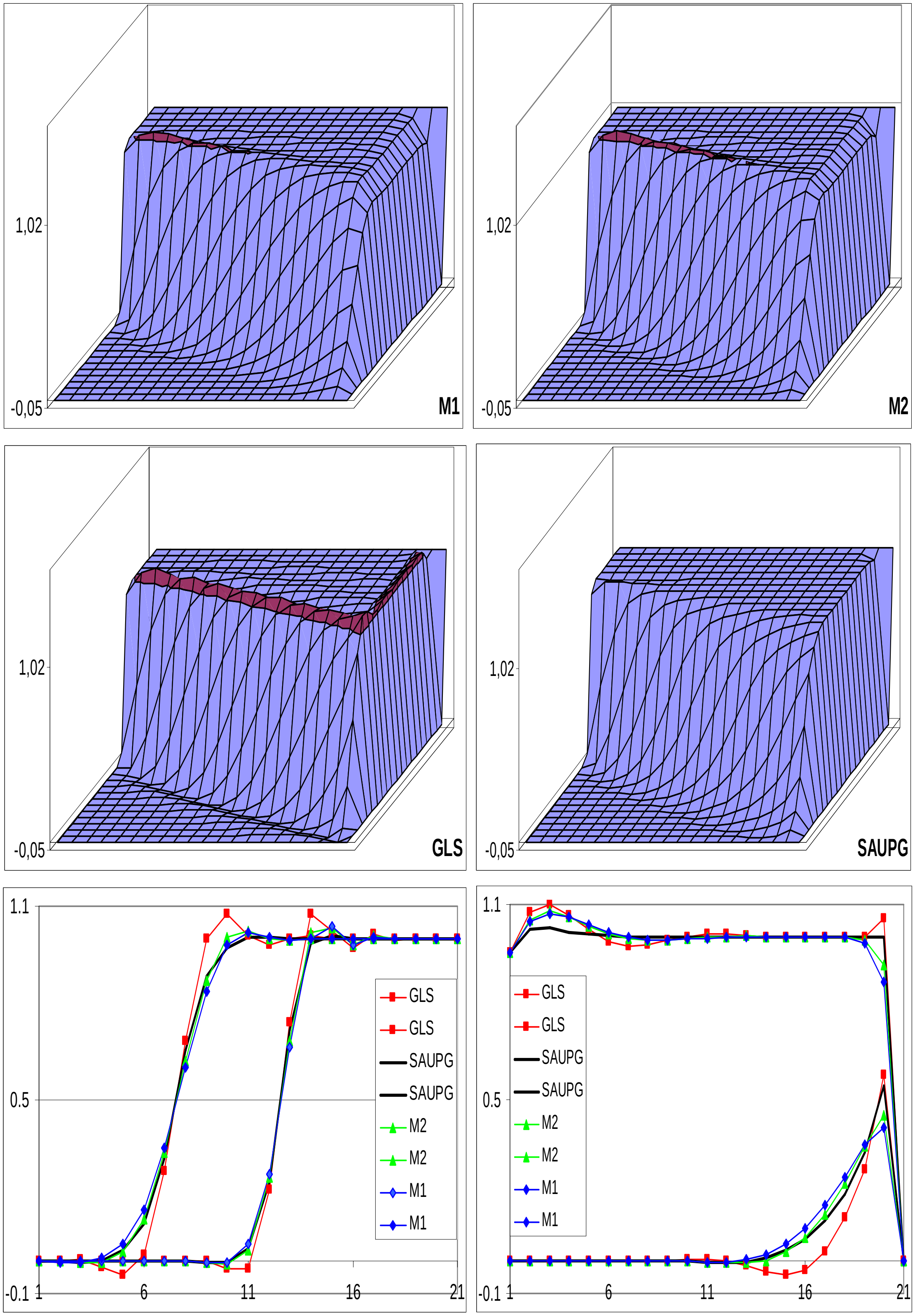

Figure 6 - Solutions for Example 2 with advection skew to mesh: case *u=(2,-1)*

The first numerical test presents sharp internal layer and the second numerical test presents sharp internal and boundary layers. The boundary conditions for Example 2 are:

$$\phi(x,0) = 0\,, \tag{20}$$

$$\phi(x,1) = 1\,, \tag{21}$$

$$\phi(1,y) = 0\,, \tag{22}$$

$$\phi(0,y) = 0 \quad \forall y \in [0,0.6]\,, \tag{23}$$

$$\phi(0,y) = y - 0.6 \quad \forall y \in [0.6,0.65]\,, \tag{24}$$

$$\phi(0,y) = 18(y - 0.65) + 0.05 \quad \forall y \in [0.65,0.7]\,, \tag{25}$$

$$\phi(0,y) = (y - 0.7) + 0.95 \quad \forall y \in [0.7,0.75]\,, \tag{26}$$

$$\phi(0,y) = 1 \quad \forall y \in [0.75,1]\,, \tag{27}$$

For Example 1 $\phi(0,y)$ and $\phi(x,1)$ are the same of the Example 2, but $\phi(x,0)$ and $\phi(1,y)$ are such that the external boundary layer is not presents. For Example 1 the Figs. 1, 2 and 3 show a comparison between the solutions of GSPR, GLS and SAUPG methods. The SAUPG (Streamline and Approximate Upwind/Petrov-Galerkin) method is a no linear formulation that adds to the GLS formulation a capture operator (Dutra do Carmo et al., 2003). Figures 4, 5 and 6 show a similar comparison for Example 2. The solution of GSPR method is presented for two generating matrix: M1 defined by Eq. 18 and M2 by Eq. 19. All figures show the four compared solutions in sections y = 0.2, y = 0.7 along the x direction (left) and in sections x = 0.2, x = 0.7 along the y direction (right). In all cases, the solution of the SAUPG and GSPR methods are very close near the internal layer. However, near the boundary layer the GSPR method present some degree of instability, but inferior to the instability of the GLS method.

5 CONCLUSIONS

We developed, based on the initial ideas of the GPR method, a new consistent FEM to be applied to advection-diffusion boundary value problems. The GSPR method introduces new ideas about the space of matrices associated with the multiple projections of the residual. The space of matrices is splitting in symmetrical and skew-symmetrical matrices. These multiple projections allow the generation of appropriate number of free stabilization parameters in the element matrix. We observed that only the symmetrical matrices are decisive to determine these free parameters. The methodology to choose the free parameters is similar to used for GPR method and consists in to postulate an element matrix with the desired stability properties (generating matrix) and the free parameters are determinate solving a least square problem at element level.

The numerical results presented in the previous section allow us to conclude that the GSPR method presents the following properties:

- it is a linear stabilized method as GLS method, different of the nonlinear SAUPG method,
- its computational algorithm can be easily implemented,
- its stability in the proximity of internal layers is similar to the SAUPG method,
- its stability in the proximity of boundary layers is inferior to the SAUPG method but superior to the GLS method.

The good performance of the GSPR formulation stimulates to improvement this methodology, in future works, in order to obtain better stabilization near of boundary layers.

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