A discontinuous finite element method at element level applied to Helmholtz equation with minimal pollution

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Time-harmonic acoustic, elastic and electromagnetic waves are governed by the Helmholtz equation. Numerical approximation of this equation is particularly challenging [1-10]. The oscillatory behavior of the exact solution and the quality of the numerical approximation depend on the wave number *k*. To approximate Helmholtz equation with acceptable accuracy the resolution of the mesh should be adjusted to the wave number according to rule of thumb [6]. The performance of the Galerkin finite element method deteriorates as *k* increases. This misbehavior, known as pollution of the finite element solution [2,8], can only be avoided after a drastic refinement of the mesh, which normally entails significant barriers for the numerical analysis of Helmholtz equation at mid and high frequencies.

A great effort has been devoted to alleviate the pollution effect [1,2,4,5,7,9,10]. In particular, the GLS method (Galerkin Least-Squares) is able to completely eliminate the phase lag in one dimension problems [7]. In two space dimensions, there is no finite element method with piecewise linear shape functions free of pollution effect [10]. Stencils with minimal pollution error are constructed in [2] through the Quasi Stabilized Finite Element Method (QSFEM). The QSFEM is really a finite difference rather than a finite element method. The modifications of the discrete operator are made on the algebraic level and no variational formulation is associated with the QSFEM presented in [2].

Recently, we introduced a discontinuous finite element formulation for Helmholtz equation depending on two stabilization parameters [1]. Several numerical experiments show the good performance and potential of this formulation to reduce the pollution effect. Completely discontinuous formulation, as presented in [1], may lead to high computational cost since the degrees of freedom associated with the discontinuity can not be eliminated. Moreover, the two parameters of this formulation (β and λ) are determined through numerical experiments.

The new method contained in the present work is also based on a discontinuous finite element formulation [1,3], but now the continuity is relaxed only on the interiors of elements instead of across the element edges as it was admitted in our previous formulation. Continuity on the interelement boundaries is enforced considering C^0 Lagrangian interpolation globally as usual. Discontinuities are introduced locally, inside each element, through C^{-1} shape functions associated with interior nodes with zero value on the element boundary. Thus, the interior shape functions can be viewed as discontinuous bubbles and the corresponding degrees of freedom can be eliminated at element level by static condensation yielding a global matrix topologically equivalent to those of classical C^0 finite element approximations. Again, a crucial point of the discontinuous formulation relies on the choice of the stabilization parameters (β and λ) related to the weak enforcement of continuity inside each element. For uniform meshes we present a methodology to determine explicitly the stabilization parameters minimizing the pollution effect. In particular, the QSFEM stencil emanates consistently from the proposed variational formulation by an appropriate choice of these parameters.

THE HELMHOLTZ EQUATION

Let $\Omega \subset R^n$ be an open bounded domain with a Lipschitz continuous smooth piecewise boundary Γ. Let Γ_g, Γ_q, Γ_r be three disjoint subsets of Γ where boundary conditions are specified, such that $\Gamma_g \cup \Gamma_q \cup \Gamma_r = \Gamma$. We shall consider the interior

Helmholtz problem:

$$-\nabla\cdot(\nabla u)-k^2u=f \quad \text{in } \Omega, \tag{1}$$

$$u=g \quad \text{on } \Gamma_g, \tag{2}$$

$$\nabla u\cdot\hat{n}=q \quad \text{on } \Gamma_q, \tag{3}$$

$$\nabla u\cdot\hat{n}+\alpha u=r \quad \text{on } \Gamma_r, \tag{4}$$

where u denotes a scalar field that describes time-harmonic acoustic, elastic or electromagnetic steady state waves. The coefficient k is the wave number, f is the source term, g, q and r are the prescribed boundary conditions. The coefficient α is positive on Γ_r and $\hat{n}$ denotes the outward normal unit vector defined almost everywhere on Γ.

The continuous Galerkin FEM

Consider $M^h=\{\Omega_1,\ldots,\Omega_{NE}\}$ a finite element partition of Ω, such that: $\overline{\Omega}=\Omega\cup\Gamma=\bigcup_{E=1}^{NE}\overline{\Omega}_E=\bigcup_{E=1}^{NE}(\Omega_E\cup\Gamma_E)$,

$\Omega_E\cap\Omega_{E'}=\emptyset$ if $E\neq E'$ and Γ_E denotes the boundary of Ω_E. The continuous finite element set and space are defined as:
$S_{h,a}^l=\{u^{h,a}\in H^1(\Omega):u_E^{h,a}\in P^l(\Omega_E),\ u^{h,a}=g^h \text{ on } \Gamma_g\}$,
$V_{h,a}^l=\{v^{h,a}\in H^1(\Omega):v_E^{h,a}\in P^l(\Omega_E),\ v^{h,a}=0 \text{ on } \Gamma_g\}$,
where $P^l(\Omega_E)$ is the space of polynomials of degree less than or equal to l, g^h denotes the interpolation of g and $u_E^{h,a}$ denotes the restriction of $u^{h,a}$ to Ω_E.

Problem Eq. (1-4) have been approximated by the following finite element methods: find $u^h\in S_{h,a}^l$ that satisfies $\forall v^h\in V_{h,a}^l$,

$$A_G(u^h,v^h)=F_G(v^h), \tag{5}$$

$$A_G=\sum_{E=1}^{NE}\int_{\Omega_E}[\nabla u_E^h\cdot\nabla v_E^h-k^2u_E^h\,v_E^h]\,d\Omega+ + \int_{\Gamma_r}\alpha u^h v^h\,d\Gamma,$$

$$F_G=\sum_{E=1}^{NE}\int_{\Omega_E}f\,v_E^h\,d\Omega+\int_{\Gamma_q}qv^h\,d\Gamma+\int_{\Gamma_r}rv^h\,d\Gamma.$$

The discontinuous FEM at element level

Consider for each element $\Omega_E\in M^h$ a subgrid $\overline{\Omega}_E=\bigcup_{e=1}^{ne}\Omega_E^e\cup\Gamma_E^e$, where Γ_E^e denotes the boundary of Ω_E^e. Introducing in each macroelement Ω_E the discontinuous finite element subspaces, $V_{h,b}^l=\{v^{h,b}\in L^2(\Omega_E): v_{E,e}^{h,b}\in P^l(\Omega_E^e)$ and $v_{E,e}^{h,b}=0$ on $\Gamma_E^e\cap\Gamma_E=\emptyset\}$, the discontinuous finite element method at element level consists in finding $u^h=(u^{h,a}+u^{h,b})\in S_{h,a}^l+S_{h,b}^l$ satisfying two equations:

$$A_{DG}(u^{h,a}+u^{h,b},v^{h,a})=F_G(v^{h,a})\ \forall v^{h,a}\in V_{h,a}^l, \tag{6}$$

$$A_{DG}(u^{h,a}+u^{h,b},v^{h,b})=F_G(v^{h,b})\ \forall v^{h,b}\in V_{h,b}^l, \tag{7}$$

where $A_{DG}(u^h,v^h)$ and $F_G(v^h)$ are given by

$$A_{DG}(u^h,v^h)=\sum_{E=1}^{NE}A_E(u_E^h,v_E^h)+\int_{\Gamma_r}\alpha u^h v^h\,d\Gamma,$$

$$F_G(v^h)=\sum_{E=1}^{NE}F_E(v_E^h)+\int_{\Gamma_q}qv^h\,d\Gamma+\int_{\Gamma_r}rv^h\,d\Gamma,$$

$$A_E(u_E^h,v_E^h)=\sum_{e=1}^{ne}\int_{\Omega_E^e}[\nabla u_{E,e}^h\cdot\nabla v_{E,e}^h-k^2u_{E,e}^h\,v_{E,e}^h]\,d\Omega+$$

$$+\sum_{e=1}^{ne}\sum_{\substack{e'>e\\ \Gamma_E^{ee'}\neq\emptyset}}^{ne}\int_{\Gamma_E^{ee'}}\left[\frac{\beta_E^{ee'}}{h_{ee'}}(u_{E,e}^h-u_{E,e'}^h)(v_{E,e}^h-v_{E,e'}^h)+\right.$$

$$+\frac{\lambda_E^{ee'}}{2}(u_{E,e}^h-u_{E,e'}^h)(\nabla v_{E,e}^h\cdot\hat{n}_E^e-\nabla_{E,e'}^h\cdot\hat{n}_E^{e'})-$$

$$\left.-\frac{1}{2}(\nabla u_{E,e}^h\cdot\hat{n}_E^e-\nabla u_{E,e'}^h\cdot\hat{n}_E^{e'})(v_{E,e}^h-v_{E,e'}^h)\right]d\Gamma,$$

$$F_E(v_E^h)=\sum_{e=1}^{ne}\int_{\Omega_E^e}f_{E,e}\,v_{E,e}^h\,d\Omega,$$

$u_{E,e}^h$ denotes the restriction u^h to element Ω_E^e, $\Gamma_E^{e,e'}=\Gamma_E^e\cap\Gamma_E^{e'}$, $\hat{n}_E^e$ is the outward normal unit vector to Γ_E^e, $h_{ee'}=\min\{h_{E,e},h_{E,e'}\}$, where $h_{E,e}$ and $h_{E,e'}$ are the subgrid mesh parameters. This formulation is consistent in the sense that the exact solution u of problem Eq.(1-4) is also solution of Eq.(6-7).

The space $V_{h,a}^l+V_{h,b}^l$ can be understood as classical finite element space $V_{h,a}^l$ enriched with discontinuous bubble functions within each macroelement. Bubbles functions are typically higher-order polynomials defined on the interiors of each element, which vanish on element boundaries. Note that, for this case, these bubble functions do not need to be higher-order polynomials ($l \geq 1$). The degrees of freedom associated with bubbles can be eliminated by the know 'static condensation'. Moreover, the continuity in this formulation is relaxed on the interiors of elements (subgrid) depending on $\beta_E^{ee'}$ and $\lambda_E^{ee'}$ parameters. Initially, these parameters were introduced in [1,3] and its choice is crucial for the quality of the numerical solution. Here, $\beta_E^{ee'}$ and $\lambda_E^{ee'}$ parameters will be determined in order to reduce the pollution effects of the numerical solution.

The finite element system Eq. (6) and Eq. (7) in matrix form is given by

$$AU_a + B(\tilde{\lambda})U_b = F_a, \quad (8)$$

$$CU_a + D(\tilde{\lambda},\tilde{\beta})U_b = F_b \quad (9)$$

where A, $B(\tilde{\lambda})$, C and $D(\tilde{\lambda},\tilde{\beta})$ are global matrices, F_a and F_b are the global vectors of source term, U_a is the vector of global unknowns of the coarse mesh, U_b is the vector of subgrid unknowns, $\tilde{\lambda},\tilde{\beta} = \{\lambda_E^{ee'}, \beta_E^{ee'}\}$ are the two set of parameters related to the weak enforcement of continuity on the interface $\Gamma_E^{ee'}$ of the elements Ω_E^e and $\Omega_E^{e'}$ in each macroelement Ω_E. For given $\tilde{\lambda}$ and $\tilde{\beta}$ the matrix $D(\tilde{\lambda},\tilde{\beta})$ can be easily inverted for being block diagonal a direct consequence of choosing $v^{h,b}$ bubble-like functions. Eliminating the vector U_b in system Eq. (9) we obtain the condensed global system

$$A^* U_a = F^*,$$

$$A^* = A - B(\tilde{\lambda})D(\tilde{\lambda},\tilde{\beta})^{-1}C,$$

$$F^* = F_a - B(\tilde{\lambda})D(\tilde{\lambda},\tilde{\beta})^{-1}F_b,$$

which is topologically equivalent to that corresponding to the classical C^0 Galerkin approximation in the macro mesh. In fact the subgrid degrees of freedom are eliminated at macroelement level, and the condensed global system is obtained by adding the corresponding macroelement contributions

$$A_E^* = A_E - B_E(\tilde{\lambda})D_E(\tilde{\lambda},\tilde{\beta})^{-1}C_E,$$

$$F_E^* = F_{a,E} - B_E(\tilde{\lambda})D_E(\tilde{\lambda},\tilde{\beta})^{-1}F_{b,E}.$$

Optimal choice of the stabilization parameters

For simplicity we now consider only bilinear polynomial interpolations and uniform mesh with square macroelements of length h composed by a subgrid with four square elements of length $\frac{h}{2}$. Thus we have 8 degrees of freedom per macroelements: 4 corresponding to $u^{h,a}$ and the other 4 corresponding to $u^{h,b}$ that will be condensed at the macroelement level. We observe that, for this particular mesh, $\lambda_E^{e,e'} = \lambda$, $\beta_E^{e,e'} = \beta$ and the local matrices are given explicitly by

$$A_E = \sum_{i=0}^{2} a_i E_i,\ B_E = \sum_{i=0}^{2} b_i(\lambda)E_i,$$

$$C_E = \sum_{i=0}^{2} c_i E_i,\ D_E = \sum_{i=0}^{2} d_i(\lambda,\beta)E_i,$$

where E_i (*i=0,1,2*) are the following *4×4* matrices

$$E_0 = I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ E_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix},\ E_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and

$$a_0 = \frac{2}{3} - \alpha_G, \quad a_1 = -\frac{1}{6} - \frac{\alpha_G}{2}, \quad a_2 = \frac{1}{3} - \frac{\alpha_G}{4},$$

$$b_0 = -\frac{\alpha_G}{4} - \frac{\lambda+1}{3}, \quad b_1 = -\frac{\alpha_G}{8} + \frac{\lambda+1}{12},$$

$$b_2 = -\frac{\alpha_G}{16} + \frac{\lambda+1}{6},$$

$$c_0 = -\frac{\alpha_G}{4}, \quad c_1 = -\frac{\alpha_G}{8}, \quad c_2 = -\frac{\alpha_G}{16},$$

$$d_0 = \gamma + 2\mu, \quad d_1 = -\mu, \quad d_2 = 0,$$

$$\alpha_G = \frac{(kh)^2}{9},\ \gamma = \frac{2}{3} - \frac{\alpha_G}{4},\ \mu = \frac{2\beta + \lambda - 1}{6}.$$

Using the matrix equation $A_E^* = \sum_{i=0}^{2} a_i^* E_i = A_E - B_E(\lambda)D_E(\lambda,\beta)^{-1}C_E$ in the diagonal form, we obtain the following algebraic system of three independent equations

$$a_0^* + 2a_1^* + a_2^* = -\frac{9\alpha_G}{4} - \left[\frac{9\alpha_G}{16}\right]^2 \frac{1}{\gamma}, \quad (10\text{-}12)$$

$$a_0^* - a_2^* = 1 - \frac{3\alpha_G}{4} - \frac{3\alpha_G}{16}\left[\frac{3\alpha_G}{16} + \frac{\lambda+1}{2}\right]\frac{1}{\gamma+2\mu},$$

$$a_0^* - 2a_1^* + a_2^* = \frac{2}{3} - \frac{\alpha_G}{4} - \frac{\alpha_G}{16}\left[\frac{\alpha_G}{16} + \frac{\lambda+1}{3}\right]\frac{1}{\gamma+4\mu},$$

relating the coefficients of the condensed matrix A_E^* and the stabilization parameters.

Now, using discrete Fourier transform in the homogeneous form of the global system corresponding to the present uniform mesh, the stencil of an interior node leads to

$$a_2^* \tilde{r} + a_0^* + a_1^* \tilde{w} = 0, \quad (13)$$

$$\tilde{r} = \cos(\tilde{k}h\cos\theta)\cos(\tilde{k}h sin\theta),$$

$$\tilde{w} = \cos(\tilde{k}h\cos\theta) + \cos(\tilde{k}h sin\theta),$$

where θ is the direction of a plane wave, and $\tilde{k}$ is the discrete wavelength. We should observe that the coefficients $a_i^*, i = 0,1,2,$ depend on k and h, and on the free parameters λ and β. Thus we have the freedom to choose λ and β to minimize the phase lag. We then choose two directions θ_1 and θ_2 such that the dispersion relation Eq. (13) is verified for $\tilde{k} = k$, yielding

$$a_2^* r_1 + a_0^* + a_1^* w_1 = 0, \quad (14)$$

$$a_2^* r_2 + a_0^* + a_1^* w_2 = 0, \quad (15)$$

with

$$r_1 = \cos(kh\cos\theta_1)\cos(kh sin\theta_1),$$

$$r_2 = \cos(kh\cos\theta_2)\cos(kh sin\theta_2),$$

$$w_1 = \cos(kh\cos\theta_1) + \cos(kh sin\theta_1),$$

$$w_2 = \cos(kh\cos\theta_2) + \cos(kh sin\theta_2),$$

Solving the algebraic system Eq. (10-12, 14,15) of five independent equations we obtain the following expressions for the stabilization parameters:

$$\lambda = -1 + \frac{(p_3 g_1 - p_1 g_3)}{(p_2 g_1 - p_1 g_2)},$$

$$\beta = 1 + 3\frac{(p_2 g_3 - p_3 g_2)}{(p_2 g_1 - p_1 g_2)} - \frac{1}{2}\frac{(p_3 g_1 - p_1 g_3)}{(p_2 g_1 - p_1 g_2)},$$

and for the coefficients of the condensed matrix A_E^*:

$$a_0^* = \frac{(r_2 w_1 - r_1 w_2)(576\alpha_G \gamma + 81{\alpha_G}^2)}{256\gamma\left[r_2 w_1 - r_1 w_2 + 2(r_1 - r_2) + w_2 - w_1\right]},$$

$$a_1^* = a_0^* \frac{(r_1 - r_2)}{(r_2 w_1 - r_1 w_2)},$$

$$a_2^* = a_0^* \frac{(w_1 - w_2)}{(r_2 w_1 - r_1 w_2)},$$

with

$$g_1 = -4(a_1^* + a_2^*) - 2\left[1 + 24 p_0 + \frac{81}{\gamma}{p_0}^2\right],$$

$$p_0 = \frac{\alpha_G}{16},$$

$$g_2 = \frac{3p_0}{2},$$

$$g_3 = \gamma\left[2(a_1^* + a_2^*) + 1 + 24 p_0 + \frac{81}{\gamma}{p_0}^2\right] - 9{p_0}^2,$$

$$p_1 = -16a_1^* - 4\left[\frac{2}{3} + 32 p_0 + \frac{81}{\gamma}{p_0}^2\right],$$

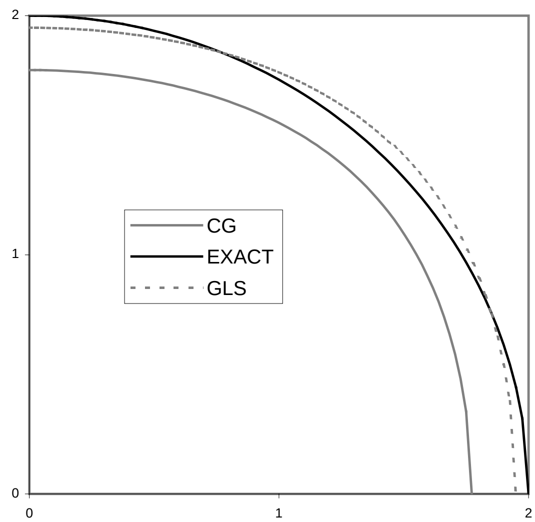
$$p_2 = \frac{p_0}{3},$$

$$p_3 = \gamma\left[4a_1^* + \frac{2}{3} + 32 p_0 + \frac{81}{\gamma}{p_0}^2\right] - {p_0}^2.$$

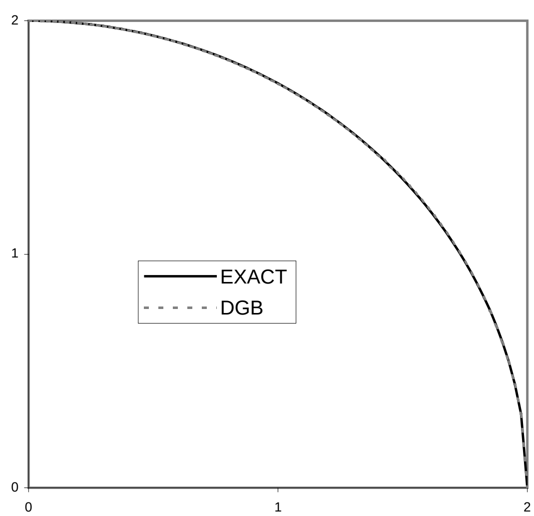
Due to the symmetry of the uniform mesh θ_1 and θ_2 should be chosen such that $\theta_1, \theta_2 \in (0, \frac{\pi}{4})$ with $\theta_1 \neq \theta_2$ and $\theta_1 - \theta_2 \neq \frac{\pi}{4}$ to avoid an indefinite system for λ and β. We should note that the approximate solution is pollution-free only if the exact solution is a plane wave in direction θ_1 or θ_2. For any other direction different from θ_1 or θ_2, when the wave number k is increased pollution effects appear.

From equations Eq. (14) and Eq. (15) we observe that choosing $\theta_1 = \frac{\pi}{16}$ and $\theta_2 = \frac{3\pi}{16}$, the condensed element matrix of the present discontinuous finite element formulation generates an interior stencil identical to that of the Quasi Stabilized Finite Element Method (QSFEM) with minimal pollution error compared to any nine point stencil (or any C^0 four node element) as presented in [2].

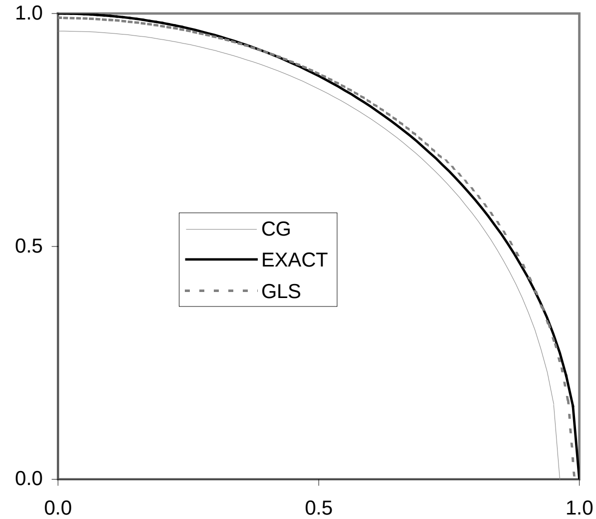
The dispersion relation Eq. (13) leads to a phase lag $\left|k-\widetilde{k}\right|$ depending on *kh* and θ. In Figure 1 we compare the exact and approximate dispersion relations corresponding to three finite element approximations: the classical Galerkin method (CG), the Galerkin Least-Squares (GLS) with $\tau(\theta=\frac{\pi}{8})$, as presented in [10], and the present discontinuous bubble formulation (DGB), for *kh=1* and *kh=2*. We observe no visual difference between the dispersion relation of DGB and the exact one (case b).



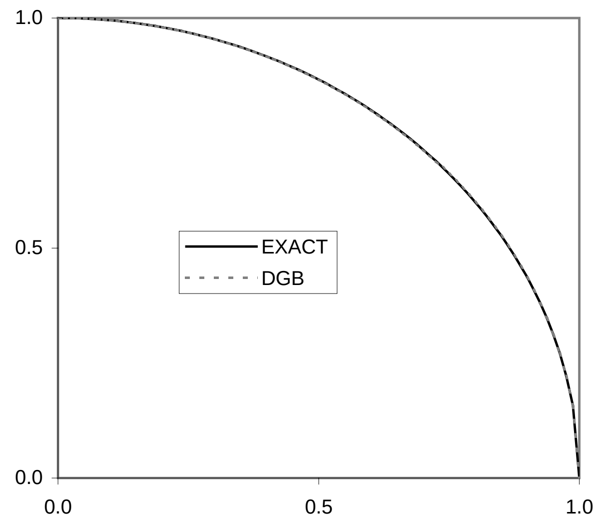
(a) *kh=2*



(b) *kh=2*



(a) *kh=1*



(b) *kh=1*

Figure 1 - Dispersion relations for *kh=1* and *kh=2*, (a) Continuous Galerkin (CG) and Galerkin Least Squares (GLS), (b) Discontinuous Bubble Galerkin formulation (DGB).

CONCLUSIONS

Herein, we present a new consistent discontinuous finite element formulation for the Helmholtz problem. Consistency derives from the fact that the exact continuous solution satisfies the discrete equation. The continuity is only relaxed on nodes placed on the interior of the elements instead across the element edges. The interior degrees of freedom are associated to discontinuous bubble functions, which are not necessarily higher-order polynomials, once those functions are to be zero on the element edges. This represents a significant reduction of costs as the interior degrees of freedom might be eliminate using standard condensation techniques.

Moreover, discontinuous formulation proposed by the authors make use of two design parameters, which are selected to enhance accuracy and stability. In the present method, departing from the stencil obtained with the internal degrees condensation, we build a strategy for choosing those parameters aiming at matching the exact wave number in two different directions. This is conducted analytically for uniform meshes. The stencil of the Quasi Stabilized Finite Element Method is retrieved for the optimal choice of the stabilization parameters. Nevertheless, it is important to remark that in our case, the final problem is derived from a consistent variational formulation, which allows the enforcement of generic boundary conditions and the use of non structured meshes as well.

Acknowledgements

The authors wish to thank the Brazilian research-funding agencies CNPq and FAPESB for their support to this work.

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