Two finite element methods with minimal pollution for Helmholtz problem in two dimensions

Gustavo B. Alvarez
UFF/EEIMVR - Depto. de Ciências Exatas
Av. dos Trabalhadores 420, Volta Redonda, RJ
E-mail: benitez.gustavo@gmail.com

Eduardo G. Dutra do Carmo, Fernando A. Rochinha
COPPE/UFRJ – Universidade Federal do Rio de Janeiro
Ilha do Fundão, 21945-970, P.B. 68509, Rio de Janeiro, RJ
E-mail: egdcarmo@hotmail.com, rochinha@adc.coppe.ufrj.br

Abimael F. D. Loula
LNCC – Laboratório Nacional de Computação Científica,
Getulio Vargas 333, Quitandinha, 25651-070, Petrópolis, RJ
E-mail: aloc@lncc.br

Time-harmonic acoustic, elastic and electromagnetic waves are governed by the Helmholtz equation. Numerical approximation of this equation is particularly challenging as reported in a vast literature [2-12]. The oscillatory behavior of the exact solution and the quality of the numerical approximation depend on the wave number k. To approximate Helmholtz equation with acceptable accuracy the resolution of the mesh should be adjusted to the wave number according to the rule of thumb [7], which prescribes a minimum number of elements per wavelength. It is well known that, despite of the adoption of this rule, the performance of the Galerkin finite element method deteriorates as k increases. This misbehavior, known as pollution of the finite element solution [9,3], can only be avoided after a drastic refinement of the mesh, which normally entails significant barriers for the numerical analysis of Helmholtz equation at mid and high frequencies.

To improve the stability and accuracy of FEM for this boundary-value problem it is necessary to have some criterion that allows obtaining the stability and/or accuracy wanted. A form of obtaining such stabilized methods consists of adding residual terms at element level to the Galerkin formulation. This is the base of the GLS (Galerkin Least-Squares), GGLS (Galerkin Gradient Least-Squares) methods and other stabilized methods. Therefore, it is desirable that a stabilized method based on these ideas with a maximum number of free parameters can be developed. In one-dimension, the GLS method is able to completely eliminate the pollution effect [8]. In two dimensions, there is no FEM with piecewise linear shape functions free of pollution effect [3,12].

We should emphasize that several methods that minimize the phase error of the approximate solution to solve the Helmholtz equation in two-dimensions exist. The Quasi-Stabilized Finite Element Method (QS) [3] is one of these methods. Though, this method is not based on a variational formulation. This hinders the extension of the QS method for no uniforms meshes, high-order polynomials and problems with source term. Three methods, originating from of variational formulations, that minimize the phase error exist: Residual-Based Finite Element Method (RBFEM) [11], Discontinuous Finite Element Method at Element Level (DGB) [10] and Galerkin Projected Residual Method (GPR)[5,6].

The RBFEM method is obtained from the Galerkin approximation by appending terms that are proportional to residuals on element interiors and inter-element boundaries. The terms that are proportional to the residuals on elements interiors can be understood as an extension of GLS method, considering the stabilization parameter dependent on the position. The free parameters are completed through residual terms on boundaries of the element. The residual on inter-element boundaries terms are motivated by the variational multiscale formulation.

These terms implicate in an extra computational effort when the RBFEM formulation is compared with a classical continuous finite element formulation. Besides, a systematic for the choice of the functions at element level, that allows obtaining these residual terms, doesn't exist.

The DGB method is a discontinuous finite element formulation, where discontinuities are introduced locally, inside each element. These discontinuities can be viewed as discontinuous bubbles and the corresponding degrees of freedom can be eliminated at element level by static condensation yielding a global matrix topologically equivalent to those of classical finite element approximations. However, the DGB method needs the condensation technique to eliminate degrees of freedom introduced by the discontinuities. The free parameters, related to the weak enforcement of continuity inside each element, need to be determined. For uniform meshes we present a methodology to determine explicitly the stabilization parameters minimizing the pollution effect. But, for no uniforms meshes this methodology can transform the DGB formulation in a non linear method, in the sense that will be necessary to solve a non linear problem at element level to determine the free parameters and to accomplish the condensation technique. Besides, the local discontinuities introduced on inter-bubbles boundaries implicate in an extra computational effort.

Recently, we developed the GPR method for second-order generic scalar linear boundary-value problems. This formulation is obtained adding appropriate multiple projections of residual of the differential equation, only at element level, to the Galerkin formulation. The formulation carries out explicitly the dependence of differential equation operator, the local approach space and is general for any dimension of the domain. Besides, this methodology allows introducing in the formulation a maximum number of free parameters, which can be determined following some criterion to improve the stability and accuracy of the approximate solution. In particular, when the GPR method is applied to Helmholtz equation is possible obtain explicit values of the free parameters minimizing the phase error via a dispersion analysis.

In this work, we introduced, concisely, two formulations that minimize the pollution effect in two-dimension: DGB and GPR methods. One numerical experiment to evaluate the performance of these formulations is presented.

THE HELMHOLTZ EQUATION

Let $\Omega \subset R^n$ be an open bounded domain with a Lipschitz continuous smooth piecewise boundary Γ. Let Γ_g, Γ_q, Γ_r be three disjoint subsets of Γ where boundary conditions are specified, such that $\Gamma_g \cup \Gamma_q \cup \Gamma_r = \Gamma$. We shall consider the interior Helmholtz problem:

$$L(u) = -\nabla \cdot (\nabla u) - k^2 u = f \quad \text{in } \Omega, \quad (1)$$

$$u = g \quad \text{on } \Gamma_g, \quad (2)$$

$$\nabla u \cdot \hat{n} = q \quad \text{on } \Gamma_q, \quad (3)$$

$$\nabla u \cdot \hat{n} + \alpha u = r \quad \text{on } \Gamma_r, \quad (4)$$

where u denotes a scalar field that describes time-harmonic acoustic, elastic or electromagnetic steady state waves. The coefficient k is the wave number, f is the source term, g, q and r are the prescribed boundary conditions. The coefficient α is positive on Γ_r and $\hat{n}$ denotes the outward normal unit vector defined almost everywhere on Γ.

The continuous Galerkin FEM

Consider $M^h = \{\Omega_1, \ldots, \Omega_{NE}\}$ a finite element partition of Ω, such that: $\overline{\Omega} = \Omega \cup \Gamma = \bigcup_{E=1}^{NE} \overline{\Omega}_E = \bigcup_{E=1}^{NE} (\Omega_E \cup \Gamma_E)$, $\Omega_E \cap \Omega_{E'} = \emptyset$ if $E \neq E'$ and Γ_E denotes the boundary of Ω_E. The continuous finite element set and space are defined as:

$S_{h,a}^l = \{u^{h,a} \in H^1(\Omega) : u_E^{h,a} \in P^l(\Omega_E),\ u^{h,a} = g^h \text{ on } \Gamma_g\}$

$V_{h,a}^l = \{v^{h,a} \in H^1(\Omega) : v_E^{h,a} \in P^l(\Omega_E),\ v^{h,a} = 0 \text{ on } \Gamma_g\}$

where $H^1(\Omega)$ is Sobolev space defined in [1], $P^l(\Omega_E)$ is the space of polynomials of degree less than or equal to l, g^h denotes the interpolation of g and $u_E^{h,a}$ denotes the restriction of $u^{h,a}$ to Ω_E.

Problem Eq. (1-4) have been approximated by the following finite element methods: find $u^h \in S^l_{h,a}$ that satisfies $\forall v^h \in V^l_{h,a}$,

$$A_G(u^h, v^h) = F_G(v^h), \tag{5}$$

$$A_G = \sum_{E=1}^{NE} \int_{\Omega_E} [\nabla u^h_E \cdot \nabla v^h_E - k^2 u^h_E v^h_E]\, d\Omega + \int_{\Gamma_r} \alpha u^h v^h \, d\Gamma,$$

$$F_G = \sum_{E=1}^{NE} \int_{\Omega_E} f\, v^h_E \, d\Omega + \int_{\Gamma_q} q v^h \, d\Gamma + \int_{\Gamma_r} r v^h \, d\Gamma.$$

Only for purely diffusive problems the solution of Galerkin FEM is the best approximation. It is well known that the Galerkin FEM is shown unstable and little accuracy for Helmholtz equation. Its numerical solution presents spurious oscillations that do not corresponding with the physical solution of problem.

The discontinuous FEM at element level

Consider for each element $\Omega_E \in M^h$ a subgrid $\overline{\Omega}_E = \bigcup_{e=1}^{ne} \Omega^e_E \cup \Gamma^e_E$, where Γ^e_E denotes the boundary of Ω^e_E. Introducing in each macroelement Ω_E the discontinuous finite element subspaces, $V^l_{h,b} = \{v^{h,b} \in L^2(\Omega_E) : v^{h,b}_{E,e} \in P^l(\Omega^e_E)$ and $v^{h,b}_{E,e} = 0$ on $\Gamma^e_E \cap \Gamma_E = \emptyset\}$, the discontinuous finite element method at element level consists in finding $u^h = (u^{h,a} + u^{h,b}) \in S^l_{h,a} + S^l_{h,b}$ satisfying $\forall v^{h,a} \in V^l_{h,a}$ and $\forall v^{h,b} \in V^l_{h,b}$ two equations:

$$A_{DG}(u^{h,a} + u^{h,b}, v^{h,a}) = F_G(v^{h,a}), \tag{6}$$

$$A_{DG}(u^{h,a} + u^{h,b}, v^{h,b}) = F_G(v^{h,b}), \tag{7}$$

where $A_{DG}(u^h, v^h)$ and $F_G(v^h)$ are given by:

$$A_{DG}(u^h, v^h) = \sum_{E=1}^{NE} A_E(u^h_E, v^h_E) + \int_{\Gamma_r} \alpha u^h v^h \, d\Gamma,$$

$$F_G(v^h) = \sum_{E=1}^{NE} F_E(v^h_E) + \int_{\Gamma_q} q v^h \, d\Gamma + \int_{\Gamma_r} r v^h \, d\Gamma,$$

$$A_E(u^h_E, v^h_E) = \sum_{e=1}^{ne} \int_{\Omega^e_E} [\nabla u^h_{E,e} \cdot \nabla v^h_{E,e} - k^2 u^h_{E,e} v^h_{E,e}]\, d\Omega$$

$$+ \sum_{e=1}^{ne} \sum_{\substack{e'>e \\ \Gamma^{ee'}_E \neq \emptyset}}^{ne} \int_{\Gamma^{ee'}_E} \left[\frac{\beta^{ee'}_E}{h_{ee'}} (u^h_{E,e} - u^h_{E,e'})(v^h_{E,e} - v^h_{E,e'}) + \right.$$

$$+ \frac{\lambda^{ee'}_E}{2} (u^h_{E,e} - u^h_{E,e'})(\nabla v^h_{E,e} \cdot \hat{n}^e_E - \nabla^h_{E,e'} \cdot \hat{n}^{e'}_E) -$$

$$\left. - \frac{1}{2} (\nabla u^h_{E,e} \cdot \hat{n}^e_E - \nabla u^h_{E,e'} \cdot \hat{n}^{e'}_E)(v^h_{E,e} - v^h_{E,e'}) \right] d\Gamma,$$

$$F_E(v^h_E) = \sum_{e=1}^{ne} \int_{\Omega^e_E} f_{E,e} v^h_{E,e} \, d\Omega,$$

$u^h_{E,e}$ denotes the restriction u^h to element Ω^e_E, $\Gamma^{e,e'}_E = \Gamma^e_E \cap \Gamma^{e'}_E$, $\hat{n}^e_E$ is the outward normal unit vector to Γ^e_E, $\beta^{ee'}_E$ and $\lambda^{ee'}_E$ are two set of free parameters related to the weak enforcement of continuity on the interface $\Gamma^{e,e'}_E$ of the elements e and e' in each macroelement E, $h_{ee'} = \min\{h_{E,e}, h_{E,e'}\}$, where $h_{E,e}$ and $h_{E,e'}$ are the subgrid mesh parameters. This formulation is consistent in the sense that the exact solution u of problem Eq.(1-4) is also solution of Eq.(6-7).

The space $V^l_{h,a} + V^l_{h,b}$ can be understood as classical finite element space $V^l_{h,a}$ enriched with discontinuous bubble functions within each macroelement. Bubbles functions are typically higher-order polynomials defined on the interiors of each element, which vanish on element boundaries. The degrees of freedom associated with bubbles can be eliminated by the 'static condensation'. Moreover, the continuity in this formulation is relaxed on the interiors of elements (subgrid) depending on $\beta^{ee'}_E$ and $\lambda^{ee'}_E$ parameters and its choice is crucial for the quality of the numerical solution. Here, $\beta^{ee'}_E$ and $\lambda^{ee'}_E$ parameters will be determined in order to reduce the pollution effects of the numerical solution.

The finite element system Eq. (6) and Eq. (7) in matrix form is given by

$$AU_a + B(\tilde{\lambda})U_b = F_a, \tag{8}$$

$$CU_a + D(\tilde{\lambda}, \tilde{\beta})U_b = F_b \tag{9}$$

where A, $B(\widetilde{\lambda})$, C and $D(\widetilde{\lambda},\widetilde{\beta})$ are global matrices, F_a and F_b are the global vectors of source term, U_a is the vector of global unknowns of the coarse mesh, U_b is the vector of subgrid unknowns, $\widetilde{\lambda},\widetilde{\beta}=\{\lambda_E^{ee'},\beta_E^{ee'}\}$ are the two set of parameters related to the weak enforcement of continuity on the interface $\Gamma_E^{ee'}$ of the elements Ω_E^e and $\Omega_E^{e'}$ in each macroelement Ω_E. For given $\widetilde{\lambda}$ and $\widetilde{\beta}$ the matrix $D(\widetilde{\lambda},\widetilde{\beta})$ can be inverted for being block diagonal a direct consequence of choosing $v^{h,b}$ bubble-like functions. Eliminating the vector U_b in system Eq. (9) we obtain the condensed global system:

$$A^*U_a = F^*,$$

$$A^* = A - B(\widetilde{\lambda})D(\widetilde{\lambda},\widetilde{\beta})^{-1}C,$$

$$F^* = F_a - B(\widetilde{\lambda})D(\widetilde{\lambda},\widetilde{\beta})^{-1}F_b,$$

which is topologically equivalent to that corresponding to the classical C^0 Galerkin approximation in the macro mesh. In fact the subgrid degrees of freedom are eliminated at macroelement level, and the condensed global system is obtained by adding the corresponding macroelement contributions

$$A_E^* = A_E - B_E(\widetilde{\lambda})D_E(\widetilde{\lambda},\widetilde{\beta})^{-1}C_E,$$

$$F_E^* = F_{a,E} - B_E(\widetilde{\lambda})D_E(\widetilde{\lambda},\widetilde{\beta})^{-1}F_{b,E}.$$

If we adopted the dispersion analysis criteria to choose of free parameters $\widetilde{\lambda}$ and $\widetilde{\beta}$, then the DGB formulation generates an interior stencil identical to the Quasi Stabilized Finite Element Method (QS) with minimal pollution error compared to any nine point stencil (or any four node element) as presented in [3].

The Galerkin projected residual method

The fundamental idea of GPR method consists of adding to the Galerkin FEM multiple projections of the residual with one free parameter associated to each projection [5,6]. The maximum number of free parameters depends on the differential operator and the local approach space. It is to say, the maximum number of linearly independent projections of residual will depend on properties of operator (such as symmetry, etc) and on order of interpolant polynomials. These free parameters would be then determined by appropriate criteria for each specific problem, seeking to obtain more accuracy and stable approximate solutions.

Then, consider for each fixed element Ω_E the space $E_{GPR}(\Omega_E)$ defined as:

$$E_{GPR}(\Omega_e)=\{\psi:\Omega_e\to R;\psi=\sum_{i=1}^{npel}\sum_{j=1}^{npel}C_{i,j}L(\eta_i)L(\eta_j),\quad C_{i,j}\in R\}$$

where $L(\cdot)$ is the differential operator defined by Eq. (1), $npel$ denotes the number of nodal points of the element Ω_E and $\eta_i\,(i=1,\ldots,npel)$ denotes the usual local shape functions associated to nodal point *i*. That is, $E_{GPR}(\Omega_E)$ is the space with dimension d_{GPR} generated by a local base: $\psi_i = L(\eta_{Lig(i,1)})L(\eta_{Lig(i,2)}), i=1,\ldots,d_{GPR}$, where $Lig(\cdot,\cdot)$ is a function of $\{1,\ldots,d_{GPR}\}\times\{1,2\}\to\{1,\ldots,npel\}\times\{1,\ldots,npel\}$.

The GPR method consists in finding $u^h\in S^h$ that satisfies $\forall v^h\in V^h$ the variational equation:

$$A_G(u^h,v^h)+\sum_{E=1}^{NE}\left(\sum_{i=1}^{d_{GPR}}\tau_i\left(\left(L(u^h),\frac{L(v^h)\psi_i}{k^2}\right)_{L^2(\Omega_E)}+\right.\right.$$

$$\left.\left.+2\int_{\Omega_E}\frac{\nabla L(u^h)\cdot\nabla L(v^h)\psi_i}{k^4}d\Omega\right)\right)=F_G(v^h)+$$

$$\sum_{E=1}^{NE}\left(\sum_{i=1}^{d_{GPR}}\tau_i\left(\left(f_E,L(v^h)\psi_i\right)_{L^2(\Omega_E)}+2\int_{\Omega_E}\frac{\nabla f_E\cdot\nabla L(v^h)\psi_i}{k^4}d\Omega\right)\right) \quad (10)$$

where f_E denote the restriction of f to Ω_E and $\tau_1,\ldots,\tau_{d_{GPR}}$ are free parameters associated to each projection of residual. Note that, the GPR formulation is consistent, in sense that the exact solution of Eq. (1-4) is also solution of Eq. (10).

Let u_E^h be the restriction of u^h to Ω_E given by $u_E^h=\sum_{i=1}^{npel}\hat{u}_E^h(i)\eta_i$, where $\hat{u}_E^h(i)$ denote the value of u_E^h in local node i of Ω_E element. Therefore, we have:

$$\left(L(u_E^h),\frac{L(\eta_i)\psi_l}{k^2}\right)_{L^2(\Omega_E)}+2\int_{\Omega_E}\frac{\nabla L(u^h)\cdot\nabla L(\eta_i)\psi_l}{k^4}d\Omega=$$

$$\sum_{m=1}^{npel}\hat{u}_E^h(m)\left[\left(L(\eta_m),\frac{L(\eta_i)\psi_l}{k^2}\right)_{L^2(\Omega_E)}+\right.$$

$$\left.+2\int_{\Omega_E}\frac{\nabla L(\eta_m)\cdot\nabla L(\eta_i)\psi_l}{k^4}d\Omega\right],$$

$i=1,\ldots,npel$ and $l=1,\ldots,d_{GPR}$. (11)

Remind that $\psi_1,\ldots,\psi_{d_{GPR}}$ is a base in $L^2(\Omega_E)$ for $E_{GPR}(\Omega_E)$ and denoting by M_{im}^E the elements matrix we obtain:

$$\left(L(u_E^h),L(\eta_i)\psi_l\right)_{L^2(\Omega_E)}+ \quad (12)$$

$$2\int_{\Omega_E}\frac{\nabla L(u^h)\cdot\nabla L(\eta_i)\psi_l}{k^4}d\Omega=\sum_{m=1}^{npel}C_{i,l}\hat{u}_E^h(m)\,,$$

$$M_{im}^E=A_G^E(\eta_m,\eta_i)+\sum_{l=1}^{d_{GPR}}\tau_l C_{i,l}\,, \quad (13)$$

$$A_G^E(\eta_m,\eta_i)=\int_{\Omega_E}[\nabla\eta_m\cdot\nabla\eta_i+k^2\eta_m\eta_i]\,d\Omega+$$

$$\int_{\Gamma_E}\alpha\eta_m\eta_i d\Gamma\,, \quad (14)$$

where $C_{i,l}$ and j are given by Eq. (12). We can notice that the element matrix is formed by the usual part of Galerkin more d_{GPR} projections of residual of the differential equation at element level. The free parameters $\tau_1,\ldots,\tau_{d_{GPR}}$, corresponding to each projection of residual, can be determined through some criterion adopted to improve the accuracy and/or stability of the approximate solution for each specific problem. For Helmholtz equation with uniform mesh and bilinear quadrilateral elements we have $d_{GPR}=9$, because:

$$\begin{aligned}
&\phi_1=+k^4\eta_1\eta_1, && \psi_1=\phi_1,\\
&\phi_2=+k^4\eta_1\eta_2, && \psi_2=\phi_2,\\
&\phi_3=+k^4\eta_1\eta_3, && \psi_3=\phi_3,\\
&\phi_4=+k^4\eta_1\eta_4, && \psi_4=\phi_4,\\
&\phi_5=+k^4\eta_2\eta_2, && \psi_5=\phi_5,\\
&\phi_6=+k^4\eta_2\eta_3, && \psi_6=\phi_6,\\
&\phi_7=+k^4\eta_2\eta_4=\phi_3, && \psi_7=\phi_8,\\
&\phi_8=+k^4\eta_3\eta_3, && \psi_8=\phi_9,\\
&\phi_9=+k^4\eta_3\eta_4, && \psi_9=\phi_{10},\\
&\phi_{10}=+k^4\eta_4\eta_4, &&
\end{aligned} \quad \text{and} \quad (15)$$

Let a_{im}^l be defined by:

$$a_{im}^l=C_{j,l}\Leftrightarrow\begin{cases}j=(m-1)npel+i & \text{if } i\geq m\\ j=(i-1)npel+m & \text{if } m\geq i\end{cases}$$

$i=1,\ldots,npel$, $m=1,\ldots,npel$ and $l=1,\ldots,d_{GPR}$. (16)

Therefore, the elements of element matrix are:

$$M_{im}^E=A_G^E(\eta_m,\eta_i)+\sum_{l=1}^{d_{GPR}}\tau_l a_{im}^l\,. \quad (17)$$

By Eq. (15), Eqs. (16) and Eq. (12) we obtain $a_{13}^l=a_{24}^l\ \forall l\in\{1,\ldots,d_{GPR}\}$. Also, by Eq. (14) proceed $A_G^E(\eta_3,\eta_1)=A_G^E(\eta_4,\eta_2)$.

Let τ_l^* defined as:

$$\tau_l^*=A_G^E(\eta_m,\eta_i)+\sum_{\mu=1}^{d_{GPR}}\tau_\mu a_{im}^\mu\,,$$

$l=(i-1)npel+m$, $i=1,\ldots,npel$,

$m=1,\ldots,npel$, (18)

and therefore the element matrix becomes:

$$M^E=\begin{bmatrix}\tau_1^* & \tau_2^* & \tau_3^* & \tau_4^*\\ \tau_2^* & \tau_5^* & \tau_6^* & \tau_3^*\\ \tau_3^* & \tau_6^* & \tau_7^* & \tau_8^*\\ \tau_4^* & \tau_3^* & \tau_8^* & \tau_9^*\end{bmatrix}. \quad (19)$$

Notice that due to symmetry of the Helmholtz operator and that is using first-order interpolant polynomials we have $E_{GPR}(\Omega_E)$ space with dimension $d_{GPR}=9$ (nine projections) and therefore nine free parameters. Here, for a uniform mesh we adopted the dispersion analysis criteria to determine the free parameters $\tau_1^*,\ldots,\tau_{d_{GPR}}^*$ that compose the element matrix M^E given by Eq. (19), through the following steps:

Step 1) Using an uniform mesh with bilinear quadrilateral elements, the interior stencil is obtained for a plane wave $e^{i\tilde{k}(x\cos\theta+y\sin\theta)}$ $(i=\sqrt{-1},\ 0\leq\theta\leq\pi)$ as a function of the components of the matrix M^E, given by:

$$\bar{\tau}_0+\bar{\tau}_1\cos(\tilde{k}h\sin\theta)+\bar{\tau}_2\cos(\tilde{k}h\cos\theta)+$$

$$\bar{\tau}_3\cos(\tilde{k}h\sin\theta)\cos(\tilde{k}h\cos\theta)=0\,, \quad (20)$$

$$\bar{\tau}_0=\tau_1^*+\tau_5^*+\tau_7^*+\tau_9^*\,, \quad (21)$$

$$\bar{\tau}_1=2(\tau_4^*+\tau_6^*)\,, \quad (22)$$

$$\bar{\tau}_2=2(\tau_2^*+\tau_8^*)\,, \quad (23)$$

$$\bar{\tau}_3 = 4\tau_3^*. \quad (24)$$

Notice that the parameters $\bar{\tau}_0, \bar{\tau}_1, \bar{\tau}_2$ and $\bar{\tau}_3$ depend on k but not on discrete wavelength $\tilde{k}$. The stencil Eq. (20) is an equation with four unknowns $\bar{\tau}_0, \bar{\tau}_1, \bar{\tau}_2$ and $\bar{\tau}_3$. Choosing two different directions θ_1 and θ_2 for the plane wave follows that the interior stencil becomes two linearly independent equations. Therefore, two unknowns will be undetermined if we used the dispersion analysis criteria.

Step 2) Let us assume the following restrictions for the free parameters:

$$\tau_1^* = \tau_5^* = \tau_7^* = \tau_9^* = \tfrac{1}{4} \Rightarrow \bar{\tau}_0 = 1, \quad (25)$$

$$\tau_4^* = \tau_6^* = \tfrac{1}{2}\bar{\tau}_1, \quad (26)$$

$$\tau_2^* = \tau_8^* = \tfrac{1}{2}\bar{\tau}_2 = \tfrac{1}{2}\bar{\tau}_1 \Rightarrow \bar{\tau}_1 = \bar{\tau}_2. \quad (27)$$

We should emphasize that for a non-uniform meshes these restrictions should not necessarily be assumed. Then the interior stencil is:

$$1 + \bar{\tau}_1(\cos(\tilde{k}h\sin\theta) + \cos(\tilde{k}h\cos\theta)) + \bar{\tau}_3\cos(\tilde{k}h\sin\theta)\cos(\tilde{k}h\cos\theta) = 0. \quad (28)$$

Step 3) To minimize the phase error of the approximate solution, following the work [3] has:

$$r_1 = \cos(kh\cos\theta_1)\cos(khsin\theta_1),$$

$$r_2 = \cos(kh\cos\theta_2)\cos(khsin\theta_2), \quad (29)$$

$$w_1 = \cos(kh\cos\theta_1) + \cos(khsin\theta_1),$$

$$w_2 = \cos(kh\cos\theta_2) + \cos(khsin\theta_2), \quad (30)$$

$$\bar{\tau}_1 = \frac{(r_1 - r_2)}{(r_2 w_1 - r_1 w_2)}, \qquad \bar{\tau}_3 = \frac{(w_2 - w_1)}{(r_2 w_1 - r_1 w_2)},$$

$$\theta_1 = \tfrac{\pi}{16} \text{ and } \theta_2 = \tfrac{3\pi}{16}. \quad (31)$$

Therefore, the matrix M^E corresponds with the matrix given by [3], the condensate matrix of DGB method and it has the following form:

$$M^E = \begin{bmatrix} \frac{1}{4} & \frac{\bar{\tau}_1}{2} & \frac{\bar{\tau}_3}{4} & \frac{\bar{\tau}_1}{2} \\ \frac{\bar{\tau}_1}{2} & \frac{1}{4} & \frac{\bar{\tau}_1}{2} & \frac{\bar{\tau}_3}{4} \\ \frac{\bar{\tau}_3}{4} & \frac{\bar{\tau}_1}{2} & \frac{1}{4} & \frac{\bar{\tau}_1}{2} \\ \frac{\bar{\tau}_1}{2} & \frac{\bar{\tau}_3}{4} & \frac{\bar{\tau}_1}{2} & \frac{1}{4} \end{bmatrix}. \quad (32)$$

This way, τ_l^* are determined by Eqs. (25-27). Finally, τ_μ are determined by Eq. (18). The determination of τ_μ becomes fundamental for problems that possess source term. We should highlight that it was not necessary any consideration as made in the reference [10], besides not needing of any term in boundary of element [11]. Beyond, we should emphasize that the GPR formulation possesses a general methodology, that it is valid for any geometry and dimension of the domain, as well as, for any local approach space.

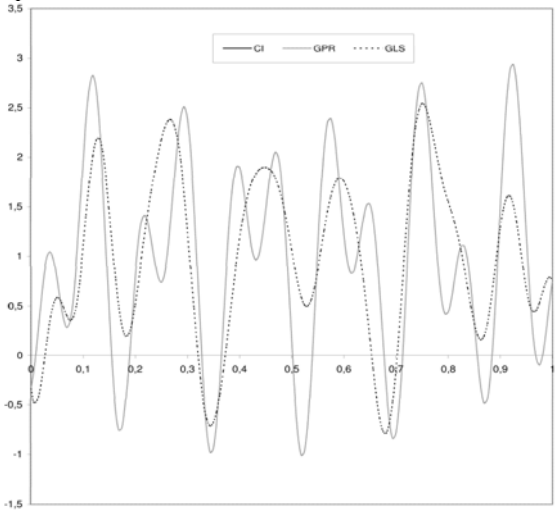

Fig. 1 Continuous interpolant CI, GPR and GLS solutions of homogeneous problem in two dimension at sections *x=0.5*:
$\theta_1 = 0,\ \theta_2 = \frac{\pi}{8}$ and $\theta_3 = \frac{\pi}{4}$.

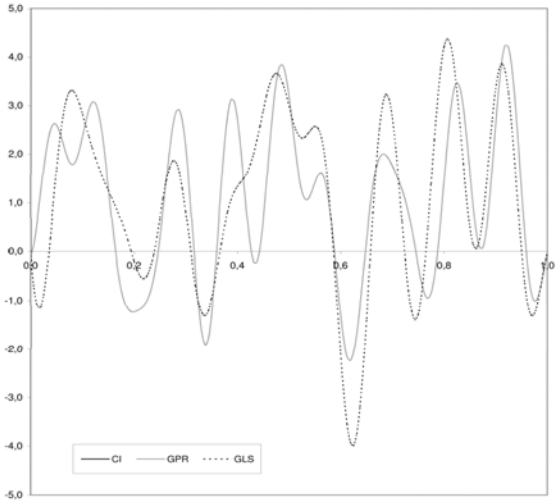

Fig. 1 Continuous interpolant CI, GPR and GLS solutions of homogeneous problem in two dimension at sections *x=0.5*
$\theta_1 = 0,\ \theta_2 = \frac{\pi}{20},\ \theta_3 = \frac{\pi}{10},\ \theta_4 = \frac{3\pi}{20},\ \theta_5 = \frac{\pi}{5}$ and $\theta_6 = \frac{\pi}{4}$.

Figures 1 and 2 present the solution of homogeneous Helmholtz equation for $k = 100$ in a unity square domain. The Dirichlet boundary conditions are such that the exact solution is a superposition of *n* mono-

energetic plane-waves propagating in n directions:

$u(x,y) = \sum_{i=1}^{n} \cos(k(x\cos\theta_i + y\sin\theta_i))$. The solution of DGB and GPR methods are equal. Both present minimum pollution.

CONCLUSION

Herein, we present concisely two consistent finite element formulations for Helmholtz equation with minimal pollution effect. The formulations are valid for any dimension of the domain and any order of local basis functions.

The GPR method introduces a maximum number of free parameters via multiple projections of the residual for each local approach space. These free parameters can be determined by appropriate criteria allowing to find the continuous finite dimensional space S^h that best approximates the infinite dimensional space S.

For bilinear shape functions the GMPR method need 3x3 Gaussian integration, since the local base of $E_{GPR}(\Omega_e)$ are quadratic polynomial functions. Even so, the GPR method only has volume terms. The RBFEM formulation and DGB method have surface terms besides the volume terms. Therefore, the extra computational effort of GPR method is minor that the extra computational effort of RBFEM and DGB methods.

For both DGB and GPR formulation explicit values of the free parameters minimizing the phase error are presented via a dispersion analysis.

Acknowledgements

The authors wish to thank the Brazilian research-funding agencies CNPq and FAPERJ for their support to this work.

Referências

[1] R. A. Adams, Sobolev spaces, New York, Academic Press, 1975.

[2] G. B. Alvarez, A. F. D. Loula, E. G. Dutra do Carmo and F. A. Rochinha, A discontinuous finite element formulation for the Helmholtz equation, *Comput. Methods Appl. Mech. Engrg.*, 195, pp 4018–4035, (2006).

[3] I. Babuška, F. Ihlenburg, E.T. Paik and S.A. Sauter, A generalized finite element method for solving the Helmholtz equation in two dimensions with minimal pollution, *Comput. Methods Appl. Mech. Engrg.*, 128, pp. 325-359, (1995).

[4] E. G. Dutra do Carmo and A. V. C. Duarte, New formulations and numerical analysis of discontinuous Galerkin methods, *Computational and Applied Mathematics* 21 No. 3, pp. 661-715, (2002).

[5] E. G. Dutra do Carmo, G. B. Alvarez, A. F. D. Loula, and F. A. Rochinha, The galerkin plus multiplies projection of residual method (GMPR) applied to helmholtz equation. *CILAMCE 2006.*

[6] E. G. Dutra do Carmo, G. B. Alvarez, A. F. D. Loula and F. A. Rochinha, A nearly optimal Galerkin projected residual finite element method for Helmholtz problem. *Comput. Methods Appl. Mech. Engrg.*, 197, pp. 1362-1375, (2008).

[7] I. Harari and T.J.R. Hughes, Finite element method for the Helmholtz equation in an exterior domain: Model problems, *Comp. Meth. Appl. Mech. Eng.*, 87, pp. 59-96, (1991).

[8] I. Harari and T.J.R. Hughes, Galerkin/least squares finite element methods for the reduced wave equation with non-reflecting boundary conditions in unbounded domains, *Comp. Meth. Appl. Mech. Eng.*, 98, pp. 411-454, (1992).

[9] F. Ihlenburg and I. Babuška, Finite element solution of the Helmholtz equation with high wave number Part I: The h-version of the FEM, *Comput. Math. Appli.*, 30, No. 9, pp. 9-37, (1995).

[10] A. F. D. Loula, G. B. Alvarez, E. G. Dutra do Carmo and F. A. Rochinha, A discontinuous finite element method at element level for Helmholtz equation. *Comput. Methods Appl. Mech. Engrg.*, 196, pp. 867–878, (2007).

[11] A. A. Oberai, P. M. Pinsky, A residual-based finite element method for the Helmholtz equation, Int. J. Numer. Methods Engrg. 49, pp. 399-419, (2000).

[12] L.L. Thompson and P.M. Pinsky, A Galerkin least squares finite element method for the two-dimensional Helmholtz equation, *Int. J. Numer. Methods Eng.*, 38, No. 3, pp. 371-397, (1995).