On Galerkin projected residual method (GPR) for two scalar and linear second-order partial differential equations: Helmholtz and diffusive-reactive singularly perturbed problems

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Abstract: The Galerkin Projected Residual Method (GPR) is applied to Helmholtz equation and to the diffusion-reaction singularly perturbed equation. The GPR method introduces an appropriate number of free stabilization parameters in the element matrix. A methodology to determine the free stabilization parameters is presented. Some numerical tests show the good performance of the GPR formulation for both equations.

1. Introduction

Boundary-value problems governed by second-order linear partial differential equations (PDE) model several physical phenomena. Usually, the Galerkin Finite Element Method (FEM) is used to numerically solve these boundary value problems. However, only for purely diffusive problems does the Galerkin method provide the optimal solution. In many other problems the Galerkin FEM is unstable and inaccurate, producing spurious oscillations that are not present in the actual solution of the problem. Stable and accuracy numerical solution via FEM for these problems has been a great challenge. The Helmholtz and reaction-diffusion equations are representative examples of the great effort that has been devoted to obtain stable and accurate FEM. Some representative works are [1-18].

Recently, a new continuous stable FEM was developed for scalar and linear second-order boundary value problems: the Galerkin Projected Residual Method [5,6]. The method is obtained by adding to the Galerkin formulation an appropriate number of projections of the residual of PDE within each element. These multiple projections allow the generation of an appropriate number of free stabilization parameters in the element matrix depending on the local space of approximation and on the differential operator. The free parameters can be determined by imposing some convergence and/or stability criteria or by postulating the element matrix with the desired stability properties. The element matrix of most stabilized methods (such as, GLS and GGLS methods [7,13,17]) can be obtained from this new method with appropriate choices of the stabilization parameters.

The GPR formulation has been applied with success to the Helmholtz problem [6] and to the diffusion-reaction singularly perturbed problem [5]. The same methodology for choosing the free parameters can be used on both problems. It consists in postulating an element matrix with the desired stability properties (GPR-generating matrix) and the free parameters are determined through the solution of a least square problem at element level.

In this work we concisely introduce the GPR formulations for both PDE. Section 2 states the model problem. The Galerkin FEM and GPR formulations are presented in

Section 3. In Section 4 we detail the element matrix of GPR formulation for each PDE and the methodology to determine the free parameters of GPR method. Some numerical experiments are presented in Section 5. Finally, Section 6 contains some conclusions and final remarks.

2. The model problem

Let $\Omega \subset R^n$ be an open bounded domain with a Lipschitz continuous smooth piecewise boundary Γ. Let Γ_g, Γ_q and Γ_r be three disjoint subsets of Γ where boundary conditions are specified, such that $\Gamma_g \cup \Gamma_q \cup \Gamma_r = \Gamma$. We shall consider

$$L(u) = -\nabla\cdot(D\nabla u) + \sigma u = f \quad \text{in } \Omega, \quad (1)$$

$$u = g \quad \text{on } \Gamma_g, \quad (2)$$

$$D\nabla u\cdot\hat{n} = q \quad \text{on } \Gamma_q, \quad (3)$$

$$D\nabla u\cdot\hat{n} + \alpha u = r \quad \text{on } \Gamma_r, \quad (4)$$

where u denotes a unknown scalar field, *f* is the source term, *g*, *q* and *r* are the prescribed boundary conditions. The coefficient α is positive on Γ_r and $\hat{n}$ denotes the outward normal unit vector defined almost everywhere on Γ. If $\sigma = -k^2$ and $D = 1$, then Eq. (1) is known as Helmholtz equation. The solution of Eq. (1) has oscillatory behavior and the coefficient k can be interpreted as the wave number. When the coefficients $\sigma = \bar{\sigma}$ (reactive) and D (diffusive) are positive coefficients and "$D << \bar{\sigma}$", then Eq. (1) is named diffusion-reaction singularly perturbed equation or diffusion-reaction dominated equation.

3. Finite element method for model problem

Consider $M^h = \{\Omega_1, \ldots, \Omega_{ne}\}$ a finite element partition of Ω, such that: $\bar{\Omega} = \Omega \cup \Gamma = \bigcup_{e=1}^{ne} \bar{\Omega}_e = \bigcup_{e=1}^{ne} (\Omega_e \cup \Gamma_e)$, $\Omega_e \cap \Omega_{e'} = \emptyset$ if $e \neq e'$ and Γ_e denotes the boundary of Ω_e. The finite element set and space are $S_h^\kappa = \{u^h \in H^1(\Omega) : u_e^h \in P^\kappa(\Omega_e),\ u^h = g^h \text{ on } \Gamma_g\}$ $V_h^\kappa = \{v^h \in H^1(\Omega) : v_e^h \in P^\kappa(\Omega_e),\ v^h = 0 \text{ on } \Gamma_g\}$, where $P^\kappa(\Omega_e)$ is the space of polynomials of degree less than or equal to κ, g^h denotes the interpolation of g and u_e^h denotes the restriction of u^h to Ω_e.

The Galerkin FEM for the model problem Eq. (1-4) consists on finding $u^h \in S_h^\kappa$ that satisfies $\forall v^h \in V_h^\kappa$,

$$A_G(u^h, v^h) = F_G(v^h), \quad (5)$$

$$A_G = \sum_{e=1}^{ne} \int_{\Omega_e} [D\nabla u^h \cdot \nabla v^h + \sigma u^h v^h]\, d\Omega + \int_{\Gamma_r} \alpha u^h v^h\, d\Gamma,$$

$$F_G = \sum_{e=1}^{ne} \int_{\Omega_e} f\, v^h\, d\Omega + \int_{\Gamma_q} q v^h\, d\Gamma + \int_{\Gamma_r} r v^h\, d\Gamma.$$

The Galerkin FEM is unstable and inaccurate for many examples of this problem, presenting spurious oscillations. A great effort has been devoted to alleviate this misbehavior [1-18]. Here we concisely introduce a stabilized FEM for both PDE, namely the Galerkin projected residual method (GPR).

The GPR method was previously introduced in [5,6]. The fundamental idea of GPR method consists of adding to the Galerkin FEM multiple projections of the residual of the PDE within each element, with one free parameter associated to each projection. The maximum number of free parameters depends on the differential operator and on the local approach space. That is, the maximum number of linearly independent projections of residual will depend on properties of operator (such as symmetry, etc) and on the order of interpolant polynomials. The element matrix then has a maximum number of free parameters, which are determined by appropriate criteria for each specific problem, seeking more accurate and more stable approximate solutions.

Other theoretical details on the method can be found in [5,6]. The GPR method can be formally stated as follows. Find $u^h \in S_h^\kappa$ satisfying $\forall v^h \in V_h^\kappa$:

a) Helmholtz equation

$$A_G(u^h, v^h) + \sum_{e=1}^{ne}\left(\sum_{l=1}^{N} \tau_l^e \left(\left(L(u_e^h), \frac{L(v_e^h)\psi_{l,e}}{k^2}\right)_{L^2(\Omega_e)} + 2\int_{\Omega_e} \frac{\nabla L(u_e^h)\cdot\nabla L(v_e^h)\psi_{l,e}}{k^4} d\Omega\right)\right) = F_G(v^h) +$$

$$\sum_{e=1}^{ne}\left(\sum_{l=1}^{\mathrm{N}}\tau_l^e\left(\left(f_e, L(v_e^h)\psi_{l,e}\right)_{L^2(\Omega_e)} + 2\int_{\Omega_e}\frac{\nabla f_e\cdot\nabla L(v_e^h)\psi_{l,e}}{k^4}d\Omega\right)\right),$$

b) Diffusion-reaction equation

$$A_G(u^h,v^h)+\underline{\sum_{e=1}^{ne}\left(\sum_{l=1}^{\mathrm{N}}\tau_l^e\left(L(u_e^h),L(v_e^h)\psi_{l,e}\right)_{L^2(\Omega_e)}\right)}$$

$$= F(v^h)+\sum_{e=1}^{ne}\left(\sum_{l=1}^{\mathrm{N}}\tau_l^e\left(f_e, L(v_e^h)\psi_{l,e}\right)_{L^2(\Omega_e)}\right),$$

where N is the dimension of a local real linear space $E_{GPR}(\Omega_e)$ generated by functions $L(\eta_i)L(\eta_j)$ with basis denoted by $\psi_{l,e}$,

$$E_{GPR}(\Omega_e)=\{\psi:\Omega_e\to R; \psi=\sum_{i=1}^{npel}\sum_{j=1}^{npel}C_{i,j}L(\eta_i)L(\eta_j), C_{i,j}\in R\},$$

$npel$ denotes the number of nodal points of the element Ω_e and $\eta_i\,(i=1,\ldots,npel)$ denotes the usual local shape functions associated to the i-th nodal point. The free stabilization parameters are denoted by τ_l^e. More details on $E_{GPR}(\Omega_e)$ and $\psi_{l,e}$ can be found in [5,6]. Note that, for each case the first and second underlined terms correspond to projections of the residual and residual gradient of the PDE respectively. These two projections are necessary to obtain a GPR method with uniform convergence properties for Helmholtz equation.

4. The element matrix

Let $\hat{u}_e^h(m)$ be the value of u_e^h at local node m of Ω_e and $u_e^h=\sum_{m=1}^{npel}\hat{u}_e^h(m)\eta_m$. Also, consider $M^l\;(l=1,\ldots,\mathrm{N})$ as being a set of $npel\times npel$ matrices defined as:

a) Helmholtz equation

$$M_{ij}^l=\left(L(\eta_j),\frac{L(\eta_i)\psi_{l,e}}{k^2}\right)_{L^2(\Omega_e)}+2\int_{\Omega_e}\frac{(\nabla L(\eta_j)\cdot\nabla L(\eta_i))\psi_{l,e}}{k^4}d\Omega$$

b) Diffusion-reaction equation

$$M_{ij}^l=\left(L(\eta_j),L(\eta_i)\psi_{l,e}\right)_{L^2(\Omega_e)}.$$

Therefore, the element matrix $\left[A_{GPR}^e\right]_{ij}$ of the GPR method will be

$$\left[A_{GPR}^e\right]_{im}=A^e(\eta_m,\eta_i)+\sum_{l=1}^{\mathrm{N}}\tau_l^eM_{im}^l .$$

We can notice that the element matrix is formed by the usual part of Galerkin plus a projected residual of the differential equation at element level. In [5] we prove that the functions $\psi_{l,e}$ are linearly independent if and only if the N matrices M^l are linearly independent. This allows choosing an appropriate base for the space of matrices generated by the GPR method. A particular GPR method is derived for each specific choice of the set of free parameters $\tau_1^e,\ldots,\tau_{\mathrm{N}}^e$, corresponding to each projection of residual. A possible criterion to determine the free parameters consists on fitting the element matrix of GPR method to a given matrix determined through some stability and/or convergence criteria. We refer to this matrix as the GPR-generating matrix and denote it by M^{gen}. Then the parameters $\tau_1^e,\ldots,\tau_{\mathrm{N}}^e$ can be determined, for example, by solving the following minimization problem at element level:

$$\frac{\partial \mathrm{F}}{\partial\tau_m^e}=0,\quad m=1,\ldots,\mathrm{N}$$

$$\mathrm{F}(M_{im}^{gen})=\sum_{i=1}^{npel}\sum_{j=1}^{npel}\left[(\sum_{l=1}^{\mathrm{N}}\tau_l^eM_{im}^l)-M_{im}^{gen}\right]^2.$$

a) Helmholtz equation

Due to the symmetry of the Helmholtz operator and to the use of first-order interpolant polynomials we have $\mathrm{N}=9$, and therefore nine free parameters. For uniform mesh, bilinear quadrilateral elements and Dirichlet boundary condition the element matrix that minimizes the phase error M^{QS} is associated to the stencil determined though standard dispersion analysis [2]

$$M^{gen}=\lambda_3M^{QS}=\lambda_3\begin{bmatrix}\frac{1}{4}&\frac{\bar\tau_1}{2}&\frac{\bar\tau_3}{4}&\frac{\bar\tau_1}{2}\\ \frac{\bar\tau_1}{2}&\frac{1}{4}&\frac{\bar\tau_1}{2}&\frac{\bar\tau_3}{4}\\ \frac{\bar\tau_3}{4}&\frac{\bar\tau_1}{2}&\frac{1}{4}&\frac{\bar\tau_1}{2}\\ \frac{\bar\tau_1}{2}&\frac{\bar\tau_3}{4}&\frac{\bar\tau_1}{2}&\frac{1}{4}\end{bmatrix},$$

where λ_3 is a parameter that should be determined and

$$\bar\tau_1=\frac{(r_1-r_2)}{(r_2w_1-r_1w_2)},\quad \bar\tau_3=\frac{(w_2-w_1)}{(r_2w_1-r_1w_2)},$$

$$r_1=\cos(kh\cos\tfrac{\pi}{16})\cos(kh\sin\tfrac{\pi}{16}),$$

$$r_2=\cos(kh\cos\tfrac{3\pi}{16})\cos(kh\sin\tfrac{3\pi}{16}),$$

$$w_1=\cos(kh\cos\tfrac{\pi}{16})+\cos(kh\sin\tfrac{\pi}{16}),$$

$$w_2=\cos(kh\cos\tfrac{3\pi}{16})+\cos(kh\sin\tfrac{3\pi}{16}).$$

Since the mesh is uniform, the following restrictions for the free parameters τ_l^e can be imposed:

$\tau_1 = \tau_5 = \tau_7 = \tau_9 = 0$,

$\tau_2 = \tau_4 = \tau_6 = \tau_8 = \lambda_1$,

$\tau_3 = \lambda_2$.

Therefore, the functional F can be written as

$$\mathrm{F} = \sum_{m=1}^{npel}\sum_{i=1}^{npel}\left[A^e(\eta_m,\eta_i) + \lambda_1(M_{im}^2 + M_{im}^4 + M_{im}^6 + M_{im}^8) + \lambda_2 M_{im}^3 - \lambda_3 M_{im}^{QS}\right]^2 .$$

b) Diffusion-reaction equation

We build the GPR generating matrix by combining the element matrix of two successful stabilized FEM: the Gradient Galerkin Least Squares (GGLS) [7] and the Unusual Stabilization (USFEM) [8,9] methods. We have

$$M^{gen,e} = K^e + B^e , \qquad (6)$$

$$K_{ij}^e = \int_{\Omega_e} \chi^{e,2}\sigma(\mathbf{J}\nabla\eta_j)\cdot(\mathbf{J}\nabla\eta_i)\,d\Omega ,$$

$$B_{ij}^e = -\int_{\Omega_e} \chi^{e,1}\sigma\eta_j\eta_i\,d\Omega ,$$

where $\chi^{e,1}$ and $\chi^{e,2}$ are dimensionless functions, understood as the weights of the nontrivial combination given by Eq. (6),

$$\chi^{e,1} = \varsigma^{e,0}\chi^e\left|1-\chi^e\right|^{\left(\frac{1}{1-\chi^e}\right)} ,$$

$$\chi^{e,2} = (\chi^e)^{\left(\frac{1}{\chi^e}\right)}\varsigma^{e,2}\varsigma^{e,0} ,$$

$$\chi^e = \frac{1}{\chi^{e,0}(P_{reat}^e) + P_{reat}^e} ,$$

$$P_{reat}^e = \frac{6D}{\overline{\sigma}(h_e)^2} ,$$

$$\chi^{e,0}(P_{reat}^e) = \begin{cases} 1, & \text{if } P_{reat}^e \le 1 \\ P_{reat}^e, & \text{if } P_{reat}^e > 1 \end{cases} ,$$

$$h_e = \left(\int_{\Omega_e} d\Omega\right)^{\frac{1}{n}} ,$$

and $\mathbf{J}$ is the Jacobian matrix corresponding to the mapping between reference and actual elements.

Based on this observation and inspired on references [7,9] we accomplished a large number of computational experiments with bilinear rectangular elements and linear triangular elements and conclude that the following expressions for the real constant $\varsigma^{e,0}$ and the dimensionless function $\varsigma^{e,2}$ present very good stability and accuracy properties:

$$\varsigma^{e,0} = \begin{cases} 0, & \text{if } \Gamma_e \cap (\Gamma \cup \Gamma_{\text{int}}) = \varnothing \\ 1, & \text{if } \Gamma_e \cap (\Gamma \cup \Gamma_{\text{int}}) \neq \varnothing \end{cases} ,$$

$$\varsigma^{e,2} = \begin{cases} \varsigma^{e,q}, & \text{for quadrilatral or hexahedron element} \\ \varsigma^{e,t}, & \text{for is a triangular or tetrahedral element} \end{cases} ,$$

where $\varsigma^{e,q}$ and $\varsigma^{e,t}$ are given as

$$\varsigma^{e,q} = \begin{cases} 2/3 + (N_{face}^e - 1)/6, & \text{if } \; meas\,(\Gamma \cap \Gamma_e) > 0 \\ 2/3, & \text{if } \; meas\,(\Gamma \cap \Gamma_e) = 0 \end{cases} ,$$

$$\varsigma^{e,t} = \begin{cases} \varsigma^{e,q}, & \text{if } \Gamma_e(\Gamma \cup \Gamma_{\text{int}}) = \varnothing \\ (P_{reat}^e)^{-(\chi^e+1)}, & \text{if } \Gamma_e(\Gamma \cup \Gamma_{\text{int}}) \neq \varnothing \end{cases} ,$$

and N_{face}^e is the number of faces of Ω_e contained in Γ. The set Γ_{int} is defined as

$$\Gamma_{\text{int}} = \bigcup_{e=1}^{ne}\left(\bigcup_{e'=1}^{ne}\Gamma_{ee'}^*\right) ,$$

$$\Gamma_{ee'}^* = \Gamma_{e'e}^* = \begin{cases} \Gamma_e \cap \Gamma_{e'} & \text{if } ([f]_{ee'} \neq 0 \text{ or } [\sigma]_{ee'} \neq 0 \text{ or } [D]_{ee'} \neq 0) \\ \varnothing & \text{if } ([f]_{ee'} = 0 \text{ and } [\sigma]_{ee'} = 0 \text{ and } [D]_{ee'} = 0) \end{cases}$$

which is the union of the external boundary with the internal edges between two elements presenting discontinuous properties or sources. For each Ω_e and for each $\Omega_{e'}$ we define $[\varphi]_{ee'} = \int_{\Gamma_e \cap \Gamma_{e'}} |\varphi_e - \varphi_{e'}|\,d\Gamma$, $\varphi \in L^2(\Omega)$ and $\varphi_e \in H^1(\Omega_e)$.

It should be observed that for diffusive reactive problems, sharp layers will only occur inside an element Ω_e if $\Gamma_e \cap (\Gamma \cup \Gamma_{\text{int}}) \neq \varnothing$.

It must be emphasized that, for the GPR method with polynomials of degree bigger than 1, additional numeric experiments need to be accomplished to validate the proposed expressions for $\varsigma^{e,q}$ and $\varsigma^{e,t}$.

5. Numerical results

Two examples to illustrate the great potential of GPR method are presented. The first deals with inhomogeneous Helmholtz equation over a unit square domain subjected to Dirichlet boundary conditions. The exact solution is a plane wave propagating in θ-direction plus a polynomial function, i.e.,

$u(x,y) = p(x,y) + \sin(k(x\cos\theta + y\sin\theta))$, where we consider two situations: case 1: $p(x,y) = x + y$; case 2: $p(x,y) = x^2 + y^2$. Figures 1 and 2 present the errors of the GPR method in L^2-norm and H^1-seminorm relative to the continuous bilinear interpolant are presented respectively.

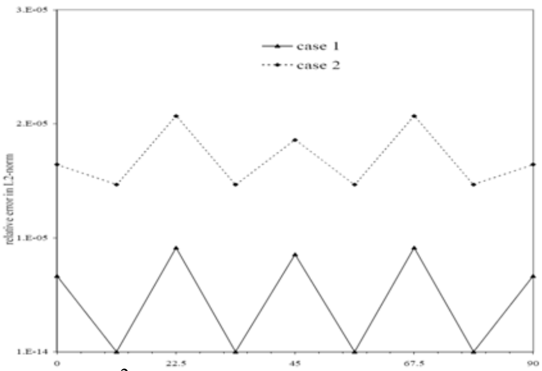

Fig. 1 L^2-norm error of the GPR solution relative to the continuous interpolant, as a function of the θ-direction, for the non homogeneous Helmholtz equation.

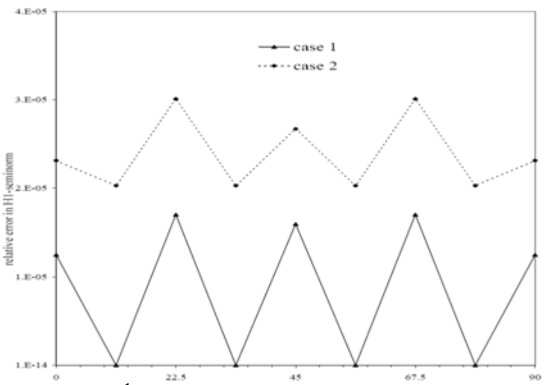

Fig. 2 H^1-norm error of the GPR solution relative to the continuous interpolant, as a function of the θ-direction, for the non homogeneous Helmholtz equation.

The second example deals with reactive dominant problem defined over a quadrilateral domain of vertexes (0.5, 0.0), (1.5, 0.0), (2.0, 2.0) and (0.0, 1.0) with $D = 10^{-6}$, $\bar{\sigma} = 1$, $f = 1$ and homogeneous Dirichlet boundary conditions. Results are presented in Fig. 3 for a non uniform mesh of quadrilateral elements. Similar results obtained for a mesh of triangular elements are shown in Fig. 4. Results obtained with the methods "USFEM" [8,9] and "ASGS" [4,10,11] are presented on both Figures 3 and 4 as well. A convergence study was also performed for this second problem with different values of diffusion coefficient ($D = 1$, $D = 10^{-3}$ and $D = 10^{-6}$), $\bar{\sigma} = 1$, $f = (2\pi^2 D + 1)\sin(\pi x)\sin(\pi y)$ and boundary conditions $u = \sin(\pi x)\sin(\pi y)$ on Γ. Results for quadrilateral elements are presented in Figure 5. The GPR method presents optimal

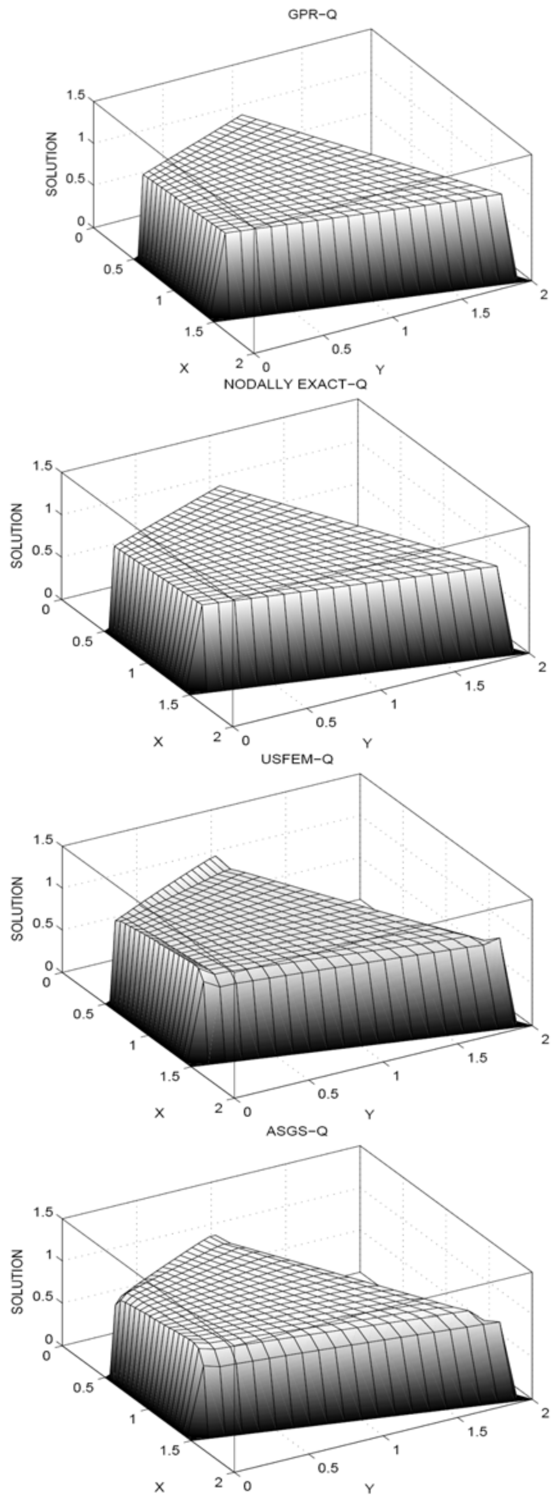

Fig. 3 Computed solutions with different methods for the reactive dominant problem in a non uniform mesh with bilinear quadrilateral elements.

rates of convergence for all tested values of *D*. Similar results are obtained for the mesh of triangular elements.

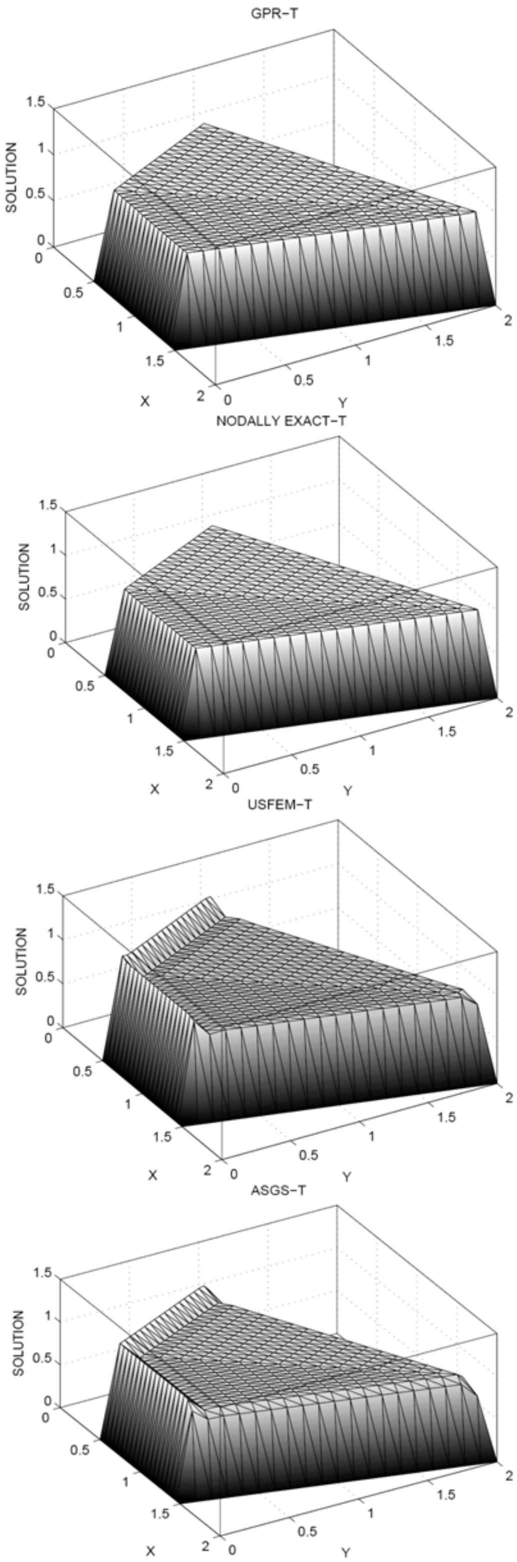

Fig. 4 Computed solutions with different methods for the reactive dominant problem in a non uniform mesh with linear triangular elements.

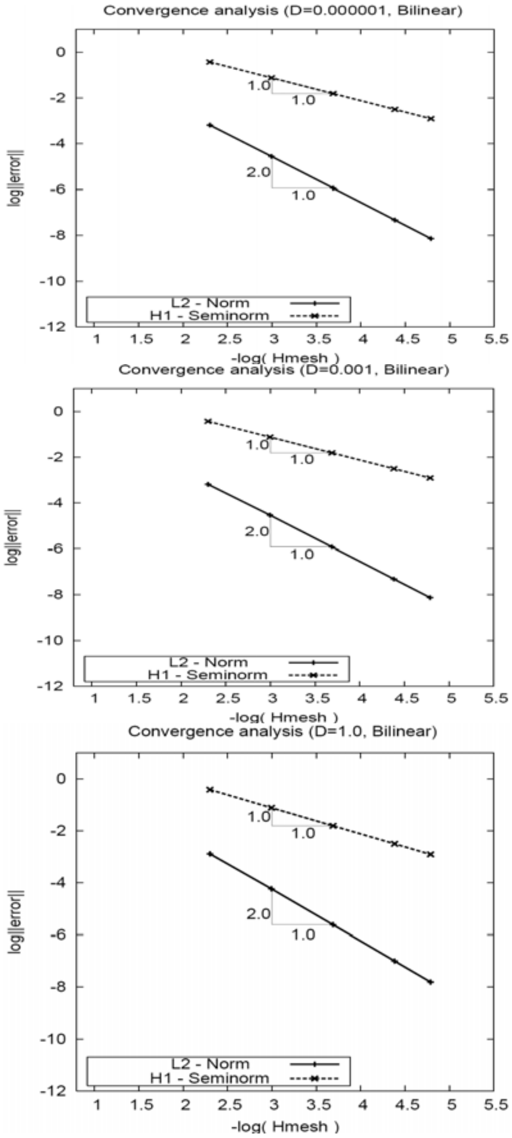

Fig. 5 A convergence study of the GPR method for the reactive dominated problem with quadrilateral elements, for different values of the diffusive coefficient D: 10^{-6}, 10^{-3} and 1.

6. Conclusion

In this work we concisely presented the Galerkin projected residual method, a new consistent FEM. This methodology allows the derivation of a family of methods through the choice of the GPR-generating matrix. The formulation is valid for any dimension of the domain and any order of local basis functions.

When compared to typical Galerkin formulations, the GPR method requires an extra computational effort related to the elements Ω_e such that $\Gamma_e(\Gamma \cup \Gamma_{int}) \neq \emptyset$. This extra effort is handled in a pre-processing phase and does not represent a real burden.

The good performance of the GPR methodology stimulates its future application to other problems, such as the diffusive-convective problems.

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