E: No 2. Calculo I 01/2006

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Favor responder com clareza e limpeza. Isto será considerado na hora da correção.

1) Encontre as primitivas da função $f(x) = x\sqrt{x-1}$. Isto é, $\int x\sqrt{x-1}\,dx$?

2) Calcule $\int e^x \cos x \, dx$.

3) Determine a área da figura plana limitada pelas curvas $y = 2 - x^2$ e $y^3 = x^2$.

4) Calcule o comprimento da cicloide dada por:

$$\begin{cases} x(t) = a(t - \operatorname{sen} t) \\ y(t) = a(1 - \cos t) \end{cases}$$ no intervalo $0 \leq t \leq 2\pi$.

5) Determine o comprimento da astroide $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Formulario: $L = \int_\alpha^\beta \sqrt{(x')^2 + (y')^2}\,dt$

$$L = \int_a^b \sqrt{1 + (y')^2}\,dx$$

Obs: Todas as questões valem 2 pontos.

Escrita No 2. Cálculo I 012006 (1)

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1) $\int x\sqrt{x-1}\,dx$ fazendo a mudança de variavel $t=\sqrt{x-1}$ segue que $x=t^2+1$ e $dx=2t\,dt$. Logo,

$$\int x\sqrt{x-1}\,dx = \int (t^2+1)\,t\cdot 2t\,dt = 2\int (t^4+t^2)\,dt =$$

$$= 2\left[\int t^4\,dt + \int t^2\,dt\right] = 2\left[\frac{t^5}{5}+C_1+\frac{t^3}{3}+C_2\right] =$$

$= \frac{2}{5}t^5 + \frac{2}{3}t^3 + C$, já que C, C_1 e C_2 são constantes arbitrarias. Subtituindo t por $\sqrt{x-1}$ obtemos:

$$\int x\sqrt{x-1}\,dx = \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C.$$

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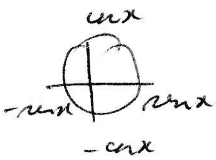
2) $\int e^x \cos x\,dx$. Aplicando integração por parte segue:



$$\int u\,dv = uv - \int v\,dv.$$

escolhendo $u=e^x$ e $dv=\cos x\,dx$ obtemos que $du=e^x\,dx$ e $v=\int dv=\int \cos x\,dx = \operatorname{sen} x$, Logo:

$$\int e^x\cos x\,dx = e^x\operatorname{sen} x - \int \operatorname{sen} x\, e^x\,dx \qquad (1)$$

Aplicando novamente integração por parte em $\int \operatorname{sen} x\, e^x\,dx$ e escolhendo $u=e^x$ e $dv=\operatorname{sen} x\,dx$ obtemos que $du=e^x\,dx$ e $v=\int dv=\int \operatorname{sen} x\,dx = -\cos x$. Subtituindo temos

$\int \operatorname{sen} x\, e^x\,dx = -e^x\cos x + \int \cos x\, e^x\,dx$, que subtituindo em (1) segue:

$$\int e^x\cos x\,dx = e^x\operatorname{sen} x + e^x\cos x - \int \cos x\, e^x\,dx \quad \text{ou}$$

$2\int e^x\cos x\,dx = e^x(\operatorname{sen} x+\cos x)$ de onde segue que

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$$\int e^x \cos x \, dx = \frac{1}{2} e^x (\operatorname{sen} x + \cos x) + C,$$ onde C é uma constante arbitrária.

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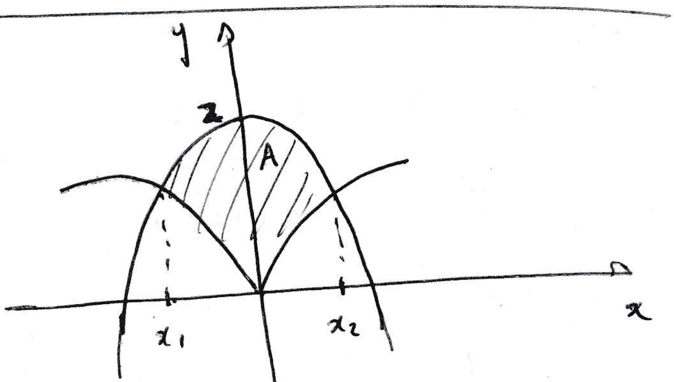
3) $y = 2 - x^2$ e $y^3 = x^2$



$(2-x^2)^3 = x^2$ ou então

$x^2 = 2 - y$ e $y^3 = 2 - y$

$y^3 + y - 2 = 0$. Logo os pontos $x_1 = -1$ e $x_2 = 1$.

$$A = \int_{-1}^{1} \left[(2-x^2) - x^{\frac{2}{3}}\right] dx = \int_{-1}^{1} \left(2 - x^2 - x^{\frac{2}{3}}\right) dx =$$

$$= \left[2x - \frac{x^3}{3} - \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1}\right]\Bigg|_{-1}^{1} = \left[2x + \frac{x^3}{3} - 3\frac{x^{\frac{5}{3}}}{5}\right]\Bigg|_{-1}^{1} =$$

$$= \left(2 + \frac{1}{3} - \frac{3}{5}\right) - \left(-2 - \frac{1}{3} + \frac{3}{5}\right) = 4 + \frac{2}{3} - \frac{6}{5} = \frac{15 \cdot 4 + 10 - 18}{15} = \frac{52}{15}$$

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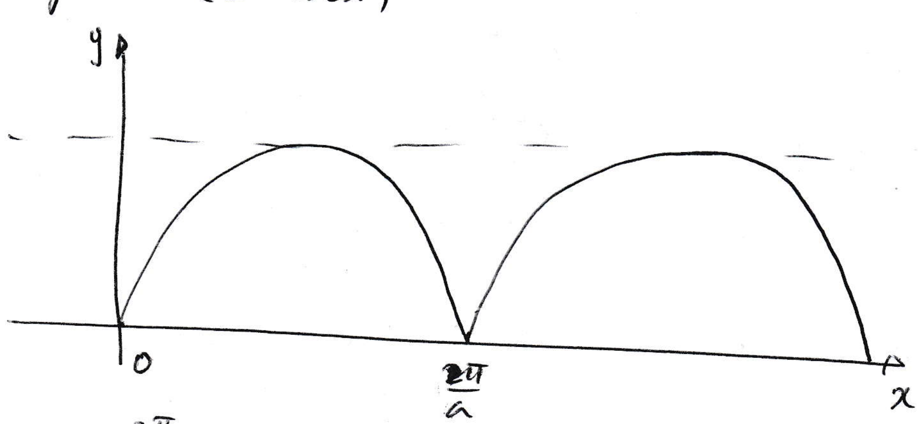
4) $x(t) = a(t - \operatorname{sen} t)$
$y(t) = a(1 - \cos t)$ $\quad 0 \leq t \leq 2\pi$



$$L = \int_0^{2\pi} \sqrt{(x')^2 + (y')^2}\, dt = \int_0^{2\pi} \sqrt{a^2(1-\cos t)^2 + a^2 \operatorname{sen}^2 t}\, dt =$$

$$= a \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt = 2a \int_0^{2\pi} \operatorname{sen}\left(\frac{t}{2}\right) dt = 8a.$$

$$\int_0^{2\pi} \sqrt{2(1-\cos t)}\, dt = \sqrt{2} \int_0^{2\pi} \sqrt{1 - 2\cos^2\left(\tfrac{t}{2}\right) + 1}\, dt = \sqrt{2} \int_0^{2\pi} \sqrt{2}\sqrt{1 - \cos^2\left(\tfrac{t}{2}\right)}\, dt$$

$$\boxed{\cos t = \cos\left(2\tfrac{t}{2}\right) = 2\cos^2\left(\tfrac{t}{2}\right) - 1}$$

$$= 2\int_0^{2\pi} \operatorname{sen}\left(\tfrac{t}{2}\right) dt = 2\cdot 2 \int_0^{2\pi} \operatorname{sen}\left(\tfrac{t}{2}\right) d\left(\tfrac{t}{2}\right) = -4\cos\left(\tfrac{t}{2}\right)\Big|_0^{2\pi} =$$

$$= -4\left[\cos\left(2\tfrac{\pi}{2}\right) - \cos 0\right] = -4\left[-1 - 1\right] = 8.$$

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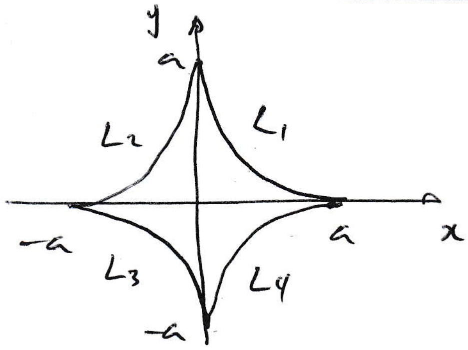
5) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$



$L = L_1 + L_2 + L_3 + L_4$

Como $L_1 = L_2 = L_3 = L_4$ temos

$L = 4L_1$

$$L_1 = \int_0^a \sqrt{1 + (y')^2}\, dx.$$

Derivando a equação da astroide obtemos:

$$\frac{d}{dx}\left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right) = \frac{d}{dx}\left(a^{\frac{2}{3}}\right) \Rightarrow \frac{d}{dx}\left(x^{\frac{2}{3}}\right) + \frac{d}{dx}\left(y^{\frac{2}{3}}\right) = 0$$

$$\frac{2}{3}x^{\frac{2}{3}-1} + \frac{2}{3}y^{\frac{2}{3}-1}\frac{dy}{dx} = 0 \text{ ou } \frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$ substituindo temos

$$L_1 = \int_0^a \sqrt{1 + \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}}\, dx = \int_0^a \sqrt{\frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{x^{\frac{2}{3}}}}\, dx = \int_0^a \sqrt{\frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}}\, dx = \int_0^a \frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}}\, dx =$$

$= \frac{3}{2}a$. Portanto, $L = 4L_1 = 4\,\frac{3}{2}\cdot a = 6a$.