A FIXED POINT CURVE THEOREM FOR FINITE ORBITS LOCAL DIFFEOMORPHISMS

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Abstract. We study local biholomorphisms with finite orbits in some neighborhood of the origin since they are intimately related to holomorphic foliations with closed leaves. We describe the structure of the set of periodic points in dimension 2. As a consequence we show that given a finite orbits local biholomorphism F, in dimension 2, there exists an analytic curve passing through the origin and contained in the fixed point set of some non-trivial iterate of F. As an application we obtain that at least one eigenvalue of the linear part of F at the origin is a root of unity. Moreover, we show that such a result is sharp by exhibiting examples of finite orbits local biholomorphisms such that exactly one of the eigenvalues is a root of unity. These examples are subtle since we show they can not be embedded in one parameter groups.

Contents

MSC-class. Primary:Primary: 32H50, 37C25; Secondary: 37F75, 34M25.
Keywords: local diffeomorphism, local dynamics, finite orbits, invariant varieties.

1. Introduction

Let $F : U \to V$ be a biholomorphism where U and V are open sets of $\mathbb{C}^n$ that contain the origin 0 and $F(0) = 0$. Fixed $p \in U$, we define $F^0(p) = p$ and, if $F^0(p), ..., F^{j-1}(p) \in U$ for $j > 0$, we define $F^j(p) = F(F^{j-1}(p))$. Given $A \subset U \cap V$, we define $\mathcal{I}^+_{F,A}(p)$ as the set of non-negative integers j such that $F^k(p) \in A$ for any $0 \le k \le j$. We define the *positive orbit of p by F in A* as

$$O^+_{F,A}(p) = \{F^j(p);\ j \in \mathcal{I}^+_{F,A}(p)\}.$$

Note that the positive F-orbit of p in A is infinite if and only if $F^j(p) \in A$ for any $j \ge 0$ and the set $\{F^j(p) : j \ge 0\}$ is infinite. We define the negative F-orbit of p in A as

$$O^-_{F,A}(p) = O^+_{F^{-1},A}(p).$$

where F^{-1} denote the inverse of F. Analogously as above, the negative F-orbit of p in A is infinite if and only if $F^{-j}(p) \in A$ for any $j \ge 0$ and $\{F^{-j}(p) : j \ge 0\}$ is infinite where $F^{-j} = (F^{-1})^j$. We define the *F-orbit of p in A* as

$$O_{F,A}(p) = O^+_{F,A}(p) \cup O^-_{F,A}(p).$$

There are two types of finite orbits $O_{F,A}(p)$, namely either $\mathcal{I}^+_{F,A}(p)$ and $\mathcal{I}^+_{F^{-1},A}(p)$ are finite or

$$\mathcal{I}^+_{F,A}(p) = \mathcal{I}^+_{F^{-1},A}(p) = \mathbb{N} \cup \{0\}$$

and p is a periodic point, i.e. there exists $k \in \mathbb{N}$ such that $F^k(p) = p$.

We say that F has *finite orbits* in A if $O_{F,A}(p)$ is a finite set for all $p \in A$. In this case, F has finite orbits in B for any subset B of A since $O_{F,B}(p) \subset O_{F,A}(p)$ for all $p \in B$. As a consequence, the finite orbits property can be defined for $F \in \mathrm{Diff}(\mathbb{C}^n, 0)$ where $\mathrm{Diff}(\mathbb{C}^n, 0)$ is the group of germs of biholomorphism fixing the origin $0 \in \mathbb{C}^n$.

Definition 1. Let $F \in \mathrm{Diff}(\mathbb{C}^n, 0)$ be a local biholomorphism. We say that F is a *finite orbits germ* or that it has *finite orbits* (and then we write $F \in \mathrm{Diff}_{<\infty}(\mathbb{C}^n, 0)$) if there exists a representative $F : U \to V$ and a neighborhood $A \subset U \cap V$ of 0 such that F has finite orbits in A.

Finite orbits local biholomorphisms appear in the study of foliations with closed leaves. In [MM80] Mattei and Moussu proved one of the most important theorems in the theory of holomorphic foliations, namely the topological characterization of the existence of a non-constant holomorphic first integral for germs of codimension 1 holomorphic foliations. More precisely, they show that a singular holomorphic foliation $\mathcal{F}$ on $(\mathbb{C}^n, 0)$ of codimension 1 has a first integral if and only if the leaves of $\mathcal{F}$ are closed subsets of the complement of the singular set and only finitely many of them accumulate at 0. A fundamental ingredient of the proof is that, in dimension 1, the finite orbits property is equivalent to periodicity. More precisely, they show that a biholomorphism $F \in \mathrm{Diff}\,(\mathbb{C}, 0)$ has finite orbits if and only if F is a finite order element of the group $\mathrm{Diff}\,(\mathbb{C},0)$. For dimension $n \geq 2$, the equivalence does not hold. For example, the local biholomorphism $F(x,y) = (x, y+x^2)$ has finite orbits but is of infinite order.

It is possible to recover the equivalence periodicity $\leftrightarrow$ finite orbits by replacing the finite order property with stronger conditions and so to obtain, in dimension greater than 1, analogues of the topological criterium of Mattei–Moussu ([RR15], [CS09], and [CS17]). In spite of this, the following elementary problems, related to the finite orbits property, were open until now:

(1) Description of the properties of the differential D_0F at the origin of a finite orbits germ F;
(2) description of the set of periodic points of $F \in \mathrm{Diff}_{<\infty}(\mathbb{C}^n,0)$.

We answer these questions in dimension two and provide partial answers for higher dimension. A natural question (that was open until now) is whether the finite orbits property for $F \in \mathrm{Diff}(\mathbb{C}^n,0)$ implies the analogue for the linear part D_0F of F at the origin. The next result provides the first counterexamples.

Theorem 1. *Suppose that $\lambda \in \mathbb{C}$ satisfies the Cremer condition and $n \geq 1$. Then there exists a global biholomorphism $F \in \mathrm{Diff}(\mathbb{C}^{n+1})$ such that $\mathrm{Spec}(D_0F) = \{\lambda, 1\}$, the algebraic multiplicity of the eigenvalue 1 of D_0F is equal to 1 and F has finite orbits in every set of the form $\mathbb{C}^n \times U$, where $U \subset \mathbb{C}$ is a bounded open set.*

Let us stress that F is a counterexample because, in the linear case, F has finite orbits if and only if the spectrum of D_0F consists of roots of unity (cf. Proposition 2). Until now it was known that finite orbits local biholomorphisms F satisfy that the eigenvalues of D_0F have modulus 1 by the Stable Manifold Theorem (cf. Corollary 5).

The next result gives an indication of why the examples of $F \in \mathrm{Diff}_{<\infty}(\mathbb{C}^n,0)$ such that $\mathrm{spec}(D_0F)$ is not contained in the group of roots of unity were missing in the literature: there are no "continuous" examples, i.e. where F belongs to a one-parameter group. Let $\mathfrak{X}(\mathbb{C}^n,0)$ denote the Lie algebra of singular local holomorphic vector fields at the origin.

Theorem 2. *Let $X \in \mathfrak{X}(\mathbb{C}^n,0)$ and let F be the time 1 map of X. Suppose that F has finite orbits. Then it satisfies*

$$\mathrm{Spec}(D_0F) \subset e^{2\pi i\mathbb{Q}}.$$

The existence of the examples provided by Theorem 1 has consequences in the problem of geometrical realization of formal invariant curves. Indeed, it was proved in [LHRRSS21] that when the multiplier of the restriction $F_{|\Gamma}$ of $F \in \mathrm{Diff}\,(\mathbb{C}^2,0)$ to a formal invariant curve Γ is not an element of $e^{2\pi i(\mathbb{R}\setminus\mathbb{Q})}$ and F is non-periodic then either the curve Γ is convergent or there are invariant analytic sets asymptotic to Γ and consisting of stable orbits, i.e. orbits of points p such that $\lim_{k\to\infty} F^k(p) = 0$. Our examples show that in general there is no systematic approach to the geometrical realization of Γ as a stable set if the multiplier of $F_{|\Gamma}$ is irrationally neutral, i.e. if it belongs to $e^{2\pi i(\mathbb{R}\setminus\mathbb{Q})}$. This completes somehow the realization program in [LHRRSS21] and [LHRSSV]. More precisely, the examples provided by Theorem 1 for dimension 2 have a formal curve Γ invariant by F, such that the multiplier λ of $F|_\Gamma$ belongs to $e^{2\pi i(\mathbb{R}\setminus\mathbb{Q})}$, but F has no stable sets, since it has finite orbits.

Theorem 1 suggests that the finite orbits property is related to small divisors. Note that the multiplier λ in Theorem 1 is very well approached by roots of unity since it is a Cremer number. Such a circumstance is not accidental; indeed we show, by applying a Theorem of Pöschel [P̈86], that there is no $F \in \mathrm{Diff}_{<\infty}(\mathbb{C}^2, 0)$ such that $\mathrm{Spec}(D_0F)$ contains a Bruno number (Proposition 5). In particular, we show that if the multiplier of $F_{|\Gamma}$ is a Bruno number, for $F \in \mathrm{Diff}\,(\mathbb{C}^2, 0)$ and a formal invariant curve Γ of F, then F is not a finite orbits germ (Corollary 7).

The natural follow up question to Theorem 1 is to understand how far is the germ of D_0F at the origin of having finite orbits if $F \in \mathrm{Diff}_{<\infty}(\mathbb{C}^n, 0)$. In particular, are there $F \in \mathrm{Diff}_{<\infty}(\mathbb{C}^n, 0)$ such that $\mathrm{Spec}(D_0F) \cap e^{2\pi i\mathbb{Q}} = \emptyset$? The answer is negative for dimension 2.

Theorem 3. *Let $F \in \mathrm{Diff}_{<\infty}(\mathbb{C}^2, 0)$. Then at least one eigenvalue of D_0F is a root of unity (and all of them belong to the unit circle).*

As a consequence, the examples of Theorem 1 have the minimal number of roots of unity eigenvalues and hence Theorem 3 is sharp. Moreover, we classify the finite orbits local biholomorphisms $F \in \mathrm{Diff}\,(\mathbb{C}^2, 0)$ such that $\mathrm{Spec}(D_0F)$ contains a non-root of unity eigenvalue: essentially they are the examples provided by Theorem 1 (Proposition 5). Theorem 3 is a consequence of the Fixed Point Curve Theorem that we discuss next. A classical result about vector fields in dimension $n = 2$ is the Camacho-Sad theorem [CS82], which states that every vector field $X \in \mathfrak{X}(\mathbb{C}^2, 0)$ admits a germ of invariant curve at the origin. Existence of invariant objects for local biholomorphisms tangent to the identity $F \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ is also well known (see [Aba01], [LS18], [BMCLH08]....). In [Aba01] Abate generalizes to $\mathbb{C}^2$ the classical Leau-Fatou flower theorem proving that if $F \in \mathrm{Diff}_1(\mathbb{C}^2, 0)$ has an isolated fixed point at 0 then F has at least one parabolic curve, that is, a F-invariant holomorphic curve, with the origin in their boundary, and whose orbits tend to 0; in particular, it is not a finite orbits germ. Thus, if $F \in \mathrm{Diff}_1(\mathbb{C}^2, 0) \cap \mathrm{Diff}_{<\infty}(\mathbb{C}^2, 0)$ then F has a non isolated fixed point at the origin. Later on, López-Hernanz and Sanz [LS18] showed that if F has a formal invariant curve Γ that is not contained in the fixed point set of F then F or F^{-1} has a parabolic curve asymptotic to Γ. In particular, if $F \in \mathrm{Diff}_1(\mathbb{C}^2, 0) \cap \mathrm{Diff}_{<\infty}(\mathbb{C}^2, 0)$ then every formal invariant curve is a fixed point curve.

In contrast to the previous approach, the second author showed that existence of germs of analytic invariant curves does not hold for biholomorphisms $F \in$

$\mathrm{Diff}\,(\mathbb{C}^2, 0)$ in general [Rib05]. Moreover, the counterexamples can be chosen to be tangent to the identity or formally linearizable.

In this work we prove the following theorem. It is a version of the Camacho-Sad Theorem for finite orbits local biholomorphisms.

Fixed Point Curve Theorem. *Let $F \in \mathrm{Diff}\,(\mathbb{C}^2, 0)$ be a finite orbits diffeomorphism. Then, there exists $m \in \mathbb{N}$ such that F^m has a germ Γ of complex analytic curve consisting of fixed points.*

We can apply The Fixed Point Curve Theorem to obtain a generalization of a Rebelo–Reis Theorem in the context of cyclic subgroups of $\mathrm{Diff}\,(\mathbb{C}^2, 0)$. More precisely, a consequence of Theorem A in [RR15] is that if, for all $m \in \mathbb{N}$, every point $p \in \mathrm{Fix}(F^m)$ in a neighborhood of 0 satisfies that either p is an isolated fixed point of F^m or the germ of F^m at p is equal to the identity map, then F is periodic. We provide a stronger version of this result in dimension 2, namely it suffices to check the condition at the origin. Thus, we obtain a negative criterium for the finite orbits property.

Corollary 1. *Let $F \in \mathrm{Diff}(\mathbb{C}^2, 0)$ such that 0 is an isolated fixed point of F^m for every $m \in \mathbb{N}$. Then F is not a finite orbits germ.*

Our approach to show the Fixed Point Curve Theorem relies on describing the connected components of the set of periodic points of $F \in \mathrm{Diff}_{<\infty}(\mathbb{C}^2, 0)$.

Theorem 4. *Let $F \in \mathrm{Diff}_{<\infty}(\mathbb{C}^2, 0)$. Let B be an open or closed ball centered at the origin such that F and F^{-1} are defined in a neighborhood U of $\overline{B}$ and F has finite orbits in U. Consider the sets*

$$\mathrm{Per}_k(F) = \{p \in B;\ p,\ F(p), \dots, F^k(p) \in B \text{ and } F^k(p) = p\}$$

for $k \in \mathbb{N}$. Let C be a connected component of $\mathrm{Per}(F) := \cup_{k=1}^{\infty} \mathrm{Per}_k(F)$. Then, C is semianalytic and there exists $m = m(C)$ such that C is a connected component of the semianalytic set $\mathrm{Per}_m(F)$. Moreover, if B is an open ball then C is complex analytic in B and the irreducible components of C have positive dimension.

Suppose that B is a closed ball since it is simpler to work in compact sets. Let $\mathcal{B}$ be the set of points $p \in B$ such that the map $q \mapsto \sharp O_{F,B}(q)$ is an unbounded function in every neighborhood of p. Such a set is the analogue for diffeomorphisms of the so called *bad set* associated to smooth foliations by compact leaves of compact manifolds; it consists of the leaves where the volume function (defined in the space of leaves) is not locally bounded. The properties of the bad set are one of the ingredients used by Edwards, Millet and Sullivan to show that, under a suitable homological condition, the volume function associated to a smooth foliation by compact leaves is uniformly bounded [EMS77]. In the finite orbits case for $n = 2$, the bad set $\mathcal{B}$ is contained in $\mathrm{Per}(F)$ and moreover, the connected components of $\mathcal{B}$ are also connected components of $\mathrm{Per}(F)$. In general, the structure of the bad set can be very complicated. However, the finite orbits property constrains the connected components of the bad set $\mathcal{B}$ to be simple for $n = 2$. Indeed they are semianalytic by Theorem 4.

Our results can be used to study holomorphic foliations of codimension 2 defined in a neighborhood of a compact leaf and whose leaves are closed. Such a problem will be contemplated in future work.

Section 2 introduces the setting of the paper along with some elementary results. Theorem 2 is proved in section 3. We show Theorems 4, 3 and the Fixed Point Curve Theorem in section 4. Finally, we provide the examples in Theorem 1 in section 5.

2. Notations and first results

As above, we denote by $\text{Diff}\,(\mathbb{C}^n, 0)$ the group of germs of biholomorphisms fixing $0 \in \mathbb{C}^n$ and by $\text{Diff}_{<\infty}(\mathbb{C}^n, 0)$ the subset of $\text{Diff}\,(\mathbb{C}^n, 0)$ consisting of those having finite orbits. In the remainder of this section we included some elementary results about the finite orbits property for the sake of completeness.

Proposition 1. Let $F \in \text{Diff}(\mathbb{C}^n, 0)$.

(i) (Invariance by analytic conjugation) If $F = HGH^{-1}$ for some $H \in \text{Diff}\,(\mathbb{C}^n, 0)$, then

$$F \in \text{Diff}_{<\infty}(\mathbb{C}^n, 0) \Leftrightarrow G \in \text{Diff}_{<\infty}(\mathbb{C}^n, 0).$$

(ii) (Invariance by iteration) The following affirmations are equivalent.
  (a) $F \in \text{Diff}_{<\infty}(\mathbb{C}^n, 0)$.
  (b) $F^m \in \text{Diff}_{<\infty}(\mathbb{C}^n, 0),\ \forall m \in \mathbb{N}$.
  (c) $F^m \in \text{Diff}_{<\infty}(\mathbb{C}^n, 0)$ for some $m \in \mathbb{N}$.

Proof. (i) Assume $H \circ G = F \circ H$ and let U be a connected open neighborhood of 0 in which all germs involved have injective representatives. There exists a neighborhood $0 \in A \subset U$ such that $G(A), H(A), H(G(A)) \subset U$. By using $H \circ G = F \circ H$, we can show by induction that, if $x, G^{\pm}(x), ..., G^{\pm k}(x) \in A$, then $F^l(H(x)) = H(G^l(x))$ for $l = \pm 1, ..., \pm k$. Therefore,

$$H(O_{G,A}(x)) = O_{F,H(A)}(H(x)) \ \text{ for any } x \in A.$$

In particular, G has finite orbits in A if and only if F has finite orbits in $H(A)$. This shows (i).

(ii) (a)$\Rightarrow$ (b): Suppose $F \in \text{Diff}_{<\infty}(\mathbb{C}^n, 0)$ and let U be a connected open neighborhood of 0 in which F e F^{-1} are defined and have finite orbits. Given $m \in \mathbb{N}$, there exists a connected open neighborhood V of 0 such that $F^{\pm j}(p) \in U$ for all $p \in V$ and $0 \leq j \leq m$. As a consequence, we obtain

$$O_{F^m,V}(p) \subset O_{F,U}(p),\ \forall p \in V,$$

and hence $F^m \in \text{Diff}_{<\infty}(\mathbb{C}^n, 0)$ for any $m \in \mathbb{N}$. It is obvious that (b) $\Rightarrow$(c). Let us show that (c)$\Rightarrow$(a). Let $m \in \mathbb{N}$ such that $F^m \in \text{Diff}_{<\infty}(\mathbb{C}^n, 0)$. As before, let V be a connected open neighborhood of 0 in which F^m has finite orbits. Up to consider a smaller V, we can assume that $F^{\pm 1},\dots,F^{\pm(m-1)}$ are defined in V. Since for each $p\in V$, the set $O_{F^m,V}(p)$ is finite and

$$O_{F,V}(p)\subset \cup_{j=-(m-1)}^{m-1}F^j(O_{F^m,V}(p)),$$

we deduce that $O_{F,V}(p)$ is finite; that is, $F\in \mathrm{Diff}_{<\infty}(\mathbb{C}^n,0)$. This concludes the proof. □

We say that a biholomorphism $F\in \mathrm{Diff}(\mathbb{C}^n,0)$ is periodic if it is a finite order element of the group $\mathrm{Diff}(\mathbb{C}^n,0)$, that is, if there exists $m\in\mathbb{N}$ such that $F^m=\mathrm{id}$. In dimension $n=1$, Mattei and Moussu proved in [MM80] that the finite orbits property is equivalent to periodicity. For dimension $n>1$, the equivalence is far from be true. For example, the biholomorphism $F(x,y)=(x,x+y)$ is non-periodic, but has finite orbits in each bounded neighborhood of $0\in\mathbb{C}^2$: the line $\{x=0\}$ is the set of fixed points of F and F is a non-trivial translation on $\{x=c\}$ with $c\neq 0$.

Remark 1. The subset $\mathrm{Diff}_{<\infty}(\mathbb{C},0)$ of $\mathrm{Diff}\,(\mathbb{C},0)$ is not a subgroup. For example, the biholomorphisms $F(x)=-x$ and $G(x)=-\frac{x}{1-x}$ are periodic of period 2 and, hence, belong to $\mathrm{Diff}_{<\infty}(\mathbb{C},0)$. However, the composition

$$H(x)=(F\circ G)(x)=\frac{x}{1-x}$$

is not periodic, since $H^n(x)=\frac{x}{1-nx}$, $n\in\mathbb{N}$. On the other hand, it is easy to check that the subset of $\mathrm{Diff}\,(\mathbb{C},0)$ formed by the linear isomorphisms with finite orbits is a subgroup of $\mathrm{Diff}\,(\mathbb{C},0)$, isomorphic to the group of roots of unity. Nevertheless, this does not hold for dimension greater than 1 (cf. Corollary 2)

Proposition 2 (Linear case). *Let $F\in \mathrm{Diff}\,(\mathbb{C}^n,0)$ be analytically linearizable. Then F has finite orbits if and only if its eigenvalues are roots of unity. Furthermore, if m is the least positive integer such that F^m is unipotent then every periodic point of F is a fixed point of F^m.*

Proof. Applying Proposition 1 (i), we can assume $F(x)=Ax$, where $A\in\mathrm{GL}(n,\mathbb{C})$. Let $\lambda\in\mathrm{Spec}(A)$ and let B be an arbitrary ball centered at 0. If $v\in B$ is a eigenvector of F associated with λ, then $F^m(v)=\lambda^m v$ for all $m\in\mathbb{Z}$. In particular, $O_{F,B}(v)$ is finite if and only if λ is a root of unity. Hence, the finite orbits property implies that all eigenvalues of F are roots of unity.

Reciprocally, suppose that the eigenvalues of F are roots of unity. It suffices to show that $\mathbb{C}^n$ admits a decomposition $\mathbb{C}^n=V_1\oplus\cdots\oplus V_r$ as F-invariant subspaces such that, for each $x_j\in V_j$, the orbit $\mathcal{O}_{F,U}(x_j)$ is finite for any bounded set U and any $x_j \in U \cap V_j$. Thus, it suffices to consider the case where A is a Jordan block

$$A = \begin{bmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ & 1 & \ddots & & \\ & & \ddots & \lambda & \\ & & & 1 & \lambda \end{bmatrix},$$

where λ is a root of unity. We can assume $n > 1$ since the remaining case is trivial. Let $x = (x_1, ..., x_n) \in \mathbb{C}^n$. By using induction on m it is easy to see that

$$F^m(x) = (\lambda^m x_1, m\lambda^{m-1}x_1 + \lambda^m x_2, \ ..., \quad P_{n,m}(x_1, ..., x_{n-2}) + m\lambda^{m-1}x_{n-1} + \lambda^m x_n),$$

with $P_{j,m}$ linear for all $j, m \in \mathbb{N}$. In other words, if π_j is the jth projection on $\mathbb{C}^n$, then

$$\begin{array}{rcl} \pi_1(F^m(x)) & = & \lambda^m x_1, \\ \pi_2(F^m(x)) & = & m\lambda^{m-1}x_1 + \lambda^m x_2, \\ \pi_3(F^m(x)) & = & P_{3,m}(x_1) + m\lambda^{m-1}x_2 + \lambda^m x_3 \\ & \vdots & \\ \pi_n(F^m(x)) & = & P_{n,m}(x_1, ..., x_{n-2}) + m\lambda^{m-1}x_{n-1} + \lambda^m x_n. \end{array}$$

Consider $x \neq 0$. Let j_0 be the first index such that $x_{j_0} \neq 0$. If $j_0 = n$ then x is periodic; so it has a finite orbit. Consider $j_0 < n$. Then

$$|\pi_{j_0+1}(F^m(x))| = |m\lambda^{m-1}x_{j_0} + \lambda^m x_{j_0+1}| \to \infty \ \text{ when } \ m \to \infty.$$

In any case, we see that x has finite positive orbit in any bounded neighborhood of 0. Since the eigenvalues of A^{-1} are the inverses of the eigenvalues of A, we show that every negative orbit of x is finite analogously. This shows that F has finite orbits. Moreover, if m is the least positive integer such that F^m is unipotent, the discussion above shows that for a Jordan block the set of periodic points coincide with $\text{Fix}(F^m)$. Therefore, $\text{Fix}(F^m)$ is the set of periodic points of F. ☐

Corollary 2. *Suppose $n \geq 2$. Then the subset of* $\text{Diff}\,(\mathbb{C}^n, 0)$ *consisting of finite orbits linear biholomorphisms is not a subgroup of* $\text{Diff}\,(\mathbb{C}^n, 0)$.

Proof. Let

$$F(x_1, ..., x_n) = (x_1, x_1 + x_2, x_3, ..., x_n) \ \text{ and } \ G(x_1, ..., x_n) = (x_1 + x_2, x_2, x_3, ..., x_n).$$

Then F and G are linear and have finite orbits, since they are unipotent. However, $F \circ G = (x_1 + x_2, x_1 + 2x_2, x_3, ..., x_n) \notin \text{Diff}_{<\infty}(\mathbb{C}^n, 0)$, since $\frac{3+\sqrt{5}}{2}$ and $\frac{3-\sqrt{5}}{2}$ are eigenvalues of $F \circ G$. ☐

2.1. **Formal diffeomorphisms.** We denote by $\mathcal{O}_{n,0}$ (resp. $\hat{\mathcal{O}}_{n,0}$) the local ring of convergent (resp. formal) power series with complex coefficients in n variables centered at the origin of $\mathbb{C}^n$. Let $\mathfrak{m}$ be the maximal ideal of $\hat{\mathcal{O}}_{n,0}$.

Definition 2. The group $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ of formal diffeomorphisms consists of the elements $F=(F_1,\ldots,F_n)$ of $\mathfrak{m}\times\ldots\times\mathfrak{m}$ such that its first jet D_0F belongs to $\mathrm{GL}(n,\mathbb{C})$.

Remark 2. *As a consequence of the inverse function theorem, we can identify* $\mathrm{Diff}\,(\mathbb{C}^n,0)$ *with the subset of* $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ *of formal diffeomorphisms* $F=(F_1,\ldots,F_n)$ *such that* $F_j\in\mathfrak{m}\cap\mathcal{O}_{n,0}$ *for every* $1\le j\le n$.

Definition 3. We denote by $\widehat{\mathrm{Diff}}_u(\mathbb{C}^n,0)$ the set of *unipotent* formal biholomorphisms, that is, the elements $F\in\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ such that $\mathrm{Spec}(D_0F)=\{1\}$. Its subset $\widehat{\mathrm{Diff}}_1(\mathbb{C}^n,0):=\{F\in\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0) : D_0F=\mathrm{id}\}$ is called the group of *tangent to the identity* formal diffeomorphisms in n variables.

2.2. **Vector fields and flows.** Let $\mathfrak{X}(\mathbb{C}^n,0)$ denote the Lie algebra of singular local holomorphic vector fields at $0\in\mathbb{C}^n$. Let $X\in\mathfrak{X}(\mathbb{C}^n,0)$ and denote by $\phi^t(z)$ its local flow, defined in a neighborhood of $\{0\}\times\mathbb{C}^n$ in $\mathbb{C}^{n+1}$. Then, for each $t\in\mathbb{C}$, the map $z\mapsto\phi^t(z)$ is defined in a neighborhood U_t of 0 in $\mathbb{C}^n$ and hence defines a biholomorphism $\phi^t\in\mathrm{Diff}(\mathbb{C}^n,0)$, the so called *time-t* map of X, also denoted by $\exp(tX)$. We also use $\exp(1X)=\exp(X)$ to denote the time-1 map of X. It turns out that if $f\in\mathcal{O}_{n,0}$ then

$$f\circ\exp(tX)(z)=f(z)+\sum_{j=1}^{\infty}\frac{t^j}{j!}X^j(f)$$

by Taylor's formula, where X is now understood as a derivation in the ring $\mathcal{O}_{n,0}$ and $X^j=X\circ X^{j-1}$. By considering $f=z_j,\ j=1,...,n$, we obtain

$$\exp(tX)(z_1,...,z_n)=\left(z_1+\sum_{j=1}^{\infty}\frac{t^j}{j!}X^j(z_1),\ \ldots,z_n+\sum_{j=1}^{\infty}\frac{t^j}{j!}X^j(z_n)\right).$$

This last identity allows us to extend the definition of flow associated to a germ of holomorphic singular vector field to a formal singular vector field $X\in\widehat{\mathfrak{X}}(\mathbb{C}^n,0)$, i.e. a derivation of the ring of formal power series that preserves the maximal ideal. Indeed $\exp(tX)\in\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ for all $X\in\widehat{\mathfrak{X}}(\mathbb{C}^n,0)$ and $t\in\mathbb{C}$. We say that two vector fields $X,Y\in\mathfrak{X}\,(\mathbb{C}^n,0)$ are analytically equivalent if there exists $H\in\mathrm{Diff}(\mathbb{C}^n,0)$ such that $H_*X=Y$, that is

$$D_xH\cdot X(x)=Y(H(x))$$

for any x in a neighborhood of the origin. The map H is called a *conjugacy* between X and Y. In this case, one could show that H is also a *conjugacy* between their time-t maps for any $t\in\mathbb{C}$.

Definition 4. *We say that a singular vector field* $X \in \mathfrak{X}(\mathbb{C}^n, 0)$ *has finite orbits (denoting by* $X \in \mathfrak{X}_{<\infty}(\mathbb{C}^n, 0)$*) if* $\exp(X) \in \mathrm{Diff}_{<\infty}(\mathbb{C}^n, 0)$.

Now we derive some properties of finite orbits vector fields.

Corollary 3. *Let* $X \in \mathfrak{X}(\mathbb{C}^n, 0)$ *be a singular vector field.*

(i) *If* $Y \in \mathfrak{X}(\mathbb{C}^n, 0)$ *is singular and analytically conjugated to* X *then*

$$X \in \mathfrak{X}_{<\infty}(\mathbb{C}^n, 0) \Leftrightarrow Y \in \mathfrak{X}_{<\infty}(\mathbb{C}^n, 0).$$

(ii) *The following statements are equivalents.*
 (a) X *has finite orbits.*
 (b) mX *has finite orbits for all* $m \in \mathbb{N}$.
 (c) mX *has finite orbits for some* $m \in \mathbb{N}$.

Proof. (i): As $\exp(X)$ is conjugated to $\exp(Y)$, we can apply Proposition 1 (i).
(ii) Set $F = \exp(X) \in \mathrm{Diff}(\mathbb{C}^n, 0)$. Then

$$F^m = \exp(mX) \quad \text{for all} \quad m \in \mathbb{Z}.$$

Then we can apply Proposition 1 (ii). □

Remark 3. The last corollary implies that the $\mathbb{Z}$-multiples of X have finite orbits. However, in general such a property is not satisfied for $\mathbb{R}$ or $\mathbb{C}$-multiples of X. In fact, the time-1 map of the one-dimensional field $X = \lambda x \frac{\partial}{\partial x}$ is $f = e^\lambda x$, $\lambda \in \mathbb{C}$. Since in dimension one finite orbits is equivalent to finite order, it follows that

$$X \in \mathfrak{X}_{<\infty}(\mathbb{C}, 0) \Leftrightarrow f \text{ has finite order } \Leftrightarrow \lambda \in 2\pi i \mathbb{Q}.$$

Corollary 4. *Let* $X \in \mathfrak{X}(\mathbb{C}^n, 0)$ *be an analytically linearizable vector field. Then*

$$X \in \mathfrak{X}_{<\infty}(\mathbb{C}^n, 0) \Leftrightarrow \mathrm{Spec}(D_0 X) \subset 2\pi i \mathbb{Q}.$$

Proof. The time-1 map $F := \exp(X)$ is linearizable and its eigenvalues have the form e^λ with $\lambda \in \mathrm{Spec}(D_0 X)$. The proof now is a consequence of Proposition 2. □

Example 1. Consider the vector field $X = x(2\pi i + y)\dfrac{\partial}{\partial x}$. Then we have

$$\exp(X) = (e^{2\pi i + y} x, y) = (e^y x, y).$$

Moreover, in each level $\{y = c\}$ with $c \notin 2\pi i \mathbb{Q}$ every point $x \neq 0$ has an infinite orbit. Since the linear part $X_0 = 2\pi i x \frac{\partial}{\partial x}$ has finite orbits, it follows that X is not analytically linearizable.

2.3. **Semisimple/nilpotent decomposition of vector fields.** We say that a singular vector field $X \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ is semisimple if it is formally conjugated to a vector field of the form $\sum_{j=1}^{n} \lambda_j x_j \frac{\partial}{\partial x_j}$. We say that X is nilpotent if the linear part of X is nilpotent. Finally, we say that

$$X = X_S + X_N$$

is the semisimple/nilpotent decomposition of X if X_S is semisimple, X_N is nilpotent and $[X_N, X_S] = 0$. Every singular formal vector field X admits a unique semisimple/nilpotent decomposition (cf. [Mar81]). We denote by $\hat{\mathfrak{X}}_N(\mathbb{C}^n, 0)$ the subset of $\hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ consisting of the formal nilpotent vector fields.

Proposition 3 (cf. [Éca75, MR83]). *The image of $\hat{\mathfrak{X}}_N(\mathbb{C}^n, 0)$ by the exponential aplication is $\widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$ and the map $\exp : \hat{\mathfrak{X}}_N(\mathbb{C}^n, 0) \to \widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$ is a bijection.*

2.4. **Poincaré-Dulac normal form.** First, let us recall that a point $\lambda = (\lambda_1, ..., \lambda_n)$ of $\mathbb{C}^n$ is said *resonant* if there exists $m \in \mathbb{N}_0^n$ with $|m| \geq 2$ and some $1 \leq k \leq n$ such that

$$\langle \lambda, m \rangle = \lambda_1 m_1 + \cdots + \lambda_n m_n = \lambda_k.$$

This equation is called a resonance and its *resonant monomial* is the vector field

$$F_{k,m} = x^m e_k = (0, ..., x_1^{m_1} \cdots x_n^{m_n}, ...0)$$

Note that $\lambda = (\lambda_1, ..., \lambda_n)$ is resonant if and only if $(\lambda_{\sigma(1)}, ..., \lambda_{\sigma(n)})$ is resonant for every permutation of the set of indexes $\{1, 2, ..., n\}$. Keeping this in mind, we say that a singular vector field $X \in \mathfrak{X}(\mathbb{C}^n, 0)$ is resonant if the the n-tuple formed by the eigenvalues of the linearization matrix D_0X of X is resonant. Finally, we say that a singular vector field $X \in \mathfrak{X}(\mathbb{C}^n, 0)$ is in the Poincaré domain if 0 does not belong to the convex hull of $\mathrm{Spec}(D_0X) = \{\lambda_1, ..., \lambda_n\}$, that is, there is no choice of $t_j \in [0, 1]$ for $1 \leq j \leq n$ such that

$$t_1 + \cdots + t_n = 1 \quad \text{and} \quad t_1\lambda_1 + \cdots + t_n\lambda_n = 0.$$

Poincaré-Dulac normal form (cf. [IY08] p. 62). *Let $X \in \mathfrak{X}(\mathbb{C}^n, 0)$ be a singular vector field in the Poincaré domain. Then X has only finitely many resonances and it is analytically conjugated to*

$$Ax + \sum c_{k,m} F_{k,m},$$

where A is in Jordan normal form, $c_{k,m} \in \mathbb{C}$ and the $F_{k,m}$ are the resonant monomials of X. In particular, if X is in the Poincaré domain and it has no resonances, then X is analytically linearizable.

2.5. **Stable manifold theorem.** We denote by (W, p) the germ of an analytic variety at a point p. A germ of analytic variety $W \subset (\mathbb{C}^n, 0)$ is said to be invariant by $F \in \mathrm{Diff}(\mathbb{C}^n, 0)$ if the germs of W and $F(W)$ coincide at the origin. Additionally, we say that W is stable if there exists a neighborhood U of 0 where F and W are defined and satisfy

(1) $F(U \cap W) \subset U \cap W$ and
(2) for each $x \in U \cap W$, the positive orbit $(F^m(x))_{m \in \mathbb{N}}$ converges to zero.

Analogously, a variety $W \subset (\mathbb{C}^n, 0)$ is invariant by $X \in \mathfrak{X}(\mathbb{C}^n, 0)$ if $X(x)$ is tangent to W at x, for every regular point $x \in (W, 0)$. In particular, W is invariant by $\exp(X)$. We say that W is stable by X if is stable for its real flow, i.e. for any $x \in U \cap W$, we have $\exp(tX)(x) \in U \cap W$ for any $t \in \mathbb{R}^+$, and $\lim_{t \to \infty} \exp(tX)(x) = 0$.

Holomorphic Stable Manifold Theorem for Diffeomorphisms (cf. [Rue89, p. 26], [IY08, p.107]). *Let $F \in \mathrm{Diff}(\mathbb{C}^n, 0)$ and $\rho \in (0, 1]$. Let*

$$L^- = \oplus_{\lambda \in A_\rho^-} \ker(D_0F - \lambda \mathrm{id})^n$$

be the sum of the generalized eigenspaces associated to the eigenvalues of D_0F in

$$A_\rho^- = \{\lambda \in \mathrm{Spec}(D_0F);\ |\lambda| < \rho\}.$$

Then there exists a unique F-stable manifold W^- whose tangent space at 0 is L^-.

Corollary 5. *Let $F \in \mathrm{Diff}_{<\infty}(\mathbb{C}^n, 0)$. Then $|\lambda| = 1$ for any $\lambda \in \mathrm{Spec}(D_0F)$.*

Proof. Suppose F has a eigenvalue λ such that $|\lambda| \neq 1$. So, up to change F by F^{-1}, we can suppose $|\lambda| < 1$. Thus F admits a stable manifold $W^- \neq \{0\}$ invariant by F and associated to A_1^-. Hence, the points of W^- close to 0 have infinite orbits. □

Holomorphic Stable Manifold Theorem for Vector Fields (cf. [CS14]). *Let $X \in \mathfrak{X}(\mathbb{C}^n, 0)$ be a singular vector field and $\theta \in \mathbb{R}_{<0}$. Suppose*

$$S_\theta^- := \{\lambda \in \mathrm{Spec}(D_0X);\ \mathrm{Re}(\lambda) \leq \theta\} \neq \emptyset$$

and denote by $L_\theta^- = \oplus_{\lambda \in S_\theta^-(X)} \ker(D_0X - \lambda \mathrm{id})^n$ the direct sum of the generalized eigenspaces associated to the eigenvalues in $S_\theta^-(X)$. Then X admits a unique germ of stable manifold W_θ^- whose tangent space at 0 is L_θ^-.

3. Finite orbits and one parameter groups

In this section we show Theorem 2.

Proof. Suppose that X has finite orbits. Applying the stable manifold theorem for X we conclude that $\mathrm{Spec}(D_0X) \subset i\mathbb{R}$. The proof now follows by induction on n.

If $n = 1$, then the finite orbits property is equivalent to periodicity and hence $\mathrm{Spec}(D_0X) = \{\lambda\} \subset 2\pi i\mathbb{Q}$.

Now, suppose $n \geq 2$ and assume the theorem holds in dimension less than n. Set

$$A_{>0} = \{\lambda \in \mathrm{Spec}(D_0X); \lambda \in i\mathbb{R}_{>0}\} \quad \text{and} \quad A_{<0} = \{\lambda \in \mathrm{Spec}(D_0X); \lambda \in i\mathbb{R}_{<0}\}.$$

Suppose $A_{>0} \neq \emptyset$. Let us prove that $A_{>0} \subset 2\pi i\mathbb{Q}$.

Setting $\tilde{X} = iX$, we obtain a vector field whose complex trajectories coincide with those of X, both interpreted as sets. The subset of $D_0\tilde{X}$ consisting of eigenvalues with negative real part is $iA_{>0}$. Therefore, $\tilde{X}$ admits a stable manifold V_0, which is invariant by X, such that $\mathrm{Spec}(D_0X_0) = A_{>0}$, where $X_0 = X_{|V_0}$. Since X has finite orbits so does X_0. If $A_{>0} \neq \mathrm{Spec}(D_0X)$, then $\dim V_0 < n$ and so by induction hypothesis we have $A_{>0} \subset 2\pi i\mathbb{Q}$. Therefore, we can suppose $A_{>0} = \mathrm{Spec}(D_0X)$ and V_0 is an open subset of $\mathbb{C}^n$. Hence $X = X_0$ is in the Poincaré domain. If X has no resonances then X is analytically linearizable and we have $A_{>0} \subset 2\pi i\mathbb{Q}$ by Corollary 4. Suppose, then, that X admits resonances. It follows that $k := \sharp(\mathrm{Spec}(D_0X))$ is greater than 1. Set $\mathrm{Spec}(D_0X) = \{\lambda_1, ..., \lambda_k\}$ with $\mathrm{Im}\,(\lambda_1) > \cdots > \mathrm{Im}\,(\lambda_k)$. By applying the Stable Manifold Theorem to iX and $\theta = i\lambda_1$, $\theta = i\lambda_2, \ldots, \theta = i\lambda_k$, we find invariant manifolds $V_1, V_2, \ldots, V_{k-1}, V_k = V_0$ such that $\mathrm{Spec}(D_0X_j) = \{\lambda_1, \ldots, \lambda_j\}$ where $X_j := X|_{V_j}$ for all $1 \leq j \leq k$. Consequently, X_1 has no resonances and so is analytically linearizable. It follows from Corollary 4 that $\lambda_1 \in 2\pi i\mathbb{Q}$.

Suppose that $\lambda_1, ..., \lambda_l \in 2\pi i\mathbb{Q}$, $l < k$. We are going to show that $\lambda_{l+1} \in 2\pi i\mathbb{Q}$. Since $\mathrm{Im}\,(\lambda_1) > \cdots > \mathrm{Im}\,(\lambda_k)$, it follows that X_{l+1} is linearizable (in which case $\lambda_1, ..., \lambda_{l+1} \in 2\pi i\mathbb{Q}$) or the possible resonances of X_{l+1} have the form

$$\lambda_j = |M_{j+1}|\lambda_{j+1} + \cdots + |M_{l+1}|\lambda_{l+1}, \quad \sum_{k=j+1}^{l+1} |M_k| \geq 2,$$

where $M_j \in \mathbb{Z}_{\geq 0}^{n_j}$, and n_j is the algebraic multiplicity of λ_j for $j = 1, ..., l$. By using the Poincaré-Dulac normal form, we have coordinates $x = (x_1, ..., x_{l+1}) \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_{l+1}} = \mathbb{C}^m$ such that $X_{l+1} \sim Ax + (*, *, ..., *, 0)$, where the first spot corresponds to the n_1 first coordinates, the second spot to the next n_2 coordinates and so on. If there exists some resonance with $M_{l+1} \neq 0$, then we can write λ_{l+1} as a combination of $\lambda_1, ..., \lambda_l$ with rationals coefficients, that is, $\lambda_{l+1} \in 2\pi i\mathbb{Q}$. If all resonances satisfies $M_{l+1} = 0$, then $X_{l+1} \sim Ax + (*, *, ...*, 0, 0)$ and we see that the manifold $W = \{x_l = 0\} \subset \mathbb{C}^m$ has dimension $n_1 + n_2 + ... + n_{l-1} + n_{l+1}$, is invariant by X_{l+1} and the restriction $X|_W$ has eigenvalues $\lambda_1, ..., \lambda_{l-1}, \lambda_{l+1}$. By hypothesis of induction, we see that $\lambda_1, ..., \lambda_{l-1}, \lambda_{l+1} \in 2\pi i\mathbb{Q}$. This shows that $A_{>0} \subset 2\pi i\mathbb{Q}$.

Analogously, $A_{<0}$ is contained in $2\pi i\mathbb{Q}$ and hence $\mathrm{Spec}(D_0X) \subset 2\pi i\mathbb{Q}$. □

The inverse proposition of the above theorem is not true, even in dimension one, as we can see in the next example.

Example 2. The vector field $X(z) = z^2\frac{\partial}{\partial z}$ has spectrum $\mathrm{Spec}(D_0X) = \{0\}$, but it is not a finite orbits germ, since $\exp(X) = z + z^2 + O(z^3)$ is clearly non-periodic.

4. Fixed Point Curve Theorem

In this section we prove that if $F \in \mathrm{Diff}_{<\infty}(\mathbb{C}^2, 0)$ then some iterate F^m admits a curve of fixed points at 0. First, we use constructions and ideas featuring in Mattei and Moussu [MM80], Rebelo and Reis [RR15] and Pérez-Marco [PM97] to show that there is a non-trivial continuum K containing the origin and satisfying $F(K) = K$. The set K consists of periodic points of F and can be obtained as a limit of compact sets where we consider the Hausdorff topology on the compact subsets of $\overline{B}$. Finally, we will use the theory of semianalytic sets (see [Loj64] [BM88]) to show that the continuum K is contained in an analytic curve which is invariant by some iterate of F.

4.1. **Continua.** For the sake of simplicity, we recall in this section the Sierpiński theorem and the Hausdorff topology on compact sets.

A topological space X is called a *continuum* if X is both connected and compact. The next result will be a key ingredient in the description of the connected components of the set of periodic points of a finite orbits local biholomorphism.

Sierpiński Theorem (see [Eng89, p.358])**.** *Let X be a continuum that has a countable cover $\{X_j\}_{j=1}^{\infty}$ by pairwise disjoint closed subsets. Then at most one of the sets X_j is non-empty.*

Now, we define the Hausdorff topology. Let (M, d) be a metric space and denote by $H(M)$ the space of bounded, non-empty closed subsets of M. Note that $H(M)$ is the set of compact subsets of M if M is compact. We define the Hausdorff metric $\rho : H(M) \times H(M) \to [0, \infty)$ by

$$\rho(A, B) = \max\{\sup_{x \in A} d(x, B),\ \sup_{y \in B} d(A, y)\}.$$

Consider $A \subset M$ and $\varepsilon > 0$. We define the ε-neighborhood of A by

$$V_\varepsilon(A) = \bigcup_{x \in A} B_\varepsilon(x),$$

where $B_\epsilon(x) = \{y \in M; d(y, x) < \epsilon\}$. The Hausdorff metric satisfies

$$\rho(A, B) = \inf\{\varepsilon > 0;\ V_\varepsilon(A) \supset B \text{ and } V_\varepsilon(B) \supset A\}$$

for all $A, B \in H(M)$. Moreover, the metric space $(H(M), \rho)$ is compact if M is compact ([Nad92, Th. 4.13, p. 59]). We assume that M is compact from now on. Given a sequence $(A_n)_{n\geq 1}$ of subsets of M, let $\liminf A_n$ be the set of points $x \in M$ such that any neighborhood of x intersects A_n for all but finitely many n. We define $\limsup A_n$ as the set of points $x \in M$ such that any neighborhood of x intersects infinitely many of the sets in the sequence $(A_n)_{n\geq 1}$. Both sets are compact and $\liminf A_n \subset \limsup A_n$. Given a sequence $(K_n)_{n\geq 1}$ of compact subsets of M, the sequence converges in the Hausdorff topology to K if and only if

$$\liminf K_n = \limsup K_n = K,$$

see [Nad92, Th. 4.11, p. 57]. Moreover, the subset of $H(M)$ consisting of continua is compact if M is compact (cf. [Nad92, Th. 4.17, p. 61]).

4.2. **Invariant curves.** A formal curve at $0 \in \mathbb{C}^2$ is a proper radical ideal $\Gamma = (f)$ of $\mathbb{C}[[x,y]]$. Such a condition is equivalent to f being reduced, i.e. f has no multiple irreducible factors.

(i) We say that Γ is invariant by $F \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^2,0)$ if $\Gamma \circ F = \Gamma$, i.e., f divides $f \circ F$.
(ii) We say that Γ is a fixed point curve of $F \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^2,0)$ if f divides $x \circ (F - \mathrm{id})$ and $y \circ (F - \mathrm{id})$.
(iii) We say that Γ is invariant by $X \in \hat{\mathfrak{X}}\,(\mathbb{C}^2,0)$ if $X(\Gamma) \subset \Gamma$, i.e., f divides $X(f)$.
(iv) We say that Γ is a singular curve of X if f divides X.

In the case where the curve $\Gamma = (f)$ is a radical ideal of $\mathbb{C}\{x,y\}$, we identify it with the germ of analytic set $V_\Gamma = (f = 0)$ and the conditions above coincide with the natural ones for $F \in \mathrm{Diff}\,(\mathbb{C}^2,0)$ and $X \in \mathfrak{X}\,(\mathbb{C}^2,0)$:

(i) We have the equality $F(V_\Gamma) = V_\Gamma$ of germs of analytic sets at 0.
(ii) $F|_{V_\Gamma} = \mathrm{id}$.
(iii) X is tangent to V_Γ at any of its regular points.
(iv) $X|_{V_\Gamma} = 0$.

Lemma 1 (cf. [Rib05]). *Let $X \in \hat{\mathfrak{X}}_N(\mathbb{C}^2,0)$ and let Γ be a formal curve at* 0.

(a) Γ *is invariant by X iff Γ is invariant by* $\exp(X)$.
(b) Γ *is singular curve of X iff Γ is a fixed point curve of* $\exp(X)$.

Lemma 2. *Let $V, W \subset (\mathbb{C}^2,0)$ be different germs of non-trivial analytic sets. Consider $\psi \in \mathrm{Diff}\,(\mathbb{C}^2,0)$ such that $V, W \subset \mathrm{Fix}(\psi)$. Then ψ is tangent to identity.*

Proof. We have $V \neq \{0\} \neq W$ and $V \neq W$ by hypothesis. If $V = (\mathbb{C}^2,0)$ or $W = (\mathbb{C}^2,0)$ then $\psi = id$. Therefore, we can suppose that V and W are analytic curves with reduced equation $f = 0$ and $g = 0$, respectively, at 0. So both f and g divide $x \circ \psi - x$ and $y \circ \psi - y$. If $\mathrm{ord}(f) \geq 2$ or $\mathrm{ord}(g) \geq 2$, then the first jet $J^1\psi$ of ψ is equal to Id. So we can assume $\mathrm{ord}(f) = \mathrm{ord}(g) = 1$. Since $V \neq W$, we deduce $fg|x \circ \psi - x$ and $fg|y \circ \psi - y$. Again, we obtain $J^1\psi = Id$. This concludes the proof. □

4.3. **Connected components of the set of periodic points.** Let us assume that $F \in \mathrm{Diff}_{<\infty}(\mathbb{C}^2,0)$, U is a neighborhood of 0 in which F is defined and injective and fix a closed ball $\overline{B}_\rho(0)$ such that $\overline{B}_\rho(0) \subset U$. Let us also set

(1) $$\mathrm{Per}_k(F) := \{p \in B_\rho(0);\ F(p), ..., F^{k-1}(p) \in B_\rho(0) \text{ and } F^k(p) = p\}$$

and

$$\overline{\mathrm{Per}}_k(F) := \{p \in \overline{B}_\rho(0);\ F(p), ..., F^{k-1}(p) \in \overline{B}_\rho(0) \text{ and } F^k(p) = p\}$$

for $k \in \mathbb{N}$ and

$$\mathrm{Per}(F) = \bigcup_{k\in\mathbb{N}} \mathrm{Per}_k(F), \quad \overline{\mathrm{Per}}(F) = \bigcup_{k\in\mathbb{N}} \overline{\mathrm{Per}}_k(F). \tag{2}$$

Lemma 3. *There is a subset $K \subset \overline{\mathrm{Per}}(F)$ with the following properties.*

1. *K is a continuum;*
2. *$0 \in K$ and $K \cap \partial B_\rho(0) \neq \emptyset$;*
3. *$F(K) = K$*

Proof. We have $\mathrm{Spec}(D_0F) \subset S^1$ as a consequence of the Stable Manifold Theorem. For each $n \in \mathbb{N}$, set $F_n = e^{-1/n}F$. Thus, F_n is a biholomorphism with $\mathrm{Spec}(D_0F_n) \subset \{\lambda \in \mathbb{C};\ |\lambda| < 1\}$.

Claim. We denote $B_r = B_r(0)$ and $\overline{B}_r = \overline{B}_r(0)$ for $r \in \mathbb{R}^+$ and $B = B_\rho$, $\overline{B} = \overline{B}_\rho$. There are sequences (k_n) of positive integer numbers and (r_n) of positive real numbers such that:

(1) $\lim_{n\to\infty} r_n = 0$ and the closed ball $\overline{B}_{r_n}$ satisfies

$$\cup_{j=1}^{\infty} F_n^j(\overline{B}_{r_n}) \subset \overline{B};$$

(2) $\overline{B}_{r_n}, F_n^{-1}(\overline{B}_{r_n}), ...,\ F_n^{-k_n}(\overline{B}_{r_n}) \subset \overline{B}$ and $F_n^{-k_n}(\overline{B}_{r_n}) \cap \partial B \neq \emptyset$.

Let us assume this for a moment to prove the lemma. We define

$$V_n = \overline{\cup_{j\geq -k_n} F_n^j(\overline{B}_{r_n})}.$$

Then V_n is connected since it is the closure of a union of connected sets that have the origin as a common point. Thus, V_n is a continuum contained in $\overline{B}$ such that $F_n^j(V_n) \subset V_n$ for all $j \geq 0$ and there exists $p_n \in V_n \cap \partial B$. Passing to a subsequence if necessary, we can assume that $V_n \to K$, in the Hausdorff topology of compact subsets of $\overline{B}$, and also $p_n \to p \in K \cap \partial B$. Since $\{0, p\} \subset \limsup V_n$, K is a continuum containing the origin such that $K \cap \partial B \neq \emptyset$. Since $(F_n)_{n\geq 1}$ converges to F uniformly in $\overline{B}$, it is easy to check that

$$F(\liminf V_n) \subset \liminf F_n(V_n) \quad \text{and} \quad F(\limsup V_n) = \limsup F_n(V_n).$$

Since $F_n(V_n) \subset V_n$ for every $n \in \mathbb{N}$ and $V_n \to K$, we deduce that

$$F(K) = F(\limsup V_n) = \limsup F_n(V_n) \subset \limsup V_n = K.$$

Therefore, since F has finite orbits, we obtain $K \subset \overline{\mathrm{Per}}(F)$. In particular, K is contained in the image of $F|_K$ and hence $F(K) = K$.

Proof of the claim.

Let us construct $0 < r_n < 1/n$ and k_n. Since the origin is an attractor for F_n, there exists $R \in (0, 1/n)$ such that the closed ball $\overline{B}_R$ is contained in the basin of attraction of 0 and satisfies $\cup_{j=1}^{\infty} F_n^j(\overline{B}_R) \subset \overline{B}$. We claim that there exists $k_n \in \mathbb{N}$ such that

$$F_n^{-1}(\overline{B}_R) \cup \ldots \cup F_n^{-(k_n-1)}(\overline{B}_R) \subset \overline{B} \quad \text{and} \quad F_n^{-k_n}(\overline{B}_R) \setminus \overline{B} \neq \emptyset.$$

Assume, aiming at contradiction, that no such k_n exists. Denote $A = D_0 F_n^{-1}$ and $A^k = (a_{ij;k})_{1 \leq i,j \leq 2}$ for $k \in \mathbb{Z}$. We have

$$a_{11;k} = \frac{1}{(2\pi i)^2} \int_{|x|=|y|=\frac{R}{2}} \frac{x \circ F_n^{-k}}{x^2 y} dxdy \implies |a_{11;k}| \leq \frac{1}{(2\pi)^2} (\pi R)^2 \rho \left(\frac{2}{R} \right)^3 = \frac{2\rho}{R}$$

for any $k \in \mathbb{N}$. Analogously, we obtain $|a_{12;k}| \leq 2\rho/R$, $|a_{21;k}| \leq 2\rho/R$ and $|a_{22;k}| \leq 2\rho/R$ for any $k \in \mathbb{N}$. We proved that the sequence $(A^k)_{k \geq 1}$ is bounded, contradicting $\operatorname{spec}(A) \subset \{z \in \mathbb{C}; |z| > 1\}$.

By defining

$$r_n = \inf\{s \in (0, R); F_n^{-k_n}(\overline{B}_s) \setminus \overline{B} \neq \emptyset\},$$

we obtain k_n and r_n satisfying the desired properties. □

Definition 5. *Let M be a real analytic manifold. A subset X of M is semianalytic if each $p \in M$ has a neighborhood V such that $X \cap V$ has the form*

$$X \cap V = \bigcup_{i=1}^{m} \bigcap_{j=1}^{n} X_{ij},$$

where $X_{ij} = \{f_{ij} = 0\}$ or $X_{ij} = \{f_{ij} > 0\}$ with f_{ij} real analytic on V.

Remark 4. Notice that every (real or complex) analytic set X is semianalytic with $m = 1$ and $X_{ij} = \{f_{ij} = 0\}$. Moreover, since $\{f_{ij} < 0\} = \{-f_{ij} > 0\}$, $\{f_{ij} \geq 0\} = \{f_{ij} = 0\} \cup \{f_{ij} > 0\}$ and $\{f_{ij} \leq 0\} = \{f_{ij} = 0\} \cup \{f_{ij} < 0\}$, we can add these types of sets to the possible options for X_{ij} to obtain an alternative definition.

Example 3. For each $r > 0$ the closed ball $\overline{B}_r(0)$ is semi-analytic in $\mathbb{C}^n$, since it can be written as $\overline{B}_r(0) = \{f \geq 0\}$, where $f(x) = r^2 - |x_1|^2 - \cdots - |x_n|^2$.

Lemma 4. *Under the hypotheses above, each set $\overline{\operatorname{Per}_k}(F)$ is semianalytic in U, has finitely many connected components, and each of its connected components is semianalytic and path-connected.*

Proof. We denote $\mathcal{P}_k = \overline{\operatorname{Per}_k}(F)$. Let $p \in \mathcal{P}_k$. The local biholomorphism F^k is well-defined in some neighborhood V of p. Moreover, we have

$$\mathcal{P}_k \cap V = \left(\cap_{j=1}^{4} \{f_j = 0\} \right) \cap \left(\cap_{l=0}^{k-1} \{f \circ F^l \geq 0\} \right),$$

where $f_1 = \operatorname{Re}(x \circ F^k - x)$, $f_2 = \operatorname{Im}(x \circ F^k - x)$, $f_3 = \operatorname{Re}(y \circ F^k - y)$, $f_4 = \operatorname{Im}(y \circ F^k - y)$, and $f(x, y) = r^2 - |x|^2 - |y|^2$. Therefore, $\mathcal{P}_k$ is semianalytic. Now, by Corollary 2.7 in [BM88], we know that each connected component of $\mathcal{P}_k$ is also semianalytic and the family of connected components of $\mathcal{P}_k$ is locally finite. Since $\mathcal{P}_k$ is compact, it has finitely many connected components. Finally, by using Theorem 1 in [Loj64], we know that $\mathcal{P}_k$ is triangulable and so is locally path connected. Thus, each connected component of $\mathcal{P}_k$ is path connected. □

Lemma 5. *Let C be a connected component of $\overline{\operatorname{Per}_{kl}}(F)$ for some $l \in \mathbb{N}$ and suppose that*

$$E = \{p \in C;\ F^k(p) = p \ \text{ and the germ } F_p^k \text{ of } F^k \text{ at } p \text{ is unipotent }\}$$

is non-empty. Then C is a subset of $\overline{\operatorname{Per}_k}(F)$ and D_pF^k is unipotent for all $p \in C$.

Proof. Denote $\psi = F^k$. We know that if $\psi(p) = p$ then the characteristic polynomial of $D_p\psi$ is

$$P_{D_p\psi}(x) = x^2 - \operatorname{tr}(D_p\psi)x + \det D_p\psi = x^2 - Sx + P,$$

where S is the sum and P is the product of the eigenvalues of $D_p\psi$. Therefore,

$$E = \{p \in C;\ \psi(p) = p,\ \operatorname{tr}(D_p\psi) = 2 \text{ and} \det(D_p\psi) = 1\}.$$

In order to prove the lemma it suffices to show that $E = C$. Since $E \neq \emptyset$, the set C is connected, and E is closed in C, it suffices to show that E is open in C. Consider $p \in E$. Let us first prove that the germ (C, p) of C at p is contained in $\overline{\operatorname{Per}_k}(F)$. Set

$$A = \{q \in U; \psi(q) = q\} \quad \text{and} \quad B = \{q \in U;\ \psi^l(q) = q\}.$$

It is obvious that A and B are analytic and $(C, p) \subset (B, p)$. Now since $\psi(p) = p$ and $D_p\psi$ is unipotent, we can consider the infinitesimal generator X_p of the germ ψ_p, i.e., the nilpotent formal vector field X_p such that $\exp(X_p) = \psi_p$. Consequently, lX_p is the infinitesimal generator of $\psi_p^l = F_p^{kl}$. Since $(\psi_p^l)|_B = \operatorname{id}$, Lemma 1, applied to the germ of B at p, implies that $(B, p) \subset \operatorname{Sing}(lX_p) = \operatorname{Sing}(X_p)$ and therefore $\psi_p|_B = \operatorname{id}$. In particular, we obtain $(C, p) \subset \overline{\operatorname{Per}_k}(F)$.

Now let us prove that $(C, p) \subset (E, p)$. Define $f : A \to \mathbb{C}$ by setting $f(q) = \det(D_q\psi)$. The restrictions of f to the irreducible components of the germ of A at p are holomorphic functions. As F has finite orbits, it follows by the Stable Manifold Theorem that the eigenvalues of $D_q\psi$ have modulus 1 and then the image of f is contained in the circle S^1, since $\det(D_q\psi)$ is the product of the eigenvalues of $D_q\psi$. In particular, the image of f does not contain any open set. We obtain that f is locally constant in a neighborhood of $C \cap A$ in A by the open mapping theorem. Since $f(p) = 1$ and $(C, p) \subset A$, we deduce that $(C, p) \subset \{f = 1\}$. Now we consider a function $g : A \to \mathbb{C}$, defined by $g(q) = \operatorname{tr}(D_q\psi)$. For each q in some neighborhood of p in A, the eigenvalues λ and μ of $D_q\psi$ satisfy $|\lambda| = |\mu| = 1$ and $\lambda\mu = 1$ since $\operatorname{Spec}(D_p\psi) \subset S^1$ and $(C, p) \subset \{f = 1\}$. Therefore, we obtain

$\mathrm{tr}(D_q\psi) = \lambda + \overline{\lambda} = 2\mathrm{Re}(\lambda) \in [-2, 2]$. Again, the restriction of g to the irreducible components of (A, p) defines holomorphic functions whose images do not contain any open set. Hence, $g \equiv g(p) = 2$ is constant in a neighborhood of p in A. It follows that $(C, p) \subset (E, p)$. This concludes the proof. □

We have already seen in Lemma 4 that each $\overline{\mathrm{Per}}_k(F)$ has finitely many connected components, say, $C_1^k, ..., C_l^k$. Now, let us consider the *family of all components of F in D*, namely,

$$\mathcal{A} := \{C_j^k;\ k \in \mathbb{N} \text{ and } C_j^k \text{ is a connected component of } \overline{\mathrm{Per}}_k(F)\ \}.$$

We say that two components $C, D \in \mathcal{A}$ are equivalent (and then we write $C \sim D$) if there are

$$C_{\alpha_1}^{k_1}, ..., C_{\alpha_r}^{k_r} \in \mathcal{A}$$

such that $C = C_{\alpha_1}^{k_1},\ D = C_{\alpha_r}^{k_r}$ and $C_{\alpha_s}^{k_s} \cap C_{\alpha_{s+1}}^{k_{s+1}} \neq \emptyset$ for all $1 \leq s < r$. Notice that $\sim$ is a relation of equivalence in $\mathcal{A}$.

Remark 5. If $C_{\alpha_1}^{k_1}, ..., C_{\alpha_r}^{k_r}$ are components in $\mathcal{A}$ such that $C_{\alpha_s}^{k_s} \cap C_{\alpha_{s+1}}^{k_{s+1}} \neq \emptyset$ for all $1 \leq s < r$, then there are $j, k \in \mathbb{N}$ such that $\bigcup_s C_{\alpha_s}^{k_s} \subset C_j^k$. If fact, since the union is connected and is contained in $\overline{\mathrm{Per}}_{k_1 \ldots k_r}(F)$, it is contained in $C_j^{k_1 \ldots k_r}$ for some j.

Lemma 6 (stability of classes)**.** *If $[C]$ is an equivalence class (possibly infinite) in $\mathcal{A}/\sim$, then*

$$\bigcup_{C_j \in [C]} C_j = C_{j_0}^{k_0}$$

for some j_0, k_0.

Proof. The result is obvious if there is at most one non-unitary component in $[C]$, i.e. a component C_j^k of $[C]$ such that $\sharp C_j^k > 1$. Thus, we can assume that there are two distinct non-unitary components C_a and C_b of $[C]$ such that $C_a \cap C_b \neq \emptyset$. Suppose that C_a is a connected component of $\mathcal{P}_a$ and C_b is a connected component of $\mathcal{P}_b$, where we denote $\mathcal{P}_j = \overline{\mathrm{Per}}_j(F)$.

Let us show that there is $p \in C_a \cap C_b$ such that D_pF^{ab} is unipotent. First of all, there is $p \in C_a \cap C_b$ such that $(C_a, p) \neq (C_b, p)$ in $\overline{B}$ since C_a and C_b are connected. Let (V_a, p) and (V_b, p) be the germs of analytic set of equation $F^a = Id$ and $F^b = Id$, respectively, defined in some neighborhood of p in U. Since C_a and C_b are non-unitary, we have $\dim(V_a, p) \geq 1$ and $\dim(V_b, p) \geq 1$.

We claim that $(V_a, p) \neq (V_b, p)$. Otherwise, we have $(C_a \cup C_b, p) \subset \mathcal{P}_a \cap \mathcal{P}_b$ and hence

$$(C_a, p) = (C_a \cup C_b, p) = (C_b, p),$$

that contradicts the choice of p. Therefore we obtain $D_pF^{ab} = \mathrm{id}$ by Lemma 2.

Consider the connected component C_ℓ^{ab} of $\mathcal{P}_{ab}$ containing p. Let A be a connected union of finitely many components of $[C]$ that contains p. Then $A \subset C_\ell^{ab}$ by Lemma 4.5. By varying A, we deduce $\cup_{C_j \in [C]} C_j = C_\ell^{ab}$. □

4.4. **Structure of the set of periodic points.** Now, we combine the previous results to show Theorem 4 and the Fixed Point Curve Theorem.

Proof of Theorem 4 and the Fixed Point Curve Theorem. Let B an open ball such that F and F^{-1} are defined in a neighborhood of $\overline{B}$. Let $\mathcal{P}$ be a connected component of $\overline{\mathrm{Per}}(F)$. It is a countable union of elements of the family $\mathcal{A}$ of components of F in $\overline{B}$. Up to consider only maximal components of F in $\overline{B}$, we can suppose that such a union is disjoint by Lemma 6. Therefore, we obtain $\mathcal{P} = C$ for some $C \in \mathcal{A}$ by Sierpinski Theorem. Hence $\mathcal{P}$ is a connected component of some $\overline{\mathrm{Per}}_k(F)$. Both C and $\overline{\mathrm{Per}}_k(F)$ are semianalytic by Lemma 4.

Let $\mathcal{Q}$ be a connected component of $\mathrm{Per}(F)$. It is contained in a connected component $\mathcal{Q}'$ of $\overline{\mathrm{Per}}(F)$. Then $\mathcal{Q}'$ is a semianalytic subset of $\overline{\mathrm{Per}}_k(F)$ for some $k \in \mathbb{N}$ by the first part of the proof. We obtain $\mathcal{Q} \subset \mathrm{Per}_k(F)$ and hence the set $\mathcal{Q}$ is given locally by the equation $F^k = \mathrm{id}$. It follows that $\mathcal{Q}$ is a complex analytic subset of the open ball. Since the set $\mathrm{Per}_k(F) \cap \mathcal{Q}'$ is semianalytic and relatively compact, it follows that it has finitely many connected components [BM88, Cor. 2.7] and they are all semianalytic. We deduce that $\mathcal{Q}$ is a semianalytic subset of $\mathbb{C}^2$.

We claim that $\dim(\mathcal{Q}, p) \geq 1$ for any $p \in \mathcal{Q}$. This is equivalent to the property $\sharp \mathcal{Q} > 1$ since $\mathcal{Q}$ is a connected component of $\mathrm{Per}_k(F)$. First, suppose $0 \in \mathcal{Q}$. Since $\mathcal{Q}' \subset \overline{\mathrm{Per}}_k(F)$, there exists a neighborhood W of the origin such that

$$W \cap \mathcal{Q} = W \cap \mathcal{Q}' = W \cap \mathrm{Fix}(F^k).$$

Note that $\mathcal{Q}'$ is a continuum that contains the non-trivial subcontinuum K obtained in Lemma 3. Since $\mathcal{Q}'$ is a non-trivial continuum, we deduce that the germ of $\mathrm{Fix}(F^k)$ and then of $\mathcal{Q}$ at the origin have positive dimension and thus contains an analytic curve Γ passing through 0. Finally, consider a general connected component $\mathcal{Q}$ of $\mathrm{Per}(F)$. Given $p \in \mathcal{Q}$, there exists a germ of analytic curve Γ' at p contained in $\mathrm{Per}_l(F)$ for some $l \in \mathbb{N}$ by the previous discussion. Since $(\Gamma', p) \subset \mathcal{Q}$, we obtain $\dim(\mathcal{Q}, p) \geq 1$. □

Proof of Theorem 3. The eigenvalues of D_0F belong to the unit circle by Corollary 5. By the Fixed Point Curve Theorem, there exists an analytic curve $\Gamma \subset \mathrm{Fix}(F^k)$ through the origin for some $k \geq 1$. Suppose $1 \notin \mathrm{spec}(D_0F^k)$. Then $D_0(F^k - \mathrm{Id})$ is a regular matrix, therefore $F^k - \mathrm{Id}$ is a local diffeomorphism at 0. But this contradicts $\Gamma \subset (F^k - \mathrm{Id})^{-1}(0)$. Therefore, we obtain $1 \in \mathrm{spec}(D_0F^k)$. Since the eigenvalues of D_0F^k are kth-powers of eigenvalues of D_0F, it follows that there exists $\lambda \in \mathrm{spec}(D_0F)$ such that $\lambda^k = 1$. □

Remark 6. Corollary 1 provides negative algebraic criteria for the finite orbits property. For instance, let $F(x,y) = (x + f_1(x,y), y + f_2(x,y)) \in \text{Diff}_1(\mathbb{C}^2, 0)$ where $f_j(x,y) = \sum_{k=m}^{\infty} P_{k,j}(x,y)$ is the expansion of $f_j \in \mathbb{C}\{x,y\}$ as a sum of homogeneous polynomials for $j \in \{1,2\}$, where $m \geq 2$. Assume that $P_{m,1}$ and $P_{m,2}$ are relatively prime. Since

$$x \circ F^k - x = kP_{m,1} + h.o.t. \text{ and } y \circ F^k - y = kP_{m,2} + h.o.t.$$

we deduce that the fixed point $(0,0)$ of F^k is isolated for any $k \in \mathbb{N}$ and hence F is not a finite orbits germ.

Later on, we will see that there exists $F \in \text{Diff}\,(\mathbb{C}^2, 0)$ with finite orbits but D_0F has no finite orbits (see Theorem 1). It makes sense to study whether the finite orbits property for other actions naturally associated to F implies $\text{spec}(D_0F) \subset e^{2\pi i \mathbb{Q}}$. We are going to consider the blow-up $\pi : \tilde{\mathbb{C}}^2 \to \mathbb{C}^2$ of the origin and the diffeomorphism $\tilde{F}$ induced by F in a neighborhood of the divisor $D := \pi^{-1}(0)$ (see [Rib05]).

Corollary 6. *Let $F \in \text{Diff}\,(\mathbb{C}^2, 0)$. Assume that the germ of $\tilde{F}$ defined in the neighborhood of D in $\tilde{\mathbb{C}}^2$ has finite orbits. Then* $\text{spec}(D_0F)$ *consists of roots of unity.*

Proof. The diffeomorphism $\tilde{F}_{|D}$ has finite orbits and hence D_0F induces an element of finite order of $\text{PGL}(2, \mathbb{C})$. Thus D_0F is diagonalizable and has eigenvalues $\lambda, \mu \in \mathbb{C}^*$ such that λ/μ is a root of unity. Since at least one eigenvalue of D_0F is a root of unity by Theorem 3, we deduce $\text{spec}(D_0F) \subset e^{2\pi i \mathbb{Q}}$. □

Remark 7. In general, a germ of biholomorphism H does not admit germs of fixed point curves, even when $F \in \text{Diff}_{<\infty}(\mathbb{C}^2, 0)$. For example, $F(x,y) = (-x, -x-y)$ is a finite orbits germ, because it is linear and $\text{Spec}(D_0F) = \{-1\}$, but the only fixed point of F is the origin. Note, however, that $F^2(x,y) = (x, 2x+y)$ and $\{x = 0\}$ is a fixed point curve of F^2. Moreover, the curve $\{x = 0\}$ is an irreducible curve invariant by F.

We conclude this section providing an example of $F \in \text{Diff}(\mathbb{C}^2, 0)$ that has finite orbits but has no irreducible germ of invariant curve. The diffeomorphism F is of the form $F = S \circ T$ where $S(x,y) = (iy, ix)$, $T = \exp(X)$ and

$$X = xy\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) + ix^2y^2\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right).$$

Note that $S^*X = X$ and hence S and T commute. Moreover S has order 4. Any germ of irreducible curve γ that is invariant by F is also invariant by F^4 and hence by $\exp(4X)$. Since X is the infinitesimal generator of a tangent to the identity local biholomorphism, we deduce that γ is invariant by X by Lemma 1. Note that the singularity of $X/(xy)$ at the origin is reduced and hence $X/(xy)$ has only two irreducible invariant curves by Briot and Bouquet theorem, namely the x and y

axes. As a consequence, the axes are the unique irreducible germs of X-invariant curves. Since S permutes the axes, it follows that F has no irreducible germ of invariant curve.

Let us show that F has finite orbits. It suffices to prove that $F^4 = T^4$ has finite orbits by Proposition 1. Indeed, it suffices to show that T has finite orbits by the same result. Next, we study the action induced by X on the leaves of the foliation $d(xy) = 0$. Such a foliation is preserved by X since $X(xy) = 2i(xy)^3$. We can relate the properties of X with those of $Z = 2iz^3\partial/\partial z$ and its time 1 map $G = \exp(Z)$. Indeed, we have

$$(xy) \circ T^k(x, y) = G^k(xy)$$

for $k \in \mathbb{Z}$.

Fix a small bounded neighborhood V' of 0 in $\mathbb{C}$ and a small bounded neighborhood V of $(0, 0)$ in $\mathbb{C}^2$ such that $(xy)(V) \subset V'$. Consider $(x_0, y_0) \in V$ and denote $z_0 = x_0y_0$. Since the axes consist of fixed points of T, we can suppose $x_0y_0 \neq 0$. Assume, aiming at a contradiction that the positive T-orbit of (x_0, y_0) in V is infinite. Therefore, $G^k(z_0)$ is well-defined and belongs to V' for any $k \geq 0$. Since G has a dynamics of flower type, we deduce

$$\lim_{k\to\infty} G^k(z_0) = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{G^k(z_0)}{|G^k(z_0)|} \in \{e^{\frac{i\pi}{4}}, -e^{\frac{i\pi}{4}}\}.$$

Assume that the latter limit is equal to $e^{\frac{i\pi}{4}}$. Let us study the variation of the monomials x^ay^b by iteration; we have

$$x^ay^b \circ T = x^ay^b(1 + (a - b)xy + O(x^2y^2))$$

for any $(a, b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. As a consequence, we get

$$|x \circ T^{k+1}(x_0, y_0)| > |x \circ T^k(x_0, y_0)|, \quad |(x^2y) \circ T^{k+1}(x_0, y_0)| > |(x^2y) \circ T^k(x_0, y_0)|$$

for any non-negative integer number k. Since $|x|$ increases along the positive T-orbit of (x_0, y_0) and $\lim_{k\to\infty}(xy)(T^k(x_0, y_0)) = 0$, we get $\lim_{k\to\infty} y(T^k(x_0, y_0)) = 0$. Moreover, since $|x^2y|$ increases along the positive T-orbit of (x_0, y_0) it follows that $\lim_{k\to\infty} |x|(T^k(x_0, y_0)) = \infty$. This property contradicts that V is bounded. The case $\lim_{k\to\infty} \frac{G^k(z_0)}{|G^k(z_0)|} = -e^{\frac{i\pi}{4}}$ is treated in a similar way. Analogously, we can show that the negative T-orbit of (x_0, y_0) is finite.

5. Non-virtually unipotent biholomorphisms with finite orbits

So far, all the examples in the literature of finite orbits local diffeomorphisms were virtually unipotent, i.e. the eigenvalues of their linear parts were roots of unity. The likely reason is revealed in Theorem 2: time 1 maps with finite orbits have roots of unity eigenvalues. In this section we construct a family of finite orbits local diffeomorphisms that are non-virtually unipotent.

Definition 6. *We say that* $\lambda \in \mathbb{C}$ *is a Cremer number if* λ *is not a root of unity, but*

$$|\lambda| = 1 \text{ and } \liminf_{m\to\infty} \sqrt[m]{|\lambda^m - 1|} = 0.$$

This equation is called Cremer's condition.

Fix $n \geq 1$ and consider coordinates $x = (x_1, ..., x_n) \in \mathbb{C}^n$ and $y \in \mathbb{C}$. Given $j \in \mathbb{N}$ we denote by $[j]$ the unique natural number $j' \in \{1, \dots, n\}$ such that $j - j'$ is a multiple of n. The proof of Theorem 1 consists in building a convergent power series

$$a(x_1, \dots, x_n) = \sum_{j=1}^{\infty} \frac{(2jx_{[j]})^{m_j}}{M_j^{m_j}} = \sum_{j=1}^{\infty} \left(\frac{2j}{M_j}\right)^{m_j} x_{[j]}^{m_j}, \tag{3}$$

where $(m_j)_{j\geq 1}$ is an increasing sequence of natural numbers and $(M_j)_{j\geq 1}$ is a sequence of positive numbers that will be chosen to ensure that

$$F(x, y) = (\lambda x_1, \lambda x_2, \dots, \lambda x_n, y + a(x_1, \dots, x_n)) \tag{4}$$

has finite orbits. We need auxiliary sequences $(k_j)_{j\geq 1}$ and $(r_j)_{j\geq 1}$ of natural numbers. They satisfy certain conditions that are provided by the following lemma.

Lemma 7. *Let* λ *be a Cremer number. There exist a sequence* $(M_j)_{j\geq 1}$ *in* $\mathbb{R}_+$ *and sequences* $(m_j)_{j\geq 1}$, $(k_j)_{j\geq 1}$ *and* $(r_j)_{j\geq 1}$ *in* $\mathbb{N}$ *such that, for any* $j \in \mathbb{N}$,

- (C1) $M_j := \frac{1}{\sqrt[m_j]{|\lambda^{m_j} - 1|}}$ *satisfies* $M_j \geq 4j^2$;
- (C2) $2^{m_j} \geq 1 + j + \sum_{\ell=1}^{j-1} 2(2\ell)^{m_\ell} j^{m_\ell}$ $(2^{m_1} \geq 2)$;
- (C3) $\min(m_j, r_j) > \max(m_{j-1}, r_{j-1})$ *if* $j \geq 2$;
- (C4) $|\lambda^{k_j m_j} - 1| \geq 1$;
- (C5) $k_j \sum_{\ell=r_j}^{\infty} \frac{1}{2^\ell} < 1$.

Proof. Since λ satisfies the Cremer condition, we can choose m_1 and M_1 such that the first three conditions hold, where $(C3)$ is an empty condition. As λ^{m_1} is not a root of unity, the sequence $(\lambda^{km_1})_{k\geq 1}$ is dense on the unit circle; hence, there exists $k_1 \in \mathbb{N}$ such that the fourth condition holds for $j = 1$. Now we can define r_1 in such a way that the last condition holds for $j = 1$. Analogously, we can define (M_2, m_2), k_2 and r_2 such that $(C1) - (C5)$ hold for $j = 2$. Indeed, we define the sequences $(M_j)_{j\geq 1}$, $(m_j)_{j\geq 1}$, $(k_j)_{j\geq 1}$ and $(r_j)_{j\geq 1}$ recursively for $j \in \mathbb{N}$. □

Remark 8. Notice that conditions $(C1)$, $(C2)$, $(C3)$, $(C4)$ and $(C5)$ still hold if we replace λ by λ^{-1}, since $\overline{\lambda} = \lambda^{-1}$ implies

$$|\lambda^{-n} - 1| = |\overline{\lambda^n - 1}| = |\lambda^n - 1|$$

for any $n \in \mathbb{Z}$.

Lemma 8. *Consider the setting provided by Lemma 7. Then $a(x)$ (cf. (3)) is an entire function of $\mathbb{C}^n$. Moreover, the map $F(x,y) = (\lambda x, y + a(x))$ is a holomorphic automorphism of $\mathbb{C}^{n+1}$ whose inverse is*

$$F^{-1}(x_1, \ldots, x_n, y) = (\lambda^{-1}x_1, \ldots, \lambda^{-1}x_n, y - a(\lambda^{-1}x_1, \ldots, \lambda^{-1}x_n)).$$

Proof. Since $M_j \geq 4j^2$ for any $j \geq 1$, it follows that

$$\lim_{j\to\infty} \sqrt[m_j]{\left(\frac{2j}{M_j}\right)^{m_j}} = \lim_{j\to\infty} \frac{2j}{M_j} = 0$$

and hence $a(x_1, \ldots, x_n)$ is an entire function. We can verify directly that $F^{-1}(x,y) = (\lambda^{-1}x, y - a(\lambda^{-1}x))$ is the inverse of F. □

Now note that if $A(x) = \sum a_{j_1\cdots j_n} x_1^{j_1} \ldots x_n^{j_n}$ is a power series and $G(x,y) = (\lambda x_1, \ldots, \lambda x_n, y + A(x_1, \ldots, x_n))$, then we can show that

$$G^k(x,y) = (\lambda^k x_1, \ldots, \lambda^k x_n, y + (L_k A)(x_1, \ldots, x_n))$$

for any $k \in \mathbb{N}$ by induction, where L_k is the linear operator of the ring of convergent power series defined by

$$(5) \qquad \begin{aligned} (L_k A)(x) &:= A(x) + A(\lambda x) + \ldots + A(\lambda^{k-1}x) \\ &= \sum_{j\in\mathbb{N}^n} (1 + \lambda^{|j|} + \lambda^{2|j|} + \ldots + \lambda^{(k-1)|j|}) a_{j_1\cdots j_n} x_1^{j_1} \ldots x_n^{j_n} \\ &= \sum_{j\in\mathbb{N}^n} \frac{\lambda^{k|j|} - 1}{\lambda^{|j|} - 1} a_{j_1\cdots j_n} x_1^{j_1} \ldots x_n^{j_n}, \end{aligned}$$

where $|j| = j_1 + j_2 + \ldots + j_n$.

Lemma 9. *Consider $j \geq 1$ and $x \in \mathbb{C}$ with $\max_{1\leq k\leq n} |x_k| \leq j$ and $|x_{[j]}| \geq 1/j$. Then we have*

$$j \leq |(L_{k_j} a)(x)| \leq 2(2j^2)^{m_j} + 2^{m_j} - j.$$

Proof. First, let us study $\left| L_{k_j} \frac{(2jx_{[j]})^{m_j}}{M_j^{m_j}} \right|$. By (5) we have

$$\left| L_{k_j} \frac{(2jx_{[j]})^{m_j}}{M_j^{m_j}} \right| = \left| \frac{\lambda^{k_j m_j} - 1}{\lambda^{m_j} - 1} \frac{(2jx_{[j]})^{m_j}}{M_j^{m_j}} \right| = |(\lambda^{k_j m_j} - 1)(2jx_{[j]})^{m_j}|,$$

where the second equality follows from the definition of M_j. Thus, the choice of k_j allows us to conclude that

$$2^{m_j} \leq \left| L_{k_j} \frac{(2jx_{[j]})^{m_j}}{M_j^{m_j}} \right| \leq 2(2j^2)^{m_j}$$

since $\frac{1}{j} \leq |x_{[j]}| \leq j$. Now, let us study

$$L_{k_j}\left(\sum_{l=1}^{j-1} \frac{(2lx_{[l]})^{m_l}}{M_l^{m_l}}\right).$$

We obtain

$$\begin{aligned}(6) \qquad \left|L_{k_j}\left(\sum_{l=1}^{j-1} \frac{(2lx_{[l]})^{m_l}}{M_l^{m_l}}\right)\right| &= \left|\sum_{l=1}^{j-1} \frac{\lambda^{k_j m_l}-1}{\lambda^{m_l}-1}\frac{(2lx_{[l]})^{m_l}}{M_l^{m_l}}\right| \\ &= \left|\sum_{l=1}^{j-1}(\lambda^{k_j m_l}-1)(2l)^{m_l}x_{[l]}^{m_l}\right| \\ &\leq \sum_{l=1}^{j-1} 2(2l)^{m_l} j^{m_l} \\ &\leq 2^{m_j}-(j+1)\end{aligned}$$

if $\max(|x_1|,\ldots,|x_n|) \leq j$, where the final inequality follows from condition (C2). Finally, let us consider

$$L_{k_j}\left(\sum_{l=j+1}^{\infty} \frac{(2lx_{[l]})^{m_l}}{M_l^{m_l}}\right).$$

The condition (C1) implies $\frac{2l}{M_l} < \frac{1}{2j}$ for any $l > j$. Therefore $\max(|x_1|,\ldots,|x_n|) \leq j$ implies

$$\begin{aligned}(7) \quad \left|L_{k_j}\left(\sum_{l=j+1}^{\infty}\left(\frac{2l}{M_l}\right)^{m_l} x_{[l]}^{m_l}\right)\right| &\leq \sum_{l=j+1}^{\infty} |1+\lambda^{m_l}+\lambda^{2m_l}+\ldots+\lambda^{(k_j-1)m_l}|\frac{1}{2^{m_l}} \\ &\leq k_j \sum_{l=j+1}^{\infty}\frac{1}{2^{m_l}} \\ &\leq k_j \sum_{l=r_j}^{\infty}\frac{1}{2^{l}} \\ &< 1\end{aligned}$$

by conditions (C3) and (C5). In particular, by combining the previous estimates we get

$$j = 2^{m_j}-(2^{m_j}-(j+1))-1 < \left|(L_{k_j}a)(x)\right| < 2(2j^2)^{m_j}+2^{m_j}-j$$

if $\max(|x_1|,\ldots,|x_n|) \leq j$ and $|x_{[j]}| \geq 1/j$. □

The next lemma concludes the proof of Theorem 1.

Lemma 10. *Let $\lambda \in \mathbb{C}$ be a Cremer number and $n \geq 1$. Consider the function $a(x)$ in (3) where $(M_j)_{j\geq 1}$ and $(m_j)_{j\geq 1}$ are provided by Lemma 7. Then the biholomorphism $F(x,y) = (\lambda x, y + a(x))$ has finite orbits in any set of the form $\mathbb{C}^n \times U$, where U is a bounded open set in $\mathbb{C}$.*

Proof. Let d be the diameter of U. Fix $(x_{1,0}, \dots, x_{n,0}, y_0) \in (\mathbb{C}^n \setminus \{0\}) \times U$. Then there exists $j \in \mathbb{N}$ such that $\max(|x_{1,0}|, \dots, |x_{n,0}|) \leq j$, $x_{[j],0} \geq 1/j$ and $j > d$. Lemma 9 implies that $F^{k_j}(x_0, y_0)$ does not belong to $\mathbb{C}^n \times U$. Notice that by Remark 8 conditions $(C1)$, $(C2)$, $(C3)$, $(C4)$ and $(C5)$ still hold if we replace λ by λ^{-1}. Since $F^{-1}(x,y) = (\lambda^{-1}x, y - a(\lambda^{-1}x))$, we have

$$-a(\lambda^{-1}x) = \sum (-\lambda^{-|j|}) a_{j_1 \dots j_n} x_1^{j_1} \dots x_n^{j_n},$$

that is, the monomials of $-a(\lambda^{-1}x)$ are obtained by multiplying those of $a(x)$ by complex numbers of modulus 1. In particular, the proof of Lemma 9 is still valid for F^{-1}. One concludes that $F^{-k_j}(x_0, y_0) \notin \mathbb{C}^n \times U$. Therefore, F has finite orbits in $(\mathbb{C}^n \setminus \{0\}) \times U$. In the other hand, as $x = 0$ is a fixed point curve of F, it is clear that F has finite orbits in $\mathbb{C}^n \times U$. □

Theorem 5. *Consider the hypotheses in Lemma 10. Then the local diffeomorphism $F(x,y) = (\lambda x, y + a(x))$ is formally linearizable. In particular, F has a first integral of the form $y + b(x)$, where $b(x)$ is a divergent power series with $b(0) = 0$. Specifically, $y - y_0 + b(x) = 0$ defines a (divergent) formal invariant hypersurface through the point $(0, y_0)$ for every $y_0 \in \mathbb{C}$.*

Proof. The conjugacy equation

$$(x, y + b(x)) \circ (\lambda x, y + a(x)) \circ (x, y - b(x)) = (\lambda x, y)$$

is equivalent to

$$b(x) - b(\lambda x) = a(x).$$

Setting $b(x) = \sum_{j \in \mathbb{N}^n} b_j x^j$ and $a(x) = \sum_{j \in \mathbb{N}^n} a_j x^j$, we see that $b(x) - b(\lambda x) = a(x)$ can be expressed as

$$\sum_{j \in \mathbb{N}^n} (b_j - \lambda^{|j|} b_j - a_j) x^j = 0.$$

Hence, the conjugacy equation has a solution

$$b(x) = \sum_{j \in \mathbb{N}^n} \frac{a_j}{1 - \lambda^{|j|}} x^j.$$

Since y is a first integral of $(x,y) \mapsto (\lambda x, y)$, the series $y + b(x)$ is a first integral of F. Note that the series $b(x)$ is divergent, otherwise, F and $(x,y) \mapsto (\lambda x, y)$ would be analytically conjugated, which is impossible, because F has finite orbits whereas $(x,y) \mapsto (\lambda x, y)$ does not. □

We want to understand the non-virtually unipotent diffeomorphisms $F \in \mathrm{Diff}\,(\mathbb{C}^2, 0)$ with finite orbits. First, we focus on the arithmetic properties of the non-root of unity eigenvalue. It is not casual that in our examples such an eigenvalue is well approximated by roots of unity.

Definition 7. *A number $\lambda \in \mathbb{C}$ is called a Bruno number if there is a sequence $1 < q_1 < q_2 < \ldots$ of integers such that*

$$|\lambda| = 1 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{q_k} \log \frac{1}{\Omega_\lambda(q_{k+1})} < +\infty, \tag{8}$$

where $\Omega_\lambda(m) = \min_{2 \le k \le m} |\lambda^k - \lambda|$ for all $m \geq 2$. Condition 8 is called Bruno Condition and it is equivalent to the following (see [Bry73]*)*

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \log \frac{1}{\Omega_\lambda(2^{k+1})} < +\infty.$$

Given $\lambda \in \mathbb{S}^1$ and $l \in \mathbb{N}$, we see that $\Omega_{\lambda^l}(m) \geq \Omega_\lambda((m-1)l + 1)$ for all $m \geq 2$. Thus we can adjust (8) to conclude that if λ is a Bruno number then so is λ^l.

Proposition 4. *Consider $F \in \mathrm{Diff}_{<\infty}(\mathbb{C}^2, 0)$ with $D_0F \notin \mathrm{Diff}_{<\infty}(\mathbb{C}^2, 0)$. Let λ be the eigenvalue of D_0F that is not a root of unity. Then λ is not a Bruno number.*

Proof. Suppose, aiming a contradiction, that one of the eigenvalues of D_0F is a Bruno number. By Theorem 3, up to replace F with a non-trivial iterate F^k, we can suppose that $\mathrm{spec}(D_0F) = \{1, \lambda\}$, where λ is a Bruno number.

Up to a linear change of coordinates, we can suppose $(D_0F)(x, y) = (\lambda x, y)$. Note that the line $y = 0$ is invariant by D_0F. Now, we apply a theorem of Pöschel that relates the invariant manifolds of D_0F and F [P8̈6]. In our context, it determines a sufficient condition for the existence of a smooth analytic curve γ, invariant by F , tangent to $y = 0$ at 0 and such that $F_{|\gamma}$ is analytically conjugated to the rotation $x \mapsto \lambda x$. The Pöschel condition is

$$\sum_{\nu \geq 0} 2^{-\nu} \log \omega^{-1}(2^{\nu+1}) < \infty \tag{9}$$

where we define

$$\omega(m) = \min_{2 \le k \le m} (|\lambda^k - \lambda|, |\lambda^k - 1|).$$

Since $\omega(m) \geq \Omega_\lambda(m + 1)$ for $m \geq 2$, it follows that the property (9) is a consequence of the Bruno condition. Thus, the intended γ exists and $F_{|\gamma}$ is an irrational rotation, contradicting that F is a finite orbits germ. □

Corollary 7. *Let $F \in \mathrm{Diff}\,(\mathbb{C}^2, 0)$. Consider a formal invariant curve Γ such that the multiplier of $F_{|\Gamma}$ is a Bruno number. Then F is not a finite orbits germ.*

Proof. Assume, aiming at a contradiction, that F has finite orbits. Let $\gamma(t)$ be a Puiseux parametrization of Γ. Since Γ is invariant, we have $F(\gamma(t)) = (\gamma \circ h)(t)$ for some $h \in \widehat{\text{Diff}}\,(\mathbb{C}, 0)$. We denote the multiplier of $F_{|\Gamma}$ by μ; it satisfies $\mu = h'(0)$. We can suppose that $1 \in \text{spec}(D_0 F)$ up to replace F with some non-trivial iterate F^k by Theorem 3. Note that the multiplier of $F^k_{|\Gamma}$ is equal to μ^k. Since μ^k is a Bruno number, the hypothesis still holds for F^k and Γ. The tangent cone of Γ is a subspace of eigenvectors of $D_0 F$, associated to an eigenvalue that we denote by λ. Moreover, we have $\mu^m = \lambda$, where m is the multiplicity of Γ. Since μ is a Bruno number, so is λ. Proposition 4 implies that λ is not a Bruno number, providing a contradiction. □

Next, we see that the diffeomorphisms provided by Theorem 1 are archetypic examples of finite orbits diffeomorphisms $F \in \text{Diff}\,(\mathbb{C}^2, 0)$ such that $D_0 F$ has no finite orbits. Indeed, next result classifies the properties of such diffeomorphisms.

Proposition 5. *Let* $F \in \text{Diff}_{<\infty}(\mathbb{C}^2, 0)$ *with* $\text{spec}(D_0 F) = \{1, \lambda\}$ *where* λ *is not a root of unity. Then* F *satisfies the following properties:*

- λ *is not a Bruno number;*
- $\text{Fix}(F)$ *is a smooth curve through the origin;*
- F *is formally conjugated to* $(x, y) \mapsto (\lambda x, y)$ *by a formal diffeomorphism that is transversally formal along* $\text{Fix}(F)$*;*
- *there exists a divergent smooth invariant curve through any point* $p \in \text{Fix}(F)$.

Proof. The eigenvalue λ is a non-Bruno number by Proposition 4. Fix a sufficiently small domain of definition $B_\rho(0)$. Let C be the connected component of the origin of $\text{Per}(F)$ (cf. equation (1)). It is complex analytic, has positive dimension and is contained in $\text{Fix}(F^m)$ for some $m \in \mathbb{N}$ by Theorem 4. The dimension of the germ of C at the origin is less than 2, since otherwise the germ of F^m at 0 is the identity map, contradicting $\lambda \notin e^{2\pi i \mathbb{Q}}$. Therefore, the germ of C at 0 is an analytic curve γ. Moreover, γ is irreducible and smooth, since otherwise F^m is tangent to the identity by Lemma 2. Since $F_{|\gamma}$ is a local biholomorphism in one variable with finite orbits, it has finite order. Therefore, its multiplier at 0 is a root of unity and thus it is necessarily equal to 1. Since the unique periodic tangent to the identity local diffeomorphism is the identity map, we deduce $\gamma \subset \text{Fix}(F)$. It is clear that the germ of $\text{Fix}(F)$ at 0 is contained in C and hence the germs of $\text{Fix}(F)$ and γ at 0 coincide.

Up to a change of coordinates in a neighborhood of the origin, we can assume $\text{Fix}(F) = \{x = 0\}$. As a consequence $1 \in \text{spec}(D_{(0,y)} F)$ for any y in a neighborhood of 0. We denote $\text{spec}(D_{(0,y)} F) = \{1, \lambda(y)\}$. The function $\lambda(y)$ is constant equal to

λ by the proof of Lemma 5. We obtain that F is of the form

$$F(x,y) = \left(\lambda x + \sum_{j=2}^{\infty} a_j(y)x^j, y + \sum_{j=1}^{\infty} b_j(y)x^j\right),$$

where a_{j+1}, b_j are defined in a common open neighborhood U of 0 in $\mathbb{C}$ for any $j \in \mathbb{N}$. We want to conjugate F with D_0F. In order to do it, we consider sequences $(G_{2,j})_{j\geq 1}, (G_{1,j+1})_{j\geq 1}$ of diffeomorphisms of the form

$$G_{1,j+1}(x,y) = (x + c_{j+1}(y)x^{j+1}, y) \text{ and } G_{2,j}(x,y) = (x, y + d_j(y)x^j),$$

where $d_j, c_{j+1} \in \mathcal{O}(U)$ for any $j \in \mathbb{N}$. We define $F_{1,1} = F$, $F_{2,j} = G_{2,j}^{-1} \circ F_{1,j} \circ G_{2,j}$ and $F_{1,j+1} = G_{1,j+1}^{-1} \circ F_{2,j} \circ G_{1,j+1}$ for $j \in \mathbb{N}$. We want $F_{2,j}$ and $F_{1,j+1}$ to be of the form

$$F_{2,j}(x,y) = (\lambda x + O(x^{j+1}), y + O(x^{j+1})), \ F_{1,j+1}(x,y) = (\lambda x + O(x^{j+2}), y + O(x^{j+1}))$$

for any $j \in \mathbb{N}$. Indeed, if $\alpha_{j+1}(y)$ is the coefficient of x^{j+1} in $x \circ F_{2,j}$, it suffices to define $c_{j+1}(y) = \alpha_{j+1}(y)/(\lambda^{j+1} - \lambda)$ for $j \in \mathbb{N}$. Analogously, if $\beta_j(y)$ is the coefficient of x^j in $y \circ F_{1,j}$, we have $d_j(y) = \beta_j(y)/(\lambda^j - 1)$ for $j \in \mathbb{N}$. The diffeomorphism

$$H_j := G_{2,1} \circ G_{1,2} \circ G_{2,2} \circ G_{1,3} \circ \ldots \circ G_{2,j-1} \circ G_{1,j}$$

conjugates F with $F_{1,j}$ for $j \geq 2$. By construction, it converges in the (x)-adic topology to some $H \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$, that is transversally formal along $x = 0$ and satisfies $H^{-1} \circ F \circ H = D_0F$.

Note that $y \circ D_0F \equiv y$ implies $(y \circ H^{-1}) \circ F \equiv y \circ H^{-1}$. Since $y \circ H^{-1}$ is transversally formal along $x = 0$, there exists a formal invariant curve γ_y through $(0, y)$, that is transverse to $\mathrm{Fix}(F)$, for any $y \in U$. We claim that γ_y is divergent for any $y \in U$. Otherwise, there exists $y_0 \in U$ such that γ_{y_0} is an analytic curve and since the multiplier of $F_{|\gamma_{y_0}}$ at $(0, y_0)$ is equal to λ, the diffeomorphism $F_{|\gamma_{y_0}}$ is non-periodic. This contradicts that the one dimensional diffeomorphism $F_{|\gamma_{y_0}}$ has finite orbits. □

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