Gabarito da P2 de Álgebra Linear - aplicada dia 25/7

1- O polinomio Caracteristico

$$\Delta_A(\lambda) = x^2 - (-5+7)x + (-35+32) = x^2 - 2x - 3 = (x+1)(x-3) = 0$$

se $x = -1 \Rightarrow A - (-1)I = \begin{bmatrix} -4 & -16 \\ 2 & 8 \end{bmatrix}$

E $\begin{bmatrix} -4 & -16 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow x + 4y = 0 \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4y \\ y \end{bmatrix} = y\begin{bmatrix} -4 \\ 1 \end{bmatrix}$.

se $x = 3 \Rightarrow A - 3I = \begin{bmatrix} -8 & -16 \\ 2 & 4 \end{bmatrix}$ daí

$\begin{bmatrix} -8 & -16 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow x + 2y = 0 \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2y \\ y \end{bmatrix} = y\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

seja $\beta = \{(-4,1), (-2,1)\}$ e α a base canônica e $D = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$

$$[A]_\alpha = \underset{\underset{P}{\parallel}}{[I]^\beta_\alpha} [D]_\beta [I]^\alpha_\beta$$

então $P = \begin{bmatrix} -4 & -2 \\ 1 & 1 \end{bmatrix} \Rightarrow P^{-1} = -\frac{1}{2}\begin{bmatrix} 1 & 2 \\ -1 & -4 \end{bmatrix}$.

$$A^n = P D^n P^{-1} = \begin{bmatrix} -4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (-1)^n & 0 \\ 0 & 3^n \end{bmatrix}\frac{1}{2}\begin{bmatrix} -1 & -2 \\ 1 & 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -4(-1)^n & \cancel{3^n(4)} \; -4(3^n) \\ (-1)^n & 3^n \end{bmatrix}\begin{bmatrix} \; \end{bmatrix}$$

$$= \begin{bmatrix} 2(-1)^n - 3^n & 4(-1)^n - 4 \times 3^n \\ -\frac{1}{2}(-1)^n + \frac{3^n}{2} & (-1)^{n+1} + 2 \times 3^n \end{bmatrix}$$

2 a) A matriz de T com respeito a base canonica é

$A = \begin{bmatrix} 4 & 4 \\ -2 & 0 \end{bmatrix} \Rightarrow \Delta_A(x) = x^2 - 4x + 8 = 0$. cálculo do $b^2 - 4ac = -16$

logo as raízes são $\lambda_1 = \frac{4 + i\sqrt{16}}{2} = 2 + 2i$ e $\lambda_2 = 2 - 2i$

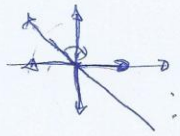
se $x = 2+2i$ $\quad A - (2+2i)I = \begin{bmatrix} 2-2i & 4 \\ -2 & -2-2i \end{bmatrix} \xrightarrow{l_2 \to -1+2i\, l_2} \begin{bmatrix} 2-2i & 4 \\ 2-2i & 4 \end{bmatrix}$

Daí $(2-2i)x + 4y = 0 \Rightarrow (1-i)x + 2y = 0 \Rightarrow x = -\frac{2}{1-i}y = (-1-i)y$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (-1-i)y \\ (1)y \end{bmatrix} = y\begin{bmatrix} -1-i \\ 1 \end{bmatrix} = y\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} + i\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$

se $x = 2-2i \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = y\begin{bmatrix} -1+i \\ 1 \end{bmatrix}$.

Considere $\beta = \{(-1,1), (-1,0)\}$



$T(-1,1) = (0,2) = x(-1,1) + y(-1,0) = 2(-1,1) + (-2)(-1,0)$

$T(-1,0) = (-4,2) = x(-1,1) + y(-1,0) = 2(-1,1) + 2(-1,0)$

$[T]^{\beta}_{\beta} = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} = \frac{2\sqrt{2}}{2\sqrt{2}}\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} = 2\sqrt{2}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ que é uma rotação de $-45°$ (~~com respeito~~ com respeito ~~a base β~~) seguido de uma expansão de $2\sqrt{2}$.
a base β

Para entender melhor o que o operador faz. Tome $(1,0)_\alpha$ na base canonica $(1,0)_\alpha = 0(-1,1) + (-1)(-1,0)$ e $\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}_\beta = -2(-1,1) + 2 - 2(-1,0)$
$= (0,-1)_\beta$ $\qquad = (4,-2)_\alpha$

Veja que com respeito a base canônica a operação não pode ser descrita desta forma.

2-b a matriz com respeito a base canonica α é

$B = [T]_\alpha^\alpha = \begin{bmatrix} 6 & 16 \\ -1 & -2 \end{bmatrix}$. $\Rightarrow \Delta_B(x) = x^2 - (6-2)x + (-12+16) = x^2 - 4x + 4 = (x-2)^2$

Logo o polinômio caracteristico tem duas raízes repetidas $x=2$.

e com certeza ~~$m_B(x)$~~ o polinômio minimo de B não pode ser $x-2$. Daí que B não é diagonalizável. Vamos determinar a base β na qual B fica na forma de Jordan

observe que $\begin{bmatrix} 6-2 & 16 \\ -1 & -2-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 16 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = -4y$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = y\begin{bmatrix} -4 \\ 1 \end{bmatrix}$

seja $v \notin \text{Span}\{(-4,1)\}$, tome $v = (1,0)$

$\beta = \{(B-2I)v, v\} = \{(4,-1), (1,0)\}$

$T(4,-1) = (8,-2) = 2(4,-1) + 0(1,0)$
$T(1,0) = (6,-1) = 1(4,-1) + 2(1,0)$ $\Rightarrow [T]_\beta^\beta = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

3 - $T(x,y,z) = x(1,0,1,0) + y(4,1,0,1) + z(0,1,-4,1)$

Daí, $Im(T) = Span\{(1,0,1,0),(4,1,0,1),(0,1,-4,1)\}$

Para minimizar a distância vamos projetar w sobre $Im(T)$

$$u_1' = (1,0,1,0)$$

$$u_2' = (4,1,0,1) - \frac{4}{2}(1,0,1,0) = (2,1,-2,1)$$

$$u_3' = (0,1,-4,1) - \left(\frac{-4}{2}\right)(1,0,1,0) - \frac{10}{10}(2,1,-2,1) = (0,0,0,0)$$

Disto temos que $(0,1,-4,1)$ é combinação de $\{(1,0,1,0),(2,1,-2,1)\}$

$$u = proj_{u_1'}(w) + proj_{u_2'}(w) = (5,1,1,1)$$

c) Para resolver $T(x,y,z) = (5,1,1,1)$

$$\begin{bmatrix} 1 & 4 & 0 & 5 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 0 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & -4 & -4 & -4 \\ 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 0 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+4z \\ 1-z \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$ e uma solução é $(1,1,0)$.

$$proj_{u_1}(x,y,z) + proj_{u_2'}(x,y,z) = \frac{1}{2}\begin{bmatrix} x+z \\ 0 \\ x+z \\ 0 \end{bmatrix} + \frac{1}{10}\begin{bmatrix} 2(t+2x+y-2z) \\ t+2x+y-2z \\ 2(-t-2x-y+2z) \\ t+2x+y-2z \end{bmatrix}$$

~~2 a) O polinomio Caracteristico é $\Delta_A(x)=$~~

4 _ $3x^2 - 4\sqrt{3}xy - y^2 + 20y - 25 = 0$

$$[x\ y]\begin{bmatrix} 3 & -2\sqrt{3} \\ -2\sqrt{3} & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + [\cancel{0}\ 20]\begin{bmatrix} x \\ y \end{bmatrix} - [25] = [0]$$

Diagonalizando $A = \begin{bmatrix} 3 & -2\sqrt{3} \\ -2\sqrt{3} & -1 \end{bmatrix}$

$\Delta_A(x) = x^2 - 2x - 15 = 0 = (x-5)(x+3)$.

Quando $x=-3$ $\quad A-(-3)I = \begin{bmatrix} 6 & -2\sqrt{3} \\ -2\sqrt{3} & 2 \end{bmatrix}$

$\Rightarrow -2\sqrt{3}x + 2y = 0 \Rightarrow y = \sqrt{3}x \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ \sqrt{3}x \end{bmatrix} = x\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$

Quando $x=5 \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = y\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$

considere $\beta = \{ \frac{1}{2}(\sqrt{3}, -1), \frac{1}{2}(1, \sqrt{3}) \}$

e $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ logo $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}x' + y'}{2} \\ \frac{-x' + \sqrt{3}y'}{2} \end{bmatrix}$

fazendo a substituição temos

$$5x'^2 - 3y'^2 - 10x' + 10\sqrt{3}y' = 25$$

$$5(x'^2 - 2x' + 1) - 3(y'^2 + \tfrac{10}{\sqrt{3}}y' + \tfrac{25}{3}) = 25 + 5 - 25$$

$$5(x'-1)^2 - \cancel{\sqrt{3}}\,3(y' - \tfrac{5}{3})^2 = 5$$

$$\frac{5(x'-1)^2}{5} - \frac{(y'-\frac{5}{3})^2}{\frac{5}{3}} = 1.$$