# On cohomological $C^0$-(in)stability

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**Abstract.** After Katok [3], a homeomorphism $f\colon M \to M$ is said to be *cohomologically $C^0$-stable* when its space of real $C^0$-coboundaries is closed in $C^0(M)$. In this short note we completely classify cohomologically $C^0$-stable homeomorphisms, showing that periodic homeomorphisms are the only ones.

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**Mathematical subject classification:** Primary: 35A20, 37B05.

## 1  Introduction

Cocycles and cohomological equations play a fundamental role in dynamical systems and ergodic theory. In this short note we shall mainly concentrate on topological dynamics. So, from now on $(M, d)$ will denote a compact metric space and the dynamics will be given by a homeomorphism $f\colon M \to M$.

In such a case, a (real) *cocycle over $f$* is just a map $\phi\colon M \to \mathbb{R}$. If $\mathcal{A} \subset M^{\mathbb{R}}$ denotes an *$f$-invariant functional space* (i.e. $\mathcal{A}$ is linear subspace of $M^{\mathbb{R}}$ such that $\psi \circ f \in \mathcal{A}$ whenever $\psi \in \mathcal{A}$), any $\phi \in \mathcal{A}$ will be called an *$\mathcal{A}$-cocycle* and we will say $\phi$ is an *$\mathcal{A}$-coboundary* whenever the cohomological equation

$$\phi = u \circ f - u$$

admits a solution $u \in \mathcal{A}$. Many questions in dynamics can be reduced to determine if certain cocycles are or not coboundaries, so it is an important problem (and in many cases rather difficult) to study the structure of the linear space of $\mathcal{A}$-coboundaries, which shall be denoted by

$$B(f, \mathcal{A}) := \{v \circ f - v : v \in \mathcal{A}\}.$$

By analogy with cohomological theories, we can define the *first cohomology space of $f$* with coefficient in $\mathcal{A}$ as the linear space

$$H^1(f, \mathcal{A}) := \mathcal{A}/B(f, \mathcal{A}).$$

To analyze the structure of $H^1(f, \mathcal{A})$ (and $B(f, \mathcal{A})$) in general we endow the space $\mathcal{A}$ with a vector space topology, and hence, $H^1(f, \mathcal{A})$ inherits the quotient one. Then, typically the analysis is divided in two steps (see [3] for a very detailed exposition):

(a) **Cohomological $\mathcal{A}$-obstructions:** very roughly, these are necessary conditions an $\mathcal{A}$-cocycle must satisfy to be an $\mathcal{A}$-coboundary. In general, these are closed conditions in $\mathcal{A}$, so typically they characterize $\overline{B(f, \mathcal{A})}^{\mathcal{A}}$ instead of $B(f, \mathcal{A})$. Some examples of cohomological obstructions:

(i) Invariant measures are the cohomological obstructions for solving cohomological equations in the topological category. In fact, if $\mathfrak{M}(f)$ denotes the space of $f$-invariant probability measures, then it holds

$$\overline{B(f, C^0(M))}^{C^0} = \left\{ \phi \in C^0(M) : \int_M \phi \, \mathrm{d}\mu = 0, \ \forall \mu \in \mathfrak{M}(f) \right\}.$$

(ii) Invariant distributions (in the sense of Schwartz) are the cohomological obstructions in the smooth category. In fact, if $M$ is closed smooth manifold and $\mathcal{D}'(M)$ denotes the topological dual space of $C^\infty(M)$, then defining $\mathcal{D}'(f) := \{\mu \in \mathcal{D}'(M) : \langle \mu, \phi \rangle = \langle \mu, \phi \circ f \rangle, \ \forall \phi \in C^\infty(M)\}$, it holds

$$\overline{B(f, C^\infty(M))}^{C^\infty} = \left\{ \phi \in C^\infty(M) : \langle \mu, \phi \rangle = 0, \ \forall \mu \in \mathcal{D}'(f) \right\}.$$

(b) **Cohomological $\mathcal{A}$-stability:** A system $f$ is said to be *cohomologically $\mathcal{A}$-stable* when $B(f, \mathcal{A})$ is closed in $\mathcal{A}$. Cohomological stability is a very desirable property because in that case, and only in that case, we can verify whether a cocycle $\phi$ is an $\mathcal{A}$-coboundary just analyzing the cohomological obstructions of item (a). Let us mention some examples:

(i) *Hyperbolic systems:* After Livšic [4] we know that hyperbolic systems are cohomologically Hölder-stable. On the other hand, de la Llave, Marco and Moriyon have shown in [2] that $C^r$ Anosov diffeomorphisms are cohomologically $C^r$-stable, for any $r \in [2, \infty]$.

(ii) *Ergodic translations on tori:* It is well-known that ergodic translations on tori are cohomologically $C^\infty$-rigid iff they are Diophantine (see [3] for details).

(iii) *Smooth circle diffeomorphisms with irrational rotation number:* In a joint work with Avila [1], we showed that a $C^\infty$-circle diffeomorphism with no periodic points is cohomologically $C^\infty$-stable iff its rotation number is Diophantine.

(iv) *Circle diffeomorphisms with irrational rotation number in low regularity:* After the very recent work of Navas and Triestino [6], we know that a $C^{2+\epsilon}$ minimal circle diffeomorphism is cohomologically $C^1$-stable iff its rotation number is Diophantine.

In this short note we completely characterize the homeomorphisms that are cohomologically $C^0$-stable. In fact, we prove the following

**Theorem A.** *A homeomorphism $f \colon M \to M$ is cohomologically $C^0$-stable if and only if $f$ is periodic, i.e. it has finite order in the group of homeomorphisms of $M$.*

## 2 Notations

As we have already mentioned in the § 1, $(M, d)$ will denote an arbitrary compact metric space. Given $x \in M$ and $r > 0$, we write $B(x, r) := \{y \in M : d(x, y) < r\}$. If $A \subset M$, $\chi_A \colon M \to \{0, 1\}$ will denote the characteristic function of $A$.

We will write $C^0(M)$ for the space of real continuous functions on $M$ endowed with the uniform norm

$$\|\phi\|_{C^0} := \sup_{x \in M} |\phi(x)|, \quad \forall \phi \in C^0(M).$$

Given a homeomorphism $f \colon M \to M$, we define the space of $C^0$-*coboundaries* by

$$B(f, C^0(M)) := \{v \circ f - v : v \in C^0(M)\}.$$

The homeomorphism $f$ is said to be *cohomologically $C^0$-stable* iff

$$B(f, C^0(M))$$

is closed in $C^0(M)$.

On the other hand, $f$ is said to be *periodic* when there exists $q \in \mathbb{N}$ satisfying $f^q = id_M$, and the number $q$ is called a *period of $f$*.

## 3 Proof of Theorem A

Let us start with the simplest part of Theorem A, i.e. let us prove that any periodic map is cohomologically $C^0$-stable:

**Lemma 3.1.** *Let us assume $f$ is periodic and let $q \in \mathbb{N}$ be a period of $f$. Then,*

$$B(f, C^0(M)) = \left\{ \phi \in C^0(M) : \sum_{j=0}^{q-1} \phi(f^j(x)) = 0, \ \forall x \in M \right\}.$$

*In particular, $f$ is cohomologically $C^0$-stable.*

**Proof.**    First of all observe that every $\phi \in B(f, C^0(M))$ satisfies $S_f^q \phi \equiv 0$. In fact, if $u \colon M \to \mathbb{R}$ is such that

$$\phi(x) = u(f(x)) - u(x), \quad \forall x \in M,$$

then it clearly holds

$$S_f^q \phi(x) = u(f^q(x)) - u(x) = 0, \quad \forall x \in M.$$

On the other hand, let us suppose $\psi \in C^0(M)$ is such that $S_f^q \psi \equiv 0$. Then, using a formula we learned from [5][1] we write

$$v(x) := -\frac{1}{q} \sum_{j=1}^{q} S_f^j \psi(x), \quad \forall x \in M.$$

Clearly, $v \in C^0(M)$ and

$$\begin{aligned}
v(f(x)) - v(x) &= -\frac{1}{q} \left( \sum_{j=1}^{q} \left( S_f^j \psi(f(x)) - S_f^j \psi(x) \right) \right) \\
&= -\frac{1}{q} \left( S_f^q \psi(f(x)) - q\psi(x) \right) = \psi(x),
\end{aligned}$$

for every $x \in M$. Thus, $\psi \in B(f, C^0(M))$, as desired.    □

## 3.1    The cohomological operator

We can define the **cohomological operator** (associated to $f$)

$$\mathcal{L}_f \colon C^0(M) \to C^0(M)$$

by

$$\mathcal{L}_f(u) := u \circ f - u, \quad \forall u \in C^0(M).$$

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[1]The author thanks A. Navas for bringing this formula to his attention.

This is clearly a linear operator, and since

$$\left\| \mathcal{L}_f(u) \right\|_{C^0} \leq 2 \left\| u \right\|_{C^0},$$

it is is also continuous.

Now, observe that the kernel of $\mathcal{L}_f$, which shall be denoted by $\ker \mathcal{L}_f$, coincides with the space of continuous $f$-invariant functions. The quotient space $C^0(M)/\ker \mathcal{L}_f$ will be denoted by $C^0_f(M)$. Defining
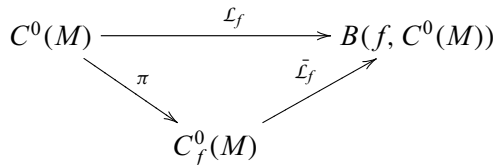
$$\left\| \phi + \ker \mathcal{L}_f \right\|_{C^0_f} := \inf_{\psi \in \ker \mathcal{L}_f} \left\| \phi + \psi \right\|_{C^0}, \quad \forall \phi \in C^0(M), \tag{1}$$

we get a norm and this turns $C^0_f(M)$ into a Banach space.

On the other hand, notice that the image of the operator $\mathcal{L}_f$ coincides with the space of continuous coboundaries $B(f, C^0(M))$, which is, by our hypothesis, a closed subspace of $C^0(M)$. We will consider $B(f, C^0(M))$ equipped with (the restriction of) the norm $\|\cdot\|_{C^0}$.

In this way, we have the following simple

**Lemma 3.2.** *Let $\bar{\mathcal{L}}_f \colon C^0_f(M) \to B(f, C^0(M))$ be the factor linear operator turning the following diagram commutative:*

$$
\begin{array}{ccc}
C^0(M) & \xrightarrow{\quad \mathcal{L}_f \quad} & B(f, C^0(M)) \\
& \searrow{\scriptstyle \pi} \qquad \nearrow{\scriptstyle \bar{\mathcal{L}}_f} & \\
& C^0_f(M) &
\end{array}
$$

*where $\pi : \phi \mapsto \phi + \ker \mathcal{L}_f$ denotes the canonical quotient projection.*

*Then, $\bar{\mathcal{L}}_f$ is continuous and bijective, and consequently, it is a Banach space isomorphism.*

**Proof.** The continuity of $\bar{\mathcal{L}}_f$ easily follows from the following estimate: for any $\phi \in C^0(M)$ and every $\psi \in \ker \mathcal{L}_f$, it holds

$$\left\| \bar{\mathcal{L}}_f(\phi + \ker \mathcal{L}_f) \right\|_{C^0} = \left\| \mathcal{L}_f(\phi + \psi) \right\|_{C^0} \leq 2 \left\| \phi + \psi \right\|_{C^0}.$$

Taking infimum over $\psi \in \ker \mathcal{L}_f$ on the right hand side, we get

$$\left\| \bar{\mathcal{L}}_f(\phi + \ker \mathcal{L}_f) \right\|_{C^0} \leq 2 \left\| \phi + \ker \mathcal{L}_f \right\|_{C^0_f}, \quad \forall \phi \in C^0(M).$$

Finally, since $\bar{\mathcal{L}}_f$ is tautologically bijective, by the open mapping theorem, $\bar{\mathcal{L}}_f$ is a Banach space isomorphism. $\qquad\square$

Now, in order to finish the proof of Theorem A, let us assume $f$ is cohomo-logically $C^0$-stable and it is not periodic. That means for every $n \in \mathbb{N}$, we can find $x_n \in M$ such that $x_n \neq f^j(x_n)$, for every $j \in \{1, \dots, 2^n\}$.

For each $n \geq 1$, let us choose $r_n > 0$ such that the ball $B_n := B(x_n, r_n)$ satisfies

$$f^j(B_n) \cap B_n = \emptyset, \quad \forall j \in \{1, \dots, 2^n\}.$$

Then, consider the function $u_n \colon M \to \mathbb{R}$ given by

$$u_n(x) := \sum_{j=-2^n+1}^{2^n-1} \chi_{f^j(B_n)}(x) \left(1 - \frac{|j|}{2^n}\right) \frac{r_n - d\big(f^{-j}(x), x_n\big)}{r_n}, \quad \forall x \in M. \quad (2)$$

One can easily check that $u_n$ is continuous, its support is equal to the disjoint union $\bigsqcup_{|j|<2^n} f^j(B_n)$,

$$\inf_{x \in M} u_n(x) = u_n\left(f^{2^n}(x_n)\right) = 0, \quad (3)$$

and

$$\sup_{x \in M} u_n(x) = u_n(x_n) = 1. \quad (4)$$

Since any function $v \in \ker \mathcal{L}_f$ must satisfy $v(x_n) = v\big(f^{2^n}(x_n)\big)$, from (3) and (4) we conclude that

$$\left\| u_n + \ker \mathcal{L}_f \right\|_{C_f^0} \geq \frac{1}{2}. \quad (5)$$

Now, consider the coboundary $\phi_n := \mathcal{L}_f(u_n) = u_n \circ f - u_n \in B(f, C^0(M))$. Thus, for every $x \in M$ it holds

$$
\begin{aligned}
\phi_n(x) &= u_n(f(x)) - u_n(x) \\
&= \sum_{j=-2^n+1}^{2^n-1} \chi_{f^j(B_n)}(f(x)) \left(1 - \frac{|j|}{2^n}\right) \frac{r_n - d\big(f^{-j+1}(x), x_n\big)}{r_n} \\
&\quad - \sum_{j=-n+1}^{n-1} \chi_{f^j(B_n)}(x) \left(1 - \frac{|j|}{2^n}\right) \frac{r_n - d(f^{-j}(x), x_n)}{r_n} \\
&= \Bigg[ \chi_{f^{-2^n}(B_n)}(x) \frac{r_n - d\big(f^{2^n}(x), x_n\big)}{2^n r_n} \\
&\quad + \sum_{j=-2^n+2}^{2^n-2} \chi_{f^j(B_n)}(x) \left(\frac{|j|}{2^n} - \frac{|j+1|}{2^n}\right) \frac{r_n - d\big(f^{-j}(x), x_n\big)}{r_n} \\
&\quad - \chi_{f^{2^n-1}(B_n)}(x) \frac{r_n - d\big(f^{-2^n+1}(x), x_n\big)}{2^n r_n} \Bigg]
\end{aligned}
\quad (6)
$$

In particular, (6) implies that

$$\|\phi_n\|_{C^0} = \left|\phi_n(f^j(x_n))\right| = \frac{1}{2^n}, \quad \forall j \in \{-2^n, \dots, 2^n - 1\}. \tag{7}$$

Finally, recalling that $\mathcal{L}_f(u_n) = \phi_n$, for every $n \in \mathbb{N}$, from (5) and (7) it follows that $\mathcal{L}_f^{-1} : B(f, C^0(M)) \to C_f^0(M)$ is not continuous, contradicting Lemma 3.2, and Theorem A is proved.

## References

[1] A. Avila and A. Kocsard. *Cohomological equations and invariant distributions for minimal circle diffeomorphisms*. Duke Math. J., **158**(3) (2011), 501–536. MR 2805066

[2] R. de la Llave, J.M. Marco and R. Moriyón. *Canonical perturbation theory of anosov systems and regularity results for the livšic cohomology equation*. Annals of Mathematics, **123**(3) (1986), 537–611. MR MR840722 (88h:58091)

[3] A. Katok. *Cocycles, cohomology and combinatorial constructions in ergodic theory*. Smooth ergodic theory and its applications (Seattle, WA, 1999), Proc. Sympos. Pure Math., vol. 69, American Mathematical Society, Providence, RI, 2001, In collaboration with E.A. Robinson, Jr., pp. 107–173. MR 1858535 (2003a:37010)

[4] A.N. Livšic. *Cohomology of dynamical systems*. Izv. Akad. Nauk SSSR Ser. Mat., **36** (1972), 1296–1320.

[5] J. Moulin-Ollagnier and D. Pinchon. *Systèmes dynamiques topologiques. I. Étude des limites de cobords*. Bull. Soc. Math. France, **105**(4) (1977), 405–414.

[6] A. Navas and M. Triestino. *On the invariant distributions of $c^2$ circle diffeomorphisms of irrational rotation number*, to appear in Mathematische Zeitschrift (2012).

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