COHOMOLOGICAL EQUATIONS AND INVARIANT DISTRIBUTIONS FOR MINIMAL CIRCLE DIFFEOMORPHISMS

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Abstract
Given any smooth circle diffeomorphism with irrational rotation number, we show that its invariant probability measure is the only invariant distribution (up to multiplication by a real constant). As a consequence of this, we show that the space of real C^∞-coboundaries of such a diffeomorphism is closed in $C^\infty(\mathbb{T})$ if and only if its rotation number is Diophantine.

1. Introduction

Cohomological equations appear very frequently in different contexts in dynamical systems. In fact, many problems, especially those concerned with certain forms of rigidity and stability, can be reduced to analyzing the existence of solutions (in certain regularity classes) of some cohomological equations (see [10], [11] for general reference; [3], [9] for applications to the study of foliations; and [4], [13] for cohomological aspects of group actions on the circle).

In the case where the dynamics are given by a diffeomorphism f on a manifold M, the most basic cohomological equation (and the only kind we shall consider from now on) is a first-order linear difference equation

$$uf - u = \phi, \tag{1.1}$$

where $\phi\colon M \to \mathbb{R}$ is given and $u\colon M \to \mathbb{R}$ is the unknown of the problem.

In this work we shall mainly concern ourselves with cohomological equations in the smooth category. In fact, most of the time we will assume that the data of the equations (i.e., the diffeomorphism f and the function ϕ in (1.1)) are C^∞, and we will be interested in the existence of smooth solutions.

By analogy with the cohomology of groups, we can consider the function ϕ in (1.1) as being a smooth *cocycle* over f, and we say that ϕ is a (smooth) *coboundary*

DUKE MATHEMATICAL JOURNAL
Vol. 158, No. 3, © 2011 DOI 10.1215/00127094-1345662
Received 22 March 2010. Revision received 8 November 2010.
2010 *Mathematics Subject Classification.* Primary 37E10; Secondary 37C55, 46F05.
Avila's work partially supported by a Clay Research Fellowship.
Kocsard's work partially supported by FAPERJ Brazil.

whenever (1.1) admits a C^∞-solution. Of course, this leads us to define the *first cohomology space* $H^1(f, C^\infty(M))$ (see Section 2.2 for details).

In general these cohomology spaces could be rather "wild" (e.g., its natural topology is non-Hausdorff), and so it is rather hard to study the structure of these spaces. However, we can distinguish two aspects that appear as the fundamental characters in the analysis of $H^1(f, C^\infty(M))$:

(i) the first one is the space of f*-invariant distributions* in the sense of Schwartz (see Section 2.2 for details); and

(ii) the second one is the concept of *cohomological stability* (see Definition 2.1).

It is important to remark that, in general, the second problem is considerably much harder than the first one. The work of Heafliger and Banghe [6] is a good testimony to this.

1.1. Cohomological equations over quasi-periodic systems

Equations like (1.1), where $f = R_\alpha : \mathbb{T}^d \to \mathbb{T}^d$ is an ergodic rigid rotation on the d-torus, appear as "linearized equations" in many Kolmogorov-Arnold-Moser problems. In such a case we have a very clear and simple description of the smooth cohomology: it can be shown that the Haar measure on $\mathbb{T}^d$ is the only (modulo multiplication by a constant) R_α-invariant distribution, and R_α is cohomologically C^∞-stable if and only if α is a Diophantine vector (see Section 2.3 for definitions).

Nevertheless, the general situation is much more complicated: when f is an arbitrary quasi-periodic diffeomorphism (i.e., $f \in \mathrm{Diff}^\infty(\mathbb{T}^d)$ is topologically conjugate to an ergodic rigid rotation), in general it is very hard to determine the space of f-invariant distributions and the cohomological stability issue seems to be even subtler.

For instance, the problem of computing the C^∞ first cohomology space of an arbitrary minimal circle diffeomorphism has been included in several compilations of open problems concerning group actions and foliations (see, e.g., [12], [5]).

In this article we solve this problem by proving the following theorem.

THEOREM A

Let $F : \mathbb{T} \to \mathbb{T}$ be an orientation-preserving C^∞-diffeomorphism with irrational rotation number, and let μ be its only invariant probability measure. Then, up to multiplication by a real constant, μ is the only F-invariant distribution.

Yoccoz, in his Ph.D. thesis, showed that within the set of smooth circle diffeomorphisms with fixed rotation number $\alpha \in (\mathbb{R}\setminus\mathbb{Q})/\mathbb{Z}$ those which are smoothly conjugate to the rotation $R_\alpha : x \mapsto x + \alpha$ form a dense subset (see [15, Chapter III]). To some extent, our Theorem A can be considered as a "cocycle version" of his result.

It is important to remark that Theorem A is absolutely one-dimensional and cannot be extended to higher dimensions. In fact, in a forthcoming article [1], we will show the existence of smooth diffeomorphisms of $\mathbb{T}^2$ which are topologically conjugate to rigid rotations and exhibit higher-order invariant distributions.

On the other hand, as an almost straightforward consequence of Theorem A, we can obtain the following corollary.

COROLLARY B
A minimal C^∞ circle diffeomorphism is cohomologically C^∞-stable if and only if its rotation number is Diophantine.

1.2. Denjoy-Koksma inequality improved
Given a circle homeomorphism F with irrational rotation number $\rho(F)$ and a real function $\phi\colon \mathbb{T} \to \mathbb{R}$, the classical Denjoy-Koksma inequality affirms that the Birkhoff sums satisfy

$$\Big|\sum_{i=1}^{q_n-1} \phi\big(f^i(x)\big) - q_n \int_{\mathbb{T}} \phi \, \mathrm{d}\mu\Big| \leq \mathrm{Var}(\phi), \quad \forall x \in \mathbb{T}, \tag{1.2}$$

whenever ϕ has bounded variation, μ is the only F-invariant probability measure, and q_n is a denominator of a rational approximation of $\rho(F)$ given by the continued fraction algorithm (see Section 2.3.2 and Proposition 4.2 for details).

Nevertheless, when F is a C^3-diffeomorphism and ϕ is the *log-derivative cocycle*, that is, $\phi = \log DF$, Herman showed (see [8, chapitre VII, corollaire 2.5.2]) that the previous estimate can be improved. In fact, he proved that $\log DF^{q_n} = \sum_{i=0}^{q_n-1} \log DF \circ F^i$ converges uniformly to zero, as $n \to \infty$. The interested reader can also find a *hard* version of this result in [15].

Here, as a consequence of Theorem 7.1, which is nothing but a finite regularity version of Theorem A, we get the following result which can be considered as a generalization of the Herman result for arbitrary cocycles.

COROLLARY C
If F is C^{11} and ϕ is C^1, it holds that

$$\Big\| \sum_{i=0}^{q_n-1} \phi \circ F^i - q_n \int_{\mathbb{T}} \phi \, \mathrm{d}\mu \Big\|_{C^0} \to 0 \quad \textit{as } n \to \infty.$$

1.3. Some open questions
At this point it seems natural to analyze the first cohomological space of higher-dimension quasi-periodic diffeomorphisms.

As we have already mentioned above, in a forthcoming article [1], we will show that there exist quasi-periodic diffeomorphisms on higher-dimensional tori exhibiting higher-order invariant distributions.

However, all the examples we know so far have Liouville rotation vectors and are cohomologically C^∞-unstable, so it is reasonable to propose the following question.

QUESTION 1.1
Let $\alpha \in \mathbb{R}^d$ be an irrational vector, and let $f \in \mathrm{Diff}_+^\infty(\mathbb{T}^d)$ be topologically conjugate to the rigid rotation $R_\alpha : x \mapsto x + \alpha$.

Is it true that f is cohomologically C^∞-stable if and only if α is Diophantine?

QUESTION 1.2
Let α and f be as above. If α is Diophantine, then does it hold that

$$\dim \mathscr{D}'(f) = 1?$$

It is interesting to remark that Question 1.2 could be a first step toward an eventually higher-dimensional version of the Herman-Yoccoz linearization theorem.

2. Preliminaries

2.1. General notation

Throughout this article M will denote an arbitrary smooth boundaryless manifold. Given $r \in \mathbb{N}_0 \cup \{\infty\}$, we write $C^r(M)$ for the space of real C^r-functions on M and $\mathrm{Diff}^r(M)$ for the group of C^r-diffeomorphisms.*

Let us recall that when r is finite, the *uniform C^r-topology* turns $C^r(M)$ into a Banach space. On the other hand, we shall consider the space $C^\infty(M)$ endowed with its usual Fréchet topology, which can be defined as the projective limit of the family of Banach spaces $(C^r(M))_{r\in\mathbf{N}}$.

Given any $f \in \mathrm{Diff}^r(M)$, $\mathrm{Fix}(f)$ and $\mathrm{Per}(f)$ stand for the set of fixed and periodic points of f, respectively. Whenever M is orientable, we write $\mathrm{Diff}_+^r(M)$ for the subgroup of C^r orientation-preserving diffeomorphisms.

The d-dimensional torus will be denoted by $\mathbb{T}^d$ and will be identified with $\mathbb{R}^d/\mathbb{Z}^d$. The canonical quotient projection will be denoted by $\pi : \mathbb{R}^d \to \mathbb{T}^d$. For simplicity, we shall just write $\mathbb{T}$ for the 1-torus, that is, the circle.

The symbol Leb_d will be used to denote the Lebesgue measure on $\mathbb{R}^d$, as well as the Haar probability measure on $\mathbb{T}^d$. Once again, for the sake of simplicity, we just write Leb, and also $\mathrm{d}x$, instead of Leb_1.

*As usual, we use the term C^0-*diffeomorphism* as a synonym of *homeomorphism*.

As usual, we shall identify $C^r(M,\mathbb{R}^k)$ with $(C^r(M))^k$ and $C^r(\mathbb{T}^d)$ with the space of $\mathbb{Z}^d$-periodic real C^r-functions on $\mathbb{R}^d$.

In the particular case of real C^r-functions on $\mathbb{T}$, we explicitly define the *C^r-norm* on $C^r(\mathbb{T})$ (with $0 \leq r < \infty$) by

$$\|\phi\|_{C^r} := \max_{x\in\mathbb{T}} \max_{0\leq j\leq r} |D^j\phi(x)|, \quad \forall \phi \in C^r(\mathbb{T}).$$

Moreover, whenever $I \subset \mathbb{R}$ is a compact interval and $\psi \in C^r(\mathbb{R})$, we define

$$\|\psi|_I\|_{C^r} := \max_{x\in I} \max_{0\leq j\leq r} |D^j\psi(x)|.$$

Next, we define the space of lifts of circle diffeomorphisms by

$$\widetilde{\mathrm{Diff}}^r_+(\mathbb{T}) := \left\{ f \in \mathrm{Diff}^r_+(\mathbb{R}) : f - id_{\mathbb{R}} \in C^r(\mathbb{T}) \right\}.$$

It can be easily shown that this space is connected and simply connected. In particular, this space can be identified with the universal covering of $\mathrm{Diff}^r_+(\mathbb{T})$. Making some abuse of notation, we will also denote by π the canonical projection $\widetilde{\mathrm{Diff}}^r_+(\mathbb{T}) \to \mathrm{Diff}^r_+(\mathbb{T})$ that associates to each $f \in \widetilde{\mathrm{Diff}}^r_+(\mathbb{T})$ the only circle diffeomorphism lifted by f.

As usual, we write $\rho\colon \widetilde{\mathrm{Diff}}^0_+(\mathbb{T}) \to \mathbb{R}$ for the *rotation number* function, and we will use the same letter to call the induced map $\rho\colon \mathrm{Diff}^0_+(\mathbb{T}) \to \mathbb{R}/\mathbb{Z}$ (see [2, Section 1.1] for the definitions).

Finally, we have two important remarks about notation. First, for the sake of simplicity, when dealing with estimates we will use the letter C to denote any positive real constant which may assume different values throughout the article, even in a single chain of inequalities.

Second, we will denote the intervals of the real line regardless of the order of the extremal points; that is, if a, b are two different points of $\mathbb{R}$, we shall write (a, b) for the only bounded connected component of $\mathbb{R} \setminus \{a, b\}$, independent of the order of the points. Of course, we will follow the same convention for the intervals $[a, b]$, $[a, b)$, and $(a, b]$.

2.2. Cocycles, coboundaries, and invariant distributions

From now on let us assume our manifold M is closed, that is, compact and boundaryless, and let $f \in \mathrm{Diff}^r(M)$, with $r \in \mathbb{N}_0 \cup \{\infty\}$. Every $\psi \in C^k(M)$, where $0 \leq k \leq r$, can be considered as a (real) *C^k-cocycle* over f by writing

$$M \times \mathbb{Z} \ni (x, n) \mapsto \mathcal{S}^n\psi(x) \in \mathbb{R},$$

where $\mathcal{S}^n\psi = \mathcal{S}^n_f\psi$ denotes the *Birkhoff sum over* f given by

$$\mathcal{S}^n\psi := \begin{cases} \sum_{i=0}^{n-1} \psi \circ f^i & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \\ -\sum_{i=1}^{-n} \psi \circ f^{-i} & \text{if } n < 0. \end{cases}$$

We say that the cocycle $\psi \in C^k(M)$ is a *C^ℓ-coboundary*, with $0 \leq \ell \leq k \leq r$, whenever there exists $u \in C^\ell(M)$ solving the following cohomological equation:

$$u \circ f - u = \psi.$$

We say that $\phi, \psi \in C^k(\mathbb{T})$ are *C^ℓ-cohomologous* whenever the function $\phi - \psi$ is a C^ℓ-coboundary.

The space of C^ℓ-coboundaries will be denoted by $B(f, C^\ell(M))$, and since it is clearly a linear subspace of $C^\ell(M)$, we can define

$$H^1\big(f, C^\ell(M)\big) := C^\ell(M)/B\big(f, C^\ell(M)\big),$$

called the *first C^ℓ-cohomology space of f*.

This space $H^1(f, C^\ell(M))$ naturally inherits the quotient topology from $C^\ell(M)$. Unfortunately, in general, $B(f, C^\ell(M))$ is not closed in $C^\ell(M)$, and therefore, this quotient topology is non-Hausdorff. So, it is reasonable to propose the following definition.

Definition 2.1
We say that f is *cohomologically C^ℓ-stable* whenever $B(f, C^\ell(M))$ is closed in $C^\ell(M)$. On the other hand, we define the *first reduced C^ℓ-cohomology space* as being

$$\tilde{H}^1\big(f, C^\ell(M)\big) := C^\ell(M)/\operatorname{cl}_\ell\big(B(f, C^\ell(M))\big),$$

where $\operatorname{cl}_\ell(\cdot)$ denotes the closure in the uniform C^ℓ-topology.

As we have already mentioned in Section 1, the study of the structure of the spaces $H^1(f, C^\infty(M))$ and $\tilde{H}^1(f, C^\infty(M))$ naturally leads us to consider the space of f-invariant *(Schwartz) distributions* on M.

So, for each $k \in \mathbb{N}_0$, let $\mathcal{D}'_k(M)$ be the topological dual space of $C^k(M)$, that is, the space of *distributions of order up to k* of M. As usual, the dual space of $C^\infty(M)$ will be simply denoted by $\mathcal{D}'(M)$.

Since all the inclusions $C^{k+1}(M) \hookrightarrow C^k(M)$ and $C^\infty(M) \hookrightarrow C^k(M)$ are continuous and have dense range, we can suppose we have the following chain of inclusions, which are defined modulo unique extensions:

$$\mathcal{D}'_0(M) \subset \mathcal{D}'_1(M) \subset \mathcal{D}'_2(M) \subset \cdots \subset \mathcal{D}'(M).$$

Moreover, since we are assuming that M is compact, it is well known that

$$\mathcal{D}'(M) = \bigcup_{k \geq 0} \mathcal{D}'_k(M).$$

On the other hand, any C^k-diffeomorphism f acts linearly on $C^k(M)$ by pullback, and the adjoint of this action is the linear operator $f_* \colon \mathcal{D}'_k(M) \to \mathcal{D}'_k(M)$ given by

$$\langle f_* T, \psi \rangle := \langle T, \psi \circ f \rangle, \quad \forall\, T \in \mathcal{D}'_k(M),\ \forall\, \psi \in C^k(M).$$

In this case, $\mathrm{Fix}(f_*)$ is the space of *f-invariant distributions of order up to k* and it will be denoted by $\mathcal{D}'_k(f)$. Of course, it holds that $\mathcal{D}'(f) = \bigcup_{k\geq 0} \mathcal{D}'_k(f)$.

As we mentioned above, there is a tight relation between the space of invariant distributions $\mathcal{D}'(f)$ and the reduced cohomology group $\tilde{H}^1(f, C^\infty(M))$. In fact, as a straightforward consequence of the Hahn-Banach theorem we get the following result.

PROPOSITION 2.2
Given any $f \in \mathrm{Diff}^k(M)$, with $k \in \mathbb{N}_0 \cup \{\infty\}$, it holds that

$$\mathrm{cl}_k\big(B(f, C^k(M))\big) = \bigcap_{T \in \mathcal{D}'_k(f)} \ker T.$$

In particular, this implies that

$$\dim \tilde{H}^1\big(f, C^k(M)\big) = \dim \mathcal{D}'_k(f).$$

2.3. Arithmetic

2.3.1. Diophantine and Liouville vectors

For any $\boldsymbol{\alpha} \in \mathbb{R}^d$ we write

$$\|\boldsymbol{\alpha}\| := \mathrm{dist}(\boldsymbol{\alpha}, \mathbb{Z}^d).$$

Notice that, since $\|\boldsymbol{\alpha} + \mathbf{n}\| = \|\boldsymbol{\alpha}\|$ for every $\mathbf{n} \in \mathbb{Z}^d$, we can naturally consider $\|\cdot\|$ as defined on $\mathbb{T}^d$.

We say $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ is *irrational* if and only if, for any $(n_1, \ldots, n_d) \in \mathbb{Z}^d$, it holds that

$$\Big\| \sum_{i=1}^{d} n_i \alpha_i \Big\| = 0 \implies n_i = 0 \quad \text{for } i = 1, \ldots, d. \tag{2.1}$$

Moreover, the vector $\boldsymbol{\alpha}$ is said to be *Diophantine* whenever there exist constants $C, \tau > 0$ satisfying

$$\Big\| \sum_{i=1}^{d} \alpha_i q_i \Big\| \geq \frac{C}{\max_i |q_i|^{\tau}},$$

for every $(q_1, \ldots, q_d) \in \mathbb{Z}^d \setminus \{0\}$. On the other hand, an irrational element of $\mathbb{R}^d$ which is not Diophantine is called *Liouville*.

2.3.2. Continued fractions

In this section we introduce some common notation and recall some elementary and well-known results about continued fractions (see [7] for details).

First of all, the *Gauss map* $A\colon (0, 1) \to [0, 1)$ is defined by

$$A(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$

For each $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ we can associate the sequences $(\alpha_n)_{n \geq 0}$ and $(a_n)_{n \geq 0}$ which are recursively defined by

$$\alpha_0 := \alpha - \lfloor \alpha \rfloor, \qquad \alpha_n := A^n(\alpha_0), \quad \forall\, n \geq 1; \tag{2.2}$$

$$a_0 := \lfloor \alpha \rfloor, \qquad a_{n+1} := \left\lfloor \frac{1}{\alpha_n} \right\rfloor, \quad \forall\, n \geq 0. \tag{2.3}$$

The *nth convergent* of α is defined by

$$p_n/q_n := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}}$$

and the sequences $(p_n)_{n \geq -2}$ and $(q_n)_{n \geq -2}$ satisfy the following recurrences:

$$p_{-2} := 0, \qquad p_{-1} := 1, \qquad p_n := a_n p_{n-1} + p_{n-2}, \quad \forall\, n \geq 0, \tag{2.4}$$

$$q_{-2} := 1, \qquad q_{-1} := 0, \qquad q_n := a_n q_{n-1} + q_{n-2}, \quad \forall\, n \geq 0. \tag{2.5}$$

A very important property about (q_n) that we will repeatedly use in the future is the following:

$$q_{n+1} = \min\big\{q \in \mathbb{N} : \|q\alpha\| < \|q_n\alpha\|\big\}, \quad \forall n \geq 1. \tag{2.6}$$

The reader can easily show that the sequences (p_n) and (q_n) satisfy the following relation:

$$p_{n-1}q_n - p_n q_{n-1} = (-1)^n, \quad \forall\, n \geq -1. \tag{2.7}$$

Now let us define the sequence $(\beta_n)_{n\geq -1}$ by

$$\beta_{-1} := 1, \tag{2.8}$$

$$\beta_n := \prod_{i=0}^{n} \alpha_i, \quad \forall\, n \geq 0. \tag{2.9}$$

By straightforward computations we can show that the sequence (β_n) satisfies

$$\beta_n = (-1)^n(q_n\alpha - p_n) > 0 \tag{2.10}$$

and

$$\frac{1}{q_n + q_{n+1}} < \beta_n < \frac{1}{q_{n+1}}, \tag{2.11}$$

for every $n \geq 0$.

On the other hand, the growth of the sequences (q_n) and (β_n) determines whether the number α is Diophantine or Liouville: if τ denotes any positive real number and we write

$$\mathcal{L}(\alpha, \tau) := \{m \in \mathbb{N} : \beta_m < \beta_{m-1}^{\tau}\}, \tag{2.12}$$

then it is very easy to verify that α is Liouville if and only if $\mathcal{L}(\alpha, \tau)$ has infinitely many elements, for every $\tau > 1$. In fact, this can be proved by rewriting (2.1) for $d = 1$: we have that α is Diophantine if and only if there exist constants $C, \tau > 0$ such that

$$\beta_n = |q_n\alpha - p_n| > \frac{C}{q_n^{1+\tau}}, \quad \forall n \geq 0; \tag{2.13}$$

and, by estimate (2.11), this is equivalent to

$$\beta_{n+1} > C\beta_n^{1+\tau}, \quad \forall n \geq 0. \tag{2.14}$$

2.4. Cohomology of minimal rotations on the torus

Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$ be an irrational vector in $\mathbb{R}^d$. It is well known that in such a case the rotation $R\colon \mathbb{T}^d \to \mathbb{T}^d$ given by

$$R : x \mapsto x + (\boldsymbol{\alpha} + \mathbb{Z}^d)$$

is minimal (i.e., all its orbits are dense in $\mathbb{T}^d$) and uniquely ergodic, where the Haar measure Leb_d is the only R-invariant Borel probability measure.

Moreover, we have the following result which belongs to the *folklore*.

PROPOSITION 2.3

Continuing with the notation we introduced above, we have

$$\mathscr{D}'(R) = \mathbb{R}\mathrm{Leb}_d.$$

On the other hand, R is cohomologically C^∞-stable if and only if $\boldsymbol{\alpha}$ is Diophantine.

Proof

Let $\psi \in C^\infty(\mathbb{T}^d)$ be such that $\int \psi \,\mathrm{dLeb}_d = 0$, and let us consider the Fourier development of ψ:

$$\psi(x) = \sum_{\mathbf{k}\in\mathbf{Z}^d\setminus\{0\}} \hat{\psi}_k e^{2\pi i \mathbf{k}\cdot x}.$$

Then, the Fourier coefficients of any (integrable) solution of the cohomological equation $\psi = uR - u$ must satisfy the following relation:

$$\hat{u}_{\mathbf{k}} := \frac{\hat{\psi}_{\mathbf{k}}}{e^{2\pi i \mathbf{k}\cdot\boldsymbol{\alpha}} - 1}, \quad \forall\, \mathbf{k} \in \mathbb{Z}^d \setminus \{0\}. \tag{2.15}$$

If $\{U_n\}_{n\geq 1}$ is any sequence of finite subsets of $\mathbb{Z}^d$ such that $\bigcup_{n\geq 1} U_n = \mathbb{Z}^d \setminus \{0\}$ and $U_n \subset U_{n+1}$, for every $n \geq 1$, then one can define the trigonometric polynomials

$$\psi_n(x) := \sum_{\mathbf{k}\in U_n} \hat{\psi}_{\mathbf{k}} e^{2\pi i \mathbf{k}\cdot x},$$
$$u_n(x) := \sum_{\mathbf{k}\in U_n} \hat{u}_{\mathbf{k}} e^{2\pi i \mathbf{k}\cdot x} \quad \text{for } n \geq 1.$$

Since $u_n R - u_n = \psi_n$, we have $\psi_n \in B(R, C^\infty(\mathbb{T}^d))$, and clearly $\psi_n \to \psi$ in the C^∞ topology. Thus, $\psi \in \mathrm{cl}_\infty\big(B(f, C^\infty(\mathbb{T}^d))\big)$. By Proposition 2.2, we conclude that $\mathscr{D}'(R) = \mathbb{R}\mathrm{Leb}_d$.

Now, when $\boldsymbol{\alpha}$ is Diophantine it is easy to verify that the Fourier coefficients $(\hat{u}_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ decay sufficiently fast at infinity to guarantee that

$$u(x) := \sum_{\mathbf{k}\in\mathbf{Z}^d\setminus\{0\}} \hat{u}_{\mathbf{k}} e^{2\pi i\mathbf{k}\cdot x}$$

defines a C^∞-function which turns out to be a solution for the cohomological equation $uR - u = \psi$. Applying Proposition 2.2 once again, we obtain

$$B\big(f, C^\infty(\mathbb{T}^d)\big) = \ker \mathrm{Leb}_d = \mathrm{cl}_\infty\big(B(f, C^\infty(\mathbb{T}^d))\big),$$

and therefore, R is cohomologically C^∞-stable.

On the other hand, when $\boldsymbol{\alpha}$ is Liouville it is possible to find a sequence $(\mathbf{n}_j)_{j\geq 1} \subset \mathbb{Z}^d \setminus \{0\}$ satisfying $\mathbf{n}_j \to \infty$, as $j \to +\infty$ and

$$\Big\| \sum_{i=1}^{d} \alpha_i n_{j,i} \Big\| \leq \frac{1}{\max_i |n_{j,i}|^j}, \quad \forall\, j \geq 1,$$

where, of course, $\mathbf{n}_j = (n_{j,1}, \ldots, n_{j,d})$.

This clearly implies that by writing

$$\psi(x) := \sum_{j\in\mathbf{N}} [(e^{2\pi i\mathbf{n}_j\cdot\alpha} - 1)e^{2\pi i\mathbf{n}_j\cdot x} + (e^{-2\pi i\mathbf{n}_j\cdot\alpha} - 1)e^{-2\pi i\mathbf{n}_j\cdot x}]$$

we get $\psi \in C^\infty(\mathbb{T}^d) \cap \ker \mathrm{Leb}_d$. However, $\psi \notin B(R, C^\infty(\mathbb{T}^d))$ because the Fourier coefficients $(\hat{u}_{\mathbf{k}})$ of an eventual solution of the cohomological equation given by (2.15) satisfy

$$\hat{u}_{\mathbf{n}_j} = 1, \quad \forall\, j \in \mathbb{N},$$

with $\mathbf{n}_j \to \infty$, as $j \to \infty$. □

3. Proofs of the corollaries

Since the proof of Theorem A is rather technical, we will start by proving Corollaries B and C, assuming Theorem A.

Proof of Corollary B

Let $F \in \mathrm{Diff}_+^\infty(\mathbb{T})$ be such that $\rho(F) \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$, let μ denote the only F-invariant probability measure, and let $f \in \widetilde{\mathrm{Diff}}_+^\infty(\mathbb{T})$ be a lift of F.

By the unique ergodicity, it easily follows that

$$\int_{\mathbb{T}} \log Df \, \mathrm{d}\mu = 0. \tag{3.1}$$

On the other hand, by Theorem A we know that $\mathscr{D}'(F) = \mathbb{R}\mu$. Hence, if we suppose that F is cohomologically C^∞-stable, by (3.1) and Proposition 2.2 we conclude that $\log Df \in B(F, C^\infty(\mathbb{T}))$; that is, there exists $u \in C^\infty(\mathbb{T})$ satisfying

$$uF - u = \log Df. \tag{3.2}$$

Now, let us write

$$h' := C^{-1} \exp(-u) \in C^\infty(\mathbb{T}),$$

where $C := \int_{\mathbb{T}} \exp(-u) \, \mathrm{dLeb}$. So, in particular h' is positive and satisfies $\int_{\mathbb{T}} h' \, \mathrm{dLeb} = 1$. Hence, defining

$$h(x) := \int_0^x h'(t) \, \mathrm{d}t, \quad \forall x \in \mathbb{R},$$

we have $h \in \widetilde{\mathrm{Diff}}_+^\infty(\mathbb{T})$, and by (3.2) we get

$$D(h \circ f) = (Dh \circ f)Df = (h' \circ f)Df = h' = Dh.$$

This implies that there exists $\rho \in \mathbb{R}$ such that $hf = h + \rho$, and by invariance of the rotation number under conjugacy we have $\rho = \rho(f)$.

Therefore, F is C^∞-conjugate to the rigid rotation $x \mapsto x + \rho(F)$, and applying Proposition 2.3 we conclude that $\rho(F)$ is Diophantine.

Reciprocally, by the Herman-Yoccoz theorem (see [8, chapitre IX, théorème 5.1], [14, p. 335]) any circle diffeomorphism with Diophantine rotation number is C^∞-conjugate to the rigid rotation, and by Proposition 2.3 it must be cohomologically C^∞-stable, as desired. $\square$

Proof of Corollary C

To prove Corollary C we will use two different arguments depending on how well we can approximate the rotation number of the diffeomorphism by rational numbers.

When the rotation number is badly approximated, we will use the finite regularity version of the Yoccoz linearization theorem (see [14, p. 335]). On the other hand, when the rotation number is not “too badly” approximated, we will use a finite regularity version of our Theorem A, which is Theorem 7.1.

So, let $F \in \mathrm{Diff}^{11}_{+}(\mathbb{T})$ be a minimal diffeomorphism, let $f \in \widetilde{\mathrm{Diff}^{11}_{+}}(\mathbb{T})$ be a lift of F, let μ be the only F-invariant probability measure, and let $\phi \in C^1(\mathbb{T})$. Let (p_n) and (q_n) be the sequences associated to $\alpha := \rho(f)$ given by (2.4) and (2.5), respectively, and let $\varepsilon > 0$ be arbitrary. Then, let us consider the set $\mathcal{L}(\alpha, 11/2)$ given by (2.12).

First, let us suppose that $\mathcal{L}(\alpha, 11/2)$ is finite. Thus, as we have already mentioned at the end of Section 2.3.2, in this case α is Diophantine. More precisely, there exists $C > 0$ such that

$$\|q\alpha\| \geq \frac{C}{q^{11/2}}, \quad \forall q \in \mathbb{N}. \tag{3.3}$$

Then, by the Yoccoz linearization theorem [14] we have that F is C^1-conjugate to the rigid rotation R_α. Therefore, applying Proposition 2.3 we can conclude that $\mathcal{D}'_1(F) = \mathbb{R}\mu$.

On the other hand, if we suppose that $\mathcal{L}(\alpha, 11/2)$ has infinite elements, we can apply Theorem 7.1 to conclude that $\mathcal{D}'_1(F) = \mathbb{R}\mu$, too.

In any case, applying Proposition 2.2 we can conclude that there exists $u \in C^1(\mathbb{T})$ such that

$$\left\|(uF - u) - \left(\phi - \int_{\mathbb{T}} \phi \,\mathrm{d}\mu\right)\right\|_{C^1} \leq \frac{\varepsilon}{2}.$$

Writing

$$\tilde{\phi} := uF - u + \int_{\mathbb{T}} \phi \,\mathrm{d}\mu,$$

it clearly holds that

$$\int_{\mathbb{T}} \tilde{\phi} \,\mathrm{d}\mu = \int_{\mathbb{T}} \phi \,\mathrm{d}\mu.$$

Hence, $\|\tilde{\phi} - \phi\|_{C^1} \leq \varepsilon/2$ and

$$\left(\tilde{\phi} - \int_{\mathbb{T}} \tilde{\phi} \,\mathrm{d}\mu\right) \in B\big(F, C^1(\mathbb{T})\big).$$

On the other hand, if we write

$$M_n := \sup_{x \in \mathbb{R}} |f^{q_n}(x) - x - p_n|,$$

by the minimality of F we get $M_n \to 0$ when $n \to \infty$, and so there exists $N \in \mathbb{N}$ such that $\|Du\|_{C^0} M_n \leq \varepsilon/2$, provided $n \geq N$.

Finally, applying the Denjoy-Koksma inequality (1.2) (see Proposition 4.2 for the precise statement) for $\tilde{\phi} - \phi$ we get

$$\begin{aligned}
&\left|\mathcal{S}^{q_n}\phi(x) - q_n \int_{\mathbb{T}} \phi \, \mathrm{d}\mu\right| \\
&\qquad \leq \left|\mathcal{S}^{q_n}(\phi - \tilde{\phi})(x) - q_n \int_{\mathbb{T}} (\phi - \tilde{\phi}) \, \mathrm{d}\mu\right| + \left|\mathcal{S}^{q_n}\tilde{\phi}(x) - q_n \int_{\mathbb{T}} \tilde{\phi} \, \mathrm{d}\mu\right| \\
&\qquad \leq \mathrm{Var}(\phi - \tilde{\phi}) + \left|u\big(F^{q_n}(x)\big) - u(x)\right| \\
&\qquad \leq \int_{\mathbb{T}} |D(\phi - \tilde{\phi})| \, \mathrm{d}\mu + \|Du\|_{C^0} M_n \leq \|\phi - \tilde{\phi}\|_{C^1} + \frac{\varepsilon}{2} \leq \varepsilon
\end{aligned}$$

for every $x \in \mathbb{T}$ and provided that n is sufficiently big. Since ε is arbitrary, Corollary C is proved. □

4. C^r-estimates for real cocycles

This section can be considered the starting point of the proof of Theorem A. The principal new result here is Proposition 4.12, which is mainly inspired by the work of Yoccoz [14].

Throughout this section, F will denote an arbitrary orientation-preserving diffeomorphism of $\mathbb{T}$ with irrational rotation number. Once and for all we fix a lift $f : \mathbb{R} \to \mathbb{R}$ of F, and to simplify the exposition, we write $\alpha := \rho(f) \in \mathbb{R} \setminus \mathbb{Q}$.

Using the notation we introduced in Section 2.3, let (a_n), (α_n), (β_n), (p_n), (q_n) be the sequences associated to α defined by (2.2), (2.4), (2.5), and (2.8).

For each $n \geq 0$ and $\phi \colon \mathbb{T} \to \mathbb{R}$, we define f_n and ϕ_n by

$$f_n := f^{q_n} - p_n \tag{4.1}$$

and

$$\phi_n := \mathcal{S}^{q_n}\phi = \sum_{i=0}^{q_n - 1} \phi \circ f^i. \tag{4.2}$$

For any $x \in \mathbb{R}$ and $n \geq 0$, we consider the following closed intervals:

$$\begin{aligned}
I_n(x) &:= [x, f_n(x)], \\
J_n(x) &:= [f_{n+1}(x), f_n(x)], \\
K_n(x) &:= [f_n^{-2}(x), f_n(x)].
\end{aligned} \tag{4.3}$$

Let us recall that, according to our notation conventions (see Section 2), we denote intervals of the real line, not taking into account the order of their extreme points.

On the other hand, for any $\hat{x} \in \mathbb{T}$, we will write $\hat{I}_n(\hat{x})$, $\hat{J}_n(\hat{x})$, and $\hat{K}_n(\hat{x})$ to denote the intervals $\pi(I_n(x))$, $\pi(J_n(x))$, $\pi(K_n(x)) \subset \mathbb{T}$, respectively, where x is any point in $\pi^{-1}(\hat{x}) \subset \mathbb{R}$.

The reader can find the proof of the following simple and classical result in [2, Chapter I, Lemma 1.3].

LEMMA 4.1
Given any $n \geq 0$ and any $x \in \mathbb{T}$, the interior of the intervals $\hat{I}_n(x)$, $\hat{I}_n(F(x)), \dots,$ $\hat{I}_n(F^{q_{n+1}-1}(x))$ are two-by-two disjoint. In particular, it holds that

$$\begin{aligned} J_n(x) &= I_{n+1}(x) \cup I_n(x), \\ K_n(x) &= I_n\big(f_n^{-2}(x)\big) \cup I_n\big(f_n^{-1}(x)\big) \cup I_n(x). \end{aligned}$$

We will need the following notation:

$$\begin{aligned} m_n(x) &= |f_n(x) - x| = \mathrm{Leb}\big(I_n(x)\big), \\ M_n &:= \max_{x \in \mathbf{R}} m_n(x). \end{aligned} \tag{4.4}$$

For the sake of completeness, let us recall the Denjoy-Koksma inequality (see, e.g., [8]).

PROPOSITION 4.2 (Denjoy-Koksma inequality)
If F is C^0 and $\phi\colon \mathbb{T} \to \mathbb{R}$ has bounded variation, then for each $n \geq 1$, it holds that

$$\left\| \phi_n - q_n \int_{\mathbb{T}} \phi \, \mathrm{d}\mu \right\|_{C^0} \leq \mathrm{Var}(\phi), \tag{4.5}$$

where μ denotes the unique F-invariant probability measure and $\mathrm{Var}(\phi)$ *denotes the total variation of ϕ over $\mathbb{T}$.*

On the other hand, for every $x \in \mathbb{R}$ and $1 \leq k \leq q_{n+1}$ it holds that

$$|\phi_k(y) - \phi_k(z)| \leq \mathrm{Var}(\phi), \quad \forall y, z \in I_n(x). \tag{4.6}$$

4.1. C^r-estimates for the log-derivative cocycle

When F is C^1, we can consider a very particular cocycle which plays a fundamental role in the analysis of the dynamical properties of F: this is $\log DF = \log Df \in C^0(\mathbb{T})$ and will be called the *log-derivative cocycle*.

The fundamental property of the log-derivative cocycle that makes it so important is the chain rule:

$$\mathcal{S}^k(\log Df) = \log Df^k, \quad \forall k \geq 1.$$

Applying Proposition 4.2 for the log-derivative cocycle, we can easily show the following result.

COROLLARY 4.3
If f is C^2, for every $n \geq 2$ we have

$$\max\left\{\|\log Df_n\|_{C^0}, \|\log Df_n^{-1}\|_{C^0}\right\} \leq \operatorname{Var}(\log Df).$$

And, on the other hand, for every $x \in \mathbb{R}$ it holds that

$$C^{-1} < \frac{m_n(x)}{m_n(y)} < C, \quad \forall y \in K_{n-1}(x),$$

where $C := \exp(3\operatorname{Var}(\log Df))$.

The following three propositions are due to Yoccoz [14].

PROPOSITION 4.4
Let f be C^2. Then for every $\ell \geq 1$, $n \geq 2$, and $1 \leq k \leq q_n$ it holds that

$$\sum_{i=0}^{k-1}\big(Df^i(x)\big)^\ell \leq C\frac{M_{n-1}^{\ell-1}}{m_{n-1}(x)^\ell}, \quad \forall x \in \mathbb{T},$$

where $C = C(f) := \exp(\operatorname{Var}(\log Df))$.

PROPOSITION 4.5
Let f be C^r, with $r \geq 3$, and let s be a natural number satisfying $1 \leq s \leq r-1$. Then there exists a real constant $C > 0$, depending only on f and s, such that for every $n \geq 2$ and $1 \leq k \leq q_n$ it holds that

$$|D^s(\mathcal{S}^k \log Df)(x)| = |D^s \log Df^k(x)| \leq C\Big(\frac{\sqrt{M_{n-1}}}{m_{n-1}(x)}\Big)^s, \quad \forall x \in \mathbb{T}.$$

PROPOSITION 4.6
Supposing f is C^3 for every $n \geq 2$ and any $x \in \mathbb{R}$, there exist $y \in I_{n-1}(x)$ and $z \in I_n(x)$ such that

$$m_n(y) = \frac{\beta_n}{\beta_{n-1}} m_{n-1}(z) = \alpha_n m_{n-1}(z).$$

We will also need the following estimate which, to some extent, can be considered as an improvement of Proposition 4.5 for $k = q_n$.

PROPOSITION 4.7
Let us assume f is C^r, with $r \geq 3$. Then there exist a constant $C > 0$ and a natural number n_0, such that for any $x \in \mathbb{R}$, every $n \geq n_0$, and $0 \leq s \leq r-2$ it holds that

$$|D^s \log Df_n(y)| \leq C \frac{\sqrt{M_{n-1}}}{(m_{n-1}(x))^s} \Big[(\sqrt{M_{n-1}})^{r-2} + \frac{m_n(x)}{m_{n-1}(x)} \Big],$$

for every $y \in K_{n-1}(x)$.

Proof
See [15, chapitre III, section 3.6]. □

The following elementary formula relates the derivatives of the Birkhoff sums of the log-derivative cocycles with the derivatives of the iterates of the diffeomorphism. It can be found in [14, p. 337, equation (A)], too.

PROPOSITION 4.8
Given any $r \in \mathbb{N}_0$ and $g \in \mathrm{Diff}_+^{r+1}(\mathbb{R})$, we have

$$D^{r+1} g = P_r(D \log Dg, D^2 \log Dg, \ldots, D^r \log Dg) Dg,$$

where P_r is the polynomial in $\mathbb{Z}[X_1, \ldots, X_r]$ defined inductively by $P_0 = 1$ and

$$P_{r+1}(X_1, \ldots, X_{r+1}) := X_1 P_r(X_1, \ldots, X_r) + \sum_{i=1}^{r} X_{i+1} \frac{\partial P_r}{\partial X_i}(X_1, \ldots, X_r),$$

for every $r \geq 0$.

In particular, all the polynomials P_r satisfy

$$P_r(tX_1, t^2 X_2, t^3 X_3, \ldots, t^r X_r) = t^r P_r(X_1, \ldots, X_r). \tag{4.7}$$

As a straightforward consequence of Propositions 4.5 and 4.8, we get the following result.

COROLLARY 4.9
Given f, s, n, and k as in Proposition 4.5, there exists a constant $C > 0$, depending only on f and s, such that

$$|D^s f^k(x)| \leq C \Big(\frac{\sqrt{M_n}}{m_n(x)} \Big)^{s-1} Df^k(x), \quad \forall x \in \mathbb{T}.$$

4.2. C^r-estimates for arbitrary real cocycles
In this section we shall concern ourselves with arbitrary real cocycles and get some estimates for the (higher-order) derivatives of them. The main difference between the

results of Yoccoz and those examined here is that his estimates hold in the whole circle and ours are instead "local."

Now we can get our first C^1-estimate for real cocycles.

PROPOSITION 4.10

Let f be C^2. Then, there exists a constant $C > 0$ such that for every $\phi \in C^1(\mathbb{T})$, any $n \geq 0$, any $x^\star \in \mathbb{R}$ satisfying $m_n(x^\star) = M_n$, and every $y \in K_n(x^\star)$, it holds that

$$|D(\mathcal{S}^k\phi)(y)| \leq C\frac{\|D\phi\|_{C^0}}{M_n}, \quad \text{for } k = 1, \dots, q_{n+1}.$$

proof

Applying estimate (4.6) to the log-derivative cocycle we get

$$|\log Df^i(y) - \log Df^i(z)| \leq \operatorname{Var}(\log Df),$$

for every $x \in \mathbb{R}$, every $y, z \in I_n(x)$, and $0 \leq i \leq q_{n+1}$. This clearly implies that

$$C^{-1} < \frac{Df^i(y)}{Df^i(z)} < C, \quad \forall y, z \in K_n(x), \tag{4.8}$$

where $C := \exp(3\|D\log Df\|_{L^1(\mathbb{T})})$.

Then, combining Proposition 4.4 and estimate (4.8) we get, for every $y \in K_n(x^\star)$ and $1 \leq k \leq q_{n+1}$,

$$\begin{aligned}|D(\mathcal{S}^k\phi)(y)| &= \Big|\sum_{i=0}^{k-1} D\phi\big(f^i(y)\big)Df^i(y)\Big| \leq \|D\phi\|_{C^0}\sum_{i=0}^{k-1} Df^i(y)\\ &\leq C\|D\phi\|_{C^0}\sum_{i=0}^{k-1} Df^i(x^\star) \leq C\frac{\|D\phi\|_{C_0}}{m_n(x^\star)} = C\frac{\|D\phi\|_{C^0}}{M_n}. \qquad \square\end{aligned}$$

Now let us recall the classical Faà di Bruno equation.

PROPOSITION 4.11

Given $g, h \in C^r(\mathbb{R})$, with $r \geq 1$, it holds that

$$D^r(g \circ h) = \sum_{j=1}^{r}(D^j g \circ h)B_{r,j}(D^1 h, \dots, D^{r-j+1}h),$$

where $B_{r,j}$ is polynomial in $r - j + 1$ variables given by

$$\begin{aligned}&B_{r,j}(x_1, \dots, x_{r-j+1})\\ &\quad = \sum_{(c_i)\in\Omega_{r,j}} \frac{r!}{c_1!\cdots c_{r-j+1}!(1!)^{c_1}\cdots((r-j+1)!)^{c_{r-j+1}}} x_1^{c_1}x_2^{c_2}\cdots x_{r-j+1}^{c_{r-j+1}}\end{aligned}$$

and where

$$\Omega_{r,j} := \Big\{(c_1, \dots, c_{r-j+1}) \in \mathbb{N}_0^{r-j+1} : \sum i c_i = r,\ \sum c_i = j\Big\}.$$

Using the formulas given in Propositions 4.8 and 4.11, together with Yoccoz's estimate of Proposition 4.5, we can extend the previous result to higher-order derivatives.

PROPOSITION 4.12

Let f be C^{r+1} with $r \geq 2$. Then there exists $C > 0$ depending only on f and r such that for every $\phi \in C^{r+1}(\mathbb{T})$, every $n \geq 0$, $1 \leq k \leq q_{n+1}$, and any $x^\star \in \mathbb{R}$ satisfying $m_n(x^\star) = M_n$, it holds that

$$|D^r(\mathcal{S}^k\phi)(y)| \leq C\|\phi\|_{C^r}\Big(\frac{1}{\sqrt{M_n}}\Big)^{r+1}, \quad \forall y \in K_n(x^\star). \tag{4.9}$$

Proof

The case $r = 1$ was already proved in Proposition 4.10, so let us assume that $r \geq 2$. Applying Proposition 4.11 and the estimate given by Corollary 4.9, for any $x \in \mathbb{T}$ we obtain that

$$\begin{aligned}
|D^r(\mathcal{S}^k\phi)(x)| &= \Big|\sum_{i=0}^{k-1} D^r(\phi \circ f^i)(x)\Big| \\
&= \Big|\sum_{i=0}^{k-1}\sum_{j=1}^{r} D^j\phi\big(f^i(x)\big)B_{r,j}\big(Df^i(x), \dots, D^{r-j+1}f^i(x)\big)\Big| \\
&\leq \sum_{j=1}^{r}\|D^j\phi\|_{C^0}\sum_{i=0}^{k-1}\big|B_{r,j}\big(Df^i(x), \dots, D^{r-j+1}f^i(x)\big)\big| \\
&\leq C\|\phi\|_{C^r}\sum_{j=1}^{r}\Big(\frac{\sqrt{M_n}}{m_n(x)}\Big)^{r-j}\sum_{i=0}^{k-1}\big(Df^i(x)\big)^j \\
&\leq C\|\phi\|_{C^r}\sum_{j=1}^{r}\Big(\frac{\sqrt{M_n}}{m_n(x)}\Big)^{r-j}\frac{M_n^{j-1}}{m_n(x)^j} \\
&= C\frac{\|\phi\|_{C^r}}{m_n(x)^r}\sum_{j=1}^{r}(\sqrt{M_n})^{r+j-2} \leq C\|\phi\|_{C^r}\frac{(\sqrt{M_n})^{r-1}}{m_n(x)^r}.
\end{aligned} \tag{4.10}$$

Finally, if $y \in K_n(x^\star)$, by Corollary 4.3 we have $m_n(y) \leq C m_n(x^\star) = C M_n$, where C is any constant bigger than $\exp(3\mathrm{Var}(\log Df))$. Thus, putting together this last estimate with (4.10), we obtain (4.9). □

5. Fibered $\mathbb{Z}^2$-actions and coboundaries

The main purpose of this section is to introduce the *fibered $\mathbb{Z}^2$-actions on* $\mathbb{R}^2$, which shall play a central role in our renormalization scheme. We will see that (a lift of) a circle diffeomorphism and a cocycle naturally induce a fibered $\mathbb{Z}^2$-action. Then, we shall extend the notion of coboundary to fibered $\mathbb{Z}^2$-actions, and in Lemma 5.1 we prove that this new definition indeed generalizes the previous one given in Section 2.2.

Then, in Proposition 5.3, we give a simple but fundamental characterization of certain coboundaries which is mainly inspired by the definition of *quasi-rotations* by Yoccoz [15].

The space $\mathcal{W}^r := \mathrm{Diff}^r_+(\mathbb{R}) \times C^r(\mathbb{R})$ can be seen as a subgroup of $\mathrm{Diff}^r_+(\mathbb{R}^2)$ defining

$$(f, \psi) : (x, y) \mapsto \big(f(x), y + \psi(x)\big),$$

for each $(f, \psi) \in \mathcal{W}^r$.

The space of *fibered $\mathbb{Z}^2$-actions* on $\mathbb{R}^2$ will be denoted by

$$\mathcal{A}^r := \mathrm{Hom}(\mathbb{Z}^2, \mathcal{W}^r) \subset \mathrm{Hom}\big(\mathbb{Z}^2, \mathrm{Diff}^r_+(\mathbb{R}^2)\big).$$

Given any $\Phi \in \mathcal{A}^r$ and any $(m, n) \in \mathbb{Z}^2$, we write

$$\Phi(m, n) = (f_\Phi^{m,n}, \psi_\Phi^{m,n}) \in \mathcal{W}^r.$$

Whenever the action is clear from the context, we shall just write $(f^{m,n}, \psi^{m,n})$ instead of $(f_\Phi^{m,n}, \psi_\Phi^{m,n})$.

There are two group actions on $\mathcal{A}^r$ that will be used in our renormalization scheme: The first one is the left $\mathcal{W}^s$-action $\mathcal{T} : \mathcal{W}^s \times \mathcal{A}^r \to \mathcal{A}^r$ (with $0 \leq s \leq r$), given by conjugation in $\mathrm{Diff}^r_+(\mathbb{R}^2)$; that is,

$$\begin{aligned}\mathcal{T}_{(g,\xi)}(\Phi)(m, n) &:= (g, \xi)\Phi(m, n)(g, \xi)^{-1}\\ &= \big(g f_\Phi^{m,n} g^{-1}, (\psi_\Phi^{m,n} + \xi f_\Phi^{m,n} - \xi) \circ g^{-1}\big),\end{aligned}$$

for every $(g, \xi) \in \mathcal{W}^s$, $\Phi \in \mathcal{A}^r$, and $(m, n) \in \mathbb{Z}^2$. The second one is the left $\mathrm{GL}(2, \mathbb{Z})$-action $\mathcal{U}\colon\ \mathrm{GL}(2, \mathbb{Z}) \times \mathcal{A}^r \to \mathcal{A}^r$ given by change of basis in $\mathbb{Z}^2$; that is,

$$\mathcal{U}_A(\Phi)(m, n) := \Phi(m', n'),$$

where $A \in \mathrm{GL}(2, \mathbb{Z})$ and

$$\begin{pmatrix} m' \\ n' \end{pmatrix} := A^{-1} \begin{pmatrix} m \\ n \end{pmatrix}.$$

A very simple but fundamental remark about these actions is that $\mathcal{T}$ and $\mathcal{U}$ commute.

Most of the time we will work on the subset $\mathcal{A}_0^r \subset \mathcal{A}^r$ given by

$$\mathcal{A}_0^r := \big\{\Phi \in \mathcal{A}^r : \mathrm{Fix}(f_\Phi^{m,n}) = \emptyset,\ \forall (m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}\big\}.$$

Observe that this subset is invariant under the actions $\mathcal{T}$ and $\mathcal{U}$.

Next, notice that each pair $(f, \phi) \in \widetilde{\mathrm{Diff}}_+^r(\mathbb{T}) \times C^r(\mathbb{T}) \subset \mathcal{W}^r$ naturally induces an action $\Gamma = \Gamma(f, \phi) \in \mathcal{A}^r$ given by

$$\Gamma : \begin{cases} (1,0) \mapsto (\tau, 0), \\ (0,1) \mapsto (f, \phi), \end{cases} \tag{5.1}$$

where τ is the translation $x \mapsto x - 1$. Notice that this action Γ belongs to $\mathcal{A}_0^r$ if and only if $\mathrm{Per}(\pi(f)) = \emptyset$, that is, $\rho(f)$ is an irrational number.

Now, taking into account that a circle diffeomorphism and a cocycle induce a fibered $\mathbb{Z}^2$-action, it is reasonable to extend the notion of *coboundary* to fibered $\mathbb{Z}^2$-actions. We say that $\Phi \in \mathcal{A}_0^r$ is a C^s*-coboundary*, with $0 \leq s \leq r$, if and only if there exist $(g, \xi) \in \mathcal{W}^s$ and $A \in \mathrm{GL}(2, \mathbb{Z})$ such that $\Phi' := \mathcal{U}_A(\mathcal{T}_{(g,\xi)}\Phi)$ satisfies the following conditions:

$$f_{\Phi'}^{1,0} = \tau,$$
$$\psi_{\Phi'}^{1,0} = \psi_{\Phi'}^{0,1} \equiv 0.$$

It is very easy to verify that this new notion of coboundary is coherent with the previous one. In fact, we have the following result.

LEMMA 5.1
Let $F \in \mathrm{Diff}_+^r(\mathbb{T})$ (with $0 \leq r \leq \infty$) be such that $\rho(F) \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$, let $\phi \in C^r(\mathbb{T})$, and let $f \in \widetilde{\mathrm{Diff}}_+^r(\mathbb{T})$ be any lift of F. Then $\phi \in B(F, C^s(\mathbb{T}))$ if and only if the induced action $\Gamma(f, \phi)$ given by (5.1) is a C^s-coboundary.

Proof
Let us start by assuming that $\Gamma = \Gamma(f, \phi)$ is a C^s-coboundary. This means that there exist $(g, u) \in \mathcal{W}^s$ and $A \in \mathrm{GL}(2, \mathbb{Z})$ such that $\Xi := \mathcal{T}_{(g,u)}(\mathcal{U}_A\Gamma)$ satisfies

$\psi_{\Xi}^{1,0} = \psi_{\Xi}^{0,1} \equiv 0$ and $f_{\Xi}^{1,0} = \tau$. This implies that

$$\psi_{\Xi}^{m,n} \equiv 0, \quad \forall\, (m,n) \in \mathbb{Z}^2.$$

In particular, if we write $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, we get

$$\psi_{\Xi}^{a,b} = (u \circ \tau - u) \circ g^{-1} \equiv 0, \tag{5.2}$$

$$\psi_{\Xi}^{c,d} = (\phi + u \circ f - u) \circ g^{-1} \equiv 0. \tag{5.3}$$

By (5.2), u is $\mathbb{Z}$-periodic, and by (5.3) ϕ is a C^s-coboundary for F.

Reciprocally, let us suppose that $\phi \in B(F, C^s(\mathbb{T}))$. So, we can find $u \in C^s(\mathbb{T})$ satisfying $uf - u = \phi$. Thus, writing $\Gamma' := \mathcal{T}_{(id,-u)}\Gamma$ we clearly have $f_{\Gamma'}^{1,0} = f_{\Gamma}^{1,0} = \tau$ and $\psi_{\Gamma'}^{1,0} = \psi_{\Gamma'}^{0,1} \equiv 0$. Therefore, Γ is a C^s-coboundary. $\square$

Our next result is a very elementary but useful characterization of coboundaries that will be our fundamental tool for constructing a coboundary as a small perturbation of a cocycle after applying our renormalization scheme. However, first we need a very simple (and well-known) lemma about cohomological equations on the real line.

LEMMA 5.2
Given any $f \in \mathrm{Diff}_+^r(\mathbb{R})$ *with* $\mathrm{Fix}(f) = \emptyset$ *and any* $\phi \in C^r(\mathbb{R})$, *the cohomological equation*

$$uf - u = \phi$$

always admits a solution $u \in C^r(\mathbb{R})$.

Proof
Let us write $z := f(0)$. Since f is fixed-point free, we do not lose any generality by assuming $z > 0$.

Next, let $u\colon [0,z] \to \mathbb{R}$ be any C^r-function satisfying

$$u\big|_{[0,z/3]} \equiv 0, \qquad u\big|_{[2z/3,z]} \equiv \phi.$$

Now, since the interval $[0,z]$ is a fundamental domain for f, for any $x \in \mathbb{R}$ there is a unique $n(x) \in \mathbb{Z}$ such that $f^{-n(x)}(x) \in [0, x_1)$, and so we can extend the function u to the whole real line by writing

$$u(x) := u\big(f^{-n(x)}(x)\big) + \mathcal{S}^{n(x)}\phi\big(f^{-n(x)}(x)\big), \quad \forall x \in \mathbb{R}.$$

By the very definition, u is a C^r-function and satisfies $uf - u = \phi$. $\square$

Now we can state our characterization of coboundaries within the context of fibered $\mathbb{Z}^2$-actions.

PROPOSITION 5.3
Let $\Phi \in \mathcal{A}_0^r$ be so that there exists $x^\star \in \mathbb{R}$ satisfying

$$\psi^{1,0}(x) = 0, \quad \forall x \in [x^\star, f^{0,1}(x^\star)], \tag{5.4}$$

$$\psi^{0,1}(x) = 0, \quad \forall x \in [x^\star, f^{1,0}(x^\star)]. \tag{5.5}$$

Then, Φ is a C^r-coboundary.

Proof
First of all, notice that we do not lose any generality by supposing that

$$f^{0,1}(x^\star) \in \big(f^{-1,0}(x^\star), f^{1,0}(x^\star)\big).$$

In this case there are two possibilities: $f^{0,1}(x^\star)$ belongs to either $(x^\star, f^{1,0}(x^\star))$ or $(f^{-1,0}(x^\star), x^\star)$. For the sake of concreteness, let us suppose that the first case holds (the second one is completely analogous).

By Lemma 5.2, we can find a function $u \in C^r(\mathbb{R})$ such that

$$\psi^{1,0} = uf^{1,0} - u. \tag{5.6}$$

Notice that by (5.4), we have

$$u\big(f^{1,0}(x)\big) = u(x), \quad \forall x \in [x^\star, f^{0,1}(x^\star)]. \tag{5.7}$$

Now, since we are supposing that $[x^\star, f^{0,1}(x^\star)] \subset [x^\star, f^{1,0}(x^\star)]$, from (5.7) we can conclude that there exists a unique function $\bar{u} \in C^r(\mathbb{R})$ satisfying

$$\begin{aligned} \bar{u}|_{[x^\star, f^{1,0}(x^\star)]} &\equiv u|_{[x^\star, f^{1,0}(x^\star)]}, \\ \bar{u}\big(f^{1,0}(x)\big) &= \bar{u}(x), \quad \forall x \in \mathbb{R}. \end{aligned} \tag{5.8}$$

Next, if we define

$$\bar{\psi} := \psi^{0,1} + u - uf^{0,1} \in C^r(\mathbb{R}), \tag{5.9}$$

it can be easily shown that $\bar{\psi}$ is $f^{1,0}$-periodic. In fact, since $(f^{1,0}, \psi^{1,0})$ and $(f^{0,1}, \psi^{0,1})$ commute, we have

$$\psi^{1,1} = \psi^{1,0} + \psi^{0,1} \circ f^{1,0} = \psi^{0,1} + \psi^{1,0} \circ f^{0,1},$$

and so

$$\begin{aligned}\bar{\psi} f^{1,0} &= (\psi^{0,1} + u - uf^{0,1}) f^{1,0} = \psi^{0,1} f^{1,0} + uf^{1,0} - u + u - uf^{1,1} \\ &= \psi^{0,1} f^{1,0} + \psi^{1,0} + u - uf^{1,1} = \psi^{0,1} + \psi^{1,0} f^{0,1} + u - uf^{1,1} \\ &= \psi^{0,1} + (uf^{1,0} - u) f^{0,1} + u - uf^{1,1} = \psi^{0,1} + u - uf^{0,1} \\ &= \bar{\psi}. \end{aligned} \tag{5.10}$$

Now, combining (5.5), (5.8), and (5.10), we can conclude that

$$\bar{\psi} = \bar{u} - \bar{u} f^{0,1}. \tag{5.11}$$

Then, taking into account equations (5.6), (5.9), and (5.11), we can easily see that

$$\mathcal{T}_{(id, \bar{u} - u)} \Phi(\mathbf{i}) = (f^{\mathbf{i}}, 0), \quad \text{for } \mathbf{i} = (1,0),\ (0,1).$$

Finally, applying Lemma 5.2 once again we can construct an orientation-preserving C^r-diffeomorphism $h \colon \mathbb{R} \to \mathbb{R}$ satisfying $hf^{1,0}h^{-1} = \tau$. Thus,

$$\mathcal{T}_{(h, \bar{u} - u)} \Phi = \begin{cases} (1,0) \mapsto (\tau, 0), \\ (0,1) \mapsto (h \circ f^{0,1} \circ h^{-1}, 0), \end{cases}$$

and the proposition is proved. □

6. Renormalization of fibered $\mathbb{Z}^2$-actions

The main aim of this section is to introduce the renormalization scheme for $\mathbb{Z}^2$-actions and to show how it can be used to construct coboundaries by perturbation of the original cocycle. Indeed, the notion of coboundary (for $\mathbb{Z}^2$-actions) turns out to be expressly renormalization-invariant, which allows us to take advantage of the smoothing effect of renormalization.

As in Section 4, F will denote an arbitrary minimal C^r-diffeomorphism of $\mathbb{T}$ (with $r \geq 3$), $f \in \widetilde{\mathrm{Diff}}^r_+(\mathbb{T})$ will denote a lift of F, and $\phi \colon \mathbb{T} \to \mathbb{R}$ will denote an arbitrary real C^r-cocycle.

To simplify the notation, we write $\alpha = \rho(f)$, and since we are assuming that $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we consider the sequences (a_n), (α_n), (β_n), (p_n), (q_n) associated to α defined by (2.2), (2.4), (2.5), and (2.8).

Then, we define the matrices

$$A_n = A_n(\alpha) := (-1)^n \begin{pmatrix} q_n & -p_n \\ -q_{n-1} & p_{n-1} \end{pmatrix}, \quad \forall n \geq -1. \tag{6.1}$$

Notice that, by (2.7), all the matrices A_n belong to $\mathrm{GL}(2, \mathbb{Z})$.

Now, for each $n \geq -1$ we define the *nth renormalized* action $\Gamma_n(\phi)$ by

$$\Gamma_n(\phi) := \mathcal{U}_{A_n}\big(\Gamma(f,\phi)\big), \quad \forall n \geq -1, \tag{6.2}$$

where, of course, $\Gamma(f,\phi)$ denotes the induced action defined by (5.1). Notice that

$$\Gamma_n(\phi) : \begin{cases} (1,0) \mapsto (f_{n-1}, \phi_{n-1}) = (f^{q_{n-1}} - p_{n-1}, \mathcal{S}_f^{q_{n-1}}\phi), \\ (0,1) \mapsto (f_n, \phi_n) = (f^{q_n} - p_n, \mathcal{S}_f^{q_n}\phi). \end{cases}$$

LEMMA 6.1
Let $n \geq 3$, and let $x^\star \in \mathbb{R}$ be arbitrary. Then, there exists $u \in C^r(\mathbb{T})$ such that the cocycle $\bar{\phi} := \phi + u - uF$ satisfies

$$\bar{\phi}_{n-1}(y) = 0, \quad \forall y \in J_{n-1}(x^\star). \tag{6.3}$$

Moreover, there exists a real constant $C > 0$ depending only on F and r such that whenever $x^\star$ satisfies $m_{n-1}(x^\star) = M_{n-1}$, the function u can be chosen so that it fulfills the following estimate:

$$|D^r u(y)| \leq C \|\phi_{n-1}|_{K_{n-1}(x^\star)}\|_{C^r}\, \Theta(\alpha, n, r)\Big(\frac{1}{M_{n-1}}\Big)^r, \tag{6.4}$$

for every $y \in J_{n-1}(x^\star)$, where

$$\Theta(\alpha, n, r) := \sum_{i=0}^{r}\Big(\frac{\beta_{n-1}}{\beta_{n-1} - \beta_n}\Big)^i. \tag{6.5}$$

Proof
First, let $\zeta : \mathbb{R} \to \mathbb{R}$ be any auxiliary smooth function satisfying the following:
- $\zeta(x) = 0$, for every $x \leq 0$;
- $0 < \zeta(x) < 1$, for every $x \in (0,1)$;
- $\zeta(x) = 1$, for every $x \geq 1$.

Let $\hat{x} := \pi(x^\star) \in \mathbb{T}$. By Lemma 4.1 we know that $\hat{I}_n(\hat{x}) \cap \hat{I}_{n-1}(\hat{x}) = \{\hat{x}\}$ and $F^{q_n+q_{n-1}}(\hat{x})$ belongs to the interior of $\hat{I}_{n-1}(\hat{x})$. In particular, this implies that $I_n(x^\star)$ and $f_{n-1}(I_n(x^\star))$ are disjoint, and the last interval is contained in $I_{n-1}(x^\star)$.

We can assume n is odd (the other case is completely analogous), so it holds that

$$f_n(x^\star) < x^\star < f_{n-1}\big(f_n(x^\star)\big) = f_n\big(f_{n-1}(x^\star)\big) < f_{n-1}(x^\star).$$

Notice that since $\Gamma_n(\phi) \in \mathcal{A}_0^r$, the previous relation holds for every $x \in \mathbb{R}$.

Now, we define $u\colon J_{n-1}(x^\star) \to \mathbb{R}$ by

$$u(y) := \zeta\Big(\frac{y - x^\star}{f_n(f_{n-1}(x^\star)) - x^\star}\Big)\phi_{n-1}\big(f_{n-1}^{-1}(y)\big), \tag{6.6}$$

for every $y \in J_{n-1}(x^\star) = [f_n(x^\star), f_{n-1}(x^\star)]$. Then, it clearly holds that

$$u\big(f_{n-1}(y)\big) - u(y) = \phi_{n-1}(y),$$

whenever y and $f_{n-1}(y)$ both belong to $J_{n-1}(x^\star)$, that is, for every $y \in I_n(x^\star)$.

Now, we can extend our function u to the whole real line $\mathbb{R}$ to get a C^r $\mathbb{Z}$-periodic function satisfying

$$u\big(F^{q_{n-1}}(y)\big) - u(y) = \phi_{n-1}(y), \quad \forall y \in \hat{J}_{n-1}(\hat{x}). \tag{6.7}$$

Notice that if we write $\bar{\phi} := \phi + u - uF$, we clearly get

$$\bar{\phi}_{n-1}(y) = \mathcal{S}_f^{q_{n-1}}\bar{\phi}(y) = \phi_{n-1}(y) + u(y) - u\big(f_{n-1}(y)\big) = 0,$$

for every $y \in J_{n-1}(x^\star)$, as desired.

So, it remains to prove that u satisfies estimate (6.4). To do this, first notice that by combining Corollary 4.3 and Proposition 4.6 we get

$$C^{-1}\frac{\beta_n}{\beta_{n-1}} \leq \frac{m_n(f_{n-1}(x^\star))}{m_{n-1}(x^\star)} \leq C\frac{\beta_n}{\beta_{n-1}}, \tag{6.8}$$

where $C > 1$ is a constant which depends on F but does not depend on either n or $x^\star$.

Now, to simplify the notation, let us write

$$\ell := \big|f_n\big(f_{n-1}(x^\star)\big) - x^\star\big|.$$

Observe that, since we are assuming n is odd, we have $\ell = m_{n-1}(x^\star) - m_n(f_{n-1}(x^\star))$. Hence, by (6.8) it holds that

$$\ell \geq \Big(1 - C^{-1}\frac{\beta_n}{\beta_{n-1}}\Big)m_{n-1}(x^\star) \geq C^{-1}\frac{\beta_{n-1} - \beta_n}{\beta_{n-1}}M_{n-1}. \tag{6.9}$$

Now, invoking the Denjoy-Koksma inequality (Proposition 4.2), Corollary 4.9, the Faà di Bruno formula (Proposition 4.11), Proposition 4.12, and estimate (6.9), we can prove (6.4). In fact, for any $y \in J_{n-1}(x^\star) \subset K_{n-1}(x^\star)$ we have

$$\begin{aligned}
|D^r u(y)| &= \Big|\sum_{i=0}^{r}\binom{r}{i} D^i\zeta\Big(\frac{y-x^\star}{\ell}\Big)\ell^{-i} D^{r-i}(\phi_{n-1}\circ f_{n-1}^{-1})(y)\Big| \\
&\le C\sum_{i=0}^{r}\Big(\frac{\beta_{n-1}}{(\beta_{n-1}-\beta_n)M_{n-1}}\Big)^i \Big|\sum_{j=1}^{r-i}(D^j\phi_{n-1})\big(f_{n-1}^{-1}(y)\big) \\
&\qquad\qquad \times B_{r-i,j}(D^1 f_{n-1}^{-1},\ldots,D^{r-i-j+1}f_{n-1}^{-1})\Big| \\
&\le C\|\phi_{n-1}|_{K_{n-1}(x^\star)}\|_{C^r}\Theta(\alpha,n,r) \\
&\quad\times\sum_{i=0}^{r}\Big(\frac{1}{M_{n-1}}\Big)^i\sum_{j=1}^{r-i}\big(Df_{n-1}^{-1}(y)\big)^j\Big(\frac{\sqrt{M_{n-1}}}{m_{n-1}(y)}\Big)^{r-i-j} \\
&\le C\|\phi_{n-1}|_{K_{n-1}(x^\star)}\|_{C^r}\Theta(\alpha,n,r)\Big(\frac{1}{M_{n-1}}\Big)^r,
\end{aligned}\tag{6.10}$$

and estimate (6.4) is proved. □

LEMMA 6.2
Let ϕ, $x^\star$, n, u, and $\bar\phi$ be as in Lemma 6.1. Then there exists $\xi \in C^r(\mathbb{T})$ such that

$$\operatorname{supp}\xi \subset \hat I_{n-1}\big(\pi(x^\star)\big)\cup \hat I_{n-1}\big(\pi(f_{n-1}(x^\star))\big),\tag{6.11}$$

$$\xi(y)+\xi\big(f_{n-1}(y)\big)=\bar\phi_n(y),\quad \forall y\in I_{n-1}(x^\star).\tag{6.12}$$

Moreover, there exists a constant $C>0$ depending only on F and r such that the function ξ can be constructed so that it fulfills the following estimate:

$$\|\xi\|_{C^r}\le C\|\bar\phi_n|_{I_{n-1}(x^\star)}\|_{C^r}\Big(\frac{1}{M_{n-1}}\Big)^r.\tag{6.13}$$

Proof
As in the proof of Lemma 6.1, we will assume n is odd, and therefore, for every $x \in \mathbb{R}$ it holds that $f_n(x) < x < f_{n-1}(x) < f_{n-1}^2(x)$.

Then, let us start by defining ξ on the interval $[x^\star, f_{n-1}^2(x^\star)] = I_{n-1}(x^\star) \cup I_{n-1}(f_{n-1}(x^\star))$ by writing

$$\xi(y) := \begin{cases} \zeta\left(\dfrac{y-x^\star}{f_{n-1}(x^\star)-x^\star}\right)\bar\phi_n(y) & \text{if } y\in I_{n-1}(x^\star), \\ \Big[1-\zeta\Big(\dfrac{f_{n-1}^{-1}(y)-x^\star}{f_{n-1}(x^\star)-x^\star}\Big)\Big]\bar\phi_n\big(f_{n-1}^{-1}(y)\big) & \text{if } y\in I_{n-1}\big(f_{n-1}(x^\star)\big),\end{cases}\tag{6.14}$$

where ζ is the auxiliary function we used in the proof of Lemma 6.1.

In this way, our function ξ is clearly C^r on the interiors of the intervals $I_{n-1}(x^\star)$ and $I_{n-1}(f_{n-1}(x^\star))$, and by the very properties of the auxiliary function ζ, we have

$$D^k\xi(x^\star) = D^k\xi\big(f_{n-1}^2(x^\star)\big) = 0, \quad \text{for } k = 0, 1, \dots, r. \tag{6.15}$$

In order to see that ξ is also continuous and has continuous derivatives up to order r at $f_{n-1}(x^\star)$, let us consider the fibered $\mathbb{Z}^2$-action $\Phi := \Gamma_n(\bar{\phi})$ (see (6.2) for the definition of Γ_n) and notice that condition (6.3) can be translated into the $\mathbb{Z}^2$-action language by stating

$$\psi_\Phi^{1,0}(y) = 0, \quad \forall y \in J_{n-1}(x^\star) = [f_n(x^\star), f_{n-1}(x^\star)]. \tag{6.16}$$

On the other hand, since $\Phi(1,0)$ and $\Phi(0,1)$ commute in $\mathrm{Diff}_+^r(\mathbb{R}^2)$, and taking into account that $f_\Phi^{1,0} = f_{n-1}$ and $f_\Phi^{0,1} = f_n$, we have

$$\psi_\Phi^{0,1}(y) + \psi_\Phi^{1,0}\big(f_n(y)\big) = \psi_\Phi^{1,0}(y) + \psi_\Phi^{0,1}\big(f_{n-1}(y)\big), \quad \forall y \in \mathbb{R}. \tag{6.17}$$

Now, putting together equations (6.16) and (6.17) and recalling that $\psi_\Phi^{0,1} = \bar{\phi}_n$, we conclude that

$$\bar{\phi}_n(y) = \bar{\phi}_n\big(f_{n-1}(y)\big), \quad \forall y \in I_{n-1}(x^\star). \tag{6.18}$$

From (6.18) we can easily show that ξ is continuous and has continuous derivatives up to order r at the point $f_{n-1}(x^\star)$.

Hence, by this remark and (6.15) we can affirm that there is a unique extension of ξ to the whole real line such that it is C^r, $\mathbb{Z}$-periodic, and satisfies

$$\operatorname{supp}\xi \subset \bigcup_{k\in\mathbb{Z}} [x^\star + k, f_{n-1}^2(x^\star) + k].$$

Of course, this is clearly equivalent to (6.11). The condition (6.12) is also satisfied by the pure construction of ξ.

Next, let us prove that ξ satisfies estimate (6.13). To do this, first let y be an arbitrary point in $I_{n-1}(x^\star)$ and notice that

$$\begin{aligned} |D^r\xi(y)| &= \Big|\sum_{j=0}^{r}\binom{r}{j}(D^j\zeta)\Big(\frac{y - x^\star}{f_{n-1}(x^\star) - x^\star}\Big)\Big(\frac{1}{M_{n-1}}\Big)^j D^{r-j}\bar{\phi}_n(y)\Big| \\ &\leq C\Big(\frac{1}{M_{n-1}}\Big)^r \|\bar{\phi}_n|_{I_{n-1}(x^\star)}\|_{C^r}. \end{aligned} \tag{6.19}$$

Now, if y denotes an arbitrary point of $I_{n-1}(f_{n-1}(x^\star))$ and $1 \leq i \leq r$, by Corollary 4.9 and Proposition 4.11 we have

$$\begin{aligned}
&\Big| D^i\Big[\zeta\Big(\frac{f_{n-1}^{-1}(y)-x^\star}{f_{n-1}(x^\star)-x^\star}\Big)\Big]\Big| \\
&\quad = \Big|\sum_{j=1}^{i}(D^j\zeta)\Big(\frac{f_{n-1}^{-1}(y)-x^\star}{f_{n-1}(x^\star)-x^\star}\Big)\Big(\frac{1}{M_{n-1}}\Big)^j \\
&\qquad \times B_{i,j}\big(Df_{n-1}^{-1}(y),\dots,D^{i-j+1}f_{n-1}^{-1}(y)\big)\Big| \\
&\quad \leq C\sum_{j=1}^{i}\Big(\frac{1}{M_{n-1}}\Big)^j\big(Df_{n-1}^{-1}(y)\big)^j\Big(\frac{\sqrt{M_{n-1}}}{m_{n-1}(y)}\Big)^{i-j} \\
&\quad \leq C\sum_{j=1}^{i}\Big(\frac{1}{\sqrt{M_{n-1}}}\Big)^{i+j} \leq C\Big(\frac{1}{M_{n-1}}\Big)^i,
\end{aligned} \tag{6.20}$$

and so

$$\begin{aligned}
|D^r\xi(y)| &\leq C\sum_{i=0}^{r}\Big|D^i\Big[\zeta\Big(\frac{f_{n-1}^{-1}(y)-x^\star}{f_{n-1}(x^\star)-x^\star}\Big)\Big]D^{r-i}\bar{\phi}_n\big(f_{n-1}^{-1}(y)\big)\Big| \\
&\leq C\sum_{i=0}^{r}\Big(\frac{1}{M_{n-1}}\Big)^i\|\bar{\phi}_n\circ f_{n-1}^{-1}|_{I_{n-1}(f_{n-1}(x^\star))}\|_{C^{r-i}} \\
&\leq C\Big(\frac{1}{M_{n-1}}\Big)^r\|\bar{\phi}_n\circ f_{n-1}^{-1}|_{I_{n-1}(f_{n-1}(x^\star))}\|_{C^{r}} \\
&= C\Big(\frac{1}{M_{n-1}}\Big)^r\|\bar{\phi}_n|_{I_{n-1}(x^\star)}\|_{C^{r}},
\end{aligned} \tag{6.21}$$

where the last equality is a consequence of (6.18).

Now, combining (6.19) and (6.21) we can easily get (6.13). □

LEMMA 6.3

Let ϕ, $\bar{\phi}$, and ξ be as in Lemma 6.2. Then the cocycle $\tilde{\phi} := \phi - \xi$ is a C^r-coboundary for F.

Proof

Since ϕ and $\bar{\phi}$ are C^r-cohomologous, this is equivalent to showing that $\bar{\phi} - \xi \in B(F, C^r(\mathbb{T}))$. To do this, we will show that the $\mathbb{Z}^2$-action $\Gamma := \Gamma_n(\bar{\phi} - \xi)$ is a C^r-coboundary.

First, observe that $\Gamma(1,0) = (f_{n-1}, \bar{\phi}_{n-1} - \xi_{n-1})$. By (6.3) we know $\bar{\phi}_{n-1}|_{I_n(x^\star)} \equiv 0$, and if we write $\hat{x} := \pi(x^\star)$, Lemma 4.1 implies that for any $y \in \hat{I}_n(\hat{x})$, it holds that

$$F^i(y) \notin \hat{I}_{n-1}(\hat{x}) \setminus \{\hat{x}\}, \quad \text{for } i = 0, 1, \dots, q_{n-1} - 1. \tag{6.22}$$

On the other hand, we affirm that

$$F^i(y) \notin \hat{I}_{n-1}\big(F^{q_{n-1}}(\hat{x})\big), \quad \text{for } i = 0, 1, \dots, q_{n-1} - 1. \tag{6.23}$$

In fact, let us suppose that (6.23) does not hold. So, there exist $y \in \hat{I}_n(\hat{x})$ and $i \in \{1, \dots, q_{n-1} - 1\}$ such that $F^i(y) \in \hat{I}_{n-1}(F^{q_{n-1}}(\hat{x}))$.

Moreover, we have

$$F^{q_{n-1}-i}\big(F^i(y)\big) = F^{q_{n-1}}(y) \in F^{q_{n-1}}\big(\hat{I}_n(\hat{x})\big) = \hat{I}_n\big(F^{q_{n-1}}(\hat{x})\big), \tag{6.24}$$

and by Lemma 4.1 we know that $F^{q_{n-1}-i}(F^{2q_{n-1}}(\hat{x})) \notin \hat{I}_{n-1}(F^{q_{n-1}}(\hat{x}))$. In particular, this last remark and (6.24) imply that $F^{q_{n-1}-i}(F^{2q_{n-1}}(\hat{x})) \in I_n(F^{q_{n-1}}(\hat{x}))$, and since F is topologically conjugate to the irrational rotation R_α, this clearly contradicts (2.6). Hence, (6.23) is proved.

Now, by (6.11), (6.22), and (6.23), we have $\xi_{n-1}|_{I_n(x^\star)} \equiv 0$, and so

$$\psi_\Gamma^{1,0}(y) = \bar{\phi}_{n-1}(y) = 0, \quad \forall y \in I_n(x^\star) = [x^\star, f_\Gamma^{0,1}(x^\star)]. \tag{6.25}$$

On the other hand, let z be an arbitrary point in $\hat{I}_{n-1}(\hat{x})$, and let us consider the set

$$A_z := \left\{ i \in \mathbb{N}_0 : F^i(z) \in \hat{I}_{n-1}(\hat{x}) \cup \hat{I}_{n-1}\big(F^{q_{n-1}}(\hat{x})\big),\ i < q_n \right\}.$$

Let us prove that $A_z = \{0,\ q_{n-1}\}$. To do this, first notice that clearly $\{0,\ q_{n-1}\} \subset A_z$. Then, consider any $i \in \mathbb{N}$ with $0 < i < q_{n-1}$. Observe that by Lemma 4.1 we have $F^i(z) \notin \hat{I}_{n-1}(\hat{x})$. On the other hand, if $F^i(z)$ belonged to $\hat{I}_{n-1}(F^{q_{n-1}}(\hat{x}))$, we would have $\{F^i(z),\ F^{q_{n-1}-i}(F^i(z))\} \subset \hat{I}_{n-1}(F^{q_{n-1}}(\hat{x}))$, which clearly contradicts Lemma 4.1.

Now let j be any natural number with $q_{n-1} < j < q_n$. Applying Lemma 4.1 once again we know $F^j(z) \notin \hat{I}_{n-1}(z)$. On the other hand, if $F^j(z)$ belonged to $\hat{I}_{n-1}(F^{q_{n-1}}(\hat{x}))$, it would hold that $\{F^{q_{n-1}}(z),\ F^{j-q_{n-1}}(F^{q_{n-1}}(z))\} \subset \hat{I}_{n-1}(F^{q_{n-1}}(\hat{x}))$, which contradicts Lemma 4.1 too. Thus, $A_z = \{0, q_{n-1}\}$.

Now, by (6.12), it holds that

$$\psi_\Gamma^{0,1}(y) = \bar{\phi}_n(y) - \xi(y) - \xi\big(f_{n-1}(y)\big) = 0, \tag{6.26}$$

for every $y \in I_{n-1}(x^\star) = [x^\star, f_\Gamma^{1,0}(x^\star)]$.

Finally, putting together (6.26), (6.25), and Proposition 5.3, we conclude that Γ is a C^r-coboundary, and by Lemma 5.1, $\tilde{\phi} \in B(F, C^r(\mathbb{T}))$, as desired. $\square$

7. Proof of Theorem A

First of all, let us suppose that $\rho(F)$ is Diophantine. Then, by the Herman-Yoccoz theorem (see [8], [14]), F is smoothly conjugate to the rigid rotation $R_{\rho(F)}$, and by Proposition 2.3, we know that $\dim \mathcal{D}'(R_{\rho(F)}) = 1$. Then, $\mathcal{D}'(F)$ is one-dimensional, too; that is, $\mathcal{D}'(F)$ is spanned by the only F-invariant probability measure.

Therefore, from now on we can assume that F exhibits a Liouville rotation number, and Theorem A will follow as a straightforward consequence of the following result, which can be considered as a finitary version of it.

THEOREM 7.1

Let $F \in \mathrm{Diff}^r_+(\mathbb{T})$ (with $r \geq 5$) be such that $\rho(F)$ satisfies the following condition: the set $\mathcal{L}(\alpha, r/2)$ given by (2.12) contains infinitely many elements for some, and hence any, $\alpha \in \pi^{-1}(\rho(F)) \subset \mathbb{R}$ (e.g., when $\rho(F)$ is Liouville).

Let $k := \lfloor (r-5)/6 \rfloor$, and let $\phi \in C^k(\mathbb{T})$ be such that

$$\int_{\mathbb{T}} \phi \, \mathrm{d}\mu = 0,$$

where μ is the only F-invariant probability measure.

Then, given any $\epsilon > 0$, there exists $\tilde{\phi} \in C^k(\mathbb{T})$ such that $\tilde{\phi} \in B(F, C^k(\mathbb{T}))$ and

$$\|\tilde{\phi} - \phi\|_{C^k} \leq \epsilon. \tag{7.1}$$

Notice that, by Proposition 2.2, the conclusion of this theorem can be briefly summarized by saying that $\mathcal{D}'_k(F) = \mathbb{R}\mu$.

Proof of Theorem 7.1

First, let us fix a lift $f \in \widetilde{\mathrm{Diff}^r_+}(\mathbb{T})$ of F, and then we can suppose that $\alpha := \rho(f)$. Let (q_n) and (β_n) be the sequences given by (2.5) and (2.8) associated to the continued fraction expansion of α.

By our arithmetical hypothesis, $\mathcal{L}(\alpha, r/2)$ is an infinite set, so we can find $n \in \mathcal{L}(\alpha, r/2)$ with $n \geq n_0$, where n_0 is the natural number given by Proposition 4.7. Let $x^\star \in \mathbb{R}$ be any point such that $m_{n-1}(x^\star) = M_{n-1}$.

Now, by combining Propositions 4.6 and 4.7, for any $0 \leq s \leq r-2$ and any $y \in K_{n-1}(x^\star)$ we have

$$\begin{aligned} |D^s \log Df_n(y)| &\leq C \frac{\sqrt{M_{n-1}}}{(m_{n-1}(x^\star))^s} \Big((\sqrt{M_{n-1}})^{r-2} + \frac{m_n(x^\star)}{m_{n-1}(x^\star)} \Big) \\ &\leq C \Big((M_{n-1})^{(r-1)/2-s} + \frac{\beta_n}{\beta_{n-1}} (M_{n-1})^{1/2-s} \Big) \\ &\leq C \big((M_{n-1})^{(r-1)/2-s} + \beta_{n-1}^{r/2-1} (M_{n-1})^{1/2-s} \big) \\ &\leq C (M_{n-1})^{(r-1)/2-s}, \end{aligned} \tag{7.2}$$

where the last inequality is a consequence of the fact that

$$\beta_{n-1} = \int_{\mathbb{T}} m_{n-1}(t)\,\mathrm{d}\mu(t) \leq M_{n-1}.$$

Then, if $P_s \in \mathbb{Z}[X_1, \dots, X_s]$ denotes the polynomial of degree s given by Proposition 4.8, applying (4.7) and (7.2) we get

$$\begin{aligned} |D^{s+1} f_n(y)| &= \big|P_s\big(D \log Df_n(y), \dots, D^s \log Df_n(y)\big) Df_n(y)\big| \\ &\leq C\,P_s\big((M_{n-1})^{(r-1)/2-1}, (M_{n-1})^{(r-1)/2-2}, \dots, (M_{n-1})^{(r-1)/2-s}\big) \\ &\leq C\Big(\frac{1}{M_{n-1}}\Big)^s P_s\big((M_{n-1})^{(r-1)/2}, \dots, (M_{n-1})^{(r-1)/2}\big) \\ &\leq C(M_{n-1})^{s((r-1)/2)-s} = C(\sqrt{M_{n-1}})^{rs-3s}, \end{aligned} \tag{7.3}$$

for every $y \in K_{n-1}(x^\star)$.

Now, let $u \in C^r(\mathbb{T})$ be the function we constructed in Lemma 6.1. We affirm that we can find a constant $C > 0$, depending only on f and r, such that

$$\|uf_n - u|_{I_{n-1}(x^\star)}\|_{C^s} \leq C\|\phi\|_{C^{s+1}}(\sqrt{M_{n-1}})^{r-3s-4}. \tag{7.4}$$

To prove this, first notice that for any $0 \leq s \leq r-2$ and $y \in I_{n-1}(x^\star)$, we have

$$\begin{aligned} &\big|D^s u\big(f_n(y)\big) - D^s u(y)\big| \leq \int_y^{f_n(y)} |D^{s+1}u(t)|\,\mathrm{dLeb}(t) \\ &\leq m_n(y)\|D^{s+1}u|_{I_{n-1}(x^\star)}\|_{C^0} \\ &\leq C\frac{\beta_n}{\beta_{n-1}} M_{n-1}\|\phi_{n-1}|_{K_{n-1}(x^\star)}\|_{C^{s+1}}\Theta(\alpha, n, s+1)\Big(\frac{1}{M_{n-1}}\Big)^{s+1} \\ &\leq C\beta_{n-1}^{r/2-1}\|\phi\|_{C^{s+1}}\Big(\frac{1}{\sqrt{M_{n-1}}}\Big)^{s+2}\Big(\frac{1}{1-\beta_{n-1}^{r/2-1}}\Big)^{s+1}\Big(\frac{1}{M_{n-1}}\Big)^s \\ &\leq C\|\phi\|_{C^{s+1}}(M_{n-1})^{r/2-1}\Big(\frac{1}{M_{n-1}}\Big)^{(3s+2)/2} = C\|\phi\|_{C^{s+1}}(\sqrt{M_{n-1}})^{r-3s-4}. \end{aligned} \tag{7.5}$$

Observe that (7.5) is indeed the proof of (7.4) for the particular case $s = 0$.

In the other cases, that is, when $s \geq 1$, we can use the Faà di Bruno equation and estimates (7.2), (7.3), and (7.5) to prove (7.4),

$$\begin{aligned}
|D^s(uf_n - u)(y)| &\leq \left|D^s u\big(f_n(y)\big)Df_n(y) - D^s u(y)\right| \\
&\quad + \left|\sum_{j=1}^{s-1} D^j u\big(f_n(y)\big)B_{s,j}\big(Df_n(y), \ldots, D^{s-j+1}f_n(y)\big)\right| \\
&\leq Df_n(y)\left|D^s u\big(f_n(y)\big) - D^s u(y)\right| + |Df_n(y) - 1||D^s u(y)| \\
&\quad + C\sum_{j=1}^{s-1} \|\phi_{n-1}|_{K_{n-1}(x^\star)}\|_{C^j}\,\Theta(\alpha, n, j)\Big(\frac{1}{M_{n-1}}\Big)^j \\
&\quad \times B_{s,j}\big(1, (\sqrt{M_{n-1}})^{r-3}, \ldots, (\sqrt{M_{n-1}})^{(r-3)(s-j)}\big) \\
&\leq C(\|\phi\|_{C^{s+1}} M_{n-1}^{(r-3s-4)/2} + M_{n-1}^{(r-1)/2}\|\phi_{n-1}|_{K_{n-1}(x^\star)}\|_{C^s} M_{n-1}^{-s}) \\
&\quad + C\sum_{j=1}^{s-1} \|\phi\|_{C^j}\Big(\frac{1}{\sqrt{M_{n-1}}}\Big)^{j+1}\Big(\frac{1}{M_{n-1}}\Big)^j(\sqrt{M_{n-1}})^{(r-3)(s-j)} \\
&\leq C\|\phi\|_{C^{s+1}}[(\sqrt{M_{n-1}})^{r-3s-4} + (\sqrt{M_{n-1}})^{r-3s-1}] \\
&\leq C\|\phi\|_{C^{s+1}}(\sqrt{M_{n-1}})^{r-3s-4},
\end{aligned} \tag{7.6}$$

and (7.4) is proved.

Now, let us consider the cocycle $\bar{\phi}$ as defined in Lemma 6.1, that is, given by $\bar{\phi} := \phi + u - uf$. Notice that whenever $0 \leq s \leq (2r-3)/3$, combining Proposition 4.12 and (7.4) we get

$$|D^s\bar{\phi}_n(y)| \leq |D^s\phi_n(y)| + |D^s(u - uf_n)(y)| \leq C\|\phi\|_{C^{s+1}}\Big(\frac{1}{\sqrt{M_{n-1}}}\Big)^{s+1}, \tag{7.7}$$

for every $y \in I_{n-1}(x^\star)$.

On the other hand, remember that in the middle of the proof of Lemma 6.2 we show that $\bar{\phi}$ satisfies (6.18); that is,

$$\bar{\phi}_n(y) = \bar{\phi}_n\big(f_{n-1}(y)\big), \quad \forall y \in I_{n-1}(x^\star). \tag{7.8}$$

This implies that there exists a unique function $\gamma \in C^r(\mathbb{R})$ which coincides with $\bar{\phi}_n$ on $I_{n-1}(x^\star)$ and is f_{n-1}-invariant on the whole real line. To estimate the C^r-norm of γ, first observe that, by the definition of γ and estimate (7.7), it holds that

$$\|\gamma|_{I_{n-1}(x^\star)}\|_{C^s} = \|\bar{\phi}_n|_{I_{n-1}(x^\star)}\|_{C^s} \leq C\|\phi\|_{C^{s+1}}\Big(\frac{1}{\sqrt{M_{n-1}}}\Big)^{s+1}.$$

On the other hand, applying estimate (7.7) and recalling $\gamma = \gamma \circ f_{n-1}$ for any $y \in I_{n-1}(f_{n-1}(x^\star))$ we get

$$
\begin{aligned}
|D^s\gamma(y)| &= |D^s(\gamma \circ f_{n-1}^{-1})(y)| = |D^s(\bar{\phi}_n \circ f_{n-1}^{-1})(y)| \\
&= \Big| \sum_{i=1}^{s} D^i\bar{\phi}_n\big(f_{n-1}^{-1}(y)\big) B_{s,i}\big(Df_{n-1}^{-1}(y), \ldots, D^{s-i+1} f_{n-1}^{-1}(y)\big)\Big| \\
&\leq \sum_{i=1}^{s} \|\bar{\phi}|_{I_{n-1}(x^\star)}\|_{C^i} \big|B_{s,i}\big(Df_{n-1}^{-1}(y), \ldots, D^{s-i+1} f_{n-1}^{-1}(y)\big)\big| \\
&\leq C\|\phi\|_{C^{s+1}} \sum_{i=1}^{s} \Big(\frac{1}{\sqrt{M_{n-1}}}\Big)^{i+1} \big(Df_{n-1}^{-1}(y)\big)^i \Big(\frac{1}{\sqrt{M_{n-1}}}\Big)^{s-i} \\
&= C\|\phi\|_{C^{s+1}} \Big(\frac{1}{\sqrt{M_{n-1}}}\Big)^{s+1}.
\end{aligned} \tag{7.9}
$$

Repeating this argument we can show that, given any $k \in \mathbb{N}$, there exists a constant $C_k > 0$ depending only on f, r, and k such that

$$
|D^s\gamma(y)| \leq C_k\|\phi\|_{C^{s+1}} \Big(\frac{1}{\sqrt{M_{n-1}}}\Big)^{s+1}, \quad \forall y \in I_{n-1}\big(f_{n-1}^k(x^\star)\big). \tag{7.10}
$$

Now, returning to (7.8), we can affirm that there exists $x_1 \in I_{n-1}(x^\star)$ satisfying

$$
D\gamma(x_1) = D\bar{\phi}_n(x_1) = 0.
$$

Moreover, since γ is (f_{n-1})-periodic and f_{n-1} is a diffeomorphism, $x_1^k := f_{n-1}^k(x_1)$ is a critical point of γ, for every $k \in \mathbb{Z}$.

This implies that, for each integer k, we can find a point $x_2^k \in [f_{n-1}^k(x_1), f_{n-1}^{k+1}(x_1)]$ such that $D^2\gamma(x_2^k) = 0$, and, inductively, we can define the sequence (of sequences) of points $(x_s^k)_{1\leq s\leq r, k\in\mathbb{Z}}$ that satisfies

$$
D^s\gamma(x_s^k) = 0 \qquad \text{and} \qquad x_{s+1}^k \in [x_s^k, x_s^{k+1}] \subset \mathbb{R}, \tag{7.11}
$$

for every $1 \leq s \leq r$ and every $k \in \mathbb{Z}$. Now, if we write

$$
\mathcal{J}_s := I_{n-1}(x^\star) \cup I_{n-1}\big(f_{n-1}(x^\star)\big) \cup \cdots \cup I_{n-1}\big(f_{n-1}^{s-1}(x^\star)\big),
$$

one can easily check that

$$
x_s^0 \in \mathcal{J}_s \qquad \text{and} \qquad \operatorname{Leb}(\mathcal{J}_s) \leq sM_{n-1}. \tag{7.12}
$$

Now, estimates (7.10), (7.11), and (7.12) can be used to improve estimate (7.7). In fact, if $0 \leq s \leq (2r-3)/3$, one can easily check that

$$\begin{aligned} |D^{s-1}\gamma(y)| &\leq \int_{x_s^0}^{y} |D^s\gamma(z)\,\mathrm{d}z| \leq C\|\phi\|_{C^{s+1}}\Big(\frac{1}{\sqrt{M_{n-1}}}\Big)^{s+1} sM_{n-1} \\ &= C\|\phi\|_{C^{s+1}} M_{n-1}^{1-(s+1)/2}, \quad \forall y \in \mathcal{I}_s. \end{aligned} \tag{7.13}$$

Iterating this procedure of integration from the appropriate point x_s^0 we get

$$\begin{aligned} |D^{\bar{s}-j}\bar{\phi}_n(y)| = |D^{\bar{s}-j}\gamma(y)| &\leq C\|\phi\|_{C^{\bar{s}+1}}\Big(\frac{1}{\sqrt{M_{n-1}}}\Big)^{\bar{s}+1} M_{n-1}^{j} \\ &= C\|\phi\|_{C^{\bar{s}+1}} M_{n-1}^{j-(\bar{s}+1)/2}, \quad \forall y \in I_{n-1}(x^\star), \end{aligned} \tag{7.14}$$

and each $j \in \{0, 1, \ldots, \bar{s}-1\}$, where $\bar{s} := \lfloor (2r-3)/3 \rfloor$.

Now recall that we are assuming that ϕ has zero average with respect to μ. Since $\bar{\phi}$ is is cohomologous to ϕ, the same holds for $\bar{\phi}$. Therefore, there must exist a point $x_0 \in I_{n-1}(x^\star)$ such that $\bar{\phi}_n(x_0) = 0$. In particular, estimate (7.14) also holds for $j = \bar{s}$. Then, recalling that the number k is equal to $\lfloor (r-5)/6 \rfloor$, we have

$$\|\bar{\phi}_n|_{I_{n-1}(x^\star)}\|_{C^k} \leq C\|\phi\|_{C^{\bar{s}+1}} M_{n-1}^{\bar{s}-k-(\bar{s}+1)/2} \leq C\|\phi\|_{C^r} M_{n-1}^{(r-3)/6}. \tag{7.15}$$

Then we apply Lemma 6.2 to construct the function $\xi \in C^r(\mathbb{T})$, and putting together estimates (6.13) and (7.15) we obtain

$$\|\xi\|_{C^k} \leq CM_{n-1}^{-k}\|\bar{\phi}_n|_{I_{n-1}(x^\star)}\|_{C^k} \leq C\|\phi\|_{C^r} M_{n-1}^{(r-3)/6-k} \leq C\|\phi\|_{C^r} M_{n-1}^{1/3}. \tag{7.16}$$

Taking into account that C is a real constant which only depends on F and r, that by the minimality $M_m \to 0$ as $m \to +\infty$, and that $\mathcal{L}(\alpha, r/2)$ has infinitely many elements, we conclude that we can choose $n \in \mathcal{L}(\alpha, r/2)$ big enough such that $C\|\phi\|_{C^r} M_{n-1}^{1/3} \leq \epsilon$.

Finally, by Lemma 6.3 the cocycle $\tilde{\phi} := \phi - \xi$ is a C^r-coboundary for F, and by the previous remark we have $\|\tilde{\phi} - \phi\|_{C^k} = \|\xi\|_{C^k} < \epsilon$, as desired. □

Acknowledgments. We are very grateful to Giovanni Forni for several useful discussions and for having pointed out a (serious) mistake in the very first proof of Theorem 6.1.

A. Kocsard would also like to thank Étienne Ghys for having brought the problem that motivated Theorem A to his attention and Anatole Katok for insightful conversations.

We thank the anonymous referees for their valuable suggestions that helped to improve the exposition of the paper.

References

[1] A. AVILA and A. KOCSARD, *Invariant distributions for higher dimensional quasiperiodic maps*, in preparation.

[2] W. DE MELO and S. VAN STRIEN, *One-dimensional Dynamics*, Ergeb. Math. Grenzgeb. (3) **25**, Springer, Berlin, 1993. MR 1239171

[3] É. GHYS, *L'invariant de Godbillon-Vey*, Astérisque **177 – 178** (1989), 155 – 181, Séminaire Bourbaki 1988/1989, no. 706. MR 1040572

[4] ———, *Groups acting on the circle*, Enseign. Math. (2) **47** (2001), 329 – 407. MR 1876932

[5] ———, *Groups acting on the circle: A selection of open problems*, conference lecture at "IIIéme cycle romand de mathématiques," Les Diablerets, Switzerland, 2008, www.unige.ch/˜tatiana/Diablerets08/ghys_diablerets.pdf.

[6] A. HAEFLIGER and L. BANGHE, "Currents on a circle invariant by a Fuchsian group", in *Geometric Dynamics (Rio de Janeiro, 1981)*, Lecture Notes in Math. **1007**, Springer, Berlin, 1983, 369 – 378. MR 0730277

[7] G. H. HARDY and E. M. WRIGHT, *An Introduction to the Theory of Numbers*, 5th ed., Clarendon Press, New York, (1979). MR 0568909

[8] M. C. HERMAN, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Inst. Hautes Études Sci. Publ. Math. **49** (1979), 5 – 233. MR 0538680

[9] S. HURDER, "Dynamics and the Godbillon-Vey class: A history and survey" in *Foliations: Geometry and Dynamics (Warsaw, 2000)*, World Sci., River Edge, N. J., 2002, 29 – 60. MR 1882764

[10] A. KATOK, "Cocycles, cohomology and combinatorial constructions in ergodic theory" in *Smooth Ergodic Theory and Its Applications (Seattle, Wash. 1999)*, Proc. Sympos. Pure Math. **69**, Amer. Math. Soc., Providence, 2001, 107 – 173.

[11] ———, *Combinatorial Constructions in Ergodic Theory and Dynamics*, Univ. Lecture Ser. **30**, Amer. Math. Soc., Providence, 2003. MR 2008435

[12] R. LANGEVIN, "A list of questions about foliations" in *Differential Topology, Foliations, and Group Actions (Rio de Janeiro, 1992)*, Contemp. Math. **161**, Amer. Math. Soc., Providence, 1994, 59 – 80. MR 1271828

[13] A. NAVAS, *Actions de groupes de Kazhdan sur le cercle*, Ann. Sci. Éc. Norm. Supé. (4) **35** (2002), 749 – 758. MR 1951442

[14] J.-C. YOCCOZ, *Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne*, Ann. Sci. Éc. Norm. Supé. (4) **17** (1984), 333 – 359. MR 0777374

[15] ———, *Centralisateurs et conjugaison différentiable des difféomorphismes du cercle*, Astérisque **231** (1995), 89 – 242. MR 1367354

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