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Cohomologically rigid vector fields: the Katok conjecture in dimension 3

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Abstract

A smooth vector field X on a closed orientable d-manifold M is said to be *cohomologically rigid* when given any $\xi \in C^\infty(M,\mathbb{R})$, there exist $u \in C^\infty(M,\mathbb{R})$ and $c \in \mathbb{R}$ satisfying

$$\mathcal{L}_X u = \xi - c,$$

where $\mathcal{L}_X$ is the Lie derivative in the X direction. In 1984, Anatole Katok conjectured that every cohomologically rigid vector field should be smoothly conjugated to a Diophantine vector field on the d-torus $\mathbb{T}^d$. In this work the validity of the Katok conjecture for 3-manifolds is proved.

Résumé

Un champ de vecteurs X sur une variété M compacte orientable de dimension d est dit *cohomologiquement rigide* si pour toute fonction $\xi \in C^\infty(M,\mathbb{R})$ il existe $u \in C^\infty(M,\mathbb{R})$ et $c \in \mathbb{R}$ tels que

$$\mathcal{L}_X u = \xi - c,$$

où $\mathcal{L}_X$ désigne la dérivée de Lie dans la direction de X. En 1984, Anatole Katok a conjecturé que tout champ de vecteurs cohomologiquement rigide devrait être conjugué par un difféomorphisme lisse à un champ linéaire diophantien sur le tore $\mathbb{T}^d$. Dans ce travail nous démontrons la conjecture de Katok pour les variétés de dimension trois.

Keywords: Vector fields; Cohomological equations; Katok conjecture

1. Introduction

When $\mathbb{G}$ is a Lie group acting on a manifold M by a smooth action $\Phi : \mathbb{G} \times M \to M$, many questions about the dynamics of Φ can be answered studying the first cohomology group with real coefficients $H^1(\Phi)$, i.e. the quotient linear space of real cocycles over Φ (from now on, cocycles for short) by the subspace of coboundaries (see Section 2

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for definitions). The problem of analyzing the structure of $H^1(\Phi)$ leads to study *cohomological equations* (see [10,11] for a great panoramic view of the subject).

When $\mathbb{G}$ is equal to $\mathbb{Z}^p$ or $\mathbb{R}^p$, it is rather easy to verify that

$$\dim H^1(\Phi) \geqslant p,$$

and very commonly $H^1(\Phi)$ is infinite-dimensional, being its natural topology (induced by the Fréchet topology of $C^\infty(M,\mathbb{R})$) typically non-Hausdorff. Therefore, the "smallness" of $H^1(\Phi)$ is usually associated with some kind of "rigidity" of Φ, and so, it seems to be natural to say that a $\mathbb{Z}^q$ or $\mathbb{R}^q$-action Φ is *cohomologically rigid* when

$$\dim H^1(\Phi) = q. \tag{1.1}$$

One of the simplest, although important, examples of Lie group actions is given by a flow (an $\mathbb{R}$-action) $\Phi_X : \mathbb{R} \times M \to M$ induced by a smooth vector field $X \in \mathfrak{X}(M)$. In this case, the space of smooth cocycles over Φ_X is canonically identified with $C^\infty(M,\mathbb{R})$, and $\xi \in C^\infty(M,\mathbb{R})$ is a coboundary if and only if there exists $u \in C^\infty(M,\mathbb{R})$, named *transfer function*, satisfying

$$\mathcal{L}_X u = \xi,$$

where $\mathcal{L}_X$ denotes the Lie derivative in the X direction.

Taking into account this identification, $H^1(\Phi_X)$ is naturally isomorphic to $C^\infty(M,\mathbb{R})/\mathcal{L}_X(C^\infty(M,\mathbb{R}))$.

The prototypical example of cohomologically rigid $\mathbb{R}$-action is given by a linear flow on a torus generated by a Diophantine vector field (see Definition 2.5). So far these are the only known examples, and Katok has conjectured [9–11] that, modulo C^∞-conjugation, these are the only ones. More precisely, we have

Katok Conjecture. *If M is a closed orientable d-manifold and X is a cohomologically rigid smooth vector field on M, then there exist a C^∞ diffeomorphism $H : M \to \mathbb{T}^d$ and a Diophantine vector $\alpha \in \mathbb{R}^d$ (see Definition* 2.5) *verifying*

$$DH(X) \equiv \alpha.$$

It is interesting to remark that an analogous statement for higher-rank actions is far from being true. In fact, Katok and Spatzier, in their seminal work [12], showed that all known Anosov $\mathbb{Z}^k$-actions, with $k \geqslant 2$, are cohomologically rigid. A completely different kind of examples were constructed by Urzúa Luz in [28], who proved the existence of cohomologically rigid affine minimal $\mathbb{Z}^d$-actions on tori, with some acting diffeomorphisms different from translations. In all these cases the suspensions of these actions generate cohomologically rigid $\mathbb{R}^d$-actions on manifolds which are not tori.

The main result of this work is the following

Theorem A. *The Katok conjecture is true for* 3*-manifolds.*

We would like to remark that, while this work was in progress, Forni [5] and Matsumoto [15] independently proved the same result.

2. Notations and preliminaries

For simplicity, in this article we will restrict ourselves to work on the C^∞ category, and we shall use the word *smooth* as a synonymous of C^∞.

The d-dimensional torus will be denoted by $\mathbb{T}^d$ and the quotient Lie group $\mathbb{R}^d/\mathbb{Z}^d$ will be our favorite model for it. We will write $\mathrm{pr}_{\mathbb{Z}^d} : \mathbb{R}^d \to \mathbb{T}^d$ for the quotient projection, and $(\theta_1, \theta_2, \ldots, \theta_d)$ for the canonical coordinates in $\mathbb{T}^d$. The Haar probability measure on $\mathbb{T}^d$, also called the Lebesgue measure, will be denoted by Leb^d. As usual, making some abuse of notation, we shall suppose that the elements of $\mathrm{SL}(d,\mathbb{Z})$ act on $\mathbb{T}^d$ by Lie group automorphisms.

For us M will always denote a smooth closed (i.e. compact and without boundary) orientable d-dimensional manifold. We will write $\beta_k(M)$ for the k-th Betti number of M, i.e. $\beta_k(M) \doteq H_k(M,\mathbb{Q})$.

The group of smooth diffeomorphisms of M, endowed with the Whitney C^∞ topology, will denoted by $\mathrm{Diff}(M)$. The subgroup of orientation-preserving diffeomorphisms will be denoted by $\mathrm{Diff}_+(M)$.

Given a smooth fibration $p : N \to M$, the p-fiber over any $x \in M$ will be denoted by N_x, and we shall write $\Gamma(N)$ for the space of smooth sections of p. The only exception for this notational convention is the tangent bundle over M: in this case $\pi : TM \to M$ denotes the canonical projection, and we write T_xM for $\pi^{-1}(x)$, and $\mathfrak{X}(M)$ for $\Gamma(TM)$.

The expression $\Lambda^k(M)$ will be used for the space of smooth k-forms on M.

Given any $X \in \mathfrak{X}(M)$, we write $\{\Phi_X^t\}$ for its induced flow, $i_X : \Lambda^k(M) \to \Lambda^{k-1}(M)$ for the usual contraction by X, and $\mathcal{L}_X$ for the Lie derivative acting on any smooth tensor field on M.

2.1. Measures and distributions

The set of all finite signed Borel measures on M shall be denoted by $\mathfrak{M}(M)$, and we will write $\mathcal{D}'(M)$ for the space of all real continuous linear functionals on $C^\infty(M, \mathbb{R})$. Of course, we assume that $C^\infty(M, \mathbb{R})$ is equipped with its usual Fréchet topology.

In order to avoid confusions, we remark that along this work we use the term *distribution* in the "sense of Schwartz", i.e. for us a distribution is any element of $\mathcal{D}'(M)$, and we reserve the term *plane field* to mean a section of the Grasmannian of *TM*.

Let $\mathbb{G}$ be an arbitrary Lie group and $\Phi : \mathbb{G} \times M \to M$ be a smooth $\mathbb{G}$-action. We define the space of Φ-invariant distributions and measures by

$$\mathcal{D}'(\Phi) \doteq \bigl\{T \in \mathcal{D}'(M) \colon \bigl\langle T, \psi\bigl(\Phi(g, \cdot)\bigr)\bigr\rangle = \langle T, \psi \rangle, \quad \forall g \in \mathbb{G},\ \forall \psi \in C^\infty(M, \mathbb{R})\bigr\}, \tag{2.1}$$

$$\mathfrak{M}(\Phi) \doteq \mathcal{D}'(\Phi) \cap \mathfrak{M}(M). \tag{2.2}$$

When Φ is an $\mathbb{R}$-action induced by $X \in \mathfrak{X}(M)$, we shall use the notations $\mathcal{D}'(X)$ and $\mathfrak{M}(X)$ to mean $\mathcal{D}'(\Phi)$ and $\mathfrak{M}(\Phi)$, respectively. As usual, given any $T \in \mathcal{D}'(M)$ we define $\mathcal{L}_X T \in \mathcal{D}'(M)$ by $\langle \mathcal{L}_X T, \psi \rangle \doteq -\langle T, \mathcal{L}_X \psi \rangle$, for all $\psi \in C^\infty(M, \mathbb{R})$.

We will suppose $\Lambda^d(M)$ canonically embedded in $\mathfrak{M}(M)$, and $\mathfrak{M}(M)$ in $\mathcal{D}'(M)$. On the other hand, since we are assuming that M is orientable, any volume form induces an isomorphism between $C^\infty(M, \mathbb{R})$ and $\Lambda^d(M)$. However, notice that this identification is not canonical at all.

2.2. Cocycles and coboundaries

As above, let $\mathbb{G}$ be an arbitrary Lie group and $\Phi : \mathbb{G} \times M \to M$ be a smooth $\mathbb{G}$-action.

For us, a *cocycle* over Φ is a smooth real function $A : \mathbb{G} \times M \to \mathbb{R}$ satisfying

$$A(g_1 g_2, x) = A\bigl(g_1, \Phi(g_2, x)\bigr) + A(g_2, x), \quad \forall x \in M,\ \forall g_1, g_2 \in \mathbb{G}.$$

The vector space of all cocycles over Φ will be denoted by $\mathcal{Z}(\Phi)$.

On the other hand, we say that a cocycle A is a *coboundary* if there exists a smooth map $B : M \to \mathbb{R}$, usually called *transfer map*, such that

$$A(g, x) = B\bigl(\Phi(g, x)\bigr) - B(x), \quad \forall x \in M,\ \forall g \in \mathbb{G}. \tag{2.3}$$

The set of coboundaries over Φ shall be denoted by $\mathcal{B}(\Phi)$ and it is clearly a linear subspace of $\mathcal{Z}(\Phi)$. The quotient space $\mathcal{Z}(\Phi)/\mathcal{B}(\Phi)$ is called the *first cohomology group of* Φ and it is denoted by $H^1(\Phi)$.

Very commonly, $H^1(\Phi)$ is infinite-dimensional and its natural topology (induced by the usual Fréchet topology of $\mathcal{Z}(\Phi)$) is non-Hausdorff.

Notice that (2.3) implies that

$$\bigl\langle T, A(g, \cdot)\bigr\rangle = 0, \quad \forall A \in \mathcal{B}(\Phi),\ \forall g \in \mathbb{G},\ \forall T \in \mathcal{D}'(\Phi). \tag{2.4}$$

On the other hand, as we have already mentioned in the introduction of this work, if $\mathbb{K}$ denotes either $\mathbb{Z}$ or $\mathbb{R}$, and Φ is a smooth $\mathbb{K}^q$-action, we can easily verify that $\dim H^1(\Phi) \geqslant q$. In fact, the group of $\mathbb{K}$-linear homomorphisms $\mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^q, \mathbb{R})$ naturally injects in $\mathcal{Z}(\Phi)$ (i.e. each $\mathbb{K}$-linear homomorphism can be considered as a cocycle which does not depend on the M-coordinate), and taking into account (2.4), we easily show that the zero cocycle is the only coboundary contained in the image of this injection. Therefore, $H^1(\Phi)$ contains a subspace algebraically isomorphic to $\mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^q, \mathbb{R})$.

Taking into account these remarks, we have the following

Definition 2.1. Let $\mathbb{G}$ be equal to $\mathbb{Z}^q$ or $\mathbb{R}^q$, and $\Phi : \mathbb{G} \times M \to M$ be a smooth $\mathbb{G}$-action. We say that Φ is *cohomologically rigid* iff

$$\dim H^1(\Phi) = q.$$

Remark 2.2. It is important to notice that some authors [21,29,15] use the term "cohomology-free" or "parameter rigid" instead of our terminology. The reader can find some examples of cohomologically rigid actions in [12,16,28,24].

2.3. Cohomologically rigid vector fields

We say that a vector field is cohomologically rigid when its induced flow is. The following is a very simple result that will be very useful in the future:

Proposition 2.3. *A vector field $X \in \mathfrak{X}(M)$ is cohomologically rigid if and only if given any $\xi \in C^\infty(M,\mathbb{R})$, there exists $c = c(\xi) \in \mathbb{R}$ and $u \in C^\infty(M,\mathbb{R})$ satisfying*

$$\mathcal{L}_X u = \xi - c. \tag{2.5}$$

From this it easily follows

Proposition 2.4. *If $X \in \mathfrak{X}(M)$ is cohomologically rigid, then*:

(1) $\dim \mathcal{D}'(X) = 1$. *In particular, $\mathcal{D}'(X) = \mathfrak{M}(X)$ and $\{\Phi_X^t\}$ is uniquely ergodic*;
(2) *There exists a smooth X-invariant volume form $\Omega \in \Lambda^d(M)$*;
(3) *$\{\Phi_X^t\}$ is minimal, i.e. $\{\Phi_X^t(x){:}\ t \in \mathbb{R}\}$ is dense in M, for every $x \in M$.*

The reader can find a proof of this result in [13], which is a simple reformulation of the ideas presented in [11,10] to prove the analogous statement for diffeomorphisms.

To introduce the first result of classification of cohomologically rigid vector fields, we need the following

Definition 2.5. We say that $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ is a *Diophantine vector* if there exist real constants $C, \tau > 0$ satisfying

$$\left| \sum_{i=1}^{d} \alpha_i p_i \right| > C\Big(\max_{1 \leqslant i \leqslant d} |p_i| \Big)^{-\tau}, \tag{2.6}$$

for every $p = (p_1, \dots, p_d) \in \mathbb{Z}^d \setminus \{0\}$.

A vector field X_α on the d-dimensional torus $\mathbb{T}^d$ verifying $X_\alpha \equiv \alpha$ will be called a *Diophantine vector field*.

Now we can easily characterize the cohomologically rigid vector fields on tori:

Proposition 2.6. *A smooth vector field on $\mathbb{T}^d$ is cohomologically rigid if and only if it is smoothly conjugated to a Diophantine one.*

Proof. The proof of Corollary 1.7 in [29] can easily be adapted to prove that every cohomologically rigid vector field on $\mathbb{T}^d$ must be smoothly conjugated to a constant one (see [13] for details).

Finally, using Fourier analysis it is not hard to verify that a constant vector field is cohomologically rigid if and only if it is Diophantine (see for instance [11]). □

2.4. Topological obstructions

Taking into account Proposition 2.6, Katok conjecture essentially affirms that the only closed orientable manifolds that support cohomologically rigid vector fields are tori.

In Proposition 2.4 we saw that every cohomologically rigid vector field was minimal. In particular, it cannot exhibit any singularity, and hence, the Euler characteristic of the supporting manifold must vanish.

For a very long time this was the only known obstruction for the existence of cohomologically rigid vector fields, until F. and J. Rodríguez-Hertz produced a breakthrough in [21], finding additional restrictions for the topology of the supporting manifold.

To recall the main result of [21], we first need to introduce the following

Definition 2.7. Given any $X \in \mathfrak{X}(M)$, we say that $q : M \to \mathbb{T}^n$ is a *Diophantine projection* for X when q is a smooth fibration and

$$Dq(X) \equiv X_\alpha \in \mathfrak{X}(\mathbb{T}^n), \tag{2.7}$$

being X_α a Diophantine vector field (Definition 2.5).

Now we can precisely state

Theorem 2.8. *(F. and J. Rodríguez-Hertz [21].) If M is a closed d-manifold and $X \in \mathfrak{X}(M)$ is cohomologically rigid, then there exists a Diophantine projection $p : M \to \mathbb{T}^{\beta_1}$ for X, where $\beta_1 = \beta_1(M)$ is the first Betti number of M.*

This result is used as the fundamental tool in the proof of Theorem 3.1.

2.5. Tangent, normal and projective flows

In this short paragraph we introduce some terminology that we shall use in Section 4.

Let $X \in \mathfrak{X}^r(M)$ $(r \geqslant 2)$ be a singularity-free vector field in M and $\{\Phi_X^t\}$ be its induced flow. Then, the *tangent flow* of X is nothing but the derivative of its induced flow and will be denoted by $\{D\Phi_X^t\}$.

Since X is singularity-free, we can define the *normal bundle* associated to X as the quotient bundle $NX \doteq TM/\mathbb{R}X \to M$. Its natural bundle projection (induced by $\pi : TM \to M$) will be denoted by $\pi_N : NX \to M$, and we shall write $\mathrm{pr}_X : TM \to NX$ for the canonical quotient projection.

Since the line bundle $\mathbb{R}X$ is invariant under the action of the tangent flow $\{D\Phi_X^t\}$, it clearly induces a linear flow on NX which will be called the *normal flow* and will be denoted by $\{N\Phi_X^t\}$.

Finally, we can projectivize each fiber of $\pi_N : NX \to M$ to get a new fiber bundle $\pi_{\mathbb{P}} : \mathbb{P}(NX) \to M$, called the *projective bundle*. The normal flow will induce a bundle flow on $\mathbb{P}(NX)$ which will be denoted by $\{P\Phi_X^t\}$. We will write $\mathrm{pr}_{\mathbb{P}} : NX \setminus \{0\} \to \mathbb{P}(NX)$ [2] for the canonical quotient projection given by $\mathrm{pr}_{\mathbb{P}} : \hat{v} \mapsto (\mathbb{R} \setminus \{0\})\hat{v}$.

2.6. Hyperbolic dynamics

In this paragraph we introduce some notations and known results on hyperbolic dynamics that will be useful in Section 4.

Given a vector bundle $\pi : E \to M$ and a singularity-free vector field $Y \in \mathfrak{X}^r(M)$ $(r \geqslant 2)$, we say that $A : \mathbb{R} \times E \to E$ is a *linear cocycle over* $\{\Phi_Y^t\}$ if, for every t, $\Phi_X^t \circ \pi = \pi \circ A(t, \cdot)$, where the maps $A(t, \cdot) : \pi^{-1}(p) \to \pi^{-1}(\Phi_X^t(p))$ are linear isomorphisms verifying

$$A(t_0 + t_1, p) = A\big(t_0, \Phi_X^{t_1}(p)\big) A(t_1, p), \quad \forall p \in M,\ \forall t_0, t_1 \in \mathbb{R}.$$

The cocycle A is said to be *Anosov* if there exist two continuous sub-bundles $E^s, E^u \subset E$, a C^0 Finsler structure $\|\cdot\|$ in E, and real constants $C > 0$ and $\rho \in (0, 1)$ verifying

- $E^s \oplus E^u = E$,
- $A(t, E_p^\sigma) = E^\sigma_{\Phi_Y^t(p)}$, for every $p \in M$, every $t \in \mathbb{R}$ and $\sigma = s, u$,
- $\|A(t, \cdot)|_{E^s}\| \leqslant C\rho^t$, and $\|A(-t, \cdot)|_{E^u}\| \leqslant C\rho^t$, for every $t > 0$.

[2] In this context $\{0\}$ means the zero section of NX.

On the other hand, the cocycle A is said to be *quasi Anosov* if, given any $v \in E$, it holds

$$\sup_{t\in\mathbb{R}} \| A(v,t) \| < \infty \Rightarrow v = 0. \tag{2.8}$$

The following result appears in different forms, and in fact with different hypothesis, in the works of Mañé [14], Sacker and Sell [23], and Selgrade [25,26]:

Theorem 2.9. *If the flow $\{\Phi_Y^t\}$ does not have any wandering point (i.e. every non-empty open set $U \subset M$ satisfies $\Phi_Y^t(U) \cap U \neq \emptyset$, for some $t > 1$), then a cocycle A over $\{\Phi_Y^t\}$ is quasi Anosov if and only if it is Anosov.*

On the other hand, we shall say that $Y \in \mathfrak{X}^r(M)$ is *Anosov* if there exists a codimension-one $D\Phi_Y$-invariant subbundle $F \subset TM$ verifying $F \oplus \mathbb{R}X = TM$ and such that $D\Phi_Y|_F : \mathbb{R} \times F \to F$ is an Anosov linear cocycle (over $\{\Phi_Y^t\}$.

Then we have the following result due to Doering:

Theorem 2.10. *(Doering [3].) Let suppose that $\{\Phi_Y^t\}$ does not have any wandering point. Then, Y is an Anosov vector field if and only if its normal flow $\{N\Phi_Y^t\}$ (see Section 2.5) is an Anosov linear cocycle (over $\{\Phi_Y^t\}$).*

3. Non-vanishing first Betti number

In this section we begin the proof of Theorem A considering the case where the supporting manifold has non-trivial real first homology group. In fact, the main purpose of this section consists in proving the following

Theorem 3.1. *Let M be a closed, orientable 3-manifold such that $\beta_1(M) \geqslant 1$, and let us suppose that $X \in \mathfrak{X}(M)$ is a cohomologically rigid vector field. Then, M is diffeomorphic to $\mathbb{T}^3$ and X is C^∞-conjugated to a Diophantine constant vector field on $\mathbb{T}^3$.*

To simplify the exposition, we will subdivide the proof of Theorem 3.1 in the following partial results:

Lemma 3.2. *Under the hypothesis of Theorem 3.1, there exists $n_0 \in \mathbb{Z}$ such that M is diffeomorphic to the torus bundle $\mathbb{T}_A^3 \doteq \mathbb{T}^2 \times \mathbb{R}/(x,t) \sim (Ax, t-1)$, where*

$$A \doteq \begin{pmatrix} 1 & 0 \\ n_0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}). \tag{3.1}$$

In particular, it holds

$$\beta_1(M) \geqslant 2. \tag{3.2}$$

Lemma 3.3. *The flow $\{\Phi_X^t\}$ is smoothly conjugated to the suspension of the affine 2-torus automorphism $A + \gamma$, for some $\gamma \in \mathbb{T}^2$ and A given by (3.1).*

Assuming Lemmas 3.2 and 3.3, now we can easily prove Theorem 3.1:

Proof of Theorem 3.1. Let us suppose that $n_0 \neq 0$ in (3.1). Then, applying a construction attributed to Katok [11], for each $m \in \mathbb{Z} \setminus \{0\}$ we define $T_m \in \mathcal{D}'(\mathbb{T}^2)$ by

$$\langle T_m, \psi \rangle \doteq \sum_{k\in\mathbb{Z}} \hat{\psi}(kn_0m, m) e^{-2\pi ikm(\gamma_2 + \frac{k-1}{2} n_0\gamma_1)}, \quad \psi \in C^\infty(\mathbb{T}^2, \mathbb{R}) \tag{3.3}$$

where γ_i are the coordinates modulo $\mathbb{Z}$ of γ. Straight forward computations show that each T_m is $(A+\gamma)$-invariant and $\{T_m: m \in \mathbb{Z} \setminus \{0\}\}$ is linearly independent.

By Lemma 3.3, there exists a C^∞ diffeomorphism $H : M \to \mathbb{T}_A^3$ conjugating $\{\Phi_X^t\}$ and the suspension of $A + \gamma$, denoted by $\{[A+\gamma]^t\}$.

Finally, for each $m \in \mathbb{Z} \setminus \{0\}$, we define $\tilde{T}_m \in \mathcal{D}'(M)$ by

$$\langle \tilde{T}_m, \psi \rangle \doteq \int_0^1 \big\langle T_m, \big(\psi \circ H^{-1} \circ [A+\gamma]^{-t}\big)\big|_{\mathbb{T}^2 \times \{t\}} \big\rangle \mathrm{d}t, \quad \forall \psi \in C^\infty(M, \mathbb{R}). \tag{3.4}$$

Notice that, by a minor abuse of notation, we are supposing that each T_m is a distribution on $\mathbb{T}^2 \times \{0\} \subset \mathbb{T}^3_A$.

Once again, simple computations let us show that the distributions $\tilde{T}_m$ are X-invariant and linearly independent, which clearly contradicts Proposition 2.4.

Therefore, it must hold $n_0 = 0$, and M is diffeomorphic to $\mathbb{T}^3$. By Proposition 2.6, X is smoothly conjugated to a Diophantine vector field on $\mathbb{T}^3$. □

Lemmas 3.2 and 3.3 will be proved in the following two paragraphs.

3.1. Torus bundle structure

In this paragraph we shall prove Lemma 3.2, so we will continue working under the hypothesis of Theorem 3.1.

Since $\beta_1(M) \geqslant 1$, we can apply Theorem 2.8 to affirm that there exists a Diophantine projection for X over $\mathbb{T}^1$, i.e. a smooth fibration $p: M \to \mathbb{T}^1$ such that $Dp(X) \equiv \alpha \in \mathbb{R} \setminus \{0\}$. This implies that, for all $\theta \in \mathbb{T}^1$, $M_\theta \doteq p^{-1}(\theta)$ is a transverse section for X, and by the minimality of $\{\Phi^t_X\}$, M_θ must be a global section. So, the Poincaré first return map to M_θ is given by the diffeomorphism

$$\mathscr{P}_\theta \doteq \Phi_X^{\alpha^{-1}}\big|_{M_\theta} : M_\theta \to M_\theta.$$

Clearly, $\mathscr{P}_\theta$ cannot exhibit any periodic point. Then, the Euler characteristic of M_θ must vanish [6]. So we can conclude that any fiber of p is diffeomorphic to a disjoint union of k copies of $\mathbb{T}^2$. Let us notice that we do not lose any generality assuming that $k = 1$. In fact, if $E_k : \theta \mapsto k\theta$ is the standard k-fold covering of $\mathbb{T}^1$, then it is easy to verify that we can find another fibration $\tilde{p}: M \to \mathbb{T}^1$ satisfying

$$\begin{tikzcd} & M \arrow[dl, "\tilde{p}"'] \arrow[dr, "p"] & \\ \mathbb{T}^1 \arrow[rr, "E_k"] & & \mathbb{T}^1 \end{tikzcd}$$

From this commutative diagram it follows that each $\tilde{p}$-fiber is a connected component of a p-fiber, and $\tilde{p}$ is clearly a Diophantine projection for X too.

To simplify our notation, we will suppose that our original fibration p has connected fibers. Therefore, p is a 2-torus bundle over $\mathbb{T}^1$, and since M is orientable, there exist a matrix $A \in \mathrm{SL}(2, \mathbb{Z})$ and a diffeomorphism $H: M \to \mathbb{T}^3_A$ verifying $\mathrm{pr}_2 \circ H = p$, where $\mathbb{T}^3_A \doteq \mathbb{T}^2 \times \mathbb{R}/(x,t) \sim (Ax, t-1)$ and $\mathrm{pr}_2 : \mathbb{T}^3_A \to \mathbb{T}^1$ is the projection on the second factor.

From this observation it easily follows that $L(\mathscr{P}_\theta)$, the Lefschetz number of $\mathscr{P}_\theta$, equals to $\det(A - \mathrm{id})$. In particular, by Lefschetz fixed point theorem, we conclude that the spectrum of A is equal to $\{1\}$. Then, we can suppose that A coincides with its Jordan form, i.e.

$$A = \begin{pmatrix} 1 & 0 \\ n_0 & 1 \end{pmatrix}.$$

Finally, let $\Sigma_A : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^3_A$ be the quotient projection induced by the suspension of $A \in \mathrm{SL}(2, \mathbb{Z}) \subset \mathrm{Diff}(\mathbb{T}^2)$. Then, Σ_A is clearly a covering map and its deck group is generated by

$$\mathbb{T}^2 \times \mathbb{R} \ni \big((\theta_1, \theta_2), t\big) \mapsto \big((\theta_1 - n_0\theta_2, \theta_2), t+1\big).$$

This implies that the forms $d\theta_2, dt \in \Lambda^1(\mathbb{T}^2 \times \mathbb{R})$ are invariant under the action of the deck group of Σ_A. Therefore, both forms can be pushed forward by Σ_A getting two closed forms on $\mathbb{T}^3_A$ which clearly induced linear independent elements of $H^1(\mathbb{T}^3_A)$. In particular, this implies $\beta_1(M) \geqslant 2$, as desired.

3.2. Smooth linearization

This paragraph is devoted to prove Lemma 3.3. The main ingredients of the proof are Theorem 2.8, Herman–Yoccoz linearization theorem for smooth circle diffeomorphisms [8,31], and the following "foliated" version of Moser isotopy theorem for volume forms [17], which is nothing but a two-dimensional reformulation of Theorem 6.1 in [29]:

Proposition 3.4. *Let $\Omega_1, \Omega_2 \in \Lambda^2(\mathbb{T}^2)$ be two volume forms and suppose they satisfy*:

$$\Omega_1\big(\mathrm{pr}_1^{-1}(C)\big) = \Omega_2\big(\mathrm{pr}_1^{-1}(C)\big),$$

for every Borel measurable set $C \subset \mathbb{T}^1$ and where we are considering Ω_1 and Ω_2 as elements of $\mathfrak{M}(\mathbb{T}^2)$. Then there exists $H \in \mathrm{Diff}(\mathbb{T}^2)$ isotopic to the identity verifying

$$H^*\Omega_1 = \Omega_2, \quad \textit{and} \quad H^*\, d\theta_1 = d\theta_1,$$

where (θ_1, θ_2) are the canonical coordinates of $\mathbb{T}^2$.

By Lemma 3.2, we know that $\beta_1(M) \geqslant 2$, and hence, by Theorem 2.8, there exists a Diophantine projection $q : M \to \mathbb{T}^2$ for X. Let us write

$$\alpha = (\alpha_1, \alpha_2) \doteq Dq(X) \in \mathbb{R}^2. \tag{3.5}$$

If $\mathrm{pr}_1 : \mathbb{T}^2 \ni (\theta_1, \theta_2) \mapsto \theta_1 \in \mathbb{T}^1$ denotes the projection on the first coordinate, we can define the map $p \doteq \mathrm{pr}_1 \circ q : M \to \mathbb{T}^1$, which is a Diophantine projection for X, too.

Repeating an argument analogous to that used in Section 3.1 we can assume that the fibers of p and q are connected, and therefore, p-fibers (q-fibers) are diffeomorphic to $\mathbb{T}^2$ ($\mathbb{T}^1$, respectively).

Let $\Omega \in \Lambda^3(M)$ be the normalized (i.e. $\int_M \Omega = 1$ when we have already fixed an orientation on M) X-invariant smooth volume form (see Proposition 2.4).

If $\bar\theta$ is any point in $\mathbb{T}^1$, and $\mathscr{P} : p^{-1}(\bar\theta) \to p^{-1}(\bar\theta)$ denotes the Poincaré first return map to $p^{-1}(\bar\theta)$, then

$$\omega \doteq i_X \Omega\big|_{p^{-1}(\bar\theta)} \in \Lambda^2\big(p^{-1}(\bar\theta)\big),$$

is a smooth $\mathscr{P}$-invariant area form on $p^{-1}(\bar\theta)$.

On the other hand, every q-fiber is contained in a p-fiber, and then, on each p-fiber we have a 1-dimensional foliation, whose leaves are all diffeomorphic to $\mathbb{T}^1$, which is invariant under the action of $\mathscr{P}$.

It is well-known that any two smooth codimension-one foliations in $\mathbb{T}^2$ with all their leaves compact are diffeomorphic. Hence, we can find a C^∞ diffeomorphism $H_0 : p^{-1}(\theta_0) \to \mathbb{T}^2$ verifying

$$DH_0\big(T\big(q^{-1}(\bar\theta, \theta)\big)\big) = \ker d\theta_1, \quad \forall \theta \in \mathbb{T}^1. \tag{3.6}$$

where (θ_1, θ_2) are the canonical coordinates in $\mathbb{T}^2$.

Notice that Eq. (3.6) lets us affirm that $H_0 \circ \mathscr{P} \circ H_0^{-1}$ is a skew-product over $\mathbb{T}^1$, i.e. there exists $h_0 \in \mathrm{Diff}_+(\mathbb{T}^1)$ and $\eta \in C^\infty(\mathbb{T}^2, \mathbb{R})$ such that

$$H_0\big(\mathscr{P}\big(H_0^{-1}(\theta_1, \theta_2)\big)\big) = \big(h_0(\theta_1), \theta_2 + n_0\theta_1 + \big(\eta(\theta_1, \theta_2) + \mathbb{Z}\big)\big). \tag{3.7}$$

On the other hand, the rotation number of h_0 is equal to $\alpha_2\alpha_1^{-1} + \mathbb{Z}$, where $\alpha = (\alpha_1, \alpha_2)$ is given by (3.5). Taking into account that α is Diophantine, it is easy to verify that there exist positive real constants C and τ satisfying

$$\left| m - n\frac{\alpha_2}{\alpha_1} \right| \geqslant \frac{C}{n^\tau}, \quad \forall m \in \mathbb{Z},\ \forall n \in \mathbb{N}. \tag{3.8}$$

Therefore, by Herman–Yoccoz linearization theorem [8,31] we know that h_0 is smoothly conjugated to the rotation $\theta \mapsto \theta + (\alpha_2\alpha_1^{-1} + \mathbb{Z})$. So, we do not lose any generality supposing that in (3.7) h_0 is indeed equal to the rotation, and this is what we will do to simplify the notation.

In particular, this implies that Leb^2 and $H_0^{-1*}\omega$ satisfy the hypothesis of Proposition 3.4, so there exists $H_1 \in \mathrm{Diff}_+(\mathbb{T}^2)$ preserving the vertical foliation $\{\{\theta_1\} \times \mathbb{T}^1\}_{\theta_1 \in \mathbb{T}^1}$ of $\mathbb{T}^2$ and such that

$$H_1^*\mathrm{Leb}^2 = H_0^{-1*}\omega. \tag{3.9}$$

From (3.7) and (3.9) we can conclude that $H_1 \circ H_0 \circ \mathcal{P} \circ H_0^{-1} \circ H_1^{-1}$ is an area-preserving skew-product in $\mathbb{T}^2$, and therefore,

$$H_1\big(H_0\big(\mathcal{P}\big(H_0^{-1}(H_1^{-1}(\theta_1,\theta_2))\big)\big)\big) = \left(\theta_1 + \left(\frac{\alpha_2}{\alpha_1} + \mathbb{Z}\right), \theta_2 + n_0\theta_1 + \big(\zeta(\theta_1) + \mathbb{Z}\big)\right), \tag{3.10}$$

for some $\zeta \in C^\infty(\mathbb{T}^1, \mathbb{R})$.

On the other hand, using Fourier series techniques analogous to those used to prove Proposition 2.6 we can easily show that, since $\alpha_2\alpha_1^{-1}$ satisfies estimate (3.8), the rotation $\theta \mapsto \theta + (\alpha_2\alpha_1^{-1} + \mathbb{Z})$ is cohomologically rigid. Therefore, we can find $u \in C^\infty(\mathbb{T}^1, \mathbb{R})$ verifying

$$u\big(\theta + \big(\alpha_2\alpha_2^{-1} + \mathbb{Z}\big)\big) - u(\theta) = -\zeta(\theta) + \int_{\mathbb{T}^1} \zeta \, \mathrm{dLeb}^1, \quad \forall \theta \in \mathbb{T}^1. \tag{3.11}$$

Finally, conjugating $H_1 \circ H_0 \circ \mathcal{P} \circ H_0^{-1} \circ H_1^{-1}$ with the diffeomorphism

$$(\theta_1, \theta_2) \mapsto \big(\theta_1, \theta_2 + u(\theta_1)\big),$$

we get an affine 2-torus automorphism whose linear part is equal to A, as desired.

4. Vanishing first Betti number

In this section we complete the proof of Theorem A, proving the following

Theorem 4.1. *If M is a closed orientable 3-manifold so that $H_1(M, \mathbb{R}) = 0$, then M does not support any cohomologically rigid smooth vector field.*

So, from now on we shall assume that M satisfies the hypothesis of Theorem 4.1, and by contradiction, we will suppose that $X \in \mathfrak{X}(M)$ is a cohomologically rigid vector field.

Our first step to prove Theorem 4.1 is the following

Proposition 4.2. *There exists $\lambda \in \Lambda^1(M)$ such that $\lambda(p) \neq 0$, for every $p \in M$, and*

$$\mathcal{L}_X\lambda \equiv 0 \quad \textit{and} \quad i_X\, d\lambda \equiv 0.$$

Proof. By Proposition 2.3 we know that there exists an X-invariant volume form $\Omega \in \Lambda^3(M)$. Hence, if we write $\omega \doteq i_X\Omega$, Cartan's formula lets us affirm

$$0 = \mathcal{L}_X\Omega = d(i_X\Omega) + i_X(d\Omega) = d\omega.$$

Notice that, by Poincaré duality, $H^2(M, \mathbb{R}) = 0$. Then, there exists a 1-form $\tilde{\lambda}$ such that $\omega = d\tilde{\lambda}$. Applying Cartan's formula once again we obtain

$$\mathcal{L}_X\tilde{\lambda} = d(i_X\tilde{\lambda}) + i_X(d\tilde{\lambda}) = d(i_X\tilde{\lambda}) + i_X(i_X\Omega) = d(i_X\tilde{\lambda}),$$

where $i_X\tilde{\lambda}$ is an element of $C^\infty(M, \mathbb{R})$. So, there exists a smooth function $u : M \to \mathbb{R}$ verifying

$$\mathcal{L}_X u = -i_X\tilde{\lambda} + \int_M (i_X\tilde{\lambda})\Omega. \tag{4.1}$$

Therefore, if we define $\lambda \doteq \tilde{\lambda} + du$, it still holds $d\lambda = \omega$, and $i_X d\lambda = i_X(i_X\Omega) \equiv 0$. Moreover,

$$\begin{aligned}\mathcal{L}_X\lambda &= \mathcal{L}_X\tilde{\lambda} + \mathcal{L}_X du = d(i_X\tilde{\lambda}) + d(i_X du) \\ &= d\big(i_X\tilde{\lambda} + \mathcal{L}_X u\big) = d\left(\int_M (i_X\tilde{\lambda})\Omega\right) = 0.\end{aligned}$$

Then, taking into account the minimality of $\{\Phi_X^t\}$, we easily see that λ exhibits a singularity if and only if $\lambda \equiv 0$. On the other hand, since $d\lambda = i_X\Omega \neq 0$, we know that $\lambda \not\equiv 0$, and so, λ does not have any singularity. $\square$

Notice that, since $\dim M = 3$, it holds

$$0 = i_X(\lambda \wedge \Omega) = (i_X\lambda)\Omega - \lambda \wedge i_X\Omega$$
$$= (i_X\lambda)\Omega - \lambda \wedge \omega = (i_X\lambda)\Omega - \lambda \wedge d\lambda.$$

That is

$$\lambda \wedge d\lambda = (i_X\lambda)\Omega, \tag{4.2}$$

where $i_X\lambda$ is a real constant.

In the following paragraphs we will analyze separately the cases where $i_X\lambda \neq 0$ and $i_X\lambda = 0$.

4.1. The contact case: $i_X\lambda \neq 0$

When $i_X\lambda \neq 0$, Eq. (4.2) implies that λ is a contact form. On the other hand, by Proposition 4.2 we know that $i_X d\lambda \equiv 0$, and therefore, $(i_X\lambda)^{-1}X$ is the Reeb vector field induced by λ. And then we easily get to a contradiction invoking the famous Weinstein conjecture [30], which was recently proven by Taubes:

Theorem 4.3. *(Taubes [27].) Let N be a closed orientable 3-manifold, $\eta \in \Lambda^1(N)$ be a smooth contact form and $Y \in \mathfrak{X}(N)$ be its Reeb vector field (i.e. $i_Y\eta \equiv 1$ and $i_Y d\eta \equiv 0$). Then, the flow induced by Y exhibits a periodic orbit.*

Clearly, the existence of a periodic orbit is not compatible with the minimality of $\{\Phi_X^t\}$, and we get the desired contradiction.

Nevertheless, at this point we must recognize that it would be very desirable to solve this case not invoking Taubes' proof of Weinstein conjecture. In fact, in a certain way the Katok conjecture is just the first step toward the comprehension of how the topology of the manifold generates obstructions when we try to solve cohomological equations. And from this point of view the sophistication of the techniques used by Taubes in [27] does not let us understand what is really happening in this case.

4.2. The completely integrable case: $i_X\lambda = 0$

When $i_X\lambda = 0$, Eq. (4.2) implies that $\ker\lambda$ is a completely integrable plane field, i.e. there exists a smooth codimension-one foliation in M tangent to $\ker\lambda$.

While this work was in progress, Forni communicated to the author that he had been able to solve this case using the foliation tangent to $\ker\lambda$ to prove that M should be diffeomorphic to a nilmanifold and $\{\Phi_X^t\}$ smoothly conjugated to a homogeneous flow. On the other hand, Greenfield and Wallach had already proved in [7] that $\mathbb{T}^3$ was the only 3-dimensional nilmanifold that supported cohomologically rigid homogeneous vector fields (this result was extended by Flaminio and Forni [4] to higher dimensional nilmanifolds).

Our techniques are completely different to those of Forni [5]. The main novelty of ours consists in using the integrability condition in a very indirect way, and this lets us believe that most of our proof could be reusable to solve the "contact case" independently of Taube's proof of Weinstein conjecture [27].

Very roughly, our strategy consists in doing a very detailed analysis of the dynamics of the tangent flow $\{D\Phi_X^t\}$, proving that the normal flow as well as the tangent flow restricted to the kernel of λ exhibit a "parabolic behavior". Then, we shall see that this implies that our original flow $\{\Phi_X^t\}$ should be positively expansive (see Definition 4.12). Finally, we show that there does not exist positively expansive flows on closed 3-manifolds.

4.2.1. Two simple properties of the normal flow

In this short paragraph we start the analysis of the dynamics of the normal flow $\{N\Phi_X^t\}$, presenting two very simple lemmas.

Let us start introducing any smooth Riemannian structure $\langle\cdot,\cdot\rangle$ in TM. This naturally induces another Riemannian structure $\langle\cdot,\cdot\rangle_{NX}$ in NX defining, for each $p \in M$,

$$\langle v_1, v_2\rangle_{NX} \doteq \langle v_1', v_2'\rangle, \quad \forall v_1, v_2 \in NX_p,$$

where v_i' is the only element of T_pM verifying simultaneously $\langle X(p), v_i'(p)\rangle = 0$ and $\mathrm{pr}_X(v_i') = v_i$. The Finsler structures induced by $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{NX}$ will be denoted by $\|\cdot\|$ and $\|\cdot\|_{NX}$, respectively. As usual, we shall also use the Riemannian structures $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{NX}$ to measure angles between non-null vectors of the same fiber. Making some abuse of notation, we shall use the symbol $\sphericalangle(\cdot,\cdot)$ for both.

Now, we can present our first result about the dynamics of our normal flow $\{N\Phi_X^t\}$:

Lemma 4.4. *There exists $\hat{v}_0 \in NX$ such that $\hat{v}_0 \neq 0$ and*

$$\sup_{t\in\mathbb{R}} \left\| N\Phi_X^t(\hat{v}_0)\right\|_{NX} < \infty. \tag{4.3}$$

Proof. Let us suppose estimate (4.3) is not satisfied by any non-vanishing vector in NX. In other words, let suppose that $N\Phi_X : \mathbb{R}\times NX \to NX$ is quasi-Anosov. By Theorem 2.9, $\{N\Phi_X^t\}$ is an Anosov cocycle. Then, Theorem 2.10 lets us affirm that X is indeed Anosov.

Finally, it is a very well-known fact that any Anosov flow exhibits (infinitely many) periodic orbits [1], which clearly contradicts the minimality of $\{\Phi_X^t\}$. $\square$

Our second result about the dynamics of the normal flow is the following

Lemma 4.5. *The normal flow $\{N\Phi_X^t\}$ is conservative. More precisely, there exists a symplectic form κ on the vector bundle $\pi_N : NX \to M$ which is invariant under the action of $\{N\Phi_X^t\}$.*

Proof. Notice that $\omega = i_X\Omega = d\lambda$ is a 2-form on TM verifying $i_X\omega \equiv 0$. This implies that we may push-forward this form by pr_X on NX, i.e. we can find a smooth 2-form κ on NX such that

$$\kappa\big(\mathrm{pr}_X(v), \mathrm{pr}_X(w)\big) = \omega(v,w), \quad \forall v,w \in T_pM,\ \forall p\in M.$$

Now, it is very easy to verify that κ is symplectic on NX and $N\Phi_X$-invariant. $\square$

4.2.2. Dynamics of the projective flow

This paragraph is devoted to prove that the dynamics of the projective flow is very simple. In fact, we shall get that the limit set of $\{P\Phi_X^t\}$ is a smooth submanifold of $\mathbb{P}(NX)$ which happens to be a graph over M, being the dynamics on this set smoothly conjugated to $\{\Phi_X^t\}$.

For this, first we need the following result due to Nakayama and Noda about the geometry and amount of minimal sets of the projective flow:

Theorem 4.6. *(Nakayama and Noda [18].) Let V be a closed 3-manifold and let $Y \in \mathfrak{X}(V)$ be such that its induced flow $\Phi_Y : \mathbb{R}\times V \to V$ is minimal.*

Let $P\Phi_Y : \mathbb{R}\times \mathbb{P}(NY) \to \mathbb{P}(NY)$ be the projective flow of Y. Hence, we have:

(1) *If $\{P\Phi_Y^t\}$ exhibits more than two minimal sets, then V is diffeomorphic to $\mathbb{T}^3$ and $\{\Phi_X^t\}$ is continuously conjugate to an irrational translation.*
(2) *If $\{P\Phi_Y^t\}$ exhibits exactly two minimal sets $M_1, M_2 \subset \mathbb{P}(NY)$ and $\{\Phi_X^t\}$ is not C^0-conjugate to an irrational translation on $\mathbb{T}^3$, then for any $z\in V$ it holds: $M_1\cap\pi_{\mathbb{P}}^{-1}(z)$ or $M_2\cap\pi_{\mathbb{P}}^{-1}(z)$ consists of a single point. Moreover, there exists a residual subset $B\subset V$ such that both sets $M_1\cap\pi_{\mathbb{P}}^{-1}(z)$ and $M_2\cap\pi_{\mathbb{P}}^{-1}(z)$ contain just a point, for every $z\in B$.*

Since we are assuming that $H_1(M,\mathbb{R}) = 0$, Theorem 4.6 lets us affirm that the flow $\{P\Phi_X^t\}$ exhibits at most two minimal sets. Then, observe that one of the minimal sets (in fact, we will prove this is the only one) is given by the $\mathrm{pr}_{\mathbb{P}}$-projection of the plane field

$$\Sigma \doteq \ker\lambda \subset TM.$$

Indeed, for any $p\in M$ we have $X(p)\in\Sigma_p$, and hence,

$$E_\Sigma \doteq \mathrm{pr}_X\big(\Sigma\setminus\{0\}\big) \subset NX, \tag{4.4}$$

is a smooth one-dimensional linear sub-bundle of NX. In this way, E_Σ determines exactly one point on each fiber of $\pi_{\mathbb{P}}:\mathbb{P}(NX)\to M$. More precisely, we may define $\theta_p \doteq \mathrm{pr}_{\mathbb{P}}(E_{\Sigma p}\setminus\{0\})\in\pi_{\mathbb{P}}^{-1}(p)$.

Notice that since the plane field Σ is invariant under the action of $\{D\Phi_X^t\}$, we have the flow $\{N\Phi_X^t\}$ leaves invariant the line field E_Σ, and therefore, it holds $P\Phi_X^t(\theta_p)=\theta_{\Phi_X^t(p)}$, for every $p\in M$ and every $t\in\mathbb{R}$. So, summarizing we have

$$K_\Sigma \doteq \{\theta_p:\ p\in M\}\subset\mathbb{P}(NX) \tag{4.5}$$

is a minimal set for $\{P\Phi_X^t\}$.

As it was already mentioned above, we aim to prove the following

Theorem 4.7. *The only minimal set for the flow $\{P\Phi_X^t\}$ is $K_\Sigma\subset\mathbb{P}(NX)$, defined in* (4.5).

To prove Theorem 4.7 we shall suppose that there exists another minimal set $K_0\subset\mathbb{P}(NX)$ (of course, different from K_Σ), and for the sake of clarity of the exposition we will divide the proof in several lemmas:

Lemma 4.8. *The sub-bundle $E_\Sigma\subset NX$ defined in* (4.4) *is orientable, and therefore, it admits a non-vanishing section $\hat{Y}_0\in\Gamma(E_\Sigma)$.*

Proof. Since M is orientable and Σ was defined as the kernel of a non-singular 1-form, the vector bundle $\pi|_\Sigma:\Sigma\to M$ is orientable. On the other hand, our vector field X can be considered as a non-singular element of $X\in\Gamma(\Sigma)$. This lets us affirm that $\Sigma\to M$ is a globally trivial vector bundle. Therefore, we can find a smooth section $Y_0\in\Gamma(\Sigma)$ verifying $\Sigma_p=\mathrm{span}\{X(p),Y_0(p)\}$, for every $p\in M$.

Finally, defining $\hat{Y}_0\doteq\mathrm{pr}_X(Y_0)$ we get our desired section of $E_\Sigma\to M$. □

Lemma 4.9. *Assuming that there exists another minimal set $K_0\subset\mathbb{P}(NX)$ (different from K_Σ), we can find a non-vanishing $\hat{Y}\in\Gamma(E_\Sigma)$ verifying*

$$N\Phi_X^t\big(\hat{Y}(p)\big)=\hat{Y}\big(\Phi_X^t(p)\big),\quad \forall p\in M,\ \forall t\in\mathbb{R}. \tag{4.6}$$

Proof. Let $L_\Sigma\in C^\infty(M,\mathbb{R})$ be defined by

$$L_\Sigma(p)\hat{Y}_0(p)=\lim_{t\to 0}\frac{N\Phi_X^{-t}(\hat{Y}_0(\Phi_X^t(p)))-\hat{Y}_0(p)}{t},\quad \forall p\in M, \tag{4.7}$$

where $\hat{Y}_0$ is the smooth section of E_Σ given by Lemma 4.8.

Using the fact that X is cohomologically rigid, we can find a function $u\in C^\infty(M,\mathbb{R})$ verifying

$$\mathcal{L}_X u=-L_\Sigma+\int_M L_\Sigma\Omega. \tag{4.8}$$

Then, if we define $\hat{Y}\doteq e^u\hat{Y}_0$, we clearly get

$$\lim_{t\to 0}\frac{N\Phi_X^{-t}(\hat{Y}(\Phi_X^t(p)))-\hat{Y}(p)}{t}=\bigg(\int_M L_\Sigma\Omega\bigg)\hat{Y}(p),\quad \forall p\in M,$$

and therefore, it holds

$$N\Phi_X^t\big(\hat{Y}(p)\big)=\exp\bigg(t\int_M L_\Sigma\Omega\bigg)\hat{Y}\big(\Phi_X^t(p)\big), \tag{4.9}$$

for every $p\in M$ and every $t\in\mathbb{R}$.

Notice that by Eq. (4.9), $\int_M L_\Sigma\Omega$ is a Lyapunov exponent of the linear cocycle $\{N\Phi_X^t\}$. So, let us suppose that $\int_M L_\Sigma\Omega\neq 0$. In this case, the one-dimensional sub-bundle $E_\Sigma\subset NX$ is uniformly hyperbolic.

On the other hand, by Theorem 4.6 we know that K_0 and K_Σ are the only two minimal sets in $\mathbb{P}(NX)$, and moreover, we can find a point $p_0\in M$ such that $\theta'\in\mathbb{P}(NX)$ is the only point in $K_0\cap\pi_{\mathbb{P}}^{-1}(p_0)$.

Observe that since K_0 and K_Σ are disjoint and closed, there exists a real constant $C > 0$ such that

$$\mathrm{dist}_{\mathbb{P}}\big(P\Phi_X^t(\theta_{p_0}), P\Phi_X^t(\theta')\big) > C, \quad \forall t \in \mathbb{R}, \tag{4.10}$$

where $\mathrm{dist}_{\mathbb{P}}$ denotes the distance function on $\mathbb{P}(NX)$ induced by the Riemannian structure $\langle\cdot,\cdot\rangle_{NX}$.

Then, by conservativeness proved in Lemma 4.5, estimate (4.10) and Eq. (4.9) we have that any vector $\hat{v} \in NX_{p_0}$ whose pr_X-projection is equal to $\theta' \in K_0 \cap \pi_{\mathbb{P}}^{-1}(p_0)$ will satisfy the following estimate:

$$\big\|N\Phi_x^t(\hat{v})\big\|_{NX} \leqslant C' \exp\biggl(-t \int\limits_M L_\Sigma \Omega\biggr)\|\hat{v}\|_{NX}, \quad \forall t \in \mathbb{R}, \tag{4.11}$$

for some real constant $C' > 0$, which just depends on constant C in (4.10).

From Eq. (4.9) and estimate (4.11) (and supposing that $\int_M L_\Sigma \Omega \neq 0$), we clearly conclude that Oseledets splitting (see [19]) of $\{N\Phi_X^t\}$ ($\{N\Phi_X^t\}$ can be thought as a linear cocycle over $\{\Phi_X^t\}$) is not just measurable, but continuous and uniformly hyperbolic. This implies that $\{N\Phi_X^t\}$ is an Anosov cocycle, and by Theorem 2.10 we know that X must be Anosov, which is clearly impossible since $\{\Phi_X^t\}$ does not have any periodic orbit.

Therefore, the contradiction arises from our assumption that $\int_M L_\Sigma \Omega \neq 0$. Finally, Eq. (4.9) lets us assure that $\hat{Y}$ is a $N\Phi_X$-invariant section, as desired. □

Now, we are ready to complete the

Proof of Theorem 4.7. Let $K_0 \subset \mathbb{P}(NX)$, $p_0 \in M$, $\hat{Y} \in \Gamma(E_\Sigma)$ and $\theta' \in K_0 \cap \pi_{\mathbb{P}}^{-1}(p_0) \subset \mathbb{P}(NX)$ as above. Let $\hat{v} \in NX_{p_0}$ such that $\mathrm{pr}_{\mathbb{P}}(\hat{v}) = \theta'$.

We can rewrite estimate (4.10) as

$$\inf_{t\in\mathbb{R}} \sphericalangle\big(\hat{Y}\big(\Phi_X^t(p)\big), N\Phi_X^t(\hat{v})\big) > 0. \tag{4.12}$$

Putting together Eq. (4.6), estimate (4.12) and Lemma 4.5 we see that there exists a real constant $C'' > 1$ so that

$$\frac{1}{C''} < \big\|N\Phi_X^t(\hat{v})\big\|_{NX} < C'', \quad \forall t \in \mathbb{R}. \tag{4.13}$$

Now, consider another vector $\hat{w} \in NX_{p_0} \setminus \{0\}$ such that $\mathrm{pr}_{\mathbb{P}}(\hat{w}) \notin K_\Sigma \cup K_0$. Since K_Σ and K_0 are the only minimal sets for $\{P\Phi_X^t\}$, we know that the ω-limit of $\mathrm{pr}_{\mathbb{P}}(\hat{w})$ must intersects either K_0 or K_Σ. Let us suppose that the positive semi-orbit of $\mathrm{pr}_{\mathbb{P}}(\hat{w})$ accumulates on K_Σ. This implies that

$$\liminf_{t\to+\infty} \sphericalangle\big(\hat{Y}\big(\Phi_X^t(p_0)\big), N\Phi_X^t(\hat{w})\big) = 0. \tag{4.14}$$

Once again, taking into account that $\{N\Phi_X^t\}$ preserves the symplectic form κ and the section $\hat{Y} \in \Gamma(NX)$, we see that Eq. (4.14) implies that

$$\limsup_{t\to+\infty} \big\|N\Phi_x^t(\hat{w})\big\|_{NX} = \infty. \tag{4.15}$$

Finally, we clearly see that estimates (4.12), (4.13) and (4.15) violate conservativeness.

Analogously we can get a contradiction supposing that the ω-limit of $\mathrm{pr}_{\mathbb{P}}(\hat{w})$ intersects K_0. In this way we conclude that K_Σ is the only minimal set for $\{P\Phi_X^t\}$. □

4.2.3. Dynamics of the normal flow

In Section 4.2.1 we begun the analysis of the dynamics of the normal flow $\{N\Phi_x^t\}$. After what we have just done in Section 4.2.2, here we shall see that some of those results can be considerably improved. In fact, we will completely characterize the dynamics of $\{N\Phi_X^t\}$, showing that it exhibits a parabolic behavior.

In Lemma 4.4 we showed that there was some non-null vector in NX such that its whole $N\Phi_X$-orbit was bounded. On the other hand, in Lemma 4.9, under the assumption that there were two different minimal sets for $\{P\Phi_X^t\}$, we proved there existed $\hat{Y} \in \Gamma(E_\Sigma)$ which was invariant under the action of $\{N\Phi_X^t\}$. Our first result of this paragraph consists in proving that we can get the same invariant section assuming, in this case, that K_Σ is the only minimal set:

Lemma 4.10. *There exists a non-vanishing section $\hat{Y} \in \Gamma(E_\Sigma)$ verifying*

$$N\Phi_X^t\big(\hat{Y}(p)\big) = \hat{Y}\big(\Phi_X^t(p)\big), \quad \forall p \in M,\ \forall t \in \mathbb{R}. \tag{4.16}$$

Proof. Continuing with the notation of Lemma 4.9, let $\hat{Y}_0 \in \Gamma(E_\Sigma)$ be a section given by Lemma 4.8, $L_\Sigma \in C^\infty(M, \mathbb{R})$ defined by Eq. (4.7), $u : M \to \mathbb{R}$ given by Eq. (4.8), and $\hat{Y} \doteq e^u \hat{Y}_0$.

Recalling equation (4.9), we have

$$N\Phi_X^t\big(\hat{Y}(p)\big) = \exp\left(t \int_M L_\Sigma \Omega\right) \hat{Y}\big(\Phi_X^t(p)\big), \quad \forall t \in \mathbb{R}.$$

On the other hand, by Lemma 4.4, there exists $\hat{v}_0 \in NX \setminus \{0\}$ such that its $N\Phi_X$-orbit is bounded, and by the arguments used in the proof of Theorem 4.7, we know

$$\liminf_{t \to \pm\infty} \operatorname{dist}_{\mathbb{P}}\left(\operatorname{pr}_{\mathbb{P}}\left(\hat{Y}\big(\Phi_X^t\big(\pi_N(\hat{v}_0)\big)\big)\right), \operatorname{pr}_{\mathbb{P}}\big(N\Phi_x^t(\hat{v}_0)\big)\right) = 0. \tag{4.17}$$

This clearly implies that $\|N\Phi_X^t(\hat{Y})\|_{NX}$ cannot exhibit exponential growth, and therefore $\int_M L_\Sigma \Omega = 0$, getting the desired invariance of $\hat{Y}$. □

Next, notice that $\pi_N : NX \to M$ is an orientable vector bundle with 2-dimensional fibers and $\hat{Y}$ is a non-singular section of this bundle. This clearly implies that $\pi_N : NX \to M$ is globally trivial, so we can find a smooth section $\hat{Z}_0 \in \Gamma(NX)$ verifying

$$\kappa\big(\hat{Y}(p), \hat{Z}_0(p)\big) = 1, \quad \forall p \in M, \tag{4.18}$$

and in particular, it holds $\operatorname{span}\{\hat{Y}, \hat{Z}_0\} = NX$.

The arguments used in the proof of Theorem 4.7 let us affirm that there exists $\sigma \in \{-1, 1\}$ satisfying

$$\begin{aligned} &\liminf_{t \to +\infty} \sphericalangle\big(N\Phi_X^t\big(\hat{Z}_0(p)\big), \sigma \hat{Y}\big(\Phi_X^t(p)\big)\big) = 0, \\ &\liminf_{t \to -\infty} \sphericalangle\big(N\Phi_X^t\big(\hat{Z}_0(p)\big), -\sigma \hat{Y}\big(\Phi_X^t(p)\big)\big) = 0. \end{aligned} \tag{4.19}$$

There is no lost of generality if we suppose that $\sigma = 1$ in (4.19).

Using $\{\hat{Y}, \hat{Z}_0\}$ as an ordered basis for NX, $N\Phi_X^t : NX_p \to NX_{\Phi_X^t(p)}$ can be represented by an element of $\mathrm{SL}(2, \mathbb{R})$, and indeed, it will have the following form:

$$N\Phi_X^t(p) = \begin{pmatrix} 1 & \hat{a}(p,t) \\ 0 & 1 \end{pmatrix}, \tag{4.20}$$

where $\hat{a} : M \times \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying $\hat{a}(\cdot, 0) = 0$.

Then, if we define $\hat{A} \in C^\infty(M, \mathbb{R})$ by $\hat{A}(p) \doteq \partial_t \hat{a}(p,t)|_{t=0}$, we can find a smooth real function $\hat{B}$ such that

$$\mathcal{L}_X \hat{B} = -\hat{A} + \int_M \hat{A}\Omega. \tag{4.21}$$

Function $\hat{B}$ can be used to define a new section

$$\hat{Z} \doteq \hat{Z}_0 - \hat{B}\hat{Y} \in \Gamma(NX),$$

and in this way we clearly have

$$N\Phi_X^t\big(\hat{Z}(p)\big) = \hat{Z}\big(\Phi_X^t(p)\big) + t\left(\int_M \hat{A}\Omega\right)\hat{Y}\big(\Phi_X^t(p)\big) \tag{4.22}$$

for any $t \in \mathbb{R}$ and $p \in M$.

From (4.19) and (4.22) we easily see that $\int_M \hat{A}\Omega > 0$, proving that, in fact, $\{N\Phi_X^t\}$ exhibits a parabolic behavior as desired.

4.2.4. Dynamics on Σ

In this short paragraph we shall analyze the dynamics of the tangent flow $D\Phi_X : \mathbb{R} \times TM \to TM$ restricted to the invariant sub-bundle $\Sigma \subset TM$.

Our main result here aims to prove that $\{D\Phi_X^t\}$ on Σ, as $\{N\Phi_X^t\}$ on NX, has a parabolic behavior. In fact, the techniques used here are very similar to those used in Section 4.2.1. The only novelty is that *a priori* we do not have any information about the projective flow induced by $D\Phi_X : \mathbb{R} \times \Sigma \to \Sigma$.

In this case we know that, for each p and t, $D\Phi_X^t(X(p)) = X(\Phi_X^t(p))$ and therefore, we should prove that all the vectors non-collinear with X have polynomial growth and their directions converge to the direction of X.

Let us start considering any smooth vector field $Y_0 \in \Gamma(\Sigma) \subset \mathfrak{X}(M)$ verifying

$$\mathrm{pr}_X\big(Y_0(p)\big) = \hat{Y}(p), \quad \forall p \in M. \tag{4.23}$$

Then, notice that putting together Eqs. (4.16) and (4.23) we can affirm that

$$\mathcal{L}_X Y_0 = AX, \tag{4.24}$$

for some $A \in C^\infty(M, \mathbb{R})$.

Once again, since X is cohomologically rigid, there exists $B \in C^\infty(M, \mathbb{R})$ satisfying

$$\mathcal{L}_X B = -A + \int_M A\Omega. \tag{4.25}$$

We use this function B to define a new vector field

$$Y \doteq Y_0 + BX \in \Gamma(\Sigma) \subset \mathfrak{X}(M). \tag{4.26}$$

Notice that it continues to hold $\mathrm{span}\{X, Y\} = \Sigma \subset TM$ and, additionally, we get

$$\mathcal{L}_X Y \equiv \bigg(\int_M A\Omega\bigg) X. \tag{4.27}$$

Thus, we have the following

Lemma 4.11. *Function $A \in C^\infty(M, \mathbb{R})$ given by Eq.* (4.24) *satisfies*

$$\int_M A\Omega \neq 0.$$

Proof. Contrarily, let us suppose that $\int_M A\Omega = 0$.

Then, Eq. (4.27) is equivalent to say that $[X, Y] \equiv 0$, i.e. X and Y commute. Since X and Y generate Σ, in particular they are everywhere linearly independent, and so, these vector fields induce a locally free $\mathbb{R}^2$-action on M.

Finally, by a result due to Rosenberg, Roussarie and Weil [22] we know that the only orientable closed 3-manifolds admitting locally free $\mathbb{R}^2$-actions are 2-torus bundles over a circle, and our manifold M clearly does not satisfy this property since we are assuming that $H_1(M, \mathbb{R}) = 0$. $\square$

As a corollary of this lemma we easily see that, given any $p \in M$, it holds $\|D\Phi_X^t(Y(p))\| \to \infty$, uniformly as $t \to \pm\infty$, and

$$\begin{aligned} &\lim_{t\to+\infty} \sphericalangle\big(D\Phi_X^t\big(Y(p)\big), \sigma_0 X\big(\Phi_X^t(p)\big)\big) = 0, \\ &\lim_{t\to-\infty} \sphericalangle\big(D\Phi_X^t\big(Y(p)\big), -\sigma_0 X\big(\Phi_X^t(p)\big)\big) = 0, \end{aligned} \tag{4.28}$$

where $\sigma_0 \doteq \mathrm{sign}(\int_M A\Omega) \in \{1, -1\}$.

For the sake of simplicity, and since we do not lose any generality, we shall assume that $\int A\Omega > 0$, and thus, $\sigma_0 = 1$.

Summarizing what we have just proved, $D\Phi_X^t : \Sigma_p \to \Sigma_{\Phi_X^t(p)}$ is a parabolic linear map, and taking the ordered set $\{X, Y\}$ as basis of $\Sigma \subset TM$, we can represent it by

$$D\Phi_X^t\big|_\Sigma = \begin{pmatrix} 1 & t(\int_M A\Omega) \\ 0 & 1 \end{pmatrix}. \tag{4.29}$$

4.2.5. Expansiveness

In this paragraph we finish the proof of Theorem 4.1. For this, let us start recalling the definition of *expansive flow* due to Bowen and Walters [2]:

Definition 4.12. Given a compact metric space (K, d), a continuous flow $\Psi : \mathbb{R} \times K \to K$ is called *expansive* if it satisfies the following property:

For every $\epsilon > 0$, there is a $\delta > 0$ such that if there exists a pair of points $x, y \in K$ and an orientation preserving homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ verifying

$$d\big(\Psi^t(x), \Psi^{h(t)}(y)\big) < \delta, \quad \forall t \in \mathbb{R}, \tag{4.30}$$

then $y = \Psi^\tau(x)$, for some $\tau \in (-\epsilon, \epsilon)$.

Moreover, we shall say that Ψ is *positively expansive* (respectively *negatively expansive*) if the above estimate (4.30) is satisfied replacing $\mathbb{R}$ by $(0, +\infty)$ (respec. $(-\infty, 0)$) in Eq. (4.30). More precisely, if it holds $y = \Psi^\tau(x)$, for some $\tau \in (-\epsilon, \epsilon)$, whenever

$$d\big(\Psi^t(x), \Psi^{h(t)}(y)\big) < \delta, \quad \forall t \in (0, +\infty)\ (\forall t \in (-\infty, 0)).$$

Then we get

Proposition 4.13. *The flow $\{\Phi_X^t\}$ is positively expansive.*

Proof. First notice that in Section 4.2.3 we have constructed a smooth section $\hat{Z} \in \Gamma(NX)$ that verifies Eq. (4.22), where $\int_M \hat{A}\Omega \neq 0$ (in fact, we have supposed that this constant is positive). Then, if $Z \in \mathfrak{X}(M)$ is any smooth vector field verifying $\mathrm{pr}_X(Z) = \hat{Z}$, we will clearly have that for every $p \in M$,

$$\big\| D\Phi_X^t\big(Z(p)\big)\big\| \to \infty, \quad \text{when } t \to \pm\infty, \tag{4.31}$$

being the convergence uniform.

On the other hand, Eqs. (4.19) and (4.28) let us affirm that (modulo our sign assumptions made there), for every p, it holds

$$\sphericalangle\big(D\Phi_X^t\big(Z(p)\big), X\big(\Phi_X^t(p)\big)\big) \to 0, \quad \text{when } t \to +\infty, \tag{4.32}$$

being this convergence uniform, too.

Then, taking into account that $\{X, Y, Z\}$ is a global basis for TM, jointly with Eqs. (4.11), (4.28), (4.31) and (4.32) we can easily conclude that $\{\Phi_X^t\}$ is positively expansive, as desired.

And then we are very close to the end of our proof. In fact, as we will shortly see, there is no closed 3-manifold supporting positively expansive flows. To get this, we will invoke the work of M. Paternain [20] about the existence of stable and unstable foliations for expansive flows on 3-manifolds.

To recall Paternain's result, first we need to introduce some additional notation. Let K be any closed manifold, $\mathrm{dist} : K \times K \to \mathbb{R}$ be any distance compatible with the topology of K and $\Psi : \mathbb{R} \times K \to K$ be a continuous expansive flow.

As usual, given any $x \in K$, we can define its *stable* and *unstable sets* writing

$$W^s(x, \Psi) \doteq \big\{y \in K:\ \mathrm{dist}\big(\Psi^t(x), \Psi^t(y)\big) \to 0,\ \text{as } t \to +\infty\big\},$$
$$W^u(x, \Psi) \doteq \big\{y \in K:\ \mathrm{dist}\big(\Psi^{-t}(x), \Psi^{-t}(y)\big) \to 0,\ \text{as } t \to +\infty\big\},$$

respectively.

Then, we can precisely state

Theorem 4.14. *(Paternain [20].) If K is a closed 3-manifold and Ψ is an expansive flow on K, then there exists a finite set (maybe empty) of periodic orbits $\gamma_1, \gamma_2, \dots, \gamma_n$ of Ψ such that the partitions*

$$\mathcal{F}^\sigma = \left\{ W^\sigma(x, \Psi)\colon\ x \in M \Bigm\backslash \bigcup_{i=1}^{n} \gamma_i \right\}, \quad \text{for } \sigma = s, u,$$

are C^0 codimension-two foliations on $M \setminus \bigcup \gamma_i$.

In our case we have prove that $\{\Phi_X^t\}$ is positively expansive, so $W^u(p, \Phi) = \{p\}$, for every $p \in M$, contradicting Theorem 4.14. $\square$

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References

[1] D. Anosov, Geodesic flows on closed riemannian manifolds of negative curvature, Proceedings of the Steklov Institute of Mathematics 90 (1967) 1–235.
[2] R. Bowen, P. Walters, Expansive one-parameter flows, Journal of Differential Equations 12 (1972) 180–193.
[3] C. Doering, Persistently transitive vector fields on three-dimensional manifolds, in: Dynamical Systems and Bifurcation Theory, in: Pitman Res. Notes Math. Ser., vol. 160, Longman Sci. Tech., 1987.
[4] L. Flaminio, G. Forni, On the cohomological equation for nilflows, Journal of Modern Dynamics 1 (1) (2007) 37–60.
[5] G. Forni, On the Greenfield–Wallach and Katok conjectures, preprint, available at: http://arxiv.org/abs/0706.3981.
[6] F. Fuller, The existence of periodic points, Annals of Mathematics 57 (2) (1953) 229–230.
[7] S. Greenfield, N. Wallach, Globally hypoelliptic vector fields, Topology 12 (1973) 247–253.
[8] M. Herman, Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations, Publications Mathématiques de l'I.H.É.S. 49 (1979) 5–233.
[9] S. Hurder, Problems of rigidity of group actions and cocycles, Ergodic Theory & Dynamical Systems 5 (1985) 473–484.
[10] A. Katok, Combinatorial Constructions in Ergodic Theory and Dynamical Systems, University Lecture Series, vol. 30, American Mathematical Society, 2003.
[11] A. Katok, E. Robinson, Cocycles, cohomology and combinatorial constructions in ergodic theory, in: Smooth Ergodic Theory and its Applications, in: Proceedings of the Symposia in Pure Mathematics, vol. 69, 2001.
[12] A. Katok, R. Spatzier, First cohomology of Anosov actions of higher rank abelian groups and applications to rigidity, Publications Mathématiques de l'I.H.É.S. 79 (1994) 131–156.
[13] A. Kocsard, Toward the classification of cohomology-free vector fields, Ph.D. thesis, IMPA, 2007.
[14] R. Mañé, Quasi-Anosov diffeomorphisms and hyperbolic manifolds, Transactions of the American Mathematical Society 229 (1977) 351–370.
[15] S. Matsumoto, The parameter rigid flows on 3-manifolds, preprint, 2007.
[16] S. Matsumoto, Y. Mitsumatsu, Leafwise cohomology and rigidity of certain Lie group actions, Ergodic Theory & Dynamical Systems 23 (2003) 1839–1866.
[17] J. Moser, On the volume elements on a manifold, Transactions of the American Mathematical Society 120 (1965) 186–294.
[18] H. Nakayama, T. Noda, Minimal sets and chain recurrent sets of projective flows induced from minimal flows on 3-manifolds, Discrete and Continuous Dynamical Systems 12 (4) (2005) 629–638.
[19] V. Oseledets, A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems, Transactions of the Moscow Mathematical Society 19 (1968) 197–231.
[20] M. Paternain, Expansive flows and the fundamental group, Boletim da Sociedade Brasileira de Matemática 24 (2) (1993) 179–199.
[21] F. Rodríguez Hertz, J. Rodríguez Hertz, Cohomology free systems and the first Betti number, Discrete and Continuous Dynamical Systems 15 (2006) 193–196.
[22] H. Rosenberg, R. Roussarie, D. Weil, A classification of closed orientable 3-manifolds of rank 2, Annals of Mathematics 91 (3) (1970) 449–464.
[23] R. Sacker, G. Sell, Existence of dichotomies and invariant splittings for linear differential equations, Journal of Differential Equations 15 (1974) 429–458.
[24] N.d. Santos, Parameter rigid actions of the Heisenberg group, Ergodic Theory & Dynamical Systems 27 (2007) 1719–1735.
[25] J. Selgrade, Isolated invariant sets for flows on vector bundles, Transactions of the American Mathematical Society 203 (1975) 359–390.
[26] J. Selgrade, Erratum to "isolated invariant sets for flows on vector bundles", Transaction of the American Mathematical Society 221 (1976) 249.

[27] C. Taubes, The Seiberg–Witten equations and the Weinstein conjecture, preprint, available at: http://arxiv.org/abs/math.SG/0611007.
[28] R. Urzúa Luz, The first cohomology of affine $\mathbb{Z}^p$-actions on tori and applications to rigidity, Bulletin of the Brazilian Mathematical Society 34 (2) (2003) 287–302.
[29] R. Urzúa Luz, N.d. Santos, Cohomology free diffeomorphisms of low-dimension tori, Ergodic Theory & Dynamical Systems 18 (1998) 985–1006.
[30] A. Weinstein, On the hypotheses of Rabinowitz' periodic orbit theorems, Journal of Differential Equations 33 (1979) 353–358.
[31] J.-C. Yoccoz, Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne, Annales Scientifiques de l'É.N.S. 17 (1984) 333–359.