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Rotational deviations and invariant pseudo-foliations for periodic point free torus homeomorphisms

Alejandro Kocsard[1] · Fernanda Pereira Rodrigues[1]

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Abstract This article deals with directional rotational deviations for non-wandering periodic point free homeomorphisms of the 2-torus which are homotopic to the identity. We prove that under mild assumptions, such a homeomorphism exhibits uniformly bounded rotational deviations in some direction if and only if it leaves invariant a *pseudo-foliation*, a notion which is a slight generalization of classical one-dimensional foliations. To get these results, we introduce a novel object called $\tilde{\rho}$*-centralized skew-product* and their associated *stable sets at infinity.*

1 Introduction

We denote by $\mathrm{Homeo}_0(\mathbb{T}^d)$ the space of homeomorphisms of the d-dimensional torus $\mathbb{T}^d$ which are homotopic to the identity. The main dynamical invariant for systems given by such a map is the so called *rotation set*. Given a lift $\tilde{f}\colon \mathbb{R}^d \circlearrowleft$ of a homeomorphism $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$, one defines its *rotation set* by

$$\rho(\tilde{f}) := \left\{ \rho \in \mathbb{R}^d : \exists n_i \uparrow +\infty,\ z_i \in \mathbb{R}^d,\ \frac{\tilde{f}^{n_i}(z_i) - z_i}{n_i} \to \rho,\ \text{as } i \to \infty \right\}.$$

This invariant was originally defined by Poincaré for the one-dimensional case (i.e. $d = 1$) in his celebrated work [23]. In such a case, the rotation set reduces to a point, the so called *rotation number*, and a simple but fundamental property holds (see for instance [10, page 21]):

$$\left| \tilde{f}^n(z) - z - n\rho(\tilde{f}) \right| \leqslant 1, \quad \forall n \in \mathbb{Z}, \forall z \in \mathbb{R}. \tag{1}$$

✉ Fernanda Pereira Rodrigues
fernandapr@id.uff.br

Alejandro Kocsard
akocsard@id.uff.br

1 IME-Universidade Federal Fluminense, Rua Prof. Marcos Waldemar de Freitas Reis, S/N. Bloco H, 4° andar. 24.210-201, Gragoatá, Niterói, RJ, Brazil

That is, every $\tilde{f}$-orbit exhibits *uniformly bounded rotational deviations* with respect to the rigid rotation (or translation) $z \mapsto z + \rho(\tilde{f})$.

In higher dimensions the situation dramatically changes: in fact, for $d \geqslant 2$ different orbits can exhibit different rotation vectors and, moreover, there can exist points with non-well-defined rotation vector. However, the rotation set $\rho(\tilde{f})$ is always non-empty, compact and connected; and for $d = 2$, Misiurewicz and Ziemian [22] showed it is also convex. So, in the two-dimensional case the elements of $\mathrm{Homeo}_0(\mathbb{T}^2)$ can be classified according to the following trichotomy:

(i) $\rho(\tilde{f})$ is just a point, and in this case we say f is a *pseudo-rotation;*
(ii) $\rho(\tilde{f})$ is a non-degenerate line segment;
(iii) $\rho(\tilde{f})$ has non-empty interior.

In this paper we shall concentrate on the study of *rotational deviations* in dimension two. In this case, as a generalization of (1), one says that f exhibits *uniformly bounded rotational deviations* when

$$\sup_{z\in\mathbb{R}^2}\sup_{n\in\mathbb{Z}} d\big(\tilde{f}^n(\tilde{z}) - \tilde{z}, n\rho(\tilde{f})\big) < \infty.$$

When the rotation set $\rho(\tilde{f})$ has non-empty interior Le Calvez and Tal [21] has recently shown that f exhibits uniformly bounded rotational deviations. This result had been previously gotten by Addas-Zanata [1] for smooth diffeomorphisms, and Dávalos [5] in some particular cases.

When the rotation set $\rho(\tilde{f})$ has empty interior, i.e. it satisfies either condition (i) or (ii) of above trichotomy, one can consider *directional rotational deviations:* if $v \in \mathbb{S}^1$ denotes a unit vector such that $\rho(\tilde{f})$ is contained in a straight line perpendicular to v, f is said to exhibit *uniformly bounded v-deviations* when there exists a constant $M > 0$ such that

$$\Big\langle \tilde{f}^n(z) - z - n\rho, v\Big\rangle \leqslant M, \quad \forall z \in \mathbb{R}^2,\ \forall n \in \mathbb{Z},$$

for some (and hence, any) $\rho \in \rho(\tilde{f})$.

When f is a pseudo-rotation, i.e. $\rho(\tilde{f})$ is just a point, one can study rotational deviations in any direction. Koropecki and Tal has shown in [19] that "generic" area-preserving rational pseudo-rotations, i.e. satisfying $\rho(\tilde{f}) \subset \mathbb{Q}^2$, exhibit uniformly bounded deviations in every direction (see also [21]), but there are some "exotic" rational pseudo-rotations with unbounded deviations in every direction [18]. Contrasting with these results, in [15] the first author and Koropecki proved that generic diffeomorphisms in the closure of the conjugacy class of rotations on $\mathbb{T}^2$ are irrational pseudo-rotations exhibiting unbounded deviations in every direction.

Regarding the remaining case (ii) of above trichotomy, Dávalos [5] have proved that any $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ whose rotation set is a non-trivial vertical segment and contains rational points, exhibits uniformly bounded horizontal deviations. This result had been proven by Guelman et al. [8] in the area-preserving setting.

As the reader could have already noticed, in all above boundedness results the homeomorphisms have periodic points, and in fact, these orbits play a fundamental role in their proofs.

The fundamental purpose of this paper consists in pursuing the study of directional rotational deviations for periodic point free homeomorphisms and its topological and geometric consequences.

It is well-known that any 2-torus homeomorphism in the identity isotopy class that leaves invariant a (non-singular one-dimensional) foliation exhibits uniformly bounded v-deviations, for some $v \in \mathbb{S}^1$ whose direction is completely determined by the asymptotic behavior of the foliation; and in particular, its rotation set has empty interior.

However, as it is shown in [2], there are smooth minimal area-preserving pseudo-rotations exhibiting uniformly bounded horizontal deviations, but not preserving any foliation.

One of the main results of this paper, that emerges from the combination of Theorems 5.4 and 5.5, establishes that, under some mild and natural conditions, an element of $\mathrm{Homeo}_0(\mathbb{T}^2)$ exhibits uniformly bounded v-deviations for some $v \in \mathbb{S}^1$ if and only if it leaves invariant a *pseudo-foliation*. Torus pseudo-foliations, which are a slight generalization of classical one-dimensional foliations, are defined in Sect. 5. Pseudo-foliations have been used in [16] to show that any minimal homeomorphism which is not a pseudo-rotation is topologically mixing, and in [17] to prove a particular case of the so called Franks-Misiurewicz conjecture (even though the authors did not explicitly use this name).

The main character of this work is a new class of dynamical systems on $\mathbb{T}^2 \times \mathbb{R}^2$, called *$\tilde{\rho}$-centralized skew-products*, that are associated to each homeomorphism of $\mathbb{T}^2$ which is isotopic to the identity. In Sect. 3.1 we define the *stable sets at infinity* associated to these skew-products and in Sect. 4 we stablish some important results about the topology of these sets regarding the directional rotational devitions of the original system. Stable sets at infinity are used to contruct the invariant pseudo-foliation of Theorem 5.5.

The paper is organized as follows: in Sect. 2 we fix the notation that will be used all along the article and recall some previous results. In Sect. 3 we introduce the so called *$\tilde{\rho}$-centralized skew-products* and their associated *stable sets at infinity*. We prove some elementary properties of these sets and use them to get some fundamental results about rotational deviations, like Theorem 3.1 and Corollary 3.2, showing that uniformly v- and $(-v)$-deviations are equivalent. Then, in Sect. 4 we establish some relations between the topology of *stable sets at infinity* and *à priori* boundedness of rotational deviations for non-wandering periodic point free homeomorphisms. Then, finally in Sect. 5 we introduce the new concepts of *torus pseudo-foliations* we show that under some natural and mild hypotheses, a non-wandering periodic point free homeomorphism exhibits uniformly bounded rotational deviations in some direction if and only if it leaves invariant a torus pseudo-foliation.

2 Preliminaries and notations

2.1 Maps, topological spaces and groups

Given a map $f\colon X \circlearrowleft$, its set of fixed points will be denoted by $\mathrm{Fix}(f)$. We shall write $\mathrm{Per}(f) := \bigcup_{n\geqslant 1} \mathrm{Fix}(f^n)$ for the set of periodic points. The map f is said to be *periodic point free* whenever $\mathrm{Per}(f) = \varnothing$. If $A \subset X$ denotes an arbitrary subset, we define its positively maximal f-invariant subset by

$$\mathscr{I}_f^+(A) := \bigcap_{n\geqslant 0} f^{-n}(A). \tag{2}$$

When f is bijective, we can also define its maximal f-invariant subset by

$$\mathscr{I}_f(A) := \mathscr{I}_f^+(A) \cap \mathscr{I}_{f^{-1}}^+(A) = \bigcap_{n\in\mathbb{Z}} f^n(A). \tag{3}$$

When X is a topological space, a homeomorphism $f\colon X \circlearrowleft$ is said to be *non-wandering* when for every non-empty open set $U \subset X$, there exists $n \geqslant 1$ such that $f^n(U) \cap U \neq \emptyset$. On the other hand, f is said to be *minimal* when every f-orbit is dense in X.

Given any $A \subset X$, we write $\operatorname{int}(A)$ to denote its interior, $\bar{A}$ for its closure and $\partial_X A$ for its boundary inside X. When A is connected, we write $\operatorname{cc}(X, A)$ for the connected component of X containing A. As usual, we write $\pi_0(X)$ to denote the set of all connected components of X.

If (X, d) is a metric space, the open ball of radius $r > 0$ and center at $x \in X$ will be denoted by $B_r(x)$. Given an arbitrary non-empty set $A \subset X$, we define its *diameter* by

$$\operatorname{diam}(A) := \sup_{x,y \in A} d(x, y),$$

and given any point $x \in X$, we write

$$d(x, A) := \inf_{y \in A} d(x, y).$$

We also consider the space of compact subsets

$$\mathcal{K}(X) := \{K \subset X : K \text{ is non-empty and compact}\}.$$

and we endow this space with its Hausdorff distance d_H (induced by d) defined by

$$d_H(K_1, K_2) := \max\left\{\max_{p \in K_1} d(p, K_2), \max_{q \in K_2} d(q, K_1)\right\}, \quad \forall K_1, K_2 \in \mathcal{K}(X).$$

It is well-known that $(\mathcal{K}(X), d_H)$ is compact whenever (X, d) is compact itself.

Given a locally compact non-compact topological space Y, we write $\widehat{Y} := Y \sqcup \{\infty\}$ for the one-point compactification of Y. If $A \subset Y$ is an arbitrary subset, $\widehat{A}$ will denote its closure inside the space $\widehat{Y}$, and given any continuous proper map $f\colon Y \circlearrowleft$, its unique extension to $\widehat{Y}$ (that fixes the point at infinity) will be denoted by $\widehat{f}\colon \widehat{Y} \circlearrowleft$.

Whenever $M_1, M_2, \ldots M_n$ denote n arbitrary sets, we shall use the generic notation $\operatorname{pr}_i\colon M_1 \times M_2 \times \cdots \times M_n \to M_i$ to denote the ith-coordinate projection map.

2.2 Euclidean spaces and tori

We consider $\mathbb{R}^d$ endowed with its usual Euclidean structure denoted by $\langle\cdot,\cdot\rangle$. We write $\|v\| := \langle v, v\rangle^{1/2}$, for any $v \in \mathbb{R}^d$. The unit $(d-1)$-sphere is denoted by $\mathbb{S}^{d-1} := \{v \in \mathbb{R}^d : \|v\| = 1\}$. For any $v \in \mathbb{R}^d \setminus \{0\}$ and each $r \in \mathbb{R}$, we define the *half-space*

$$\mathbb{H}_r^v := \left\{z \in \mathbb{R}^d : \langle z, v\rangle > r\right\}. \tag{4}$$

For $d = 2$, given any $v = (a, b) \in \mathbb{R}^2$, we define $v^\perp := (-b, a)$. We also introduce the following notation for straight lines: given any $r \in \mathbb{R}$ and $v \in \mathbb{S}^1$,

$$\ell_r^v := rv + \mathbb{R}v^\perp = \{rv + tv^\perp : t \in \mathbb{R}\}. \tag{5}$$

We say that $v \in \mathbb{S}^1$ has *rational slope* when there exists some $t \in \mathbb{R}\setminus\{0\}$ such that $tv \in \mathbb{Z}^2$; otherwise, v is said to have *irrational slope.*

We will need the following notation for strips on $\mathbb{R}^2$: given $v \in \mathbb{S}^1$ and $s > 0$ we define the strip

$$\mathbb{A}_s^v := \mathbb{H}_{-s}^v \cap \mathbb{H}_{-s}^{-v} = \{z \in \mathbb{R}^2 : -s < \langle z, v\rangle < s\}. \tag{6}$$

Given any $\alpha \in \mathbb{R}^d$, T_α denotes the translation $T_\alpha\colon z \mapsto z + \alpha$ on $\mathbb{R}^d$.

The d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$ will be denoted by $\mathbb{T}^d$ and we write $\pi\colon \mathbb{R}^d \to \mathbb{T}^d$ for the natural quotient projection. Given any $\alpha \in \mathbb{T}^d$, we write T_α for the torus translation $T_\alpha\colon \mathbb{T}^d \ni z \mapsto z+\alpha$.

As usual, a point $\alpha \in \mathbb{R}^d$ is said to be *rational* when $\alpha \in \mathbb{Q}^d$ and is said to be *totally irrational* when $T_{\pi(\alpha)}$ is a minimal homeomorphism of $\mathbb{T}^d$.

2.3 The boundary at infinity of planar sets

Given a non-empty set $A \subset \mathbb{R}^2$ and a point $v \in \mathbb{S}^1$, we say that *A accumulates in the direction v at infinity* if there is sequence $\{x_n\}_{n\geqslant 0}$ of points in A such that

$$\lim_{n\to\infty} \|x_n\| = \infty, \quad \text{and} \quad \lim_{n\to\infty} \frac{x_n}{\|x_n\|} = v.$$

Then we can define the *boundary of A at infinity* as the set $\partial_\infty A \subset \mathbb{S}^1$ consisting of all $v \in \mathbb{S}^1$ such that A accumulates in the direction v at infinity.

2.4 Surface topology

Let S denote an arbitrary connected surface, i.e. a two-dimensional topological connected manifold.

An open subset of S is said to be a *topological disk* when it is homeomorphic to the open unitary disk $\{z \in \mathbb{R}^2 : \|z\| < 1\}$. Similarly, a *topological annulus* is an open subset of S homeomorphic to $\mathbb{T} \times \mathbb{R}$.

An *arc* on S is a continuous map $\alpha\colon [0,1] \to S$ and a *loop* on S is a continuous map $\gamma\colon \mathbb{T} \to S$.

An open non-empty subset $U \subset S$ is said to be *inessential* when every loop in U is contractible in S; otherwise it is said to be *essential.* An arbitrary subset $V \subset S$ is said to be *inessential* when there exists an inessential open set $U \subset S$ such that $V \subset U$. On the other hand, we say that V is *fully essential* when $S\backslash V$ is inessential.

We say a subset $A \subset S$ is *annular* when it is an open topological annulus and none connected component of $S\backslash A$ is inessential.

Finally, let us recall two classical results about fixed point free orientation preserving homeomorphisms:

Theorem 2.1 (Brouwer's translation theorem [6]) *Let $f\colon \mathbb{R}^2 \circlearrowleft$ be an orientation preserving homeomorphism such that* $\mathrm{Fix}(f) = \varnothing$*. Then, every $x \in \mathbb{R}^2$ is wandering for f, i.e. there exists a neighborhood U of x such that $f^n(U) \cap U = \varnothing$, for every $n \in \mathbb{Z}\backslash\{0\}$.*

Theorem 2.2 (Corollary 1.3 in [7]) *Let $f\colon \mathbb{R}^2 \circlearrowleft$ be an orientation preserving homeomorphism such that* $\mathrm{Fix}(f) = \varnothing$*, and $D \subset \mathbb{R}^2$ be an open topological disk. Let us suppose $f(D) \cap D = \varnothing$. Then, $f^n(D) \cap D = \varnothing$, for every $n \in \mathbb{Z}\backslash\{0\}$.*

2.5 Groups of homeomorphisms

Given any topological manifold M, $\mathrm{Homeo}(M)$ denotes the group of homeomorphisms from M onto itself. The subgroup formed by those homeomorphisms which are homotopic to the identity map id_M will be denoted by $\mathrm{Homeo}_0(M)$.

We define the subgroup $\widetilde{\mathrm{Homeo}}_0(\mathbb{T}^d) < \mathrm{Homeo}_0(\mathbb{R}^d)$ by

$$\widetilde{\mathrm{Homeo}}_0(\mathbb{T}^d) := \left\{ \tilde{f} \in \mathrm{Homeo}_0(\mathbb{R}^d) : \tilde{f} - id_{\mathbb{R}^d} \in C^0(\mathbb{T}^d, \mathbb{R}^d) \right\}.$$

Notice that in this definition, we are identifying the elements of $C^0(\mathbb{T}^d, \mathbb{R}^d)$ with those $\mathbb{Z}^d$-periodic functions from $\mathbb{R}^d$ into itself.

Making some abuse of notation, we also write $\pi : \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^d) \to \mathrm{Homeo}_0(\mathbb{T}^d)$ for the map that associates to each $\tilde{f}$ the only torus homeomorphism $\pi \tilde{f}$ such that $\tilde{f}$ is a lift of $\pi \tilde{f}$. Notice that with our notations, it holds $\pi T_\alpha = T_{\pi(\alpha)} \in \mathrm{Homeo}_0(\mathbb{T}^d)$, for every $\alpha \in \mathbb{R}^d$.

Given any $\tilde{f} \in \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^d)$, we define its *displacement function* by

$$\Delta_{\tilde{f}} := \tilde{f} - id_{\mathbb{R}^d} \in C^0(\mathbb{T}^d, \mathbb{R}^d). \tag{7}$$

Observe this function can be naturally considered as a cocycle over $f := \pi \tilde{f}$, so we will use the usual notation

$$\Delta_{\tilde{f}}^{(n)} := \Delta_{\tilde{f}^n} = \mathcal{S}_f^n \Delta_{\tilde{f}}, \quad \forall n \in \mathbb{Z}, \tag{8}$$

where $\mathcal{S}_f^n$ denotes the *Birkhoff sum,* i.e. given any $n \in \mathbb{Z}$ and $\phi \colon \mathbb{T}^d \to \mathbb{R}$, $\mathcal{S}_f^n(\phi)$ is given by

$$\mathcal{S}_f^n \phi := \begin{cases} \sum_{j=0}^{n-1} \phi \circ f^j, & \text{if } n \geqslant 1; \\ 0, & \text{if } n = 0; \\ -\sum_{j=1}^{-n} \phi \circ f^{-j}, & \text{if } n < 0. \end{cases} \tag{9}$$

The map $\mathbb{R}^d \ni \alpha \mapsto T_\alpha \in \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^d)$ defines an injective group homomorphism, and hence, $\mathbb{R}^d$ naturally acts on $\widetilde{\mathrm{Homeo}}_0(\mathbb{T}^d)$ by conjugacy. However, since every element of $\widetilde{\mathrm{Homeo}}_0(\mathbb{T}^d)$ commutes with $T_{\boldsymbol{p}}$, for all $\boldsymbol{p} \in \mathbb{Z}^d$, we conclude $\mathbb{T}^d$ itself acts on $\widetilde{\mathrm{Homeo}}_0(\mathbb{T}^d)$ by conjugacy, i.e. the map $\mathrm{Ad} \colon \mathbb{T}^d \times \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^d) \to \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^d)$ given by

$$\mathrm{Ad}_t(\tilde{f}) := T_{\tilde{t}}^{-1} \circ \tilde{f} \circ T_{\tilde{t}}, \quad \forall (t, \tilde{f}) \in \mathbb{T}^d \times \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^d), \quad \forall \tilde{t} \in \pi^{-1}(t), \tag{10}$$

is well-defined.

2.6 Invariant measures

Given any topological space M, we shall write $\mathfrak{M}(M)$ for the space of Borel probability measures on M. We say $\mu \in \mathfrak{M}(M)$ is a *topological measure* when $\mu(U) > 0$ for every non-empty open subset $U \subset M$.

Given any $f \in \mathrm{Homeo}(M)$, the space of f-invariant probability measures will be denoted by $\mathfrak{M}(f) := \{\nu \in \mathfrak{M}(M) : f_\star \nu = \nu \circ f^{-1} = \nu\}$.

2.7 Rotation set and rotation vectors

Let $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$ denote an arbitrary homeomorphism and $\tilde{f} \in \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^d)$ be a lift of f. We define the *rotation set of* $\tilde{f}$ by

$$\rho(\tilde{f}) := \bigcap_{m \geqslant 0} \overline{\bigcup_{n \geqslant m} \left\{ \frac{\Delta_{\tilde{f}^n}(z)}{n} : z \in \mathbb{R}^d \right\}}. \tag{11}$$

It can be easily shown that $\rho(\tilde{f})$ is non-empty, compact and connected. We say that f is a *pseudo-rotation* when $\rho(\tilde{f})$ is just a point. Notice that whenever $\tilde{f}_1, \tilde{f}_2 \in \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^d)$ are such that $\pi \tilde{f}_1 = \pi \tilde{f}_2$, then $\pi(\rho(\tilde{f}_1)) = \pi(\rho(\tilde{f}_2))$. Thus, given any $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$, we can just define

$$\rho(f) := \pi(\rho(\tilde{f})) \subset \mathbb{T}^d, \tag{12}$$

where $\tilde{f} \in \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^d)$ denotes any lift of f. In particular, the fact of being a pseudo-rotation depends just on f and not on the chosen lift.

By (8), the rotation set is formed by accumulation points of Birkhoff averages of the displacement function, so given any $\mu \in \mathfrak{M}(\pi \tilde{f})$ we can define the *rotation vector of* μ by

$$\rho_\mu(\tilde{f}) := \int_{\mathbb{T}^d} \Delta_{\tilde{f}} \, \mathrm{d}\mu, \tag{13}$$

and it clearly holds $\rho_\mu(\tilde{f}) \in \rho(\tilde{f})$, whenever μ is ergodic.

When $d = 1$, by classical Poincaré theory of circle homeomorphisms we know that $\rho(\tilde{f})$ reduces to a point. This does not hold in higher dimensions, but when $d = 2$, after Misiurewicz and Ziemian [22] we know that $\rho(\tilde{f})$ is not just connected but also convex. In fact, in the two-dimensional case it holds

$$\rho(\tilde{f}) = \left\{ \rho_\mu(\tilde{f}) : \mu \in \mathfrak{M}\left(\pi \tilde{f}\right) \right\}, \quad \forall \tilde{f} \in \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^2). \tag{14}$$

2.8 Rational rotation vectors and periodic points

Let $\tilde{f} \in \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^2)$ be a lift of $f := \pi \tilde{f} \in \mathrm{Homeo}_0(\mathbb{T}^2)$. For any $z \in \mathrm{Per}(f)$, there is $(p_1, p_2, q) \in \mathbb{Z}^2 \times \mathbb{N}$ such that $\tilde{f}^q(\tilde{z}) = T_{(p_1,p_2)}(\tilde{z})$, for every $\tilde{z} \in \pi^{-1}(z)$. Thus, the point $(p_1/q, p_2/q) \in \rho(\tilde{f}) \cap \mathbb{Q}^2$, and in this case one says the periodic point z realizes the rational rotation vector $(p_1/q, p_2/q)$.

The following result due to Handel asserts that the rotation set of a periodic point free homeomorphism has empty interior:

Theorem 2.3 *(Handel [9]) Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ be a periodic point free homeomorphism and $\tilde{f} \in \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^2)$ be a lift of f. Then, there exists $v \in \mathbb{S}^1$ and $\alpha \in \mathbb{R}$ so that*

$$\frac{\left\langle \tilde{f}^n(z) - z, v \right\rangle}{n} \to \alpha, \quad \text{as } n \to \infty,$$

where the convergence is uniform in $z \in \mathbb{R}^2$. In other words, it holds

$$\rho(\tilde{f}) \subset \ell_\alpha^v, \tag{15}$$

where the straight line ℓ_α^v is given by (5).

In general, not every rational point in $\rho(\tilde{f})$ is realized by a periodic orbit of f. However, in the non-wandering case the following holds:

Theorem 2.4 *If $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is non-wandering and periodic point free, then*

$$\rho(f) \cap \mathbb{Q}^2/\mathbb{Z}^2 = \varnothing. \tag{16}$$

Proof Let $\tilde{f} \in \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^2)$ be a lift of f. By Theorem 2.3, $\rho(\tilde{f})$ has empty interior. So, $\rho(\tilde{f})$ is either a point or line segment.

By a result Misiurwicz and Ziemian [22] we know that every rational extreme point of the rotation set is realized by a periodic orbit. So, if f is a periodic point free pseudo-rotation, then condition (16) necessarily holds.

Hence, the only remaining case to consider is when $\rho(\tilde{f})$ is a non-degenerate line segment. In such a case, since f is non-wandering, every rational point of $\rho(\tilde{f})$ would be also realized by a periodic orbit (see [4,13,14] for details), and thus (16) holds, too. ◻

2.9 Directional rotational deviations

Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ and $\tilde{f} : \mathbb{R}^2 \circlearrowleft$ be a lift of f. If the rotation set $\rho(\tilde{f})$ has empty interior, then there exists $\alpha \in \mathbb{R}$ and $v \in \mathbb{S}^1$ such that

$$\rho(\tilde{f}) \subset \ell_\alpha^v = \ell_{-\alpha}^{-v}, \tag{17}$$

where ℓ_α^v is the straight line given by (5).

In such a case we say that a point $z_0 \in \mathbb{T}^d$ exhibits *bounded* v*-deviations* when there exists a real constant $M = M(z_0, f) > 0$ such that

$$\left\langle \Delta_{\tilde{f}}^{(n)}(z_0), v \right\rangle - n\alpha \leqslant M, \quad \forall n \in \mathbb{Z}. \tag{18}$$

Moreover, we say that f exhibits *uniformly bounded* v*-deviations* when there exists $M = M(f) > 0$ such that

$$\left\langle \Delta_{\tilde{f}}^{(n)}(z), v \right\rangle - n\alpha \leqslant M, \quad \forall z \in \mathbb{T}^d, \ \forall n \in \mathbb{Z}. \tag{19}$$

Remark 2.5 Notice that the lines $\ell_\alpha^v = \alpha v + \mathbb{R}v^\perp$ and $\ell_{-\alpha}^{-v} = (-\alpha)(-v) + \mathbb{R}(-v)^\perp$ coincide as subsets of $\mathbb{R}^2$. However *à priori* there is no obvious relation between v-deviation and $(-v)$-deviation.

Remark 2.6 Once again, notice that this concept of rotational deviation does just depend on the torus homeomorphism and not on the chosen lift.

Remark 2.7 Let us make a final comment about the negation of the above concept: we will say that f *does not exhibit uniformly bounded* v*-deviations* when for every $M > 0$, there exist $z \in \mathbb{T}^2$, $n \in \mathbb{Z}$ and $\rho \in \rho(\tilde{f})$ such that

$$\left\langle \Delta_{\tilde{f}}^{(n)}(z) - n\rho, v \right\rangle > M.$$

That means that in such a case we are also considering the possibility that there exists no $\alpha \in \mathbb{R}$ such that $\rho(\tilde{f}) \subset \ell_\alpha^v$.

2.10 Annular and strictly toral dynamics

Here we recall the concepts of *annular* and *strictly toral* homeomorphisms that have been introduced by Koropecki and Tal in [20].

To do this, let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ denote an arbitrary homeomorphism.

We say that f is *annular* when there exists a lift $\tilde{f} : \mathbb{R}^2 \circlearrowleft$, $p/q \in \mathbb{Q}$ and $v \in \mathbb{S}^1$ with rational slope such that $\rho(\tilde{f}) \subset \ell_{p/q}^v$ and f exhibits uniformly bounded v-deviations.[1]

On the other hand, f is said to be *strictly toral* when f is not annular and $\mathrm{Fix}(f^k)$ is not fully essential, for any $k \in \mathbb{Z}\setminus\{0\}$.

In order to state the main properties of strictly toral homeomorphisms, we first need to introduce some notations: for any $x \in \mathbb{T}^2$ and any $r > 0$ let us consider the set

$$U_r(x, f) := \mathrm{cc}\left(\bigcup_{n\in\mathbb{Z}} f^n(B_r(x)), x \right). \tag{20}$$

[1] Our definition of annular map is slightly more general than the one given in [20]. In fact, they just consider the case $p/q = 0$.

Then, a point $x \in \mathbb{T}^2$ is said to be *inessential for* f if there is some $r > 0$ such that $U_r(x, f)$ is inessential; otherwise x is said to be *essential for* f.

As a consequence of Theorem 2.1, we get the following

Proposition 2.8 *If f is periodic point free and non-wandering, then every point of $\mathbb{T}^2$ is essential for f.*

Proof Let us suppose there is an inessential point $x \in \mathbb{T}^2$, i.e. there exists $r > 0$ such that $U_r(x, f)$ is inessential. Since, $U_r(x, f)$ is a connected component of an f-invariant set, and f is non-wandering, there is a positive integer $n_0 \in \mathbb{N}$ such that $f^{n_0}\big(U_r(x, f)\big) = U_r(x, f)$.

On the other hand, since $U_r(x, f)$ is inessential, its *filling* (i.e. the union of $U_r(x, f)$ with all the inessential connected components of $\mathbb{T}^2 \setminus U_r(x, f)$), which will be denoted by U_F, is a open topological disk; and it can be easily shown that U_F is f^{n_0}-invariant itself.

By Theorem 2.1, f^{n_0} has a fixed point in U_F, contradicting the hypothesis that f is periodic point free. □

We shall need the following result about strictly toral dynamics [20, Proposition 1.4]:

Proposition 2.9 *If f is strictly toral, then the set $U_r(x, f)$ is fully essential, for any essential point $x \in \mathbb{T}^2$ and every $r > 0$.*

3 The $\tilde{\rho}$-centralized skew-product

Given a lift $\tilde{f} \in \widetilde{\mathrm{Homeo}_0}(\mathbb{T}^2)$ of a torus homeomorphism $f := \pi \tilde{f}$ and any vector $\tilde{\rho} \in \rho(\tilde{f})$, we will define the $\tilde{\rho}$-*centralized skew-product* induced by $\tilde{f}$ which shall play a key role in this work.

To do that, we first define the map $H \colon \mathbb{T}^2 \to \widetilde{\mathrm{Homeo}_0}(\mathbb{T}^2)$ by

$$H_t := \mathrm{Ad}_t \left(T_{\tilde{\rho}}^{-1} \circ \tilde{f} \right), \quad \forall t \in \mathbb{T}^2, \tag{21}$$

where Ad denotes the $\mathbb{T}^2$-action given by (10).

Considering H as a cocycle over the torus translation $T_\rho \colon \mathbb{T}^2 \circlearrowleft$, where $\rho := \pi(\tilde{\rho})$, one defines the $\tilde{\rho}$-*centralized skew-product* as the skew-product homeomorphism $F \colon \mathbb{T}^2 \times \mathbb{R}^2 \circlearrowleft$ given by

$$F(t, z) := \big(T_\rho(t), H_t(z)\big), \quad \forall (t, z) \in \mathbb{T}^2 \times \mathbb{R}^2.$$

One can easily show that

$$F(t, z) = \left(t + \rho, z + \Delta_{\tilde{f}}\big(t + \pi(z)\big) - \tilde{\rho}\right), \quad \forall (t, z) \in \mathbb{T}^2 \times \mathbb{R}^2, \tag{22}$$

where $\Delta_{\tilde{f}} \in C^0(\mathbb{T}^2, \mathbb{R}^2)$ is the displacement function given by (7). We will use the following classical notation for cocycles: given $n \in \mathbb{Z}$ and $t \in \mathbb{T}^2$, we write

$$H_t^{(n)} := \begin{cases} id_{\mathbb{T}^2}, & \text{if } n = 0; \\ H_{t+(n-1)\rho} \circ H_{t+(n-2)\rho} \circ \cdots \circ H_t, & \text{if } n > 0; \\ H_{t+n\rho}^{-1} \circ \cdots \circ H_{t-2\rho}^{-1} \circ H_{t-\rho}^{-1}, & \text{if } n < 0. \end{cases}$$

Using such a notation, it holds $F^n(t, z) := \big(T_\rho^n(t), H_t^{(n)}(z)\big)$, for all $(t, z) \in \mathbb{T}^2 \times \mathbb{R}^2$ and every $n \in \mathbb{Z}$.

This skew-product F will play a fundamental role in our analysis of rotational deviations and the following simple formula for iterates of F represents the main reason:

$$\begin{aligned} F^n(t,z) &= \Big(T_\rho^n(t), H_{t+(n-1)\rho}\Big(H_{t+(n-2)\rho}\Big(\cdots\Big(H_t(z)\Big)\Big)\Big)\Big) \\ &= \Big(t+n\rho, \mathrm{Ad}_{t+(n-1)\rho}\left(T_{\tilde{\rho}}^{-1}\circ\tilde{f}\right)\circ\cdots\circ\mathrm{Ad}_t\left(T_{\tilde{\rho}}^{-1}\circ\tilde{f}\right)(z)\Big) \\ &= \Big(t+n\rho, \mathrm{Ad}_t\left(T_{\tilde{\rho}}^{-n}\circ\tilde{f}^n\right)(z)\Big), \end{aligned} \tag{23}$$

for every $(t,z)\in\mathbb{T}^2\times\mathbb{R}^2$ and every $n\in\mathbb{N}$.

Notice that assuming inclusion (17), from (23) it easily follows that a point $z\in\mathbb{R}^2$ exhibits bounded v-deviations (as in (18)) if and only if

$$\left\langle H_0^{(n)}(z)-z, v\right\rangle \leqslant M, \quad \forall n\in\mathbb{Z}, \tag{24}$$

where $M=M(z,f)$ is the constant given in (18).

3.1 Fibered stable sets at infinity

Continuing with the notation we have introduced at the beginning of § 3, let $\widehat{\mathbb{R}^2}$ denote the one-point compactification of $\mathbb{R}^2$, and $\widehat{F}\colon\mathbb{T}^2\times\widehat{\mathbb{R}^2}\circlearrowleft$ be the unique continuous extension of F, that clearly satisfies $\widehat{F}(t,\infty):=(T_\rho(t),\infty)$, for every $t\in\mathbb{T}^2$.

Hence, for every $r\in\mathbb{R}$ and each $t\in\mathbb{T}^2$ we define the *fibered* (r,v)*-stable set at infinity* of F by

$$\widehat{\Lambda_r^v}\big(\tilde{f},t\big) := \mathrm{cc}\left(\{t\}\times\widehat{\mathbb{R}^2}\cap\mathscr{I}_{\widehat{F}}\big(\mathbb{T}^2\times\widehat{\mathbb{H}_r^v}\big), (t,\infty)\right), \tag{25}$$

where $\mathbb{H}_r^v$ is the semi-plane given by (4), $\widehat{\mathbb{H}_r^v}$ denotes its closure in $\widehat{\mathbb{R}^2}$ and $\mathscr{I}(\cdot)$ is the maximal invariant set given by (3).

We also define

$$\Lambda_r^v\big(\tilde{f},t\big) := \mathrm{pr}_2\left(\widehat{\Lambda_r^v}\big(\tilde{f},t\big)\setminus\{(t,\infty)\}\right)\subset\mathbb{R}^2, \quad \forall t\in\mathbb{T}^2, \tag{26}$$

where $\mathrm{pr}_2\colon\mathbb{T}^2\times\mathbb{R}^2\to\mathbb{R}^2$ denotes the projection on the second coordinate. Finally, we define the (r,v)*-stable set at infinity* by

$$\Lambda_r^v\big(\tilde{f}\big) := \bigcup_{t\in\mathbb{T}^2}\{t\}\times\Lambda_r^v\big(\tilde{f},t\big)\subset\mathbb{T}^2\times\mathbb{R}^2. \tag{27}$$

For the sake of simplicity, if there is no risk of confusion we shall just write $\widehat{\Lambda_r^v}(t)$, $\Lambda_r^v(t)$ and Λ_r^v instead of $\widehat{\Lambda_r^v}\big(\tilde{f},t\big)$, $\Lambda_r^v\big(\tilde{f},t\big)$ and $\Lambda_r^v\big(\tilde{f}\big)$, respectively.

By the very definitions, stable sets at infinity are closed and F-invariant, but at first glance they might be empty. In Theorem 3.4 we will prove this is never the case and we shall use them to get some results about rotational v-deviations for the original homeomorphism f.

Our first application of these stable sets at infinity is the following result which concerns the symmetry of boundedness of v- and $(-v)$-deviations and is just a simple consequence of the F-invariance:

Theorem 3.1 *Assuming inclusion* (17)*, the homeomorphism f exhibits uniformly bounded v-deviations if and only if it exhibits uniformly bounded $(-v)$-deviations. More precisely, it holds*

$$\sup_{n\in\mathbb{Z}}\sup_{z\in\mathbb{T}^2}\left\langle \Delta_{\tilde f}^{(n)}(z)-n\tilde\rho, v\right\rangle-\sqrt{2}\leqslant \sup_{n\in\mathbb{Z}}\sup_{z\in\mathbb{T}^2}\left\langle \Delta_{\tilde f}^{(n)}(z)-n\tilde\rho, -v\right\rangle$$
$$\leqslant \sup_{n\in\mathbb{Z}}\sup_{z\in\mathbb{T}^2}\left\langle \Delta_{\tilde f}^{(n)}(z)-n\tilde\rho, v\right\rangle+\sqrt{2},$$

for $\tilde\rho\in\rho(\tilde f)$.

Proof By Remark 2.5, the statement is completely symmetric with respect to v and $-v$. Thus, let us assume f exhibits uniformly bounded v-deviations and let us prove uniformly boundedness for $(-v)$-deviations. So let us define

$$M:=\sup_{n\in\mathbb{Z}}\sup_{z\in\mathbb{T}^2}\left\langle \Delta_{\tilde f}^{(n)}(z)-n\tilde\rho, v\right\rangle\in\mathbb{R},$$

and notice that $M\geqslant 0$. Then, by (24) we know

$$\left\langle H_0^{(n)}(z)-z, v\right\rangle\leqslant M,\quad \forall z\in\mathbb{R}^2,\ \forall n\in\mathbb{Z}, \tag{28}$$

and consequently $\mathbb{H}_{M-r}^{-v}\subset\Lambda_{-r}^{-v}(0)$.

For an arbitrary $t\in\mathbb{T}^2$, notice that

$$H_t^{(n)}(z)-z=\Delta_{\tilde f}^{(n)}\big(\pi(z)+t\big)-n\tilde\rho,\quad \forall(t,z)\in\mathbb{T}^2\times\mathbb{R}^2,\ \forall n\in\mathbb{Z}. \tag{29}$$

Then, putting together (28) and (29) we conclude that

$$\left\langle H_t^{(n)}(z)-z, v\right\rangle\leqslant M,\quad \forall(t,z)\in\mathbb{T}\times\mathbb{R}^2,\ \forall n\in\mathbb{Z},$$

and hence, $\mathbb{H}_{M-r}^{-v}\subset\Lambda_{-r}^{-v}(t)$, for any $t\in\mathbb{T}^2$.

Thus, we have

$$\mathbb{H}_M^{-v}\subset\Lambda_0^{-v}(t)\subset\mathbb{H}_0^{-v},\quad \forall t\in\mathbb{T}^2,$$

and since $\Lambda_0^{-v}\subset\mathbb{T}^2\times\mathbb{H}_0^{-v}$ is an F-invariant set, this implies

$$F^n\big(\mathbb{T}^2\times\mathbb{H}_\varepsilon^v\big)\cap\mathbb{T}^2\times\mathbb{H}_M^{-v}=\varnothing,\quad \forall n\in\mathbb{Z}, \tag{30}$$

and for any $\varepsilon>0$.

In particular, if $D\subset\mathbb{R}^2$ denotes a squared fundamental domain for the covering map $\pi\colon\mathbb{R}^2\to\mathbb{T}^2$ such that $D\subset\mathbb{H}_\epsilon^v$, by (22) and (30) it follows that

$$\left\langle \Delta_{\tilde f^n}(z)-n\tilde\rho, -v\right\rangle<M+\varepsilon+\operatorname{diam}(D)=M+\varepsilon+\sqrt{2},\quad \forall z\in D. \tag{31}$$

Since the displacement function $\Delta_{\tilde f^n}$ is $\mathbb{Z}^2$-periodic and (31) holds for any $\varepsilon>0$, we conclude that

$$\sup_{n\in\mathbb{Z}}\sup_{z\in\mathbb{T}^2}\left\langle \Delta_{\tilde f^n}(z)-n\tilde\rho, -v\right\rangle\leqslant \sup_{n\in\mathbb{Z}}\sup_{z\in\mathbb{T}^2}\left\langle \Delta_{\tilde f^n}(z)-n\tilde\rho, v\right\rangle+\sqrt{2}.$$

In particular, f exhibits uniformly bounded $(-v)$-deviations. □

For the sake of concreteness, we explicitly state the following corollary which is a straightforward consequence of Theorem 3.1:

Corollary 3.2 *The following assertions are all equivalent:*

(i) f *exhibits uniformly bounded* v*-deviations;*

(ii) *f exhibits uniformly bounded $(-v)$-deviations;*

(iii) *there exists $M > 0$ such that*

$$\mathbb{H}^{v}_{r+M} \subset \Lambda^{v}_{r}(t), \quad \forall r \in \mathbb{R}, \ \forall t \in \mathbb{T}^2;$$

(iv) *there exists $M > 0$ such that*

$$\mathbb{H}^{-v}_{r+M} \subset \Lambda^{-v}_{r}(t), \quad \forall r \in \mathbb{R}, \ \forall t \in \mathbb{T}^2.$$

Our next result describes some elementary equivariant properties of (r, v)-stable sets at infinity:

Proposition 3.3 *For each $t \in \mathbb{T}^2$ and any $r \in \mathbb{R}$, the following properties hold:*

(i) $\Lambda^{v}_{r}(t) \subset \Lambda^{v}_{s}(t)$, *for every $s < r$;*

(ii)

$$\Lambda^{v}_{r}(t) = \bigcap_{s<r} \Lambda^{v}_{s}(t);$$

(iii)

$$\Lambda^{v}_{r+\langle \tilde{t}, v\rangle}\big(t' - \pi\big(\tilde{t}\big)\big) = T_{\tilde{t}}\big(\Lambda^{v}_{r}(t')\big), \quad \forall \tilde{t} \in \mathbb{R}^2, \ \forall t' \in \mathbb{T}^2;$$

(iv)

$$T_{\boldsymbol{p}}\big(\Lambda^{v}_{r}(t)\big) = \Lambda^{v}_{r+\langle \boldsymbol{p}, v\rangle}(t), \quad \forall \boldsymbol{p} \in \mathbb{Z}^2.$$

Proof Inclusion (i) trivially follows from the inclusion $\mathbb{H}^{v}_{r} \subset \mathbb{H}^{v}_{s}$, for $s < r$.

To show (ii), let $\widehat{\Lambda^{v}_{s}(t)} = \Lambda^{v}_{s}(t) \sqcup \{\infty\}$ denote the closure of $\Lambda^{v}_{s}(t)$ inside the one-point compactification $\widehat{\mathbb{R}^2}$. Since $\widehat{\Lambda^{v}_{s}(t)}$ is compact and connected for every $s \in \mathbb{R}$, by (i) we conclude

$$\bigcap_{s<r} \widehat{\Lambda^{v}_{s}(t)}$$

is compact and connected itself and contains $\widehat{\Lambda^{v}_{r}(t)}$. On the other hand, it clearly holds $\bigcap_{s<r} \Lambda^{v}_{s}(t) \subset \mathscr{I}_F(\mathbb{T}^2 \times \mathbb{H}^{v}_{r})$ and thus, $\bigcap_{s<r} \Lambda^{v}_{s}(t) = \Lambda^{v}_{r}(t)$.

To prove (iii), taking into account (10) and (21), we get

$$\begin{aligned} H^{(n)}_{t'-\pi(\tilde{t})}(z) &= \mathrm{Ad}_{t'-\pi(\tilde{t})}\left(T^{-n}_{\tilde{\rho}} \circ \tilde{f}^{n}\right)(z) \\ &= T_{\tilde{t}} \circ \mathrm{Ad}_{t'}\left(R^{-n}_{\rho} \circ \tilde{f}^{n}\right) \circ T^{-1}_{\tilde{t}}(z) \\ &= H^{(n)}_{t'}(z - \tilde{t}) + \tilde{t}, \end{aligned}$$

for every $z \in \mathbb{R}^2$ and every $n \in \mathbb{Z}$. Hence, for any $z \in \Lambda^{v}_{r+\langle \tilde{t}, v\rangle}\big(t' - \pi(\tilde{t})\big)$, it holds

$$\begin{aligned} r &\leqslant \left\langle H^{(n)}_{t'-\pi(\tilde{t})}(z), v\right\rangle - \langle \tilde{t}, v\rangle \\ &= \left\langle H^{(n)}_{t'}(z - \tilde{t}) + \tilde{t}, v\right\rangle - \langle \tilde{t}, v\rangle = \left\langle H^{(n)}_{t'}(z - \tilde{t}), v\right\rangle, \end{aligned}$$

for any $n \in \mathbb{Z}$. This implies $z \in T_{\tilde{t}}\big(\Lambda^{v}_{r}(t')\big)$, and so, $\Lambda^{v}_{r+\langle \tilde{t}, v\rangle}\big(t' - \pi(\tilde{t})\big) \subset T_{\tilde{t}}\big(\Lambda^{v}_{r}(t')\big)$. The other inclusion is symmetric.

Finally, relation (iv) is just a particular case of (iii). $\square$

Now we can show the main result of this section:

Theorem 3.4 *Assuming there exist $\alpha \in \mathbb{R}$ and $v \in \mathbb{S}^1$ such that condition (17) holds, the fibered (r, v)-stable set at infinity $\Lambda_r^v(t)$ is non-empty, for all $r \in \mathbb{R}$ and every $t \in \mathbb{T}^2$.*

Proof of Theorem 3.4 Our strategy to show $\Lambda_r^v(t)$ is non-empty is mainly inspired by some ideas due to Birkhoff [3].

By Corollary 3.2, we know that $\Lambda_r^v(t) \neq \varnothing$ whenever f exhibits uniformly bounded $(-v)$-deviations.

Hence we can suppose this is not the case, so either

$$\sup_{n\geqslant 0}\sup_{z\in\mathbb{T}^2}\left\langle \Delta_{\tilde{f}}^{(n)}(z) - n\tilde{\rho}, -v\right\rangle = +\infty, \tag{32}$$

or

$$\sup_{n\leqslant 0}\sup_{z\in\mathbb{T}^2}\left\langle \Delta_{\tilde{f}}^{(n)}(z) - n\tilde{\rho}, -v\right\rangle = +\infty. \tag{33}$$

For the sake of concreteness, let us suppose (33) holds. Then, consider the set

$$\widehat{B^+} := \bigcap_{j=0}^{+\infty} \widehat{F}^{-j}\left(\mathbb{T}^2 \times \widehat{\mathbb{H}_r^v}\right) \subset \mathbb{T}^2 \times \widehat{\mathbb{R}^2}.$$

We will show that $\widehat{B^+}$ exhibits unbounded connected components along the fibers. More precisely, for each $t \in \mathbb{T}^2$ we define

$$B^+(t) := \mathrm{pr}_2\left(\mathrm{cc}\left(\widehat{B^+} \cap \{t\} \times \widehat{\mathbb{R}^2}, (t, \infty)\right)\right) \setminus \{(t, \infty)\} \subset \mathbb{R}^2, \tag{34}$$

where $\mathrm{pr}_2 \colon \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ denotes the projection on the second factor, and we shall show that $B^+(t)$ is non-empty, for all t.

Since we are assuming (33) holds, for every $a > 0$ we can define the following natural number:

$$n(a) := \min\left\{n \in \mathbb{N} : F^{-n}\left(\mathbb{T}^2 \times \ell_{a+r}^v\right) \cap \left(\mathbb{T}^2 \times \mathbb{H}_{-r}^{-v}\right) \neq \varnothing\right\}. \tag{35}$$

Then, there exists a point $t_a \in \mathbb{T}^2$ such that

$$F^{-n(a)}\left(\{t_a + n(a)\rho\} \times \mathbb{H}_{r+a}^v\right) \cap \{t_a\} \times \mathbb{H}_{-r}^{-v} \neq \varnothing.$$

So, we can find a simple continuous arc $\gamma_a \colon [0, 1] \to \widehat{\mathbb{R}^2}$ such that

$$\begin{aligned} &\gamma_a(0) \in \ell_r^v = \partial\mathbb{H}_r^v, \\ &\gamma_a(1) = \infty, \\ &\gamma_a[0, 1) \subset \mathrm{pr}_2\left(F^{-n(a)}\left(\{t + n(a)\rho\} \times \mathbb{H}_{r+a}^v\right)\right) \cap \mathbb{H}_r^v, \end{aligned} \tag{36}$$

and a lattice point $\boldsymbol{p}_a \in \mathbb{Z}^2$ such that

$$\begin{aligned} &\left\langle \boldsymbol{p}_a, v\right\rangle \geqslant 0, \\ &\left\|\gamma_a(0) + \boldsymbol{p}_a - rv\right\| \leqslant \sqrt{2}. \end{aligned} \tag{37}$$

Let us define the set

$$\Gamma_a := \left\{\gamma_a(s) + \boldsymbol{p}_a : s \in [0, 1)\right\} \subset \mathbb{R}^2. \tag{38}$$

Putting together (35) and (36), we conclude that

$$F^j\left(t_a, \Gamma_a\right) \subset \mathbb{T}^2 \times \mathbb{H}_r^v, \quad \text{for } 0 \leqslant j \leqslant n(a) - 1.$$

Now, since $\widehat{\mathbb{R}^2}$ is compact, its space of compact subsets $\mathcal{K}(\widehat{\mathbb{R}^2})$ is compact, too. Hence, we can find a strictly increasing sequence of natural numbers (m_j), a point $t_\infty \in \mathbb{T}^2$ and $\widehat{\Gamma} \in \mathcal{K}(\widehat{\mathbb{R}^2})$ such that $t_{m_j} \to t_\infty$ and

$$\widehat{\Gamma_{m_j}} \to \widehat{\Gamma}, \quad \text{as } j \to \infty,$$

where $\widehat{\Gamma_{m_j}}$ denotes the closure of Γ_{m_j} inside the space $\widehat{\mathbb{R}^2}$ and the convergence is in the Hausdorff distance.

Then observe that $n(m_j) \to +\infty$ as $m_j \to +\infty$. So, $\widehat{\Gamma} \subset \widehat{B^+}$. Then, the set $\Gamma := \widehat{\Gamma}\setminus\{(t_\infty, \infty)\}$ is closed in $\mathbb{R}^2$, connected, and by (37), is non-empty and unbounded. In particular, $\Gamma \subset B^+(t_\infty)$ and so, $B^+(t_\infty)$ is non-empty.

Now, consider an arbitrary point $t \in \mathbb{T}^2$ and let us show that $B^+(t)$ is non-empty, too. To do this, let us choose two points $\tilde{t}_\infty \in \pi^{-1}(t_\infty)$ and $\tilde{t} \in \pi^{-1}(t)$ so that $\langle \tilde{t}_\infty - \tilde{t}, v\rangle \geqslant 0$. Hence, it holds

$$T_{\tilde{t}_\infty - \tilde{t}}\big(\mathbb{H}_r^v\big) \subset \mathbb{H}_r^v. \tag{39}$$

On the other hand, invoking (21) one easily see that

$$\begin{aligned}
H_t^{(n)}\left(T_{\tilde{t}_\infty-\tilde{t}}\big(B^+(t_\infty)\big)\right) &= \mathrm{Ad}_t(T_{\tilde{\rho}}^{-n} \circ \tilde{f}^n)\left(T_{\tilde{t}_\infty-\tilde{t}}\big(B^+(t_\infty)\big)\right) \\
&= T_{\tilde{t}}^{-1} \circ T_{\tilde{\rho}}^{-n} \circ \tilde{f}^n \circ T_{\tilde{t}} \circ T_{\tilde{t}_\infty-\tilde{t}}\big(B^+(t_\infty)\big) \\
&= T_{\tilde{t}_\infty-\tilde{t}} \circ \mathrm{Ad}_{t_\infty}(T_{\tilde{\rho}}^{-n} \circ \tilde{f}^n)\big(B^+(t_\infty)\big) \\
&\subset T_{\tilde{t}_\infty-\tilde{t}}(\mathbb{H}_r^v) \subset \mathbb{H}_r^v,
\end{aligned}$$

for every $n \geqslant 0$. So, we conclude $T_{\tilde{t}_\infty-\tilde{t}}(B^+(\tilde{t}_\infty)) \subset B^+(t)$ and in particular, $B^+(t)$ is non-empty.

Finally, let us show that $\Lambda_r^v(t) \neq \varnothing$, for every $t \in \mathbb{T}^2$. To do that, let us consider the set

$$\mathscr{B}^+ := \bigcup_{t\in\mathbb{T}^2} \{t\} \times B^+(t) \subset \mathbb{T}^2 \times \mathbb{R}^2.$$

Notice that $F(\mathscr{B}^+) \subset \mathscr{B}^+$ and

$$\Lambda_r^v = \bigcap_{n\geqslant 0} F^n(\mathscr{B}^+).$$

So, if we suppose that $\Lambda_r^v = \varnothing$, then there should exist $\boldsymbol{p} \in \mathbb{Z}^2$ and $n_0 \geqslant 0$ such that $\langle \boldsymbol{p}, v\rangle > 0$ and

$$F^{n_0}(\mathscr{B}^+) \subset \mathbb{T}^2 \times T_{\boldsymbol{p}}(\mathbb{H}_r^v) \subsetneq \mathbb{T}^2 \times \mathbb{H}_r^v.$$

But since F commutes with the map $id \times T_{\boldsymbol{p}} \colon \mathbb{T}^2 \times \mathbb{R}^2 \circlearrowleft$, it follows that

$$F^{jn_0}(\mathscr{B}^+) \subset \mathbb{T}^2 \times T_{\boldsymbol{p}}^j(\mathbb{H}_r^v), \quad \forall j \in \mathbb{N},$$

contradicting inclusion (17). So, we have showed Λ_r^v is non-empty. In particular, there exists $t' \in \mathbb{T}^2$ such that $\Lambda_r^v(t') \neq \varnothing$. Invoking the very same argument we used to show that for all t, $B^+(t)$ was non-empty provided $B^+(t_\infty) \neq \varnothing$, one can prove that $\Lambda_r^v(t) \neq \varnothing$, for every $t \in \mathbb{T}^2$. $\square$

The last result of this section is an elementary fact about the topology of fibered invariant sets at infinity under the assumption of unbounded v-deviations:

Proposition 3.5 *If f does not exhibits uniformly bounded v-deviations, then for any $r \in \mathbb{R}$, any $t \in \mathbb{T}^2$ and any $z \in \Lambda_r^v(t)$, it holds*

$$\mathrm{cc}\left(\Lambda_r^v(t), z\right) \not\subset \mathbb{A}_s^v, \quad \forall s > 0, \tag{40}$$

where $\mathbb{A}_s^v$ is the strip given by (6).

Proof Let us suppose (40) is not true. Then, by (iii) of Proposition 3.3, we know there exist $s > 0$ and $z \in \Lambda_0^v(0)$ such that

$$\Lambda_z := \mathrm{cc}\left(\Lambda_0^v(0), z\right) \subset \mathbb{A}_s^v.$$

Since every connected component of $\Lambda_0^v(0)$ is unbounded in $\mathbb{R}^2$, without loss of generality we can assume there exists a sequence $(z_n)_{n\geqslant 0} \subset \Lambda_z$ such that

$$\lim_{n\to+\infty} \left\langle z_n, v^{\perp}\right\rangle = +\infty. \tag{41}$$

On the other hand, there exists a sequence $(\boldsymbol{p}_n)_{n\geqslant 0}$ in $\mathbb{Z}^2$ such that

$$0 \leqslant \langle \boldsymbol{p}_n, v\rangle \leqslant 1, \quad \forall n \geqslant 0, \tag{42}$$

$$\lim_{n\to+\infty} \left\langle \boldsymbol{p}_n, v^{\perp}\right\rangle = -\infty. \tag{43}$$

By Proposition 3.3 and (42), for such a sequence it holds

$$T_{\boldsymbol{p}_n}(\Lambda_z) \subset \Lambda_0^v(0) \cap \mathbb{A}_{s+1}^v, \quad \forall n \geqslant 0. \tag{44}$$

Then, since $\tilde{f}$ preserves orientation and Λ_z is connected and unbounded, as a consequence of (43) and (44) we get that $\bigcup_{n\geqslant 0} T_{\boldsymbol{p}_n}(\Lambda_z)$ disconnects $\mathbb{R}^2$, and hence, $\mathbb{H}_{s+1}^v \subset \Lambda_0^v(0)$. By Corollary 3.2, this implies f exhibits uniformly bounded v-deviations, contradicting our hypothesis. □

4 Directional deviations for periodic point free homeomorphisms

In this section we analyze the topological properties of stable sets at infinity for non-wandering periodic point free homeomorphisms.

So, let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ be a non-wandering homeomorphism with $\mathrm{Per}(f) = \varnothing$ and $\tilde{f} \colon \mathbb{R}^2 \circlearrowleft$ a lift of f. By Theorem 2.3 we know that $\rho(\tilde{f})$ has empty interior. Thus, without any further assumption, there are some $\alpha \in \mathbb{R}$ and $v \in \mathbb{S}^1$ such that condition (17) holds.

Continuing with the notation we introduced in § 3, first we will prove a density result:

Theorem 4.1 *For every $t \in \mathbb{T}^2$ the set*

$$\bigcup_{r\geqslant 0} \Lambda_{-r}^v(t)$$

is dense in $\mathbb{R}^2$.

Proof By Corollary 3.2, Theorem 4.1 clearly holds when f exhibits uniformly bounded v-deviations. So, from now on we shall suppose this is not the case.

Consider the set

$$\tilde{\Lambda} := \overline{\bigcup_{r\geqslant 0} \Lambda_{-r}^v\left(\tilde{f}, 0\right)} \subset \mathbb{R}^2$$

We claim that $\Lambda := \pi(\tilde{\Lambda}) \subset \mathbb{T}^2$ is closed, f-invariant, and it holds $\tilde{\Lambda} = \pi^{-1}(\Lambda)$.

In order to prove our claim, first observe that $\tilde{\Lambda}$ is $\mathbb{Z}^2$-invariant. In fact, this easily follows from (iv) of Proposition 3.3, noticing

$$T_{\boldsymbol{p}}\Big(\Lambda^v_{-r}\big(\tilde{f}, 0\big)\Big) = \Lambda^v_{\langle \boldsymbol{p}, v\rangle - r}\big(\tilde{f}, 0\big), \quad \forall r \geqslant 0, \ \forall \boldsymbol{p} \in \mathbb{Z}^2.$$

This immediately implies that Λ is closed, too, and $\tilde{\Lambda} = \pi^{-1}(\Lambda)$.

Secondly, we will prove $\tilde{\Lambda}$ is $\tilde{f}$-invariant. To do that, observe that for every $r \in \mathbb{R}$ and every point $z \in \Lambda^v_r(0)$, it holds

$$r \leqslant \Big\langle \tilde{f}^n(z) - n\tilde{\rho}, v\Big\rangle = \Big\langle \tilde{f}^{n-1}\big(\tilde{f}(z)\big) - (n-1)\tilde{\rho}, v\Big\rangle - \langle \tilde{\rho}, v\rangle\,,$$

for every $n \in \mathbb{Z}$. This implies that

$$\tilde{f}(z) \in \Lambda^v_{r + \langle \tilde{\rho}, v\rangle}\big(\tilde{f}, 0\big),$$

and analogously one can show that

$$\tilde{f}^{-1}(z) \in \Lambda^v_{r - \langle \tilde{\rho}, v\rangle}\big(\tilde{f}, 0\big).$$

Thus, $\tilde{\Lambda}$ is totally $\tilde{f}$-invariant, and we finish the proof of our claim.

Now, let us suppose $\tilde{\Lambda} \neq \mathbb{R}^2$. This implies that $\Lambda \neq \mathbb{T}^2$. First, observe that none connected component of Λ can be inessential in $\mathbb{T}^2$ (see Sect. 2.4 for definitions). In fact, every connected component of the pre-image by $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ of a compact inessential set in $\mathbb{T}^2$ is bounded in $\mathbb{R}^2$, and by definition, every connected component of $\tilde{\Lambda}$ is unbounded.

Henceforth, if A denotes an arbitrary connected component of $\mathbb{T}^2 \backslash \Lambda$, then A is either a topological disk or annular (see Sect. 2.4 for definitions). Since f is non-wandering and A is a connected component of an open f-invariant set, there exists $n_0 = n_0(A) \in \mathbb{N}$ such that $f^{n_0}(A) = A$.

So, if A were a topological disk, then $f^{n_0}\big|_A : A \circlearrowleft$ would be conjugate to a plane orientation-preserving non-wandering homeomorphism, and by Theorem 2.1, we know that $\mathrm{Fix}\,\big(f^{n_0}\big|_A\big) \neq \varnothing$, contradicting our hypothesis that f is periodic point free.

Thus, A should be annular, i.e. an essential open topological annulus. Let $\tilde{A}$ be any connected component of $\pi^{-1}(A) \subset \mathbb{R}^2$. Since A is annular, $\tilde{A}$ separates the plane in two different connected components, such that both of them are unbounded, and has a well-defined integral homological direction, i.e. there exists a rational slope vector $v' \in \mathbb{S}^1$ (which is unique up to multiplication by (-1)) and $s = s(\tilde{A}) > 0$ such that

$$\tilde{A} \subset \mathbb{A}^{v'}_s.$$

Then we have to consider two possible cases: when v and v' are co-linear; and when they are linearly independent in $\mathbb{R}^2$.

In the first case, let $z \in \Lambda^v_0(0)$ be an arbitrary point, Λ_z be the connected component of $\Lambda^v_0(0)$ containing z and $\boldsymbol{p} \in \mathbb{Z}^2$ such that $T_{\boldsymbol{p}}(z) \in \mathbb{H}^v_{-s-1}$. By Proposition 3.3 we know that $T_{\boldsymbol{p}}(\Lambda_z)$ is a connected component of $\Lambda^v_{\langle \boldsymbol{p}, v\rangle}(0) \subset \tilde{\Lambda}$, and since we are assuming f does not exhibit uniformly bounded v-deviations, we can apply Proposition 3.5 to conclude that $T_{\boldsymbol{p}}(\Lambda_z)$ is not contained in any v-strip. Hence, $T_{\boldsymbol{p}}(\Lambda_z)$ intersects both semi-spaces $\mathbb{H}^v_{s+1}$ and $\mathbb{H}^v_{-s-1}$. Now, since $\tilde{A}$ separates $\mathbb{H}^v_{s+1}$ and $\mathbb{H}^v_{-s-1}$, we conclude $T_{\boldsymbol{p}}(\Lambda_z)$ should intersect $\tilde{A}$, too. In particular $\tilde{\Lambda} \cap \tilde{A} \neq \varnothing$ and henceforth, $\Lambda \cap A \neq \varnothing$, contradicting our hypothesis that A was connected component of $\mathbb{T}^2 \backslash \Lambda$.

Let us analyze the second case, i.e. when v and v' are linear independent in $\mathbb{R}^2$. We know there exists $n_0 \geqslant 1$ such that $f^{n_0}(A) = A$. Hence, there exists $\boldsymbol{p}_0 \in \mathbb{Z}^2$ such that $\tilde{f}^{n_0}(\tilde{A}) = T_{\boldsymbol{p}_0}(\tilde{A})$.

Let us consider the homeomorphism $\tilde{g} := T_{\boldsymbol{p}_0}^{-1} \circ \tilde{f}^{n_0} \in \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^2)$, and notice that $\tilde{g}$ is a lift of f^{n_0} and $\tilde{g}(\tilde{A}) = \tilde{A}$. Then, $n_0\rho(\tilde{f}) - \boldsymbol{p}_0 = \rho(\tilde{g}) \subset \ell_0^{v'}$, and since v and v' are not co-linear, we conclude that $\tilde{f}$ is pseudo-rotation, i.e. $\rho(\tilde{f}) = \{\tilde{\rho}\}$.

Then, let $G\colon \mathbb{T}^2 \times \mathbb{R}^2 \circlearrowleft$ be the $(n_0\tilde{\rho} - \boldsymbol{p}_0)$-centralized skew-product induced by $\tilde{g}$, as defined at the beginning of Sect. 3. Since $\tilde{g}(\tilde{A}) = \tilde{A}$, it easily follows that the open set

$$\hat{A} := \left\{(t, z) : t \in \mathbb{T}^2,\ z \in T_{\boldsymbol{p}-\tilde{t}}(\tilde{A}),\ \tilde{t} \in \pi^{-1}(t),\ \boldsymbol{p} \in \mathbb{Z}^2\right\} \subset \mathbb{T}^2 \times \mathbb{R}^2$$

is G-invariant. Now, we are supposing v and v' are not co-linear, so we can repeat the argument used in the proof Theorem 3.4 inside the set $\hat{A}$ to show that $\Lambda_r^v\big(\tilde{g}\big) \cap \hat{A} \neq \varnothing$. Moreover, it can be shown that $\Lambda_r^v\big(\tilde{g}, 0\big) \cap \tilde{A} \neq \varnothing$. Finally observe that

$$\Lambda_r^v\big(\tilde{g}, 0\big) \subset \Lambda_{r-M}^v\big(\tilde{f}, 0\big),$$

where $M := n_0 \sup_{z\in\mathbb{R}^2} \left|\Delta_{\tilde{f}}(z)\right|$. In particular, we have shown that $\tilde{\Lambda} \cap \tilde{A} \neq \varnothing$, and thus, $\Lambda \cap A \neq \varnothing$, contradicting our hypothesis that A is a connected component of $\mathbb{T}^2\setminus\Lambda$. □

In Corollary 3.2 we have shown that f exhibits uniformly bounded $(\pm v)$-deviations if and only if the fibered (r, v)-stable sets at infinity contain whole semi-planes.

Here, we will improve this result for periodic point free systems:

Theorem 4.2 *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ be a periodic point free and non-wandering homeomorphism, $\tilde{f}\colon \mathbb{R}^2 \circlearrowleft$ a lift of f, and $v \in \mathbb{S}^1$ and $\alpha \in \mathbb{R}$ such that inclusion (17) holds. If the set $\Lambda_r^v(t)$ has non-empty interior for some (and hence, any) $t \in \mathbb{T}^2$ and $r \in \mathbb{R}$, then there exists $v' \in \mathbb{S}^1$ such that f exhibits uniformly bounded v'-deviations.*

Proof If f is an annular homeomorphism, (see Sect. 2.10 for the definition), then the conclusion automatically holds. So, we can assume f is non-annular.

On the other hand, since f is periodic point free, it clearly holds $\mathrm{Fix}(f^k) = \varnothing$, for every $k \in \mathbb{Z}\setminus\{0\}$. So, according to the classification given in § 2.10, we can assume f is a strictly toral homeomorphism,

Now, since f is non-wandering and periodic point free, by Proposition 2.8 we know that every point of $\mathbb{T}^2$ is essential for f.

Let us suppose $\Lambda_r^v(0)$ has non-empty interior. Let x be a point in interior of $\Lambda_r^v(0)$, and $\varepsilon > 0$ such that the ball $B_\varepsilon(x) \subset \mathrm{int}\big(\Lambda_r^v(0)\big)$. Since $\pi(x)$ is an essential point for f and we are assuming f is strictly toral, by Proposition 2.9 the open set $U_\varepsilon(\pi(x), f)$ given by (20) is fully essential.

So, there are simple closed curves $\gamma_1, \gamma_2\colon [0, 1] \to \mathbb{T}^2$ whose images are contained in $U_\varepsilon(\pi(x), f)$ and such that they generate the fundamental group of $\mathbb{T}^2$. Since γ_1 and γ_2 are compact, there exists $m \in \mathbb{N}$ such that

$$\gamma_1[0, 1] \cup \gamma_2[0, 1] \subset \bigcup_{j=-m}^{m} f^j\Big(\pi\big(B_\varepsilon(x)\big)\Big).$$

Now, recalling that $\tilde{f}^n(\Lambda_r^v(0)) \subset \Lambda_{r+n\langle\tilde{\rho},v\rangle}^v(0)$ for every $n \in \mathbb{Z}$ and the covering $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ is an open map, we conclude that

$$\gamma_1[0, 1] \cup \gamma_2[0, 1] \subset \mathrm{int}\big(\pi(\Lambda_{r-m|\langle\tilde{\rho},v\rangle|})(0)\big).$$

So, we can construct a bi-sequence of simple curves $\left(\tilde{\gamma}^n : [0,1] \to \mathbb{R}^2\right)_{n\in\mathbb{Z}}$ satisfying the following properties:

(1) $\tilde{\gamma}^n(t) \in \Lambda_{r-m|\langle\tilde{\rho},v\rangle|}(0)$, for every $n \in \mathbb{Z}$ and all $t \in [0,1]$;
(2) $\tilde{\gamma}^n$ is a lift either of γ_1 or γ_2, for each $n \in \mathbb{Z}$;
(3) $\tilde{\gamma}^n(1) = \tilde{\gamma}^{n+1}(0)$, for every $n \in \mathbb{Z}$;
(4) there exists a constant $M > 0$ such

$$d(\tilde{\gamma}^n(t), \ell^v_{r-m|\langle\tilde{\rho},v\rangle|}) \leqslant M, \quad \forall n \in \mathbb{Z}, \ \forall t \in [0,1].$$

Then, the set $\bigcup_n \tilde{\gamma}^n[0,1]$ clearly separates the plane, and its complement has exactly two unbounded connected components. In particular, this implies $\Lambda^v_{r-m|\langle\tilde{\rho},v\rangle|}$ contains a semi-plane, and by Corollary 3.2, this implies f exhibits uniformly bounded v-deviations. □

5 Torus pseudo-foliations and rotational deviations

In this section we introduce the concepts of *pseudo-foliation* which is a generalization of singularity-free one-dimensioanl plane foliation.

A *plane pseudo-foliation* is a partition $\mathscr{F}$ of $\mathbb{R}^2$ such that every atom (also called *pseudo-leaf*) is closed, connected, has empty interior and separates the plane in exactly two connected components.

Given a map $h : \mathbb{R}^2 \circlearrowleft$, we say that the plane pseudo-foliation $\mathscr{F}$ is h-invariant when

$$h(\mathscr{F}_z) = \mathscr{F}_{h(z)}, \quad \forall z \in \mathbb{R}^2.$$

where $\mathscr{F}_z$ denotes the atom of $\mathscr{F}$ containing z; and $\mathscr{F}$ is said to be $\mathbb{Z}^2$-invariant when it is $T_{\boldsymbol{p}}$-invariant for any $\boldsymbol{p} \in \mathbb{Z}^2$.

A *torus pseudo-foliation* is a partition $\mathscr{F}$ of $\mathbb{T}^2$ such that there exists a $\mathbb{Z}^2$-invariant plane pseudo-foliation $\tilde{\mathscr{F}}$ satisfying

$$\pi(\tilde{\mathscr{F}}_z) = \mathscr{F}_{\pi(z)}, \quad \forall z \in \mathbb{R}^2.$$

In such a case, the plane pseudo-foliation $\tilde{\mathscr{F}}$ is unique and will be called the *lift* of $\mathscr{F}$.

Notice that a homeomorphism $f : \mathbb{T}^2 \circlearrowleft$ leaves invariant a torus pseudo-foliation $\tilde{\mathscr{F}}$ if and only if its lift $\tilde{\mathscr{F}}$ is $\tilde{f}$-invariant, for any lift $\tilde{f} : \mathbb{R}^2 \circlearrowleft$ of f.

Some geometric and topological properties of "classical" plane and torus foliations can be extended to pseudo-foliations:

Proposition 5.1 *If $\mathscr{F}$ is a plane pseudo-foliation, both connected components of $\mathbb{R}^2 \backslash \mathscr{F}_z$ are unbounded, for every $z \in \mathbb{R}^2$.*

Proof This easily follows from Zorn's Lemma. In fact, let us suppose there exists $z_0 \in \mathbb{R}^2$ such that a connected component of $\mathbb{R}^2 \backslash \mathscr{F}_{z_0}$, called B_0, is bounded. So, for every $w \in B_0$, $\mathscr{F}_w \subset B_0$ and thus, it is bounded itself. This implies that, for all $w \in B_0$, there exists a bounded connected component, called B_w, of $\mathbb{R}^2 \backslash \mathscr{F}_w$, and it clearly holds $B_w \subset B_0$, for any $w \in B_0$.

Now, if we consider the set $\{B_w : w \in B_0\}$ endowed with the partial order given by set inclusion, one can easily check that any totally decreasing chain admits a lower bound, and consequently, there exists a minimal element. In fact, given a strictly decreasing sequence $B_{w_1} \supset B_{w_2} \supset \cdots$ we have that $\partial B_{w_n} \cap \partial B_{w_{n+1}} = \varnothing$ because different pseudo-leaves are disjoint, and therefore, it holds $\bigcap_{n\geqslant 1} B_{w_n} = \bigcap_{n\geqslant 1} \overline{B_{w_n}} \neq \varnothing$. Now, taking any point

$w \in \bigcap_{n \geqslant 1} B_{w_n}$, B_w happens to be a lower bound for the chain. So, by Zorn's Lemma there exists a minimal element B_z but of course, $B_w \subsetneq B_z$, for any $w \in B_z$, getting a contradiction. □

In order to show that torus pseudo-foliations exhibit some properties similar to classical foliations, we will use a geometric result due to Koropecki and Tal [19]. To do that, first we need to introduce some terminology.

Given a closed discrete set $\Sigma \subset \mathbb{R}^2$, a subset $U \subset \mathbb{R}^2$ is said to be Σ*-free* when $T_{\boldsymbol{p}}(U) \cap U = \varnothing$, for all $\boldsymbol{p} \in \Sigma$.

A *chain* is a decreasing sequence $\mathscr{C} = (U_n)_{n \in \mathbb{N}}$ of arcwise connected subsets of $\mathbb{R}^2$, i.e. $U_{n+1} \subset U_n$, for all $n \in \mathbb{N}$. We say that $\mathscr{C}$ is *eventually* Σ*-free* if for every $\boldsymbol{p} \in \Sigma$, there is $n \in \mathbb{N}$ such that $T_{\boldsymbol{p}}(U_n) \cap U_n = \varnothing$.

Let us write $\mathbb{Z}^2_* := \mathbb{Z}^2 \backslash \{(0,0)\}$ and recall the following result of [19, Theorem 3.2]:

Theorem 5.2 *If $\mathscr{C} = (U_n)_{n \in \mathbb{N}}$ is an eventually $\mathbb{Z}^2_*$-free chain of arcwise connected sets, then one of the following holds:*

(i) there exist $n \in \mathbb{N}$ and $\boldsymbol{q} \in \mathbb{Z}^2$ such that U_n is $\mathbb{Z}^2 \backslash (\mathbb{R}\boldsymbol{q})$-free;
(ii) there is a unique $v \in \mathbb{S}^1$ such that $\bigcap_{n \in \mathbb{N}} \partial_\infty U_n = \{v\}$, where ∂_∞ denotes the boundary at infinity defined in Sect. 2.3;
(iii) there are $v \in \mathbb{S}^1$ and $r > 0$ such that $\bigcap_{n \in \mathbb{N}} \overline{U_n}$ is contained in the strip $\mathbb{A}^v_r$ and it separates the boundary components of $\mathbb{A}^v_r$.

Notice that, if property (iii) holds, then we have $\bigcap_{n \in \mathbb{N}} \partial_\infty U_n = \{-v, v\}$.

Now we will use Theorem 5.2 to prove our main result about torus pseudo-foliations:

Theorem 5.3 *If $\mathscr{F}$ is a $\mathbb{Z}^2$-invariant pseudo-foliation, then there exist $v \in \mathbb{S}^1$ and $r > 0$ such that*

$$\mathscr{F}_z \subset T_z(\mathbb{A}^v_r), \quad \forall z \in \mathbb{R}^2. \tag{45}$$

In such a case, v is unique up to multiplication by (-1) and we say that $v^\perp$ is an asymptotic direction *of $\mathscr{F}$.*

Moreover, there is $w \in \mathbb{R}^2$ such that $\mathscr{F}_w$ separates the boundary components of the strip $T_w(\mathbb{A}^v_r)$.

Proof Let z denote an arbitrary point of $\mathbb{R}^2$ and, for each $n \in \mathbb{N}$, consider the set

$$U_n(z) := \left\{ x \in \mathbb{R}^2 : d(x, \mathscr{F}_z) < \frac{1}{n} \right\}.$$

Notice all the sets $U_n(z)$ are arcwise connected, $U_{n+1}(z) \subset U_n(z)$ and it holds $\mathscr{F}_z = \bigcap_{n \in \mathbb{N}} U_n(z)$.

First, let us suppose that there exists $z \in \mathbb{R}^2$ such that the chain $\big(U_n(z)\big)_{n \in \mathbb{N}}$ is not eventually $\mathbb{Z}^2_*$-free. So, there is $\boldsymbol{p} \in \mathbb{Z}^2_*$, such that $T_{\boldsymbol{p}}\big(U_n(z)\big) \cap U_n(z) \neq \varnothing$, for every $n \in \mathbb{N}$. By Proposition 5.1, each pseudo-leaf of $\mathscr{F}$ is unbounded in $\mathbb{R}^2$. So, this implies that for n sufficiently big, $T_{\boldsymbol{p}^\perp}\big(U_n(z)\big) \cap U_n(z) = \varnothing$. Let us fix such an n, and consider the set

$$\Theta := \bigcup_{j \in \mathbb{Z}} T^j_{\boldsymbol{p}}\big(U_n(z)\big).$$

Then Θ is open, connected, $T_{\boldsymbol{p}}$-invariant and $\Theta \cap T_{\boldsymbol{p}^\perp}(\Theta) = \varnothing$. In particular, it is contained in the lift of an annular subset of $\mathbb{T}^2$, and this implies there exists $r > 0$ such that $\mathscr{F}_z \subset \mathbb{A}^v_r$.

The uniform result for every pseudo-leaf easily follows from the fact that $\Theta \cap T_{\boldsymbol{p}^\perp}(\Theta) = \varnothing$ and the definition of plane pseudo-foliations.

So, now we can assume that $(U_n(z))_{n\in\mathbb{N}}$ is eventually $\mathbb{Z}^2_*$-free for every $z \in \mathbb{R}^2$, and hence, Theorem 5.2 can be applied.

First suppose property (i) of Theorem 5.2 holds for some $z \in \mathbb{R}^2$, i.e. there are $n_0 \in \mathbb{N}$ and $\boldsymbol{q} \in \mathbb{Z}^2$ such that $U_{n_0}(z)$ is $\mathbb{Z}^2\backslash(\mathbb{R}\boldsymbol{q})$-free. Since $U_n(z) \subset U_{n_0}(z)$ for $n > n_0$, the same holds for every $n > n_0$. Without loss of generality we can assume $\boldsymbol{q}$ generates the cyclic group $\mathbb{Z}^2 \cap \mathbb{R}\boldsymbol{q}$. Then, since the chain $(U_n(z))_{n\in\mathbb{N}}$ is eventually $\mathbb{Z}^2_*$-free, there is $n' \in \mathbb{N}$ such that $T_{\boldsymbol{q}}\big(U_{n'}(z)\big) \cap U_{n'}(z) = \varnothing$.

By Theorem 2.2, we have

$$T_{\boldsymbol{q}}^j\left(U_{n'}(z)\right) \cap U_{n'}(z) = \varnothing, \quad \forall j \in \mathbb{Z}\backslash\{0\}.$$

So, given any integer $n \geqslant \max\{n_0, n'\}$, we conclude the chain $\big(U_n(z)\big)_{n\geqslant 1}$ is $\mathbb{Z}^2_*$-free.

On the other hand, by Proposition 5.1 we know the pseudo-leaf $\mathscr{F}_z$ is unbounded. Then, there is a sequence $(z_j)_{j\in\mathbb{N}} \subset \mathscr{F}_z$ such that the balls $B_{1/n}(z_1), B_{1/n}(z_2), \ldots$ are pairwise disjoints. Then, there is a sequence $(\boldsymbol{p}_j)_{j\in\mathbb{N}} \subset \mathbb{Z}^2$ such that $T_{\boldsymbol{p}_j}(z_j) \in [0,1]^2$, for every $j \in \mathbb{N}$. Since $\big(U_n(z)\big)_{n\geqslant 1}$ is $\mathbb{Z}^2_*$-free and $B_{1/n}(z_j) \subset U_n(z)$ for every $j \in \mathbb{N}$, we get that the balls $\big(T_{\boldsymbol{p}_j}(B_{1/n}(z_j))\big)_{j\geqslant 1}$ are pairwise disjoint, and this is clearly impossible. So, the chain $\big(U_n(z)\big)_{n\in\mathbb{N}}$ does not verify condition (i) of Theorem 5.2, for every $z \in \mathbb{R}^2$.

So, let us suppose the chain $\big(U_n(z)\big)_{n\geqslant 1}$ satisfies condition (ii) of Theorem 5.2, for every $z \in \mathbb{R}^2$. Then, for each $z \in \mathbb{R}^2$, let $v_z \in \mathbb{S}^1$ denote the only point such that $\partial_\infty \mathscr{F}_z = \{v_z\}$. Let Ω_z be the connected component of $\mathbb{R}^2\backslash\mathscr{F}_z$ such that $\partial_\infty(\Omega_z) = \{v_z\}$. Notice that Ω_z is an open disk, and since the boundary of Ω_z is contained in the pseudo-leaf $\mathscr{F}_z$, we conclude that $\overline{\Omega_z}$ is simply connected, for all $z \in \mathbb{R}^2$.

Then we define the set

$$\mathcal{M} := \left\{w \in \mathbb{R}^2 : \forall z \in \mathbb{R}^2,\ w \notin \Omega_z\right\}.$$

Since every Ω_z is open, $\mathcal{M}$ is clearly closed and might be empty.

Let us define set

$$\Xi := \mathbb{R}^2\backslash\left(\bigcup_{w\in\mathcal{M}} \overline{\Omega_w}\right)$$

Observe that if w and z are two elements of $\mathcal{M}$ such that $\mathscr{F}_w \neq \mathscr{F}_z$, then it holds $\overline{\Omega_z} \cap \overline{\Omega_w} = \varnothing$. So, since $\mathbb{R}^2$ cannot be partitioned into countably many (and more than two) closed sets, this implies that Ξ is non-empty; and noticing $\overline{\Omega_z}$ is simply-connected for every $z \in \mathcal{M}$, we can conclude that Ξ is connected. On the other hand, since $\mathcal{M}$ is $\mathbb{Z}^2$-invariant, so is Ξ.

Hence, Ξ is a non-empty open connected $\mathbb{Z}^2$-invariant set and we shall consider the following relation on it: given $w_1, w_2 \in \Xi$, we define

$$w_1 \sim w_2 \iff \exists z \in \mathbb{R}^2,\ w_1, w_2 \in \Omega_z.$$

Let us show $\sim$ is an equivalent relation. First notice it is clearly reflexive and symmetric. To prove that is transitive too, it is enough to observe that given two arbitrary points $z_1, z_2 \in \mathbb{R}^2$ such that $\overline{\Omega_{z_1}} \cap \overline{\Omega_{z_2}} \neq \varnothing$, then either $\Omega_{z_1} \subset \Omega_{z_2}$, or $\Omega_{z_2} \subset \Omega_{z_1}$. On the other hand, since Ω_z is open for every $z \in \mathbb{R}^2$, we conclude that $\sim$ is an open equivalent relation (i.e. every equivalent class is open). But we had already shown that Ξ is connected, so this implies that there is just one equivalent class. Thus, taking any point $w \in \Xi$ and any $\boldsymbol{p} \in \mathbb{Z}^2_*$, we get that $w \sim T_{\boldsymbol{p}}(w) \sim T_{\boldsymbol{p}^\perp}(w)$. So, there should exist a point $z \in \mathbb{R}^2$ such that Ω_z contains the points w, $T_{\boldsymbol{p}}(w)$ and $T_{\boldsymbol{p}^\perp}(w)$. This clearly contradicts the fact that $\partial_\infty \Omega_z$ is a singleton.

Therefore, it is not possible that condition (ii) holds for every $\big(U_n(z)\big)_{n\geqslant 1}$. So, there exists $z_0 \in \mathbb{R}^2$, $v \in \mathbb{S}^1$ and $r > 0$ such that $\mathscr{F}_{z_0} \subset \mathbb{A}^v_{r/2}$ and it separates the connected components of the boundary of the strip $\mathbb{A}^v_{r/2}$. Since, the pseudo-foliation $\mathscr{F}$ is $\mathbb{Z}^2$-invariant, this clearly implies that condition (45) holds, as desired. □

Let us relate the existence of pseudo-foliations with the boundedness of rotational deviations:

Theorem 5.4 *If $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ leaves invariant a torus pseudo-foliation $\mathscr{F}$ and $v^\perp \in \mathbb{S}^1$ is an asymptotic direction of (the lift of) $\mathscr{F}$, f exhibits uniformly bounded v-deviations.*

In particular the rotation set $\rho(f)$ has empty interior.

Proof This theorem follows from the combination of Theorem 5.3 and the argument used to prove Proposition 4.2 of [15].

Let $\tilde{f} \in \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^2)$ be a lift of f, and $\tilde{\mathscr{F}}$ be the lift of an f-invariant torus pseudo-foliation $\mathscr{F}$. So, $\tilde{\mathscr{F}}$ is $\tilde{f}$-invariant. Let $v \in \mathbb{S}^1$ such that $v^\perp$ determines the asymptotic direction of $\tilde{\mathscr{F}}$. Without loss of generality we can assume that $v \in \mathbb{S}^1$ is not vertical, i.e. $\mathrm{pr}_1(v) \neq 0$.

Let r be a positive real number given by Theorem 5.3 and let $\tilde{\mathscr{F}}_w$ be a pseudo-leaf of $\tilde{\mathscr{F}}$ such that $\tilde{\mathscr{F}}_w \subset T_w(\mathbb{A}^v_r)$ separates the boundary components of $T_w(\mathbb{A}^v_r)$.

Since $v^\perp \neq (1,0)$, we know that $T_{(1,0)}(\tilde{\mathscr{F}}_w) \cap \tilde{\mathscr{F}}_w = \varnothing$. So, we can consider the strip

$$S := \mathrm{cc}\Big(\mathbb{R}^2 \backslash \tilde{\mathscr{F}}_w, T_{(1,0)}\big(\tilde{\mathscr{F}}_w\big)\Big) \cap \mathrm{cc}\Big(\mathbb{R}^2 \backslash T_{(1,0)}\big(\tilde{\mathscr{F}}_w\big), \tilde{\mathscr{F}}_w\Big).$$

Note that $\overline{S} \subset T_w(\mathbb{A}^v_{r+1})$ and

$$\bigcup_{n\in\mathbb{Z}} T_{(n,0)}(\overline{S}) = \mathbb{R}^2. \tag{46}$$

This implies that for each $n \in \mathbb{Z}$, there exists $m_n \in \mathbb{Z}$ such that

$$\tilde{f}^n\big(\tilde{\mathscr{F}}_w\big) \subset T_{(m_n,0)}(\overline{S}),$$

and consequently,

$$\tilde{f}^n\Big(T_{(1,0)}\big(\tilde{\mathscr{F}}_w\big)\Big) = T_{(1,0)}\Big(\tilde{f}^n\big(\tilde{\mathscr{F}}_w\big)\Big) \subset T_{(m_n+1,0)}(\overline{S}).$$

Hence, we conclude that

$$\tilde{f}^n(\overline{S}) \subset T_{(m_n,0)}\big(\overline{S} \cup T_{(1,0)}(\overline{S})\big) \subset T_{(m_n,0)}(\mathbb{A}^v_{2r+2}), \quad \forall n \in \mathbb{Z}. \tag{47}$$

This fact implies that $\tilde{f}$ satisfies condition (17), i.e. , there exists $\alpha \in \mathbb{R}$ such that

$$\rho(\tilde{f}) \subset \ell^v_\alpha.$$

So we can define the $\tilde{\rho}$-centralized skew-product and the $\pm v$-stable sets at infinity as in § 3. By Theorem 4.1, there exists $r' > 0$ such that $\Lambda^{-v}_{-r'}(0) \cap S \neq \varnothing$. Let z_0 be any point in $\Lambda^{-v}_{-r'}(0) \cap S$ and z be an arbitrary point of $\mathbb{R}^2$. By (46), there exists $m \in \mathbb{Z}$ such that $T_{(m,0)}(z) \in S$.

Since $z_0 \in \Lambda^{-v}_{-r'}(0)$, it holds

$$\Big\langle \tilde{f}^n(z_0) - z_0, v \Big\rangle - n\alpha \leqslant r', \quad \forall n \in \mathbb{Z}. \tag{48}$$

So putting together (47) and (48), we conclude that

$$\begin{aligned}\left\langle \tilde{f}^n(z) - z, v\right\rangle - n\alpha &= \left\langle \tilde{f}^n \circ T_{(m,0)}(z) - T_{(m,0)}(z), v\right\rangle - n\alpha \\ &\leqslant \left\langle \tilde{f}^n(z_0) - z_0, v\right\rangle - n\alpha + 2r + 2 \leqslant 2r + 2 + r',\end{aligned}$$

and hence, every point exhibits uniformly bounded v-deviations. □

The following result is a partial reciprocal of Theorem 5.4:

Theorem 5.5 *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ be an area-preserving non-annular periodic point free homeomorphism, $\tilde{f}\colon \mathbb{R}^2 \circlearrowleft$ a lift of f, $\alpha \in \mathbb{R}$, $v \in \mathbb{S}^1$ and $M > 0$ such that*

$$\left|\left\langle \tilde{f}^n(z) - z, v\right\rangle - n\alpha\right| \leqslant M, \quad \forall z \in \mathbb{R}^2.$$

Then, there exists an f-invariant torus pseudo-foliation with asymptotic direction equal to $v^\perp$.

As it was already shown in [12, §4], the hypothesis is non-annularity is essential to guaranty the existence of the invariant pseudo-foliation, but the area-preserving assumption might be relaxed.

Proof of Theorem 5.5 First observe that if f is an area-preserving irrational pseudo-rotation with uniformly bounded rotational deviations in every direction, then Jäger showed [11, Theorem C] that f is a topological extension of totally irrational torus rotation. In such a case, the pre-image by the semi-conjugacy on any linear torus foliation will yield an f-invariant pseudo-foliation.

So, we can assume f is not a pseudo-rotation with uniformly bounded rotational deviations in every direction.

If v has rational slope, taking into account f is non-annular we can conclude α is an irrational number and this case is essentially considered in Theorem 3.1 of [12]. In fact, under these hypotheses it can be easily proved that the the family of circloids constructed there, which are nothing but the fibers of the factor map over the irrational circle rotation, is an f-invariant pseudo-foliation.

So, from now on let us assume v has irrational slope. Let $\tilde{f}\colon \mathbb{R}^2 \circlearrowleft$ be a lift of f and choose an arbitrary point $\tilde{\rho} \in \rho(\tilde{f})$. Then we consider the induced $\tilde{\rho}$-centralized skew-product $F\colon \mathbb{T}^2 \times \mathbb{R}^2 \circlearrowleft$. For each $r \in \mathbb{R}$ and $t \in \mathbb{T}^2$, consider the (r, v)-fibered stable set at infinity $\Lambda_r^v(t)$ given by (25).

For simplicity, let us fix $t = 0$. By Corollary 3.2, there is $M > 0$ such that $\mathbb{H}_{r+M}^v \subset \Lambda_r^v(0)$, for all $r \in \mathbb{R}$. Then we can define

$$\begin{aligned} U_r &:= \mathrm{cc}\Big(\mathrm{int}\big(\Lambda_r^v(0)\big), \mathbb{H}_{r+M}^v\Big), \\ C_r &:= \partial U_r, \quad \forall r \in \mathbb{R}, \end{aligned}$$

where $\partial(\cdot)$ denotes the boundary operator in $\mathbb{R}^2$. Observe that $\overline{U_r} = U_r \cup C_r \subset \Lambda_r^v(0)$, for any $r \in \mathbb{R}$. Hence, $T_{\boldsymbol{p}}(U_r) = U_{r+\langle \boldsymbol{p}, v\rangle}$ and consequently, $T_{\boldsymbol{p}}(C_r) = C_{r+\langle \boldsymbol{p}, v\rangle}$, too, for every $\boldsymbol{p} \in \mathbb{Z}^2$ and any r.

Then we claim the sets C_r are pairwise disjoint. To prove this, reasoning by contradiction, let us suppose this is not the case and so there exist $s_0 < s_1$ such that $C_{s_0} \cap C_{s_1} \neq \varnothing$. By monotonicity of the family $\{\Lambda_r^v(0) : r \in \mathbb{R}\}$, we get that

$$\varnothing \neq C_{s_0} \cap C_{s_1} \subset \big(C_{s_0} \cap C_r\big) \cap \big(C_{s_1} \cap C_r\big), \quad \forall r \in (s_0, s_1). \tag{49}$$

Let r_0 and r_1 be any pair of real numbers such that $s_0 < r_0 < r_1 < s_1$ and consider the set

$$L := \left\{ \boldsymbol{p} \in \mathbb{Z}^2 : \frac{s_0 - r_0}{2} < \langle \boldsymbol{p}, v \rangle < \frac{s_1 - r_1}{2} \right\}.$$

Notice that L has *bounded gaps in both coordinates*, i.e. there exists $N \in \mathbb{N}$ such that for every $m \in \mathbb{Z}$ it holds

$$\{\mathrm{pr}_i(\boldsymbol{p}) : \boldsymbol{p} \in L\} \cap \{n \in \mathbb{Z} : |m - n| \leqslant N\} \neq \varnothing, \quad \text{for } i \in \{1, 2\}. \tag{50}$$

On the other hand, consider the open set $\Omega := U_{r_0} \backslash \overline{U_{r_1}}$, and observe that

$$T_{\boldsymbol{p}}(\Omega) \subset U_{s_0} \backslash \overline{U_{s_1}}, \quad \forall \boldsymbol{p} \in L. \tag{51}$$

Putting together (49), (50) and (51) we can see that the diameter of the connected components of Ω must be uniformly bounded, i.e. there exists a real number $D > 0$ such that

$$\mathrm{diam}(\Omega') \leqslant D, \quad \forall \Omega' \in \pi_0(\Omega). \tag{52}$$

On the other hand, we know that

$$\tilde{f}^n(C_r) = C_{r+n\alpha}, \quad \text{and} \quad T_{\boldsymbol{p}}(C_r) = C_{r+\langle \boldsymbol{p}, v \rangle}, \quad \forall n \in \mathbb{Z}, \ \forall r \in \mathbb{R}. \tag{53}$$

So, putting together (52) and (53) we can conclude that the set

$$\left\{ n \in \mathbb{Z} : \exists \boldsymbol{p} \in \mathbb{Z}^2, \ T_{\boldsymbol{p}}\big(\tilde{f}^n(\Omega)\big) \in U_{s_0} \backslash \overline{U_{s_1}} \right\}$$

has bounded gaps, and consequently, there exists $D' > 0$ such that

$$\mathrm{diam}\left(\tilde{f}^n(\Omega')\right) \leqslant D', \quad \forall n \in \mathbb{Z}. \tag{54}$$

Then, putting together the fact that f is strictly toral and property (54), we immediately conclude that f exhibits uniformly bounded rotational deviations in every direction, contradicting our original assumption.

So, we have shown that the open set Ω separates the closed sets C_{s_0} and C_{s_1}, and then they do not intersect.

Then we define the function $H : \mathbb{R}^2 \to \mathbb{R}$ by

$$H(z) := \sup \{r \in \mathbb{R} : z \in U_r\}, \quad \forall z \in \mathbb{R}^2.$$

Since we have shown that the family $\{C_r : r \in \mathbb{R}\}$ is pairwise disjoint, it easily follows that the function H is continuous. By (53), it follows that

$$H\big(\tilde{f}^n(z)\big) = H(z) + n\alpha, \quad \forall n \in \mathbb{Z}, \ \forall z \in \mathbb{R}^2,$$

so the partition given by the H-level sets is f-invariant. On the other hand, by the topological properties of the set U_r and C_r, it clearly follows that each H-level set is connected and disconnects the plane in two connected components.

So, order to show that the level sets of H determines a pseudo-foliation, it just remains to show that, for each $r \in \mathbb{R}$, the set $H^{-1}(r)$ has empty interior. To do that, let us suppose this is not the case. Thus there exists r such that $H^{-1}(r)$ has non-empty interior in $\mathbb{R}^2$. Let W be a connected component of the interior of $H^{-1}(r)$. Since $H^{-1}(r)$ separates the plane in exactly two connected components, we conclude that W is an open topological disc. And since the covering map $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ is an open map, $\pi(W)$ will be an open itself.

Then, taking into account f is non-wandering, there exists $n_0 \geqslant 1$ such that

$$f^{n_0}(\pi(W)) \cap \pi(W) \neq \varnothing. \tag{55}$$

That means there exists $\boldsymbol{q} \in \mathbb{Z}^2$ such that

$$\tilde{f}^{n_0}\big(W\big) \cap T_{\boldsymbol{q}}(W) \neq \varnothing. \tag{56}$$

Since the partition in level sets of H is $\tilde{f}$-invariant, this implies $\tilde{f}^{n_0}\big(H^{-1}(r)\big) = T_{\boldsymbol{q}}\big(H^{-1}(r)\big)$, and taking into account W is a connected component of the interior of $H^{-1}(r)$, we conclude that $\tilde{f}^{n_0}(W) = T_{\boldsymbol{q}}(W)$. So, (56) can be improved: in fact, it holds $f^{n_0}(W) = W$. On the other hand, since v has irrational slope, we know that

$$T_{\boldsymbol{p}}\big(H^{-1}(r)\big) \cap H^{-1}(r) = \varnothing, \quad \forall \boldsymbol{p} \in \mathbb{Z}^2 \backslash \{0\}.$$

This implies $\pi(W)$ is an open disc in $\mathbb{T}^2$, and since f is non-wandering, by Theorem 2.1 f^{n_0} should have a fixed point on $\pi(W)$, contradicting the fact that f is periodic point free. □

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