Periodic point free homeomorphisms of $\mathbb{T}^2$

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Closing Lemma

Question A

Is the set

$$\left\{f \in \mathrm{Diff}^r(\mathbb{T}^2) : \mathrm{Per}(f) \neq \varnothing\right\}$$

C^r-dense in $\mathrm{Diff}^r(\mathbb{T}^2)$, for $r \geqslant 2$?

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Yes, for $r = 1$ (Pugh's closing lemma), still open for $r \geqslant 2$.

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  - $k \neq 0$: $\Delta_{\tilde{f}} := \tilde{f} - I_k \in C^0(\mathbb{T}^2, \mathbb{R}^2)$,

$$\rho_V(\tilde{f}) := \left\{ \int_{\mathbb{T}^2} \mathrm{pr}_2 \circ \Delta_{\tilde{f}} \, \mathrm{d}\mu : \mu \in \mathfrak{M}(f) \right\} \text{ compact segment}$$

Examples

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  - $\rho(\tilde{f}) = \{\alpha\}$;
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- Reparametrizations of linear flows: For $\alpha \in \mathbb{R}$, $t \in \mathbb{R} \backslash \{0\}$, $\phi\colon \mathbb{T}^2 \to \mathbb{R}_+$, $X := (\phi, \alpha\phi) \in \mathbb{R}^2$, $\tilde{f} := \Phi_X^t =$ time-t X-flow

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  - [Franks-Misiurewicz]: $\forall \alpha \in \mathbb{R} \backslash \mathbb{Q},\ \exists r \neq 0$, s.t. $\rho(\tilde{f}) = \{(r, r\alpha)\}$

Examples

- Skew-products over irrational rotations: For $\alpha \in \mathbb{R} \backslash \mathbb{Q}$ and $\phi \colon \mathbb{T}^2 \to \mathbb{R}$ with $\tilde{f}(x, y) := \big(x + \alpha, y + \phi(x, y)\big)$

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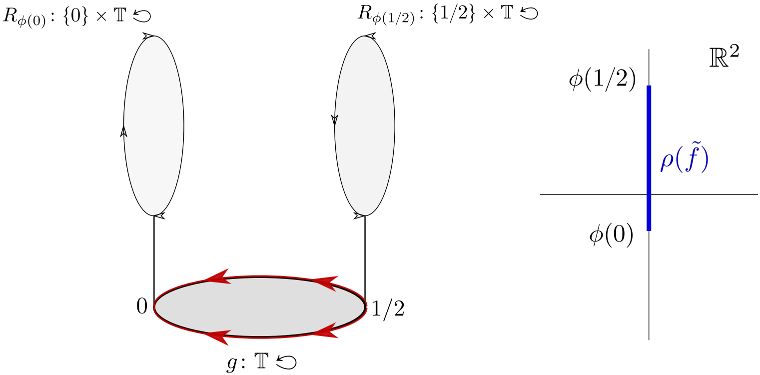
Example

$$\rho(\tilde{f}) \cap \mathbb{Q}^2 \neq \varnothing, \quad \text{and} \quad \mathrm{Per}(f) = \varnothing$$



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In all above case there was an invariant foliation.
In most of them $\rho(\tilde{f})$ was a singleton.

Main result

Theorem [K]

If $f \in \mathrm{Homeo}(\mathbb{T}^2)$ is a minimal homeomorphism, then:

(a) either f is topologically weak mixing;

(b) or f is a topological exstension of an irrational circle rotation.

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Recalling...

$f : M \circlearrowleft$ is topologically weak mixing if $\forall U \subset M$ open, $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ s.t. $f^n(U)$ is ε-dense in M

Rotation set

$f \in \mathrm{Homeo}_0(\mathbb{T}^2)$, $\tilde{f} : \mathbb{R}^2 \circlearrowleft$ a lift and $\Delta_{\tilde{f}} := \tilde{f} - id \in C^0(\mathbb{T}^2, \mathbb{R}^2)$:

$$\rho(\tilde{f}) := \left\{ \int_{\mathbb{T}^2} \Delta_{\tilde{f}} \, \mathrm{d}\mu : \mu \in \mathfrak{M}(f) \right\}$$

Question

How is $\rho(\tilde{f})$ if $\mathrm{Per}(f) = \varnothing$?

Rotation set

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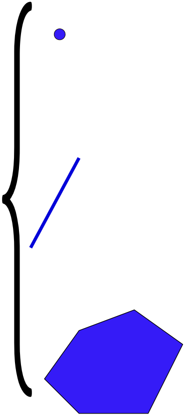

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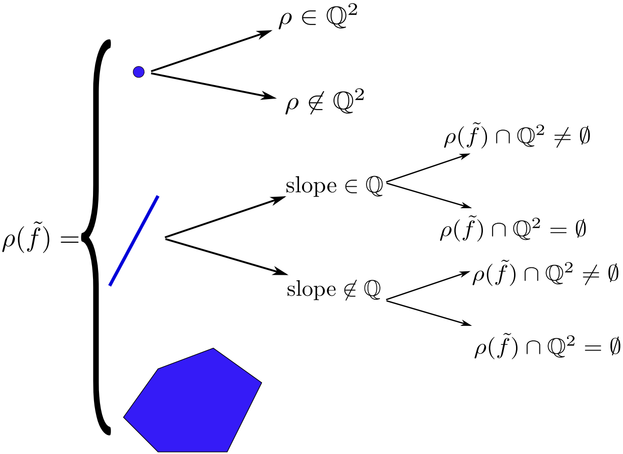

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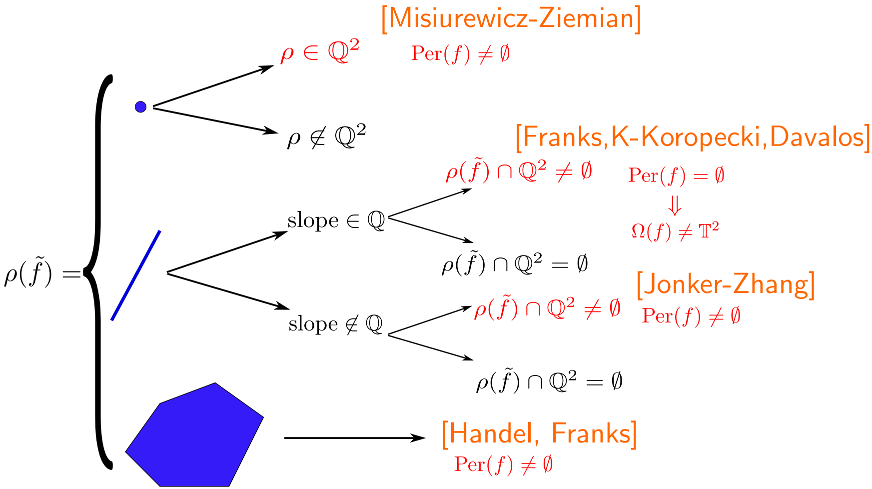

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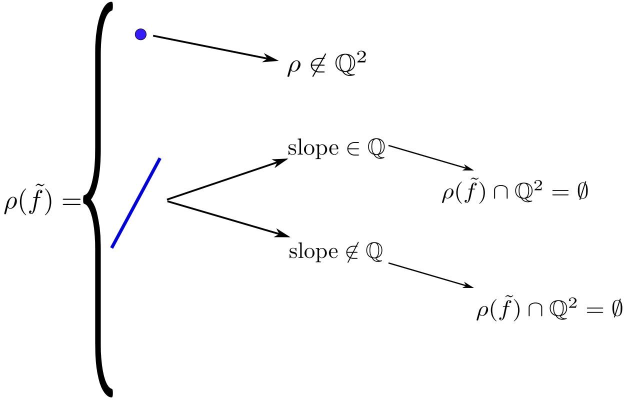

One-dimensional rotation set

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Conjecture [Franks-Misiurewicz]

If $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ and $\rho(\tilde{f})$ is a segment, then

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Theorem [K]

There is no f minimal with slope $\rho(\tilde{f}) \in \mathbb{Q}$.

Theorem wip [K-Tal]

There is no f non-wandering with slope $\rho(\tilde{f}) \in \mathbb{Q}$.

Invariant foliations

If $\overline{\mathcal{O}} := \overline{\{hR_\alpha h^{-1} : h \in \mathrm{Diff}^\infty(\mathbb{T}^2),\ \alpha \in \mathbb{T}^2\}}^{C^\infty}$, then:

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Key argument

Invariant foliation $\implies$ bounded directional rotational deviations

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Invariant foliation $\implies$ bounded directional rotational deviations

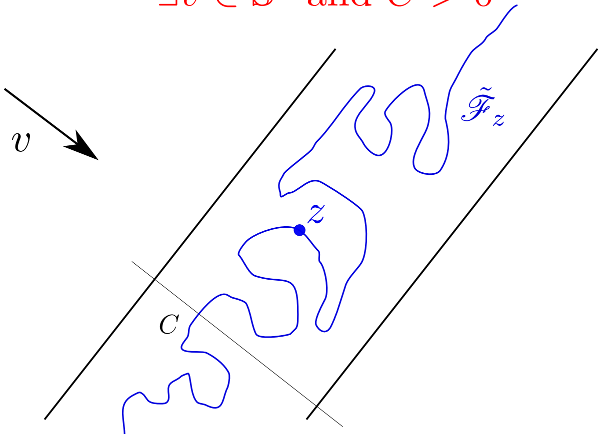
Remember, if $h \in \mathrm{Homeo}_0(\mathbb{T})$, then

$$\left|\tilde{h}^n(x) - x - n\rho(\tilde{h})\right| \leqslant 1, \quad \forall x \in \mathbb{R},\ \forall n \in \mathbb{Z}$$

Torus foliations

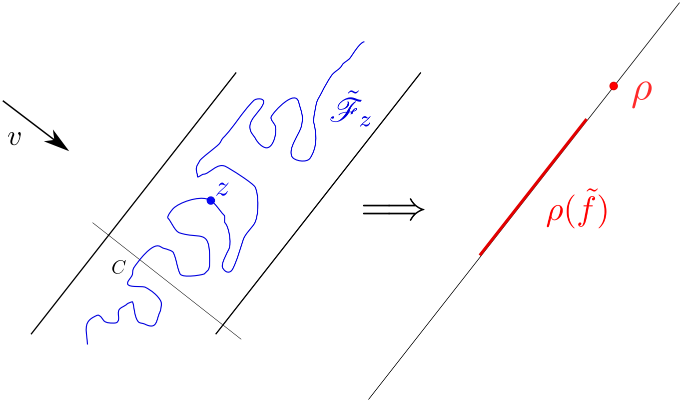
$\mathscr{F}$ foliation of $\mathbb{T}^2$, $\tilde{\mathscr{F}}$ lift to $\mathbb{R}^2$

$\exists v \in \mathbb{S}^1$ and $C > 0$



Rotational deviations

If f preserves $\mathscr{F}$, then $\exists v \in \mathbb{S}^1$ and $C' > 0$



$$\left|\langle \tilde{f}^n(z) - z - n\rho, v\rangle\right| \leq C', \forall n \in \mathbb{Z}, \ \forall z \in \mathbb{R}^2$$

Bounded rotational deviations vs. Invariant foliations

We have seen invariant folation $\implies$ bounded v-deviations

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Reciprocally,

Theorem [K-Rodrigues, K]

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A torus pseudo-foliation is $\mathbb{Z}^2$-equivariant plane pseudo-foliation.

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- **Bounded v-deviations just for one v with irrational slope:**
  - [K-Rodrigues]: If $\Omega(f) = \mathbb{T}^2 \implies \exists$ invariant pseudo-foliation
  - [K-Rodrigues]: f minimal $\implies$ f is topologically weak mixing.

Obrigado!