THE BURNSIDE PROBLEM FOR $\mathrm{Diff}_\omega(\mathbb{S}^2)$

SEBASTIÁN HURTADO, ALEJANDRO KOCSARD,
and FEDERICO RODRÍGUEZ-HERTZ

Abstract
A group G is periodic of bounded exponent if there exists $k \in \mathbb{N}$ such that every element of G has order at most k. We show that every finitely generated periodic group of bounded exponent $G < \mathrm{Diff}_\omega(\mathbb{S}^2)$ is finite, where $\mathrm{Diff}_\omega(\mathbb{S}^2)$ denotes the group of diffeomorphisms of $\mathbb{S}^2$ that preserve an area form ω.

1. Introduction

A group G is said to be *periodic* if every element of G has finite order. If there exists $N \in \mathbb{N}$ such that $g^N = \mathrm{id}$ for every $g \in G$ (where $\mathrm{id} \in G$ denotes the identity element), then G is said to be a periodic group of *bounded exponent*. The so-called *Burnside problem* is a famous question in group theory originally considered by Burnside in [3], and which can be stated as follows.

Question 1.1 (Burnside, 1905)
Let G be a finitely generated periodic group. Is G necessarily finite? What if G is periodic of bounded exponent?

Burnside himself proved in his article [3] that if G is a linear finitely generated periodic group of bounded exponent (i.e., $G < \mathrm{GL}_n(\mathbb{C})$), then G must be finite. In 1911, Schur in [31] improved Burnside's result, removing the bounded exponent hypothesis. However, in general the answer to Question 1.1 turned out to be "no," as counterexamples were later discovered in the 1960s by Golod and Schafarevich [12] and [13] and by Adian and Novikov [28]. Since then, many more examples have been constructed by Olshanskii, Ivanov, and Grigorchuk, among others, and there is a vast literature on the subject (see, e.g., [29]).

Nonlinear transformation groups like homeomorphism groups, diffeomorphism groups, volume-preserving diffeomorphism groups, groups of symplectomorphisms,

DUKE MATHEMATICAL JOURNAL
Vol. 169, No. 17, © 2020 DOI 10.1215/00127094-2020-0028
Received 10 April 2019. Revision received 6 March 2020.
First published online 15 October 2020.
2010 *Mathematics Subject Classification*. Primary 37A05; Secondary 37B05.

and so on are conjectured to have many common features with linear groups (see Fisher's survey [9] on the Zimmer program). For example, the following question, attributed to Ghys and Farb independently, is stated in [9] (see also [8, Question 13.2]).

Question 1.2 (Burnside problem for homeomorphism groups)
Let M be a connected compact manifold, and let G be a finitely generated subgroup $G < \mathrm{Homeo}(M)$ such that all the elements of G have finite order. Then, is G necessarily finite?

At this point it is important to note that compactness is an essential hypothesis in Question 1.2. In fact, it is widely known that, given any finitely presented group G', there exists a connected smooth 4-manifold M such that its fundamental group $\pi_1(M)$ is isomorphic to G'. Hence, (any subgroup of) G' clearly acts faithfully on the universal cover $\tilde{M}$ of M. To the best of our knowledge, it is not known whether there exists a finitely presented infinite periodic group, However, free Burnside groups and Grigorchuk's group from [14] are known to be recursively presented and, consequently, can be embedded by Higman's embedding theorem into a finitely presented group G' and so act faithfully on a noncompact 4-manifold smoothly.

For the time being, we know Question 1.2 has a positive answer just in a few cases and no negative one is known. For instance, in the 1-dimensional case, that is, when $M = \mathbb{S}^1$, this is an easy consequence of the following theorem attributed to Hölder (see, e.g., [26, Theorem 2.2.32]): Any group $G < \mathrm{Homeo}_+(\mathbb{S}^1)$, where $\mathrm{Homeo}_+(\mathbb{S}^1)$ denotes the group of orientation-preserving homeomorphisms which acts freely on $\mathbb{S}^1$ (i.e., the identity is the only element exhibiting fixed points), is an abelian group. Therefore, as any periodic group $G < \mathrm{Homeo}_+(\mathbb{S}^1)$ must act freely on $\mathbb{S}^1$, G must be abelian and therefore finite, being finitely generated.

In higher dimensions, Rebelo and Silva in [30] give a positive answer to Question 1.2 for groups of symplectomorphisms on certain symplectic 4-manifolds. Guelman and Liousse in [16] and [17] do so for hyperbolic surfaces and groups of homeomorphisms of $\mathbb{T}^2$ exhibiting an invariant probability measure. In the Appendix we extend these last results for higher dimensions.

The main result of this article is the following.

THEOREM 1.3
Let ω be an area form on $\mathbb{S}^2$, and let $\mathrm{Diff}^\infty_\omega(\mathbb{S}^2)$ be the group C^∞-diffeomorphisms of $\mathbb{S}^2$ that preserve ω. Then, any finitely generated periodic subgroup of bounded exponent of $\mathrm{Diff}^\infty_\omega(\mathbb{S}^2)$ is finite.

We should point out that Conejeros in [4] recently proved some results about periodic groups in $\mathrm{Homeo}_+(\mathbb{S}^2)$, using the theory of rotation sets and methods different than ours that are closer to the methods used by Guelman and Lioussse in [16] and [17].

In the Appendix, we also prove some further results about actions on hyperbolic manifolds and tori, extending to higher dimensions previous results of Guelman and Liousse in [16] and [17].

1.1. *Outline of this article*

In Section 2, we fix some notation we use throughout the paper, and we recall some previous known results on differential geometry and topology. Sections 3, 4, and 5 are dedicated to the proof of Theorem 1.3. For the sake of readability, let us roughly explain the strategy of that proof.

To do that, let G be a finitely generated periodic subgroup of $\mathrm{Diff}^\infty_\omega(\mathbb{S}^2)$ with bounded exponent. In Section 3 we show that the group G exhibits *subexponential growth of derivatives*; that is, the norm of the derivatives of the elements of G grows subexponentially with respect to their word length in G. The proof of the subexponential growth of derivatives is based on encoding the group action as a $\mathrm{Diff}^\infty_\omega(\mathbb{S}^2)$-cocycle over the full-shift k symbols, where k denotes the number of generators of G. This is a classical construction in random dynamics (see, e.g., [23]), and we use it to show that the exponential growth of the derivative implies the existence of an element of G exhibiting a hyperbolic periodic point, which contradicts the fact that G is periodic. These ideas are closely related to Livšic's theorem for diffeomorphism group cocycles (see [1], [24]) and to Katok's closing lemma (see Lemma 3.4).

Then, invoking a rather elementary argument, in Section 5 we show that the subexponential growth of derivatives is incompatible with the exponential growth of the group G itself, with respect to the word length. It is interesting to remark that this is the only part of the proof where the boundedness of the exponent is indeed used.

So, in Section 4, which contains the more elaborate and intricate arguments of the proof, we assume the group G has subexponential growth (with respect to the word length). The rough idea here is to try to show the existence of a smooth invariant Riemannian metric m on $\mathbb{S}^2$. If such a metric exists, then by the uniformization theorem of surfaces (Theorem 4.3) there exists a diffeomorphism $h\colon \mathbb{S}^2 \circlearrowleft$ such that the pullback metric $h^\star(m)$ is conformally equivalent to the standard metric m_0 on $\mathbb{S}^2$. Consequently, $h^{-1}Gh$ is a group of conformal maps and so $h^{-1}Gh < \mathrm{SL}_2(\mathbb{C})$. Then, as a consequence of Schur's theorem in [31], G must be finite.

With the idea of finding an invariant metric, we define for each $\varepsilon > 0$ an "almost-invariant" Riemannian metric given by

$$m^{\varepsilon} := \sum_{g \in G} e^{-\varepsilon |g|_S} g^* m',$$

where $S \subset G$ is a finite set of generators of G and $|\cdot|_S$ denotes the associated word length function on G.

Using the assumption that G has subexponential growth and the subexponential growth of derivatives, we show these metrics are well defined and smooth for every $\varepsilon > 0$. Then, we show in Lemma 4.2 that each element of the generating set $S \subset G$ is e^{ε}-Lipschitz with respect to m^{ε}. Using the uniformization theorem we construct a sequence of conjugacies $g_\varepsilon \in \mathrm{Diff}^{\infty}(\mathbb{S}^2)$ such that each diffeomorphism $g_\varepsilon^{-1} \circ s \circ g_\varepsilon$ G^{ε}, with $s \in S$, is an e^{ε}-quasiconformal map. Invoking rather classical facts about the compactness of the space of quasiconformal maps (Lemma 4.5) and some elementary arguments, we obtain a contradiction with our assumption that G is infinite, and this completes the proof of Theorem 1.3.

2. Preliminaries

In this section we fix some notation we will use throughout the paper and recall some concepts and results.

2.1. *Finitely generated groups*

Let G be a finitely generated group, and let $S \subset G$ be a finite set of generators of G. We say S is *symmetric* when $s^{-1} \in S$ for every $s \in S$. Then given a symmetric set of generators S, we define the *word length function* $|\cdot|_S \colon G \to \mathbb{N}_0$ by $|\mathrm{id}|_S = 0$ and

$$|g|_S := \min\{n \in \mathbb{N} : g = s_{j_1} s_{j_2} \cdots s_{j_n}, \text{for } s_{j_i} \in S\}, \tag{1}$$

for every $g \in G \setminus \{\mathrm{id}\}$. We say that the group G has *subexponential growth* when it holds that

$$\lim_{n \to +\infty} \frac{\log \sharp\{g \in G : |g|_S \leq n\}}{n} = 0, \tag{2}$$

where $\sharp\{\cdot\}$ denotes the number of elements of the set. It is well known that this concept does not depend on the finite set of generators. Observe that, by classical subadditive arguments, the above limit (2) always exists.

2.2. *Groups of diffeomorphisms and C^r-norms*

Let M be a closed smooth manifold. The group of C^r-diffeomorphisms of M will be denoted by $\mathrm{Diff}^r(M)$. The subgroup of C^r-diffeomorphisms which is isotopic to the identity will be denoted by $\mathrm{Diff}^r_0(M)$. When M is orientable and ω is a smooth volume form on M, we write $\mathrm{Diff}^r_\omega(M)$ for the group of C^r-diffeomorphisms that leaves ω invariant. When M is endowed with a Riemannian metric m, we apply the *unit tangent bundle (UTM)*; that is,

$$UTM := \big\{v \in TM : |v| = 1\big\},$$

where $|\cdot|$ denotes the norm induced by m. Then, given any $f \in \mathrm{Diff}^1(M)$, we define

$$\big\|D(f)\big\|^+ := \sup_{v \in UTM} \big|Df(v)\big| \tag{3}$$

and

$$\big\|D(f)\big\| := \max\big\{\big\|D(f)\big\|^+, \big\|D(f^{-1})\big\|^+\big\}. \tag{4}$$

Next, we will define the C^r-norm $\|\cdot\|_r$ (see [10, Section 4] and the references therein for a more complete discussion). Let us start by noting that the Riemannian metric m induces an isomorphism (by duality) between $T(M)$ and $T^*(M)$, that is, the tangent and cotangent bundles of M. So, g induces an inner product on both bundles and, consequently, on every tensor bundle over M. For instance, g induces a metric in the space of symmetric i-tensors, which is denoted by $S^i(T^*(M))$, for any i. So, we consider the space of Riemannian metrics endowed with the topology induced by the ambient space $S^2(T^*(M))$.

The norms $\|\cdot\|_r$ are defined using the language of jets. Given a C^r bundle E over M, let $J^r(E)$ be the vector bundle of r-jets on E (for more details, see [10], [6, Chapter 1], and references therein). So, a C^r-section of E gives a continuous section of $J^r(E)$ (but not the other way around) and two such sections coincide at some point in M if and only if the derivatives of the original sections agree up to order r at that point. Observe there is a natural identification

$$J^r(E) \cong \bigoplus_{i=1}^{r} S^i(T^*M) \otimes E$$

as proven, for example, in [10, Section 4].

Let $J^r(M)$ be the bundle of r-jets of sections of the trivial bundle $E := M \times \mathbb{R}$. Then notice that any C^r-diffeomorphism $\phi\colon M \circlearrowleft$ naturally induces a linear map $j^r(\phi)(x)\colon J^r(M)_x \to J^r(M)_{\phi^{-1}(x)}$ that sends each C^r-section $s\colon M \to \mathbb{R}$ to $s \circ \phi$. Therefore, we can define

$$\|\phi\|_r := \max_{x \in M} \big\|j^r(\phi)(x)\big\|, \tag{5}$$

where $\|j^r(\phi)(x)\|$ is the operator norm defined from the norms on the vector spaces $J^r(M)_x$ and $J^r(M)_{\phi^{-1}(x)}$.

One can also define norms $\|\cdot\|'_r$ on the bundles $J^r(M)$ using coordinate charts as follows. Consider a finite covering of M by coordinate charts (U_i, ψ_i). For a C^r-section s of $E = M \times \mathbb{R}$, one defines

$$\|s\|'_r := \max_{x,j,i} \left\| D^j_{\psi_i(x)}(s \circ \psi_i^{-1}) \right\|,$$

where this maximum is taken by considering the functions $s \circ \psi_i^{-1} : \mathbb{R}^n \to \mathbb{R}$ in each coordinate chart and then calculating the maximum absolute value of a partial derivative of degree j less than or equal to r of such functions over $x \in U_i$. It is easy to verify that the norms $\|\cdot\|_r$ and $\|\cdot\|'_r$ are equivalent on $J^r(M)$. Given a diffeomorphism $\phi \in \mathrm{Diff}^r(M)$, similarly to how we define the norm $\|\phi\|_r$, we can define a norm $\|\phi\|'_r$, and these norms must be equivalent.

We will need the following basic fact relating $\|\cdot\|_1$ and $\|D(\cdot)\|$, which we defined in (4).

PROPOSITION 2.1
Let M be a closed smooth manifold. Then there exists a constant $C > 0$ such that

$$\|\phi\|_1 \leq C \left\| D(\phi) \right\|, \quad \forall \phi \in \mathrm{Diff}^1(M).$$

Proof
This follows from the fact that the norms $\|\cdot\|'_1$ and $\|\cdot\|_1$ are equivalent, and $\|\phi\|'_1$ is bounded above by the maximum derivative of ϕ in coordinate charts, which is also bounded by a $C\|D(\phi)\|$, for some constant C just depending on M. □

The following notion plays a fundamental role in our work.

Definition 1 (Subexponential growth of derivatives)
Let M be a smooth closed manifold, let $G < \mathrm{Diff}^\infty(M)$ be a finitely generated subgroup, and let $S \subset G$ be a finite symmetric set of generators. Then we say that G has *subexponential growth of derivatives* when, for every $\varepsilon > 0$ and any $r \in \mathbb{N}$, there exists $N_{\varepsilon,r} \in \mathbb{N}$ such that

$$\|w_n\|_r \leq e^{n\varepsilon}, \quad \forall n \geq N_{\varepsilon,r},$$

and any $w_n \in G$ such that $|w_n|_S \leq n$, where $|\cdot|_S$ denotes the word length function given by (1).

2.3. Riemannian metrics

In Section 4 we will deal with the convergence of Riemannian metrics, so here we recall some basic facts about the space of metrics. Recall that if M is a closed smooth manifold, a Riemannian structure m on M induces a metric on $S^2(T^*(M))$. If s denotes a section of the tensor bundle $S^2(T^*(M))$ and $\phi \in \mathrm{Diff}^{r+1}(M)$, there is a natural action of ϕ in the bundle of r-jets of sections of $S^2(T^*(M))$ which is defined by sending s to the pullback $\phi^*(s)$. We will need the following basic result.

PROPOSITION 2.2
There exists a constant $C > 0$ just depending on M such that

$$\left\|\phi^*(s)\right\|_r \leq C \|\phi\|_{r+1} \|s\|_r.$$

Proof
By taking coordinate charts, we can also define norms on $J^r(S^2(T^*(M)))$ as we did for $J^r(M)$, coinciding with the norms for metrics defined in [20]. Then, Proposition 2.2 follows from Lemma 3.2 in [20]. □

We will also need to make use of the following.

PROPOSITION 2.3
Let $r \geq 1$, and let $(s_n)_n$ be a sequence of sections of $S^2(T^*(M))$ which is a Cauchy sequence in the space of r-jets of $S^2(T^*(M))$. Then there exists a section s of the bundle of r-jets in $S^2(T^*(M))$ such that $\|s_n - s\|_r \to 0$ as $n \to +\infty$.

Proof
This is consequence of the definition of the norms $\|\cdot\|'_r$ in terms of coordinate charts and the fact that a Cauchy sequence of C^r real functions which are supported in a compact set converges to a C^r real function in the C^r-topology, which is an easy consequence of Arzela–Ascoli theorem. More details can be found in [20]. □

3. Subexponential growth of derivatives

Throughout this section, M will denote a closed orientable surface, and ω will denote a smooth area form on M. Given any $f \in \mathrm{Diff}^1(M)$, we say $p \in M$ is a *hyperbolic fixed point* of f when $f(p) = p$ and the spectrum of Df_p does not contain complex numbers of modulus equal to 1.

Definition 2
We say that a group $G < \mathrm{Diff}^1(M)$ is *elliptic* if there is no element in G having a hyperbolic fixed point.

For example, any subgroup of SO(3) is an elliptic subgroup of $\mathrm{Diff}^\infty_\omega(\mathbb{S}^2)$, where ω denotes the smooth area form induced from the Euclidean structure of

$\mathbb{R}^3$. There exist other examples of elliptic subgroups of $\mathrm{Diff}^\infty_\omega(\mathbb{S}^2)$. For example, one can construct abelian groups of commuting pseudorotations using the so-called *Anosov–Katok method* (see, e.g., [7]).

A natural question about elliptic groups of diffeomorphisms of $\mathbb{S}^2$ is the following.

Question 3.1
Are all elliptic subgroups of $\mathrm{Diff}^\infty_\omega(\mathbb{S}^2)$ either solvable or conjugate to a subgroup of $\mathrm{SO}(3)$?

The main result of this section is the following.

LEMMA 3.2 (Subexponential growth of derivatives)
Let M be a closed orientable surface, let ω be an area form on M, and let G be a finitely generated elliptic subgroup of $\mathrm{Diff}^\infty_\omega(M)$. Then, G has subexponential growth of derivatives (see Definition 1).

We will begin the proof of Lemma 3.2 by proving a similar weaker result that just considers the first derivative. In fact, we will start by proving the following.

LEMMA 3.3
Let G be a finitely generated elliptic subgroup of $\mathrm{Diff}^2_\omega(M)$, and let S be a finite symmetric generating set of G. Then, for any $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that

$$\|D(w_n)\| \leq e^{\varepsilon n}, \quad \forall n \geq N_\varepsilon,$$

and every $w_n \in G$ with $|w_n| \leq n$.

To prove Lemma 3.3, we first need some definitions. Let $S = \{s_1, s_2, \ldots, s_r\}$ be the generating symmetric set of G, and consider the space $\Sigma := S^{\mathbb{Z}}$ consisting of bi-infinite sequences of elements of S. There is a natural *shift map* $\sigma \colon \Sigma \circlearrowleft$ given by $\sigma : (\ldots, g_{-1}, g_0, g_1, \ldots) \mapsto (\ldots, g'_{-1}, g'_0, g'_1, \ldots)$, where $g'_i := g_{i+1}$, for every $i \in \mathbb{Z}$. Then consider the map $F \colon \Sigma \times M \circlearrowleft$ given by

$$F(w,x) = \big(\sigma(w), g_0(x)\big), \quad \forall w = (\ldots, g_1, g_0, g_1 \ldots) \in \Sigma, \forall x \in M.$$

The map F encodes the group action, and we have the obvious commutative diagram:

$$\begin{array}{ccc} \Sigma \times M & \xrightarrow{F} & \Sigma \times M \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \Sigma & \xrightarrow[\sigma]{} & \Sigma \end{array}$$

To prove Lemma 3.3, let us start by recalling some classical facts about non-uniformly hyperbolic dynamics (see [22, Supplement] for more details). Given a C^1-diffeomorphism $f : M \circlearrowleft$ and an ergodic f-invariant probability measure μ, there are a measurable set $X \subset M$ with $\mu(X) = 1$, real numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ with $k \leq \dim M$, and a measurable splitting $TM|_X = \bigoplus_{i=1}^k E^i$ such that $Df_x(E^i_x) = E^i_{f(x)}$, for every $x \in X$ and any $i \in \{1, \ldots, k\}$, and such that

$$\lambda_i = \lim_{n\to\pm\infty} \frac{1}{n} \log \| Df^n_x(v_i) \|, \quad \forall x \in X, \forall v_i \in E^i \setminus \{0\},$$

and every $i \in \{1, \ldots, k\}$. The numbers λ_i are called the *Lyapunov exponents* of f and μ. The measure μ is said to be a *hyperbolic measure* when all its Lyapunov exponents are different from zero.

Then the idea of the proof of Lemma 3.3 goes as follows: reasoning by contradiction, we suppose subexponential growth of derivatives does not hold. Then, we show the existence of an ergodic F-invariant probability measure μ exhibiting nonzero top Lyapunov exponent along the fiber, that is, on M. Since we are assuming the fiber M has dimension 2 and the action preserves area,[1] the lower Lyapunov exponent must be negative. On the other hand, the dynamics on the base, given by the full shift, are uniformly hyperbolic. So, μ is essentially a hyperbolic measure (see [22, p. 659] for details).

We now recall the following result due to Katok.

THEOREM 3.4 ([21], [22])
Let N be a compact manifold, let F be a $C^{1+\alpha}$-diffeomorphism of N, and let μ be an ergodic hyperbolic measure for F. Then, there exists a hyperbolic periodic point of F. Moreover, the periodic point (and its orbit) can be chosen as close to supp(μ) *as one wants.*

We will prove that Theorem 3.4 is also true if one considers the space $N := \Sigma \times M$ and the map F as before (observe that $\Sigma \times M$ is not a manifold and F is not a diffeomorphism), and so we will obtain a hyperbolic periodic point (w, x) for F of, say, order k. The element $w \in \Sigma$ is determined by the infinite biconcatenation of a word of length k that defines an element of G having x as a hyperbolic fixed point, giving a contradiction.

To avoid re-proving Katok's theorem for our space $\Sigma \times M$ and F, we will embed the dynamics of F into the dynamics of a diffeomorphism F' of a 4-manifold N. Then, we will construct a hyperbolic measure μ for F' and directly apply Katok's theorem as stated above to F'.

[1] In fact, this is the only point where the volume-preserving assumption in Theorem 1.3 is crucial.

Now, we prove Lemma 3.3.

Proof of Lemma 3.3

First we embed the dynamics of the shift $\sigma\colon \Sigma \circlearrowleft$ into the dynamics of a linear 2-dimensional horseshoe map (see [22, Chapter 2, Section 5] for details). More formally, there exists a C^∞-diffeomorphism $h\colon \mathbb{S}^2 \circlearrowleft$ such that h acts on an open set $U \subset \mathbb{S}^2$ as the linear Smale's horseshoe map, where $\Lambda := \bigcap_{n\in\mathbb{Z}} h^n(U)$ is a hyperbolic Cantor set for h. So, there is an embedding $E'\colon \Sigma \to \mathbb{S}^2$ such that $E'(\Sigma) = \Lambda$ and the following commutative diagram holds:

$$\begin{array}{ccc} \Sigma & \xrightarrow{E'} & \mathbb{S}^2 \\ \sigma\big\downarrow & & \big\downarrow h \\ \Sigma & \xrightarrow[E']{} & \mathbb{S}^2 \end{array}$$

The embedding E' can be naturally extended to an embedding $E\colon \Sigma \times M \to \mathbb{S}^2 \times M$. Then we will extend the homeomorphism $E \circ F \circ E^{-1}\colon E(\Sigma \times M) \to E(\Sigma \times M)$ to a smooth diffeomorphism $F'\colon \mathbb{S}^2 \times M \circlearrowleft$, which is a skew product over the diffeomorphism $h\colon \mathbb{S}^2 \circlearrowleft$. This extension exists only if $G \subset \mathrm{Diff}^2_0(M)$; that is, every element of G is isotopic to the identity. For the sake of simplicity and since we are mainly interested in the case $M = \mathbb{S}^2$, we will assume G is contained in the identity isotopy class.

Then such an extension is constructed considering a smooth map $f\colon \mathbb{S}^2 \circlearrowleft \mathrm{Diff}^2_0(M)$ (i.e., $\mathbb{S}^2 \times M \ni (z,x) \mapsto f_z(x) \in M$ is C^2) such that $f_z = g_0$, for every $z \in \Lambda \subset \mathbb{S}^2$, and $(\ldots, g_{-1}, g_1, g_1, \ldots) = E^{-1}(z) \in \Sigma$. In fact, given such a map f, we can simply define

$$F'(z,x) := \big(h(z), f_z(x)\big), \quad \forall (z,x) \in \mathbb{S}^2 \times M,$$

and we clearly get the following commutative diagram:

$$\begin{array}{ccc} \Sigma \times M & \xrightarrow{E} & \mathbb{S}^2 \times M \\ F\big\downarrow & & \big\downarrow F' \\ \Sigma \times M & \xrightarrow[E]{} & \mathbb{S}^2 \times M \end{array}$$

In conclusion, we have embedded the dynamics of F into the dynamics of a C^2-diffeomorphism F' of the 4-manifold $N := \mathbb{S}^2 \times M$ that fibers over a diffeomorphism $h\colon \mathbb{S}^2 \circlearrowleft$. We will now construct the ergodic hyperbolic measure μ for F'.

Let UTM be the unit tangent bundle of M. Then we consider the map $\partial F' \colon \mathbb{S}^2 \times UTM \circlearrowleft$ that fibers over F' which is given by

$$\partial F'\big(z,(x,v)\big) := \Big(h(z), \frac{Df_z(x)v}{\|Df_z(x)v\|}\Big), \quad \forall z \in \mathbb{S}^2,\ \forall (x,v) \in UTM.$$

Analogously, one can define the map $\partial F \colon \Sigma \times UTM \circlearrowleft$ that fibers over F and is given by

$$\partial F\big(w,(x,v)\big) := \Big(\sigma(w), \frac{Dg_0(x)v}{\|Dg_0(x)v\|}\Big),$$

for every $w = (\ldots, g_{-1}, g_0, g_1, \ldots) \in \Sigma$ and every $(x,v) \in UTM$, and the map $\partial E \colon \Sigma \times UTM \circlearrowleft$ just given by $\partial E := E' \times \mathrm{id}_{UTM}$. Notice that by our definitions, it holds that $\partial E \circ \partial F = \partial F' \circ \partial E$.

Then, let us suppose there is no subexponential growth of derivatives. So, there exist $\varepsilon > 0$, a sequence of words w_n of elements of S of length smaller than or equal to n, and vectors $(x_n, v_n) \in UTM$ such that $\|D_{x_n} w_n(v_n)\| \geq e^{\varepsilon n}$, for each $n \geq 1$. Let us define $\hat{w}_n \in \Sigma$ to be the bi-infinite periodic word $\hat{w}_n = \cdots w_n w_n w_n \cdots$, and let $\delta_{(\hat{w}_n,(x_n,v_n))}$ be the Dirac measure on $\Sigma \times UTM$ supported on the point $(\hat{w}_n,(x_n,v_n))$. We consider the sequence of measures

$$\nu_n := \frac{1}{n}\sum_{i=1}^{n}\big((\partial F)^i\big)_*(\delta_{(\hat{w}_n,(x_n,v_n))}), \quad \forall n \geq 1.$$

Then, consider the function $\psi \colon \mathbb{S}^2 \times UTM \to \mathbb{R}$ given by

$$\psi\big(z,(x,v)\big) = \log\Big(\frac{\|Df_z(v)\|_{f_z(x)}}{\|v\|_z}\Big), \quad \forall z \in \mathbb{S}^2, \forall (x,v) \in UTM,$$

and observe that

$$\begin{aligned}
\int_{\mathbb{S}^2 \times UTM} \psi \,\mathrm{d}(\partial E_* \nu_n) &= \int_{\Sigma \times UTM} (\psi \circ \partial E)\,\mathrm{d}\nu_n \\
&= \frac{1}{n}\sum_{i=1}^{n} \psi \circ \partial E\big(\partial F^i\big(\hat{w}_n,(x_n,v_n)\big)\big) \\
&\geq \frac{1}{n}\log\Big(\frac{\|D_{x_n} w_n(v_n)\|_{w_n(x_n)}}{\|v_n\|_{x_n}}\Big) \geq \varepsilon,
\end{aligned} \tag{6}$$

for every $n \geq 1$. So, by the Banach–Alaoglu theorem, there exists a subsequence $(\partial E_* \nu_{n_j})_{n_j}$ that converges in the weak-star topology to a measure ν', which is clearly $\partial F'$-invariant, and by (6), it holds that

$$\int_{\mathbb{S}^2 \times UTM} \psi \,\mathrm{d}\nu' \geq \varepsilon.$$

Then, if we consider the ergodic decomposition of ν', there exists an ergodic $\partial F'$-invariant probability measure ν such that

$$\int_{\mathbb{S}^2 \times UTM} \psi \, \mathrm{d}\nu \geq \varepsilon. \tag{7}$$

Now, let μ be the pushforward measure of ν by the projection on the $\mathbb{S}^2 \times M$ factor; that is, $\mu := \mathrm{pr}_* \nu$, where $\mathrm{pr}\colon \mathbb{S}^2 \times UTM \to \mathbb{S}^2 \times M$ denotes the natural projection. Observe μ is an ergodic F'-invariant measure. We claim μ is a hyperbolic measure for F'; that is, all its Lyapunov exponents are different from zero.

In order to prove that, first observe that the measure μ is supported on the subset $\Lambda \times M$, where Λ is the horseshoe of diffeomorphism h, and consequently, it is a uniform hyperbolic set (see [22, Chapter 2, Section 5] for details). So, to show that μ is a hyperbolic measure, it is enough to show that its both Lyapunov exponents along the vertical fibers (i.e., on the M factor) are different from zero.

By combining (7) and the result [25, Proposition 5.1], we conclude that the top Lyapunov exponent of μ along vertical fibers is positive. Since the diffeomorphism $f_z \in \mathrm{Diff}_0^2(M)$ leaves invariant the area form ω, for every $z \in \Lambda$, and μ is supported on $\Lambda \times M$, this implies that the bottom Lyapunov exponent of μ along vertical fibers is negative. Thus, μ is an ergodic hyperbolic measure.

We can apply Theorem 3.4 to conclude that, for every open neighborhood V of Λ, there is a hyperbolic periodic point $p' \in V \times M$ for F'. Since Λ is a locally maximal invariant set for h, that is, $\Lambda = \bigcap_{n\in\mathbb{Z}} h^n(V)$ for every sufficiently small neighborhood V of Λ, this implies $p' \in \Lambda \times M$. Hence, we can consider the point $(w^p, x^p) := E^{-1}(p') \in \Sigma \times M$, which is a periodic point for F. Since $\partial F' \circ \partial E = \partial E \circ \partial F$, we conclude (w^p, x^p) is hyperbolic along the vertical fibers, contradicting the fact that there is a natural number k such that $F^k(w^p, x) = (w^p, x)$, for every $x \in M$. ☐

In order to finish the proof of Lemma 3.2, first we need to recall the following result, which is an easy consequence of the chain rule in dimension 1 and, according to Fisher and Margulis in [10], a well-known estimate used in KAM theory.

LEMMA 3.5 ([10, Lemma 6.4])
Let M be a compact smooth manifold, let $\phi_1, \phi_2, \dots, \phi_n \in \mathrm{Diff}^r(M)$, let $N_1 := \max_{1\leq i\leq n} \|\phi_i\|_1$, and let $N_r := \max_{1\leq i\leq n} \|\phi_i\|_r$. Then, there exists a polynomial Q (just depending on the dimension of M) such that

$$\|\phi_1 \circ \phi_2 \circ \dots \circ \phi_n\|_r \leq N_1^{rn} Q(nN_r).$$

We can now give a proof of Lemma 3.2.

Proof of Lemma 3.2
From Proposition 2.1 and Lemma 3.3, one observes that Lemma 3.2 holds for $r = 1$. Then, given any $r > 1$ and $\varepsilon > 0$, we apply Lemma 3.2 to guarantee the existence of a natural number $N_{\frac{\varepsilon}{3r},1}$ so that

$$\|w_n\|_1 \leq e^{\frac{\varepsilon n}{3r}}, \quad \forall n \geq N_{\frac{\varepsilon}{3r},1},$$

for any C^r-diffeomorphism $w_n : M \circlearrowleft$ that can be written as a composition of at most n elements of S. Then, Lemma 3.2 easily follows from Lemma 3.5 and a properly chosen constant $N_{\varepsilon,r}$. $\square$

4. The subexponential growth case

In this section we prove Theorem 1.3 for periodic groups of subexponential growth. So, we will assume $G < \mathrm{Diff}^\infty(\mathbb{S}^2)$ is a finitely generated periodic subgroup with bounded exponent and subexponential growth; that is, if $S \subset G$ is a finite generating set of G, then condition (2) holds.

THEOREM 4.1
Let ω be an area form on $\mathbb{S}^2$, and let $G < \mathrm{Diff}^\infty_\omega(\mathbb{S}^2)$ be a finitely generated periodic subgroup with bounded exponent and subexponential growth (i.e., if $S \subset G$ is a finite generating set of G, then condition (2) holds). Then, G is finite.

The idea of the proof of Theorem 4.1 goes as follows. Combining the subexponential growth of G and the subexponential growth of derivatives we proved in Lemma 3.2, we find, for each $\varepsilon > 0$, a C^∞ "à la Pesin" Riemannian metric m^ε on $\mathbb{S}^2$ such that each element of S is e^ε-bi-Lipschitz for m^ε (Lemma 4.2). By the uniformization theorem of surfaces, each metric m^ε is conformally equivalent to the standard metric m_0 in $\mathbb{S}^2$, which implies there are conjugates $G_\varepsilon = g_\varepsilon^{-1} G g_\varepsilon$ of the group G such that each element of the generating set $S_\varepsilon := g_\varepsilon^{-1} S g_\varepsilon$ is an e^ε-quasiconformal homeomorphism of $\mathbb{S}^2$. Then, as $\varepsilon \to 0$, we show that up to conjugation G_ε converges to a Burnside group G_0 which acts conformally on $\mathbb{S}^2$; that is, the group G_0 is a periodic subgroup of $\mathrm{PSL}_2(\mathbb{C})$ which, by the Schur theorem, must be finite. We will show this implies that G is finite as a consequence. We start by constructing the family of Riemannian metrics $(m^\varepsilon)_{\varepsilon>0}$, which is mainly motivated by Pesin's work on nonuniform hyperbolicity (see, e.g., [22] for details).

LEMMA 4.2
Let M be a closed smooth manifold, let $G < \mathrm{Diff}^\infty(M)$ be a finitely generated group with subexponential growth and subexponential growth of derivatives (see Definition 1), and let S be a finite set of generators of G. Then for any $\varepsilon > 0$ there exists a

C^∞-metric m^ε such that

$$e^{-\varepsilon}|v|_x^\varepsilon \leq \left|D_x s(v)\right|_{s(x)}^\varepsilon \leq e^\varepsilon |v|_x^\varepsilon, \tag{8}$$

for every $s \in S$ and every $(x,v) \in TM$, where $|v|^\varepsilon := \sqrt{m_x^\varepsilon(v,v)}$.

Proof
Let us start by considering an arbitrary C^∞ Riemannian metric m on M, and let ε be any positive number. For each integer $n > 0$, let us write

$$m_n^\varepsilon = \sum_{g \in B_n} e^{-\varepsilon |g|_S} g^* m,$$

where $B_n := \{g \in G : |g|_S \leq n\}$ and g^*m denotes the pullback metric of m by g; that is, $g^*m(v,w) := m(Dg(v), Dg(w))$. The metric m^ε will be constructed as the limit of the sequence $(m_n^\varepsilon)_n$ as $n \to \infty$.

By fixing an integer $r > 0$, by Lemma 3.2 there is a natural number $N_{\varepsilon/2,r+1} > 0$ so that, for any $g \in G$ satisfying $|g|_S \geq N_{\varepsilon/3,r+1}$, we have

$$\|g\|_{r+1} \leq e^{\frac{\varepsilon |g|_S}{3}}.$$

On the other hand, since we are assuming G is a group of subexponential growth, there exists a constant $K_\varepsilon > 0$ so that

$$\sharp\big\{g \in G : |g|_S \leq n\big\} \leq e^{\frac{\varepsilon n}{3}}, \quad \forall n \geq K_\varepsilon.$$

Therefore for any $n \geq \max\{N_{\varepsilon/3,r}, M_\varepsilon\}$, by Proposition 2.3 we have

$$\begin{aligned}
\|m_n^\varepsilon - m_{n-1}^\varepsilon\|_r &= \Big\| \sum_{g \in B_n \setminus B_{n-1}} e^{-\varepsilon n} g^* m \Big\|_r \\
&\leq \sharp(B_n \setminus B_{n-1}) e^{-\varepsilon n} \max_{g \in B_n \setminus B_{n-1}} \|g^* m\|_r \\
&\leq \sharp(B_n \setminus B_{n-1}) e^{-\varepsilon n} C e^{\frac{\varepsilon n}{3}} \|m\|_r \\
&\leq C \|m\|_r e^{\frac{-\varepsilon n}{3}}.
\end{aligned}$$

Therefore, the sequence $\{m_n^\varepsilon\}_{n \geq 1}$, when its elements are considered as r-jets of $S^2(T^*(M))$, is a Cauchy sequence. Then, by Proposition 2.3, $\{m_n^\varepsilon\}_{n \geq 1}$ converges to a C^r-tensor m^ε in $S^2(T^*(M))$. The tensor m^ε is easily seen to be a nondegenerate metric, and as previous estimates are true for any $r > 0$, the metric m^ε is in fact C^∞, too.

To prove the estimates in (8), observe that, given any vector $v \in T_x(M)$, we have

$$\begin{aligned}(s^* m_n^\varepsilon)(v,v) &= \sum_{g \in B_n} e^{-\varepsilon |g|_S} (gs)^* m(v,v) \\ &\le e^\varepsilon \Big(\sum_{g \in B_{n+1}} e^{-\varepsilon |g|_S} g^* m(v,v) \Big) \\ &= e^\varepsilon m_{n+1}^\varepsilon (v,v).\end{aligned}$$

Taking limits in the preceding inequality, we obtain $|D_x s(v)|_{s(x)}^\varepsilon \le e^\varepsilon |v|_x^\varepsilon$. The other inequality follows in a completely analogous way. □

Now, we invoke the uniformization theorem for the 2-sphere $\mathbb{S}^2$ for finding nice conjugates of our group G. We recall the statement of the uniformization theorem in the following form (see [5, Chapter 10] for more details).

THEOREM 4.3 (Uniformization of $\mathbb{S}^2$)
Any C^∞ Riemannian metric m on $\mathbb{S}^2$ is conformally equivalent to the standard metric m_0 in $\mathbb{S}^2$. That is, there exist a C^∞-diffeomorphism $g\colon \mathbb{S}^2 \circlearrowleft$ and a C^∞-function $h\colon \mathbb{S}^2 \to \mathbb{R}$ such that

$$g^* m = e^h m_0.$$

By invoking Lemma 4.2, we can construct a family of Riemannian metrics $(m^\varepsilon)_{\varepsilon>0}$ satisfying estimate (8) for each element of the generating set S. Then, by applying Theorem 4.3 to this family of metrics,we can construct a family of C^∞-diffeomorphisms $(g_\varepsilon)_{\varepsilon>0}$ and C^∞ real functions h_ε such that

$$g_\varepsilon^* m^\varepsilon = e^{h_\varepsilon} m_0, \quad \forall \varepsilon > 0. \tag{9}$$

Now, for each generator $s \in S$ we define

$$s_\varepsilon := g_\varepsilon^{-1} \circ s \circ g_\varepsilon. \tag{10}$$

We will prove that, as $\varepsilon \to 0$, the diffeomorphisms s_ε get closer to being conformal with respect to the standard metric m_0 on $\mathbb{S}^2$. To do that, we first recall the purely metric definition of quasiconformality (see, e.g., [19] for more details).

Definition 3
Let (X, d_X), (Y, d_Y) be two metric spaces, and let $f\colon X \to Y$ be a homeomorphism. For each $r > 0$ and $x \in X$ we define

$$H_f(x,r) := \frac{\sup\{d_Y(f(x), f(y)) : y \in X,\ d_X(x,y) < r\}}{\inf\{d_Y(f(x), f(y)) : y \in X,\ d_X(x,y) > r\}}.$$

Then we say that f is K*-quasiconformal*, for some $K \geq 1$, whenever

$$\limsup_{r \to 0} H_f(x,r) \leq K, \quad \forall x \in X.$$

Then we have the following.

LEMMA 4.4
For every number $\varepsilon > 0$ and every $s \in S$, the diffeomorphism $s_\varepsilon \colon \mathbb{S}^2 \circlearrowleft$ given by (10) is $e^{2\varepsilon}$-quasiconformal with respect to the standard Riemannian metric m_0 on $\mathbb{S}^2$.

Proof
Let s be an arbitrary element of S, and let ε be a fixed positive number. Given any point $x \in \mathbb{S}^2$ and any vector $v \in T_x(\mathbb{S}^2)$, we write $|v|_x^0$ and $|v|_x^\varepsilon$ for the norms of v in the standard metric m_0 and the metric m^ε, respectively. Observe that to prove that s_ε is $e^{2\varepsilon}$- quasiconformal it is enough to show that

$$e^{-2\varepsilon} \leq \frac{|D_x s_\varepsilon(v)|^0_{s_\varepsilon(x)}}{|D_x s_\varepsilon(w)|^0_{s_\varepsilon(x)}} \leq e^{2\varepsilon}, \quad \forall x \in \mathbb{S}^2, \forall v, w \in T_x(\mathbb{S}^2), \tag{11}$$

such that $|v|_x = |w|_x = 1$.

Let y be an arbitrary point in $\mathbb{S}^2$, and let v', w' be two vectors in $T_y(\mathbb{S}^2)$. By the inequalities (8), we know that

$$e^{-\varepsilon}|v'|_y^\varepsilon \leq \left|D_y s(v')\right|^\varepsilon_{s(y)} \leq e^{\varepsilon}|v'|_y^\varepsilon.$$

So it follows that

$$e^{-2\varepsilon}\frac{|v'|_y^\varepsilon}{|w'|_y^\varepsilon} \leq \frac{|D_y s(v')|^\varepsilon_{s(y)}}{|D_y s(w')|^\varepsilon_{s(y)}} \leq e^{2\varepsilon}\frac{|v'|_y^\varepsilon}{|w'|_y^\varepsilon}. \tag{12}$$

We also have that $g_\varepsilon^* m^\varepsilon = e^{h_\varepsilon} m_0$. Then, $|D_y g_\varepsilon(v')|^\varepsilon_{g_\varepsilon(y)} = e^{2h_\varepsilon(y)}|v'|_y^0$, and therefore, if we take $y = g_\varepsilon(x)$, $v' = D_x g_\varepsilon(v)$, and $w' = D_x g_\varepsilon(w)$ in inequality (12), then we obtain

$$\begin{aligned} e^{-2\varepsilon} &= e^{-2\varepsilon}\frac{e^{2h_\varepsilon g_\varepsilon(x)}|v|_x^0}{e^{2h_\varepsilon g_\varepsilon(x)}|w|_x^0} \\ &\leq \frac{|D_{g_\varepsilon(x)} s(D_x g_\varepsilon v)|^\varepsilon_{sg_\varepsilon(x)}}{|D_{g_\varepsilon(x)} s(D_x g_\varepsilon w)|^\varepsilon_{sg_\varepsilon(x)}} \\ &\leq e^{2\varepsilon}\frac{e^{2h_\varepsilon g_\varepsilon(x)}|v|_x^0}{e^{2h_\varepsilon g_\varepsilon(x)}|w|_x^0} = e^{2\varepsilon}. \end{aligned} \tag{13}$$

On the other hand, we have that

$$\begin{aligned}\frac{|D_{g_\varepsilon(x)}s(D_x g_\varepsilon v)|^\varepsilon_{sg_\varepsilon(x)}}{|D_{g_\varepsilon(x)}s(D_x g_\varepsilon w)|^\varepsilon_{sg_\varepsilon(x)}} &= \frac{|D_{s_\varepsilon(x)}g_\varepsilon(D_x s_\varepsilon v)|^\varepsilon_{g_\varepsilon s_\varepsilon(x)}}{|D_{s_\varepsilon(x)}g_\varepsilon(D_x s_\varepsilon v)|^\varepsilon_{g_\varepsilon s_\varepsilon(x)}}\\ &= \frac{e^{2h_\varepsilon s_\varepsilon(x)}|D_x s_\varepsilon v|^0_{s_\varepsilon(x)}}{e^{2h_\varepsilon s_\varepsilon(x)}|D_x s_\varepsilon w|^0_{s_\varepsilon(x)}}\\ &= \frac{|D_x s_\varepsilon v|^0_{s_\varepsilon(x)}}{|D_x s_\varepsilon w|^0_{s_\varepsilon(x)}}.\end{aligned}$$

Therefore, putting together the previous inequality and (13), we obtain (11). □

We will use the following known fact about quasiconformal maps on surfaces (see, e.g., [11, Theorem 1.3.13]).

LEMMA 4.5
Let x_1, x_2, x_3 be three different points on $\mathbb{S}^2$, and let $(f_\varepsilon\colon \mathbb{S}^2 \circlearrowleft)_{\varepsilon>0}$ be a family of homeomorphisms satisfying:
(1) *f_ε is K_ε-quasiconformal;*
(2) *$K_\varepsilon \to 1$, as $\varepsilon \to 0$;*
(3) *for each $\varepsilon > 0$, f_ε fixes x_1, x_2, and x_3.*
Then $f_\varepsilon \to \mathrm{Id}$, as $\varepsilon \to 0$, in the C^0-topology.

Remark 4.6
The same result holds true if the three points x_i are allowed to depend on ε under the additional assumption that they remain at a bounded distance away from each other.

Proof
This follows from the compactness of K-quasiconformal maps of $\mathbb{S}^2$, which states that the family of K-quasiconformal maps of $\mathbb{S}^2$ fixing three points x_1, x_2, x_3 is compact in the C^0-topology (see [11, Theorem 1.3.13] for further details). Therefore, any subsequence $\{f_{\varepsilon_n}\}$ of maps f_ε has a convergent subsequence converging to a map f' which must be K_ε-quasiconformal for every K_ε and, therefore, conformal. So, $f' \in \mathrm{PSL}_2(\mathbb{C})$. Besides, this map f' must fix x_1, x_2, and x_3, and so, f' must be the identity. □

In what follows, we will make repeated use of the following classical fact.

PROPOSITION 4.7
If f is a diffeomorphism of $\mathbb{S}^2$ which preserves the orientation of finite order k, then f is conjugate to a rotation of order k.

Proof
Let m' be any Riemannian metric on $\mathbb{S}^2$, and consider the metric

$$m := \frac{1}{k}\sum_{i=1}^{k}(g^i)^*(m').$$

The metric m is g-invariant and, by the uniformization theorem, conformally equivalent to the standard metric m_0 on $\mathbb{S}^2$. This implies that g is conjugate to a diffeomorphism of $\mathbb{S}^2$ which preserves orientation and acts conformally on $\mathbb{S}^2$, and therefore, $g \in \mathrm{SL}_2(\mathbb{C})$. As g has order k, the element g is conjugate to a rotation of order k in $\mathrm{SO}(3)$. $\square$

We will need the following consequence of Lemma 4.5.

PROPOSITION 4.8
Let $s\colon \mathbb{S}^2 \circlearrowleft$ be a periodic homeomorphism different from the identity, and let $\{s_\varepsilon\colon \mathbb{S}^2 \circlearrowleft\}_{\varepsilon>0}$ be a family of homeomorphisms such that the following conditions hold:

(1) *for each $\varepsilon > 0$, s_ε is topologically conjugate to s;*
(2) *there is a family of positive real numbers $\{K_\varepsilon\}_{\varepsilon>0}$ such that s_ε is K_ε-quasiconformal, for every $\varepsilon > 0$ and $K_\varepsilon \to 1$, as $\varepsilon \to 0$;*
(3) *there exists $\delta > 0$ such that if p_ε, q_ε are the two fixed points of s_ε, then it holds that $d(p_\varepsilon, q_\varepsilon) > \delta$, for every $\varepsilon > 0$.*

Then the family $\{s_\varepsilon\}_{\varepsilon>0}$ is precompact with the uniform C^0-topology, and if $s'\colon \mathbb{S}^2 \circlearrowleft$ is a homeomorphism such that there is a sequence $s_{\varepsilon_n} \to s'$, with $\varepsilon_n \to 0$, then $s' \in \mathrm{PSL}_2(\mathbb{C})$ and s' has the same period as s.

Proof
Given any $\varepsilon > 0$, the diffeomorphism s_ε is conjugate to a finite-order rotation. So s_ε has exactly two fixed points, which are denoted by p_ε and q_ε. Therefore, there exists a conformal map $A_\varepsilon \in \mathrm{PSL}_2(\mathbb{C})$ sending p_ε to $Z := (0,0,1) \in \mathbb{S}^2 \subset \mathbb{R}^3$ and q_ε to $-Z = (0,0,-1) \in \mathbb{S}^2 \subset \mathbb{R}^3$. Moreover, we can suppose that the family $\{\|A_\varepsilon\|\}_{\varepsilon>0}$ is bounded and consequently precompact.

Let us define $S_\varepsilon := A_\varepsilon s_\varepsilon A_\varepsilon^{-1}$. If we consider the great circle C in $\mathbb{S}^2$ determined by the xy-plane in $\mathbb{R}^3$, then there is a point $x_\varepsilon \in C$ such that $S_\varepsilon(x_\varepsilon) \in C$. Then, there is a rotation $R_{\theta_\varepsilon} \in \mathrm{SO}(3)$ fixing $Z, -Z$ and sending x_ε to $S_\varepsilon(x_\varepsilon)$. Therefore, the map

$S_\varepsilon R_{\theta_\varepsilon}^{-1}$ is still K_ε-quasiconformal and fixes the three points Z, $-Z$, and $S_\varepsilon(x_\varepsilon)$. By Remark 4.6, we have $S_\varepsilon R_{\theta_\varepsilon}^{-1} \to \mathrm{Id}$ in the C^0-topology, as $\varepsilon \to 0$. By the compactness of SO(3), there are a subsequence $\{\varepsilon_n\}_n$, with $\varepsilon_n \to 0$ as $n \to \infty$, and an angle θ such that $R_{\theta_{\varepsilon_n}} \to R_\theta$ as $n \to \infty$. So, $S_{\varepsilon_n} \to R_\theta$.

To finish the proof, we must show that R_θ has order k. Since $S_{\varepsilon_n} \to R_\theta$ as $n \to \infty$, we have that $S_{\varepsilon_n}^k \to R_\theta^k$, and so, $R_\theta^k = \mathrm{Id}$. This implies that R_θ has order k' dividing k. If $k' < k$, then $S_{\varepsilon_n}^{k'} \to \mathrm{Id}$ as $n \to \infty$, where $S_{\varepsilon_n}^{k'}$ is conjugate to a rotation of order $\alpha := \frac{k}{k'}$. But this is impossible as any conjugate of a rotation of order α must move some point in $\mathbb{S}^2$ at least distance α in the standard metric on $\mathbb{S}^2$. $\square$

We will now begin the proof of Theorem 4.1. We start by considering the case where G is 2-generated.

PROPOSITION 4.9

Let S be a generating set of a group G as in Theorem 4.1. For any pair of elements $s, t \in S$, the subgroup $G' = \langle s, t\rangle$ of G is finite.

Proof

First let us consider the case where s, t have a common fixed point p in $\mathbb{S}^2$. The derivative map at p gives a homomorphism $\Phi\colon G' \to \mathrm{GL}_2(\mathbb{R})$. As the image of Φ is a finitely generated periodic linear group, by Schur's theorem this homomorphism must have finite image. On the other hand, $\ker(\Phi)$ is trivial because all the elements of G' are smoothly conjugate to rotations. Then, G' must be finite (and, in fact, G' must be cyclic). We will then assume that s, t have no common fixed point.

Conjugating by the diffeomorphisms g_ε given by (9), we obtain diffeomorphisms s_ε and t_ε which are $e^{2\varepsilon}$-quasiconformal. Furthermore, by conjugation with a Möbius map we can suppose the fixed points of s_ε are $Z := (0,0,1)$ and $-Z = (0,0,-1)$ of $\mathbb{S}^2 \subset \mathbb{R}^3$ and one fixed point of t_ε is the point $X = (1,0,0)$.

Therefore, by Proposition 4.8, there exists a sequence $\{\varepsilon_n\}_n$ of positive numbers, with $\varepsilon_n \to 0$ as $n \to \infty$, so that the sequence $\{s_{\varepsilon_n}\}_n$ converges to a nontrivial rotation $R_\theta \in \mathrm{SO}(3)$, fixing the points Z and $-Z$. Recall that each diffeomorphism t_ε fixes the point X. Let us write X_{ε_n} for the other fixed point of t_{ε_n}. Then we have the following.

LEMMA 4.10

Either G' is finite, or it holds that $\lim_{\varepsilon_n \to 0} X_{\varepsilon_n} = X$.

Proof

Let us assume the sequence $\{X_{\varepsilon_n}\}_n$ does not converge to X. So we can invoke Proposition 4.8 to guarantee the existence of a subsequence $\{t_{\varepsilon_{n_j}}\}_{n_j}$ converging in the

C^0-topology to a Möbius transformation $T \in \mathrm{PSL}_2(\mathbb{C})$. In such a case, the sequences of diffeomorphisms $\{s_{\varepsilon_n}\}_n$ and $\{t_{\varepsilon_{n_j}}\}_{n_j}$ converge to R_θ and T, respectively. So, the group $\langle R_\theta, T\rangle$ is a periodic subgroup of $\mathrm{PSL}_2(\mathbb{C})$, and by Schur's theorem, it is finite. Let us show this implies that G' is finite as well.

To do that, let $\mathbb{F}_2$ denote the free group on two elements, and let $\{a,b\} \subset \mathbb{F}_2$ be a generating set of $\mathbb{F}_2$. Let us write $h\colon \mathbb{F}_2 \to G'$ and $h_0\colon \mathbb{F}_2 \to \mathrm{PSL}_2(\mathbb{C})$ for the two unique group homomorphisms such that $h(a) = s$, $h(b) = t$, $h_0(a) = R_\theta$, and $h_0(b) = T$. If G' were infinite, then there would be an element $w \in \mathbb{F}_2$ such that $h(w) \neq \mathrm{id}$ and $h_0(w) = \mathrm{id}$. However, for each ε_n we can consider the only group homomorphism $h_{\varepsilon_n} : \mathbb{F}_2 \to \mathrm{Diff}^\infty(\mathbb{S}^2)$ such that $h_{\varepsilon_n}(a) = s_{\varepsilon_n}$ and $h_{\varepsilon_n}(b) = t_{\varepsilon_n}$.

Since each s_{ε_n} and t_{ε_n} is conjugate to s and t, respectively, we get $h_{\varepsilon_n}(w) \neq \mathrm{id}$, for every n. But on the other hand, $h_{\varepsilon_n}(w) \to h_0(w) = \mathrm{id}$ as $n \to \infty$, which contradicts Proposition 4.8. So, G' is finite. □

We will now deal with the case when $X_{\varepsilon_n} \to X$ as $n \to \infty$. In order to simplify the notation, we will denote ε_n simply by ε, and any statement about $\varepsilon \to 0$ should be understood to be true up to passing to the sequence $\{\varepsilon_n\}_n$. We will show the following.

PROPOSITION 4.11
If $X_\varepsilon \to X$, then the group G' contains an element of infinite order, contradicting the fact that $G' = \langle s,t\rangle$ is periodic.

Proof
Since the point X_ε is different from X for every $\varepsilon > 0$, for each ε we can consider the great circle C_ε in $\mathbb{S}^2$ passing through the points X_ε and X. Let $M_\varepsilon \in C_\varepsilon$ be the midpoint between X_ε and X on the shortest geodesic segment determined by these points (see Figure 1). Then there is a Möbius transformation A_ε (a loxodromic element in $\mathrm{PSL}_2(\mathbb{C})$) such that $A_\varepsilon(M_\varepsilon) = M_\varepsilon$, $A_\varepsilon(-M_\varepsilon) = -M_\varepsilon$, and $A_\varepsilon(X_\varepsilon) = -A_\varepsilon(X)$. Let us define $Y_\varepsilon := A_\varepsilon(X_\varepsilon)$.

By the compactness of $\mathbb{S}^2$ and Proposition 4.8, there exist a point $Y \in \mathbb{S}^2$, a rotation $S_\alpha \in \mathrm{SO}(3)$, and a subsequence of $\{\varepsilon_n\}$ such that $Y_\varepsilon \to Y \in S^2$ and $A_\varepsilon t_\varepsilon A_\varepsilon^{-1} \to S_\alpha$ is the C^0-topology as $\varepsilon \to 0$. Notice the rotation S_α has a strictly positive finite order and fixes the points Y and $-Y$.

Our purpose now is to construct an open disk $D \subset \mathbb{S}^2$ such that $s_\varepsilon t_\varepsilon(D) \subset D$, for $\varepsilon > 0$ sufficiently small enough, where the inclusion is strict. This shows the element $s_\varepsilon t_\varepsilon$ cannot have finite order, and this immediately implies st has infinite order, as well.

In order to prove that, let $\delta > 0$ be small number (how small δ is will be determined later), and let $B_\delta(X)$ be the ball in $\mathbb{S}^2$ of radius δ with center at X.

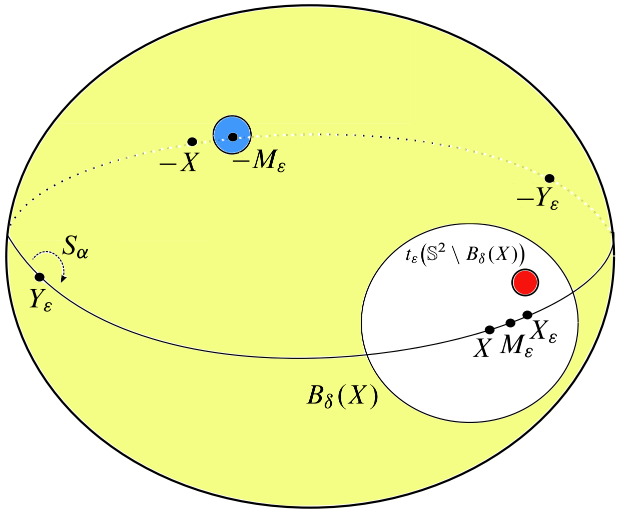

Figure 1. (Color online) The white ball corresponds to $B_\delta(X)$, the blue ball to $A_\varepsilon(\mathbb{S}^2 \setminus B_\delta(X))$, and the red ball to $t_\varepsilon(\mathbb{S}^2 \setminus B_\delta(X))$.

By fixing δ, there exists a positive number $\varepsilon(\delta) > 0$ such that $A_\varepsilon(\mathbb{S}^2 \setminus B_\delta(X)) \subset B_\delta(-M_\varepsilon)$, $\|A_\varepsilon t_\varepsilon A_\varepsilon^{-1} - S_\alpha\|_{C^0} < 0.01\delta$, and $A_\varepsilon^{-1} S_\alpha(B_{1.01\delta}(-M_\varepsilon)) \subset B_\delta(X)$, for any $\varepsilon \in (0, \varepsilon(\delta))$. And so we have

$$\begin{aligned} t_\varepsilon\big(\mathbb{S}^2 \setminus B_\delta(X)\big) &= A_\varepsilon^{-1} \circ A_\varepsilon t_\varepsilon A_\varepsilon^{-1} \circ A_\varepsilon\big(\mathbb{S}^2 \setminus B_\delta(X)\big) \\ &\subset A_\varepsilon^{-1} \circ A_\varepsilon t_\varepsilon A_\varepsilon^{-1}\big(B_\delta(-M_\varepsilon)\big) \\ &\subset A_\varepsilon^{-1} \circ S_\alpha\big(B_{1.01\delta}(-M_\varepsilon)\big) \subset B_\delta(X). \end{aligned}$$

Now, for $\varepsilon > 0$ sufficiently small,we have that s_ε is close enough to the rotation R_θ SO(3) constructed in the proof of Lemma 4.10, so for such ε we can also assume that $s_\varepsilon(B_\delta(X)) \subset \mathbb{S}^2 \setminus B_\delta(X)$, and thus, we get

$$s_\varepsilon t_\varepsilon\big(\mathbb{S}^2 \setminus B_\delta(X)\big) \subset \mathbb{S}^2 \setminus B_\delta(X),$$

where this last inclusion is strict and therefore $s_\varepsilon t_\varepsilon$ has infinite order. $\square$

Now we can finish the proof of Theorem 4.1.

Proof of Theorem 4.1

Let s, t be two arbitrary elements of the generating set S. We can suppose that s, t do not have a common fixed point, because otherwise they would generate a finite group which must be cyclic because s, t would be conjugate to rotations and in such a case we could reduce the number of generators. By the proof of Proposition 4.9, the conjugates s_ε, t_ε converge to Möbious transformations $A_s, A_t \in \mathrm{PSL}_2(\mathbb{C})$, as $\varepsilon \to 0$, generating a finite group. It also follows from the proof of Proposition 4.9 that A_s and A_t cannot have a common fixed point.

We will show that there exists a subsequence $\varepsilon_n \to 0$ such that, for any $h \in S$, the sequence of conjugates $h_{\varepsilon_n} := g_{\varepsilon_n}^{-1} h g_{\varepsilon_n}$ converges to a nontrivial Möbious map $A_h \in \mathrm{PSL}_2(\mathbb{C})$. To show this, let p_ε, q_ε denote the fixed points of h_ε. Arguing as in Lemma 4.10, if for some $h \in S$ there is a subsequence h_{ε_n} such that p_{ε_n} and q_{ε_n} converge to two different points p, q, then h_{ε_n} must converge to a finite-order element $A_h \in \mathrm{PSL}_2(\mathbb{C})$ fixing p and q.

If no such subsequence of h_ε exists, then the sequences of points p_ε, q_ε have a common limit point $X \in \mathbb{S}^2$. By possibly replacing A_s with A_t, one can suppose that X is not a fixed point of A_s. Therefore, we are in the same situation as in Proposition 4.11 for the elements s_ε, h_ε. Applying the very same argument we obtain an element of G of infinite order, getting a contradiction.

In conclusion, via a diagonal argument, we can find a subsequence $\varepsilon_n \to 0$ so that each of the conjugates h_{ε_n} converges to an element of $\mathrm{PSL}_2(\mathbb{C})$. This implies that the conjugate w_{ε_n} of every element $w \in G$ converges to an element A_w of $\mathrm{PSL}_2(\mathbb{C})$. Moreover, since the orders of w and A_w coincide, by Lemma 4.8 we know that $A_w \neq \mathrm{id}$ provided $w \neq \mathrm{id}$.

The previous discussion implies that there is an injective homomorphism $\Phi\colon G \to \mathrm{PSL}_2(\mathbb{C})$ sending g to A_g. As G is periodic and $\mathrm{PSL}_2(\mathbb{C})$ is linear, the group G must be finite. ☐

5. The exponential growth case

In this section we finish the proof of Theorem 1.3. By Theorem 4.1, we can now assume that G has exponential growth. The idea of the proof under this additional hypothesis goes as follows. As G has exponential growth, the pigeonhole principle implies there exist a constant $c \in (0,1)$ and $N_1 > 0$ such that, for any point $x \in \mathbb{S}^2$ and any $j > N_1$, there are two elements $g_j, h_j \in G$ with $|g_j|_S, |h_j|_S \leq j$ such that $d(g_j(x), h_j(x)) < c^{-j}$.

On the other hand, by Lemma 3.3 for any $\varepsilon > 0$ and j sufficiently large, the derivatives of g_j and h_j are bounded by $e^{\varepsilon j}$. Therefore, the element $f_j := h_j^{-1} g_j$ moves x exponentially close to itself. Then, since the group G has bounded exponent, the orbit $\{f_j^i(x)\}_{i=1}^k$ of x has exponentially small diameter, where $k \in \mathbb{N}$ is an exponent for the whole group.

So, we will prove that this implies that f_j has a fixed point exponentially close to x. This argument also applies if instead of a single point x we consider a finite collection of points $x_i \in \mathbb{S}^2$. This implies that there are nontrivial elements of G with as many fixed points as one wants, contradicting the fact that every element of G is conjugate to some rotation.

LEMMA 5.1
Let us suppose $G < \mathrm{Diff}^{\infty}(\mathbb{S}^2)$ is a finitely generated periodic group of bounded exponent and has subexponential growth of the derivative; that is, the conclusion of Lemma 3.3 holds. Then, G does have subexponential growth.

Remark 5.2
It is important to notice that this is the only part of Theorem 1.3 where we use the bounded exponent assumption on the group G.

Observe that Theorem 1.3 just follows as a straightforward combination of Theorem 4.1 and Lemma 5.1. We will now begin the proof of Lemma 5.1. Throughout the proof we will use the classical Vinogradov "O" notation, which states that given two sequences $\{f_n\}$, $\{g_n\}$ we have $f_n = O(g_n)$ if there exist constants $C, n_0 > 0$ such that $\frac{1}{C} f_n < g_n < C f_n$ for $n \geq n_0$.

Proof of Lemma 5.1
Reasoning by contradiction, let us suppose that G has exponential growth. Let S be a finite generating set of G. By replacing S with a larger generating set, we can suppose that

$$\sharp\{g \in G : |g| \leq j\} \geq 2^{7j}, \quad \forall j \geq 1.$$

Observe that the group G naturally acts on $(\mathbb{S}^2)^3 := \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$ via the diagonal action. Then consider a fixed triple of distinct points of $\mathbb{S}^2$ $\bar{x} = (x_1, x_2, x_3) \in (\mathbb{S}^2)^3$, let us define

$$O_j := \{(g(x_1), g(x_2), g(x_3)) \in (\mathbb{S}^2)^3 : g \in G,\ |g|_S \leq j\},$$

and observe that the G-orbit of $\bar{x}$ is equal to $\bigcup_j O_j$. Notice that if we endow $(\mathbb{S}^2)^3$ with the product Riemannian structure, then the volume of a ball of radius 2^{-n} in $(\mathbb{S}^2)^3$ is $O(2^{-6n})$.

Then, as there are at least 2^{7j} elements in $B_j := \{g \in G : |g|_S \leq j\}$, by the pigeonhole principle, for each $j \geq 1$ there are at least two different elements g_j, h_j in B_j such that their corresponding images of $\bar{x} \in (\mathbb{S}^2)^3$ satisfy $g_j(\bar{x}), h_j(\bar{x}) \in O_j$ and $d(g_j(\bar{x}), h_j(\bar{x})) < 2^{-j}$, for every $j \geq 1$.

On the other hand, by Lemma 3.3 we know that, given any $\varepsilon > 0$, there exists N_2 such that, for any $j \geq N_2$ and any element $g \in B_n$, it holds that $\|D(g)\| \leq e^{\frac{\varepsilon}{2} n}$. If we define $f_j := h_j^{-1} g_j$, we have $f_j \in B_{2j}$ and so $\|D(f_j)\| \leq e^{\varepsilon j}$ for every $j \geq N_2$. So, for each $i \in \{1,2,3\}$ and any $j \geq 1$ such that $j \geq N_2$, we have

$$d\big(f_j(x_i), x_i\big) \leq \big\|D(h_j^{-1})\big\| d\big(g_j(\bar{x}), h_j(\bar{x})\big) \leq e^{\varepsilon j} 2^{-j}. \tag{14}$$

We will show that (14) implies that, for j sufficiently large, f_j has at least three fixed points, and consequently, it is the identity, contradicting the fact that g_j and h_j were different.

PROPOSITION 5.3
For each $j \geq N_2$ and each $i \in \{1,2,3\}$, there exists $p_{i,j} \in \mathbb{S}^2$ such that $f_j(p_{i,j}) = p_{i,j}$ and

$$d(x_i, p_{i,n}) = O(2^{-n} e^{(k+2)\varepsilon n}).$$

Consequently, $f_j = \mathrm{id}$ for j sufficiently large.

Proof
It is clearly enough to prove the statement just for the point x_1. Let S_j be the shortest geodesic segment on $\mathbb{S}^2$ joining the points $f_j(x_1)$ and x_1. Then we define

$$K_j := \bigcup_{i=1}^{k} f_j^i(S_j),$$

where k is the exponent of G; that is, $g^k = \mathrm{id}$, for every $g \in G$. Observe that K_j is a compact f_j-invariant set. Also, as $\|D(f_j)\| \leq e^{\varepsilon j}$, the compact set K_j has diameter of order $O(2^{-j} e^{\varepsilon(k+1)j})$.

Therefore, for each j there exists an open disk $D_j \subset \mathbb{S}^2$ whose radius is of order $O(2^{-n} e^{\varepsilon(k+1)n})$ and containing K_j. If j is sufficiently large, then the set $\mathbb{S}^2 \setminus K_j$ has exactly one connected component R_j containing the set $\mathbb{S}^2 \setminus D_j$. Therefore, since $\|D(f_j)\| \leq e^{\varepsilon n}$, the area of $f_j(R_j)$ is greater than $O(e^{-2\varepsilon n})$, and so the connected component R_j must be f_j-invariant, provided j is large enough.

Let $K'_j := S^2 \setminus R_j$, and observe that K'_j is an f_j-invariant closed set. The set K'_j is a connected set as it is the union of K_j with the closure of the connected components of $S^2 \setminus K_j$. (The union of connected sets with nonempty intersection is connected.) Consequently, K'_j is a nonseparating continuum, and by the Cartwright–Littlewood theorem (see, e.g., [2]), there exists a fixed $p_{1,j} \in K'_j$ of f_j. As $K'_j \subset D_j$, we have that the distance between $p_{1,j}$ and x_1 should be of order $O(2^{-n} e^{\varepsilon(k+1)n})$. ∎

Therefore, Lemma 5.1 is proved. ∎

Appendix. The Burnside problem on $\mathbb{T}^q$ and manifolds with hyperbolic fundamental group

In this appendix we discuss and prove some results about actions on manifolds of nonpositive curvature. We will prove the following.

THEOREM A.1
Let M be a closed manifold of dimension $q = \dim M \geq 3$ such that its universal cover is homeomorphic $\mathbb{R}^q$ and its fundamental group $\pi_1(M)$ is a nonelementary Gromov hyperbolic group. Then, any periodic subgroup of $\mathrm{Homeo}(M)$ *is finite.*

Observe that in Theorem A.1 we are not a priori assuming the subgroup is finitely generated. Remember that Gromov hyperbolic groups were first introduced in [15], and any compact manifold admitting a Riemannian structure of nonpositive curvature satisifies the hypotheses of the theorem.

THEOREM A.2
Let μ be a Borel probability measure on the q-dimensional torus $\mathbb{T}^q$, and let $\mathrm{Homeo}_\mu(\mathbb{T}^q)$ *denote the group of homeomorphisms of $\mathbb{T}^q$ that preserves the measure μ. Then, any finitely generated periodic subgroup of* $\mathrm{Homeo}_\mu(\mathbb{T}^q)$ *is finite.*

Theorems A.1 and A.2 extend previous results of Guelman and Liousse in [16] and [17] to higher dimensions.

A.1. Displacement functions and Newman's theorem

In this paragraph we introduce some notation we will need in the proofs of Theorems A.1 and A.2. Let (X, d) be an arbitrary metric space. We define the *displacement function* $\mathcal{D}\colon \mathrm{Homeo}(X) \to [0, \infty]$ by

$$\mathcal{D}(f) := \sup_{x \in X} d\big(f(x), x\big), \quad \forall f \in \mathrm{Homeo}(X), \tag{15}$$

where $\mathrm{Homeo}(X)$ denotes the group of homeomorphisms of X.

Then, let us recall a classical result due to Newman from [27].

THEOREM A.3
Let M be a connected complete manifold, let d be a distance function compatible with the topology of M, and let $k \geq 1$ be a natural number. Then, there exists a real number $\varepsilon(M, d, k) > 0$, depending on M, d, and k, such that, for every $f \in$ $\mathrm{Homeo}(M)$ *satisfying $f^k = \mathrm{id}$ and $f \neq \mathrm{id}$, it holds that $\mathcal{D}(f) \geq \varepsilon$.*

An easy consequence of Theorem A.3 is the following.

COROLLARY A.4
Let d denote the standard Euclidean distance on $\mathbb{R}^q$, and let $f \in \mathrm{Homeo}(\mathbb{R}^q)$ be a homeomorphism such that $f \neq \mathrm{id}$ and $f^k = \mathrm{id}$, for some $k \geq 1$. Then it holds that $\mathcal{D}(f) = \infty$.

Proof
Let $\varepsilon := \varepsilon(\mathbb{R}^q, d, k) > 0$ be the real constant given by Theorem A.3 for $M = \mathbb{R}^q$, and let us suppose $\mathcal{D}(f) < \infty$. Now, if we define $f_\lambda \in \mathrm{Homeo}(\mathbb{R}^q)$ by

$$f_\lambda(x) := \lambda f(\lambda^{-1}x), \quad \forall x \in \mathbb{R}^q,$$

with $\lambda := \varepsilon(2\mathcal{D}(f))^{-1}$, then it holds that

$$\mathcal{D}(f_\lambda) = \sup_{x\in\mathbb{R}^q} \| f_\lambda(x) - x\| = \sup_{x\in\mathbb{R}^q} \lambda \| f(\lambda^{-1}x) - \lambda^{-1}x\| = \lambda\mathcal{D}(f) = \frac{\varepsilon}{2},$$

and $f_\lambda^k = \mathrm{id}$. Then, by Theorem A.3 it holds that $f_\lambda = \mathrm{id}$, contradicting the fact that $f \neq \mathrm{id}$. □

A.2. *Proof of Theorem A.1*
Let us consider M endowed with a Riemannian structure, and let d be its induced distance function on M. Let $\pi\colon \tilde{M} \to M$ denote the universal cover of M, and consider $\tilde{M}$ equipped with the Riemannian structure given by the pullback by π of the corresponding structure of M. Let $\tilde{d}$ be its induced distance function on $\tilde{M}$. Recall that $\tilde{M}$ is homeomorphic to $\mathbb{R}^q$.

Let $\mathrm{Homeo}_0(M)$ be the group of homeomorphisms of M which are homotopic to the identity, and consider the quotient group $\mathrm{MCG}(M) := \mathrm{Homeo}(M)/\mathrm{Homeo}_0(M)$. It is well known that $\mathrm{MCG}(M)$ is isomorphic to $\mathrm{Out}(\pi_1(M))$, that is, the group of outer automorphisms of the fundamental group (see, e.g., [18, Proposition 1B.9]). Since M is closed and aspherical, $\dim M \geq 3$, and $\pi_1(M)$ is a Gromov hyperbolic group, by [15, Theorem 5.4.A] we know that $\mathrm{Out}(\pi_1(M))$ is finite. So, $\mathrm{MCG}(G)$ is finite as well.

Now, let $G < \mathrm{Homeo}(M)$ be a periodic group, and define $G_0 := G \cap \mathrm{Homeo}_0(M)$. Since $\mathrm{MCG}(G)$ is finite, we know that G_0 has finite index in G; so in order to prove G is finite it is enough to show that G_0 is finite. In fact, we will prove that $G_0 = \{\mathrm{id}_M\}$. To do that, let $g \in G_0$ be an arbitrary element, and let $H\colon [0,1] \times M \to M$ be a homotopy such that $H(0,\cdot) = \mathrm{id}_M$ and $H(1,\cdot) = g$. Let $\tilde{H}\colon [0,1] \times \tilde{M} \to \tilde{M}$ denote the only lift of H such that $\tilde{H}(0,\cdot) = \mathrm{id}_{\tilde{M}}$, and let us write $\tilde{g} := H(1,\cdot) \in \mathrm{Homeo}(\tilde{M})$.

Since g has finite order, there is $n \geq 1$ and $T \in \mathrm{Deck}(\pi)$, the group of deck transformations of the covering $\pi\colon \tilde{M} \to M$ such that $\tilde{g}^n = T$.

Then note that, since $\tilde{g}$ is homotopic to the identity, its displacement

$$\mathcal{D}(\tilde{g}) = \sup_{x\in\tilde{M}} \tilde{d}\big(\tilde{g}(x), x\big) < \infty,$$

and hence $\mathcal{D}(\tilde{g}^n) = \mathcal{D}(T) < \infty$ as well. However, since $\mathrm{Deck}(\pi)$ is isomorphic to the fundamental group $\pi_1(M)$ and this is a Gromov hyperbolic group, this implies that every element of $\pi_1(M)$ different from the identity exhibits infinite displacement. Since $\mathcal{D}(T) < \infty$, we conclude T must be equal to the identy $\mathrm{id}_{\tilde{M}}$. So, $\tilde{g}$ is indeed a periodic map, and since $\mathcal{D}(\tilde{g}) < \infty$ and $\tilde{M}$ is homeomorphic to $\mathbb{R}^n$, by Corollary A.4 we know that $\tilde{g} = \mathrm{id}_{\tilde{M}}$. □

A.3. Proof of Theorem A.2

Let $\pi\colon \mathbb{R}^q \to \mathbb{T}^q := \mathbb{R}^q/\mathbb{Z}^q$ be the natural projection map. For each $f \in \mathrm{Homeo}(\mathbb{T}^q)$, there is a unique $A_f \in \mathrm{GL}_q(\mathbb{Z})$ such that the map $\Delta_{\tilde{f}} := \tilde{f} - A_f\colon \mathbb{R}^q \to \mathbb{R}^q$ is $\mathbb{Z}^q$-periodic, for any lift $\tilde{f}\colon \mathbb{R}^q \circlearrowleft$ of f. The map

$$\mathcal{I}\colon \mathrm{Homeo}(\mathbb{T}^q) \ni f \mapsto A_f \in \mathrm{GL}_q(\mathbb{R})$$

is clearly a group homomorphism. Then we define $\mathrm{Homeo}_0(\mathbb{T}^q) := \ker \mathcal{I}$.

Let $G < \mathrm{Homeo}_\mu(\mathbb{T}^q)$ denote a finitely generated periodic subgroup. Then, $\mathcal{I}(G) < \mathrm{GL}_q(\mathbb{R})$ is a finitely generated periodic group. By Schur's theorem we know $\mathcal{I}(G)$ is finite. So, the subgroup

$$G_0 := G \cap \mathrm{Homeo}_0(\mathbb{T}^q) < G$$

has finite index in G. Thus, it is enough to show G_0 is finite.

Then, let us consider the group $\mathrm{Homeo}_{0,\mu}(\mathbb{T}^q) := \mathrm{Homeo}_\mu(\mathbb{T}^q) \cap \mathrm{Homeo}_0(\mathbb{T}^q)$ and the *μ-mean rotation vector* $\rho_\mu\colon \mathrm{Homeo}_{0,\mu}(\mathbb{T}^q) \to \mathbb{T}^q$ given by

$$\rho_\mu(f) := \pi\Big(\int_{\mathbb{T}^q} \Delta_{\tilde{f}}\,\mathrm{d}\mu\Big), \quad \forall f \in \mathrm{Homeo}_{0,\mu}(\mathbb{T}^q),$$

where $\tilde{f}\colon \mathbb{R}^q \circlearrowleft$ is any lift of f and the function $\Delta_{\tilde{f}} = \tilde{f} - \mathrm{id}_{\mathbb{R}^q}$ is considered as an element of $C^0(\mathbb{T}^q, \mathbb{R}^q)$. Notice that ρ_μ is well defined and does not depend on the choice of lift.

We claim ρ_μ is a group homomorphism. To prove this, let $f, g \in \mathrm{Homeo}_{0,\mu}(\mathbb{T}^q)$ be two arbitrary homeomorphisms, and let $\tilde{f}, \tilde{g}\colon \mathbb{R}^q \circlearrowleft$ be two lifts of f and g, respectively. Then we have

$$\begin{aligned}\rho_\mu(f\circ g) &= \pi\Big(\int_{\mathbb{T}^q} \Delta_{\tilde{f}\circ\tilde{g}}\,\mathrm{d}\mu\Big) = \pi\Big(\int_{\mathbb{T}^q} \Delta_{\tilde{f}}\circ g + \Delta_{\tilde{g}}\,\mathrm{d}\mu\Big) \\ &= \pi\Big(\int_{\mathbb{T}^q} \Delta_{\tilde{f}} + \Delta_{\tilde{g}}\,\mathrm{d}\mu\Big) = \rho_\mu(f) + \rho_\mu(g),\end{aligned}$$

where the first equality is a consequence of the identity $\Delta_{\tilde{f}\circ\tilde{g}} = \Delta_{\tilde{f}} \circ g + \Delta_{\tilde{g}}$ and the second one follows from the fact that μ is g-invariant.

So, $\rho_\mu(G_0) < \mathbb{T}^q$ is a finitely generated periodic group; since $\mathbb{T}^q$ is abelian, $\rho_\mu(G_0)$ is finite. This implies that $G_{0,0} := G_0 \cap \ker \rho_\mu$ is a finite-index subgroup of G_0. Thus, it is enough to show $G_{0,0}$ is finite, and we indeed show it just reduces to the identity map. In order to do that, let f be an arbitrary element of $G_{0,0}$. Notice there is a unique lift $\tilde{f}\colon \mathbb{R}^q \circlearrowleft$ such that

$$\int_{\mathbb{T}^q} \Delta_{\tilde{f}} \, \mathrm{d}\mu = 0.$$

Since $G_{0,0}$ is periodic, there exists $n \geq 1$ such that $f^n = \mathrm{id}$, and consequently, there is $\boldsymbol{p} \in \mathbb{Z}^q$ such that $\tilde{f}^n(x) = x + \boldsymbol{p}$, for every $x \in \mathbb{R}^q$. However, since μ is f-invariant, it holds that

$$\boldsymbol{p} = \int_{\mathbb{T}^q} \Delta_{\tilde{f}^n} \, \mathrm{d}\mu = \int_{\mathbb{T}^q} \sum_{j=0}^{n-1} \Delta_{\tilde{f}} \circ f^j \, \mathrm{d}\mu = n \int_{\mathbb{T}^q} \Delta_{\tilde{f}} \, \mathrm{d}\mu = 0.$$

So, $\tilde{f}$ is a periodic homeomorphism of $\mathbb{R}^q$, and since $\Delta_{\tilde{f}}$ is $\mathbb{Z}^q$-periodic, it holds that

$$\mathcal{D}(\tilde{f}) = \sup_{x \in \mathbb{R}^q} \left\| \Delta_{\tilde{f}}(x) \right\| < \infty,$$

where $\mathcal{D}(\tilde{f})$ is the displacement function given by (15). Then, by Corollary A.4 we get that $\tilde{f}$ equals the identity, and Theorem A.2 is proven. $\square$

Acknowledgments. We want to thank Clark Butler, Amie Wilkinson, and Aaron Brown for useful comments and Ian Agol for encouragement. Kocsard's work was partially supported by FAPERJ-Brazil and CNPq-Brazil.

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Hurtado
University of Chicago, Chicago, Illinois, USA; shurtado@chicago.edu

Kocsard
Universidade Federal Fluminense, Rio de Janeiro, Brazil; akocsard@id.uff.br

Rodríguez-Hertz
Pennsylvania State University, State College, Pennsylvania, USA; hertz@math.psu.edu