Semistable modifications of families of curves and compactified Jacobians

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Abstract

Given a family of nodal curves, a semistable modification of it is another family made up of curves obtained by inserting chains of rational curves of any given length at certain nodes of certain curves of the original family. We give comparison theorems between torsion-free, rank-1 sheaves in the former family and invertible sheaves in the latter. We apply them to show that there are functorial isomorphisms between the compactifications of relative Jacobians of families of nodal curves constructed through Caporaso's approach and those constructed through Pandharipande's approach.

1 Introduction

Compactifications of (generalized) Jacobians of (reduced, connected, projective) curves have been considered by several authors. Igusa [16] was likely the first to study the degeneration of Jacobians of smooth curves when these specialize to nodal curves. Later, Mayer and Mumford [20] suggested realizing Igusa's degenerations using torsion-free, rank-1 sheaves to represent boundary points of compactifications of Jacobians. This was carried out by D'Souza [12] for irreducible curves, and by Oda and Seshadri [23] for reducible, nodal curves. In full generality, moduli spaces for torsion-free, rank-1 (and also higher rank) sheaves on curves were constructed by Seshadri [27].

As far as families are concerned, Altman and Kleiman [2], [3], [4], constructed relative compactifications of Jacobians for families of irreducible curves (and also higher dimension varieties). The author [13] continued their work, considering relative compactifications for any family of curves. The most general work in this respect is that of Simpson's [28], who constructed moduli spaces of coherent sheaves for families of schemes.

It is also natural to ask whether a compactification of the relative Jacobian can be constructed over the moduli space $\overline{M}_g$ of Deligne–Mumford stable curves of genus g, for any $g \geq 2$. This is not a direct consequence of the works cited above for families, as there is no universal family of curves over $\overline{M}_g$. Such a compactification was constructed by Caporaso [6].

Caporaso's compactification represented a departure from the approach suggested by Mayer and Mumford, as the boundary points did not correspond to torsion-free, rank-1 sheaves on stable curves, but rather invertible sheaves on semistable curves of a special type, called quasistable curves, where the exceptional components are isolated. The connection with the then usual approach was established one year later by Pandharipande [26], who constructed a compactification of the relative Jacobian (and also moduli spaces of vector bundles of any rank) over $\overline{M}_g$ using torsion-free, rank-1 sheaves, and showed that his compactification was isomorphic to Caporaso's.

More precisely, Caporaso produced a scheme $P^b_{d,g}$ coarsely representing the functor $\mathcal{P}^b_{d,g}$ that associates to each scheme S the set of isomorphism classes of pairs $(Y/S, \mathcal{L})$ of a family Y/S of quasistable curves of (arithmetic) genus g, and an invertible sheaf $\mathcal{L}$ on Y whose restrictions to the fibers of Y/S have degree d and satisfy certain "balancing" conditions; see Section 6. On the other hand, Pandharipande produced a scheme $J^{ss}_{d,g}$ coarsely representing the functor $\mathcal{J}^{ss}_{d,g}$ that associates to each scheme S the set of isomorphism classes of pairs $(X/S, \mathcal{I})$ of a family X/S of stable curves of genus g, and a coherent S-flat sheaf $\mathcal{I}$ on X whose restrictions to the fibers of X/S are torsion-free, rank-1 sheaves of degree d satisfying certain "semistability" conditions; see Section 6.

Essentially, Pandharipande constructed in [26], 10.2, p. 465, a map of functors $\Phi^b\colon \mathcal{P}^b_{d,g} \to \mathcal{J}^{ss}_{d,g}$, and showed that the corresponding map of schemes $\phi\colon P^b_{d,g} \to J^{ss}_{d,g}$ is bijective in 10.3, p. 468. Then he used the normality of $J^{ss}_{d,g}$, which he had proved in Prop. 9.3.1, p. 464, to conclude that ϕ is an isomorphism in Thm. 10.3.1, p. 470.

In the present article we prove that Φ^b is itself an isomorphism of functors, our Theorem 6.3, which thus entails that ϕ is an isomorphism. We do so by describing the inverse map. In fact, our Proposition 6.2 implies that Φ^b is the restriction of a map $\Phi\colon \mathcal{P}_{d,g} \to \mathcal{J}_{d,g}$ between "larger" functors, without the extra conditions of "balancing" and "semistability." And our Theorem 6.1 claims that Φ is an isomorphism, describing its inverse.

We feel that these results are of interest, not only because they give another proof of the existence of the isomorphism ϕ, but also because of the immediate application to stacks. The point of view of stacks was applied to the problem of compactifying the relative Jacobian over $\overline{M}_g$ in [18] and in [7], the latter in the special situation where Deligne–Mumford stacks arise, and in more generality in [21]. See [22] as well, for compactifications over the stacks of pointed stable curves. It is a natural point of view, and should be further studied. We give here a small contribution to this study.

Furthermore, we go beyond showing that Φ^b is an isomorphism. More generally, we study families of semistable curves, and give comparison theorems between torsion-free rank-1 sheaves on nodal curves and invertible sheaves on semistable modifications of them; see Theorems 3.1, 3.2 and 4.1. Though technical, we believe these are useful theorems to have when working in the field. Indeed, they have already proved fundamental in our study of Abel maps; see [10], from which [1], [11], [24] and [25] derive. In [10] we study the construction of degree-2 Abel maps for nodal curves, and we need to deal with invertible sheaves on semistable curves containing chains of two exceptional components; it is expected that longer chains will occur in the study of higher degree Abel maps.

Some of the results in these notes may be well-known to the specialists. For instance, Propositions 5.4 and 5.5, are essentially stated in [8], Prop. 4.2.2, p. 3754, for whose proof the reader is mostly refered to [15] and [17]. However, detailed statements and proofs are given here, together with generalizations and a more global approach, which works over general base schemes.

In short, here is how the paper is structured. In Section 2 we describe our basic objects, torsion-free, rank-1 sheaves on families of curves, present the notion of stability, and give cohomological characterizations for the existence of certain inequalities for degrees of invertible sheaves on chains of rational curves.

In Section 3, we prove our main result, Theorem 3.1, which gives necessary and sufficient conditions under which the pushforward $\psi_*\mathcal{L}$ of an invertible sheaf $\mathcal{L}$ under a map of curves $\psi\colon Y \to X$ contracting exceptional components is torsion-free, rank-1. We give as well sufficient conditions for when two invertible sheaves have the same pushforward. In Section 4, we prove Theorem 4.1, which compares the various notions of stability for $\mathcal{L}$ with those for $\psi_*\mathcal{L}$. In Section 5, we apply these theorems in the special situation where the exceptional components of Y are isolated. In addition, we show how to do the opposite construction, that is, how to get an invertible sheaf $\mathcal{L}$ on Y from a torsion-free, rank-1 sheaf $\mathcal{I}$ on X in such a way that $\mathcal{I} = \psi_*\mathcal{L}$. All the constructions apply to families of curves. Then, in Section 6, we apply them to produce an inverse to Φ^b.

2 Sheaves on curves

2.1 Curves

A *curve* is a reduced, connected, projective scheme of pure dimension 1 over an algebraically closed field K. A curve may have several irreducible components, which will be simply called *components*. We will always assume our curves to be *nodal*, meaning that the singularities are *nodes*, that is, analytically like the origin on the union of the coordinate axes of the plane $\mathbb{A}^2_K$.

We say that a curve X has *genus* g if $h^1(X, \mathcal{O}_X) = g$. This is in fact the so-called arithmetic genus, but the geometric genus will play no role here.

A *subcurve* of a curve X is the reduced union of a nonempty collection of its components. A subcurve is a curve if and only if it is connected. Given a proper subcurve Y of X, we will let Y' denote the *complementary subcurve*, that is, the reduced union of the remaining components of X. Also, we let k_Y denote the number of points of $Y \cap Y'$. Since X is connected, $k_Y \geq 1$. A component E of a curve X is called *exceptional* if E is smooth, rational, $E \neq X$ and $k_E \leq 2$.

We will call a curve X *semistable* if all exceptional components E have $k_E = 2$; *quasistable* if, in addition, no two exceptional components meet; and *stable* if there are no exceptional components.

A *chain of rational curves* is a curve whose components are smooth and rational and can be ordered, $E_1, \dots, E_n$, in such a way that $\#E_i \cap E_{i+1} = 1$ for $i = 1, \dots, n-1$ and $E_i \cap E_j = \emptyset$ if $|i - j| > 1$. If n is the number of components, we say that the chain has *length* n. Two chains of the same length are isomorphic. The components E_1 and E_n are called the *extreme curves* of the chain. A connected subcurve of a chain is also a chain, and is called a *subchain*.

Let $\mathcal{N}$ be a collection of nodes of a curve X, and $\eta\colon \mathcal{N} \to \mathbb{N}$ a function. Denote by $\widetilde{X}_{\mathcal{N}}$ the partial normalization of X along $\mathcal{N}$. For each $P \in \mathcal{N}$, let E_P be a chain of rational curves of length $\eta(P)$. Let X_η denote the curve obtained as the union of $\widetilde{X}_{\mathcal{N}}$ and the E_P for $P \in \mathcal{N}$ in the following way: Each chain E_P intersects no other chain, but intersects $\widetilde{X}_{\mathcal{N}}$ transversally at two points, the branches over P on $\widetilde{X}_{\mathcal{N}}$ on one hand, and nonsingular points on each of the two extreme curves of E_P on the other hand. There is a natural map $\mu_\eta\colon X_\eta \to X$ collapsing each chain E_P to a point, whose restriction to $\widetilde{X}_{\mathcal{N}}$ is the partial normalization map. The curve X_η and the map μ_η are well-defined up to X-isomorphism.

All schemes are assumed locally Noetherian. A point s of a scheme S is a map $\mathrm{Spec}(K) \to S$, where K is a field, denoted $\kappa(s)$. If $\kappa(s)$ is algebraically closed, we say that s is geometric.

A *family of* (*connected*) *curves* is a proper and flat morphism $f\colon X \to S$ whose geometric fibers are connected curves. If s is a geometric point of S, put $X_s := f^{-1}(s)$. If T is a S-scheme, put $X_T := X \times_S T$; the second projection $X_T \to T$ is also a family of curves.

If all the geometric fibers of f are (semistable, quasistable, stable) curves (of genus g), we will say that f or X/S is a *family of* (*semistable, quasistable, stable*) *curves* (*of genus* g).

If X is a curve over an algebraically closed field K, a *regular smoothing* of X is the data (f, ξ) consisting of a generically smooth family of curves $f\colon Y \to S$, where Y is regular and S is affine with ring of functions $K[[t]]$, the ring of formal power series over K, and an isomorphism $\xi\colon X \to Y_0$, where Y_0 is the special fiber of f. A *twister* of X is an invertible sheaf on X of the form $\xi^*\mathcal{O}_Y(Z)|_{Y_0}$, where $(f\colon Y \to S, \xi)$ is a regular smoothing of X, and Z is a Cartier divisor of Y supported in Y_0, so a formal sum of components of Y_0. A twister has degree 0 by continuity of the degree, since $\mathcal{O}_Y(Z)$ is trivial away from Y_0.

If Z is a formal sum of the components of X, we define

$$\mathcal{O}_X(Z) := \xi^*\mathcal{O}_Y(\xi(Z))|_{Y_0}.$$

This definition depends on the choices of f and ξ. However, for our purposes here, the definition is good enough as it is.

2.2 Sheaves

Let $f\colon X \to S$ be a family of curves. Given a coherent sheaf $\mathcal{F}$ on X and a geometric point s of S, we will let $\mathcal{F}_s := \mathcal{F}|_{X_s}$. More generally, given any

S-scheme T, denote by $\mathcal{F}_T$ the pullback of $\mathcal{F}$ to X_T under the first projection $X_T \to X$.

Let $\mathcal{I}$ be a S-flat coherent sheaf on X. We say that $\mathcal{I}$ is *torsion-free* on X/S if, for each geometric point s of S, the associated points of $\mathcal{I}_s$ are generic points of X_s. We say that $\mathcal{I}$ is *of rank* 1 or *rank*-1 on X/S if, for each geometric point s of S, the sheaf $\mathcal{I}_s$ is invertible on a dense open subset of X_s. We say that $\mathcal{I}$ is *simple* on X/S if, for each geometric point s of S, we have $\operatorname{Hom}(\mathcal{I}_s, \mathcal{I}_s) = \kappa(s)$.

Since X is flat over S, with reduced and connected fibers, each invertible sheaf on X is torsion-free, rank-1 and simple on X/S. In particular, so is the relative dualizing sheaf of X/S.

We say that $\mathcal{I}$ has degree d on X/S if $\mathcal{I}_s$ has degree d for each $s \in S$, that is,

$$d = \chi(\mathcal{I}_s) - \chi(\mathcal{O}_{X_s})$$

for each $s \in S$.

Given a geometric point s of S and a subcurve Y of X_s, let $\mathcal{I}_Y$ denote the restriction of $\mathcal{I}$ to Y modulo torsion. In other words, if $\xi_1, \dots, \xi_m$ are the generic points of Y, let $\mathcal{I}_Y$ denote the image of the natural map

$$\mathcal{I}|_Y \longrightarrow \bigoplus_{i=1}^{m} (\mathcal{I}|_Y)_{\xi_i}.$$

Also, let $\deg_Y(\mathcal{I})$ denote the degree of $\mathcal{I}_Y$, i.e.

$$\deg_Y(\mathcal{I}) := \chi(\mathcal{I}_Y) - \chi(\mathcal{O}_Y).$$

Let X be a (connected) curve over an algebraically closed field K and denote by $X_1, \dots, X_p$ its components. Fix an integer d. Since X is a proper scheme over K, by [5], Thm. 8.2.3, p. 211, there is a scheme, locally of finite type over K, parameterizing degree-d invertible sheaves on X; denote it by J_X^d. It decomposes as

$$J_X^d = \coprod_{\substack{\underline{d}=(d_1,\dots,d_p)\\ d_1+\dots+d_p=d}} J_X^{\underline{d}}, \tag{1}$$

where $J_X^{\underline{d}}$ is the connected component of J_X^d parameterizing invertible sheaves $\mathcal{L}$ such that $\deg(\mathcal{L}|_{X_i}) = d_i$ for $i = 1, \dots, p$. The $J_X^{\underline{d}}$ are quasiprojective varieties.

The scheme J_X^d is in a natural way an open subscheme of $\overline{J}_X^d$, the scheme over K parameterizing torsion-free, rank-1, simple sheaves of degree d on X; see [13] for the construction of $\overline{J}_X^d$ and its properties. The scheme $\overline{J}_X^d$ is universally closed over K but, in general, not separated and only locally of finite type. Moreover, in contrast to J_X^d, the scheme $\overline{J}_X^d$ is connected, hence not easily decomposable. Thus, to deal with a manageable piece of it, we resort to polarizations.

Let $\mathcal{E}$ be a locally free sheaf on X of constant rank, and $\mathcal{I}$ a torsion-free, rank-1 sheaf on X. We say that $\mathcal{I}$ is *semistable* (resp. *stable*, resp. X_i-*quasistable*) *with respect to* $\mathcal{E}$ if

1. $\chi(\mathcal{I} \otimes \mathcal{E}) = 0$,

2. $\chi(\mathcal{I}_Y \otimes \mathcal{E}|_Y) \geq 0$ for each proper subcurve $Y \subset X$ (resp. with equality never, resp. with equality only if $X_i \not\subseteq Y$).

Notice that it is enough to check Property 2 above for connected subcurves Y. Also, Property 1 is equivalent to the numerical condition that

$$\mathrm{rk}(\mathcal{E})(\deg(\mathcal{I}) + \chi(\mathcal{O}_X)) + \deg(\mathcal{E}) = 0.$$

The X_i-quasistable sheaves are simple, what can be easily proved using for instance [13], Prop. 1, p. 3049. Their importance is that they form an open subscheme $\overline{J}_X^{\mathcal{E},i}$ of $\overline{J}_X^d$ that is projective over K.

Let $f\colon X \to S$ be a family of curves. Let $\mathcal{E}$ be a locally free sheaf on X of constant rank and $\mathcal{I}$ a torsion-free, rank-1 sheaf on X/S. Let $\sigma\colon S \to X$ be a section of f through its smooth locus. We say that $\mathcal{I}$ is *semistable* (resp. *stable*, resp. *σ-quasistable*) *with respect to* $\mathcal{E}$ if, for each geometric point s of S, the sheaf $\mathcal{I}_s$ is semistable (resp. stable, resp. $X_{s,\sigma}$-quasistable) with respect to $\mathcal{E}_s$. Here, $X_{s,\sigma}$ is the component of X_s containing $\sigma(s)$.

There is an algebraic space $\overline{J}_{X/S}$ parameterizing torsion-free, rank-1, simple sheaves on X/S, containing the locus $J_{X/S}$ parameterizing invertible sheaves as an open subset. Remarkable facts are that, first, up to an étale base change, $\overline{J}_{X/S}$ is a scheme; second, the locus of $\overline{J}_{X/S}$ parameterizing the sheaves on X/S which are σ-quasistable with respect to $\mathcal{E}$ is an open subspace which is proper over S.

2.3 Chains of rational curves

If E is a chain of rational curves and $\mathcal{L}$ is an invertible sheaf on E, then $\mathcal{L}$ is determined by its restrictions to the components of E, and thus by the degrees of these restrictions. In particular, $\mathcal{L} \cong \mathcal{O}_E$ if and only if $\deg(\mathcal{L}|_F) = 0$ for each component $F \subseteq E$. Also, $\mathcal{L}$ is the dualizing sheaf of E if its degree on each component is zero, but for the extreme curves, where the degree is -1.

Lemma 2.1. *Let E be a chain of rational curves of length n. Let E_1 and E_n denote the extreme curves. Let $\mathcal{L}$ be an invertible sheaf on E. Then the following statements hold:*

1. $\deg(\mathcal{L}|_F) \geq -1$ *for every subchain $F \subseteq E$ if and only if $h^1(E, \mathcal{L}) = 0$.*

2. $\deg(\mathcal{L}|_F) \leq 1$ *for every subchain $F \subseteq E$ if and only if*

$$h^0(E, \mathcal{L}(-P - Q)) = 0$$

for any two points $P \in E_1$ and $Q \in E_n$ on the nonsingular locus of E.

Proof. Let $E_1, \dots, E_n$ be the components of E, ordered in such a way that $\#E_i \cap E_{i+1} = 1$ for $i = 1, \dots, n-1$. We prove the statements by induction on

n. If $n=1$ all the statements follow from the knowledge of the cohomology of the sheaves $\mathcal{O}_{\mathbb{P}^1_K}(j)$.

Suppose $n>1$. We show Statement 1. Assume that $\deg(\mathcal{L}|_F)\geq -1$ for every subchain $F\subseteq E$. Consider the natural exact sequence

$$0\to\mathcal{L}|_{E_1}(-N)\to\mathcal{L}\to\mathcal{L}|_{E_1'}\to 0,$$

where $E_1' := \overline{E-E_1}$ and N is the unique point of $E_1\cap E_1'$. By induction, $h^1(E_1',\mathcal{L}|_{E_1'})=0$. If $\deg\mathcal{L}|_{E_1}\geq 0$ then $h^1(E_1,\mathcal{L}|_{E_1}(-N))=0$ as well, and hence $h^1(E,\mathcal{L})=0$ from the long exact sequence in cohomology.

Suppose now that $\deg\mathcal{L}|_{E_1}<0$. If $\deg\mathcal{L}|_{E_n}\geq 0$, we invert the ordering of the chain, and proceed as above. Thus we may suppose $\deg\mathcal{L}|_{E_n}<0$ as well. Since $\deg\mathcal{L}|_E\geq -1$, there is $i\in\{2,\dots,n-1\}$ such that $\deg\mathcal{L}|_{E_i}\geq 1$. Let $F_1:=E_1\cup\cdots\cup E_{i-1}$ and $F_2:=E_{i+1}\cup\cdots\cup E_n$. Consider the natural exact sequence

$$0\to\mathcal{L}|_{E_i}(-N_1-N_2)\to\mathcal{L}\to\mathcal{L}|_{F_1}\oplus\mathcal{L}|_{F_2}\to 0,$$

where N_1 and N_2 are the two points of intersection of E_i with $E_i' := \overline{E-E_i}$. By induction, $h^1(F_1,\mathcal{L}|_{F_1})=h^1(F_2,\mathcal{L}|_{F_2})=0$. Also, since $\deg\mathcal{L}|_{E_i}\geq 1$, we have $h^1(E_i,\mathcal{L}|_{E_i}(-N_1-N_2))=0$, and thus it follows from the long exact sequence in cohomology that $h^1(E,\mathcal{L})=0$ as well.

Assume now that $h^1(E,\mathcal{L})=0$. Then $h^1(F,\mathcal{L}|_F)=0$ for every subchain $F\subseteq E$. By induction, $\deg(\mathcal{L}|_F)\geq -1$ for every proper subchain $F\subsetneq E$. Since E is the union of two proper subchains, it follows that $\deg(\mathcal{L})\geq -2$. Assume by contradiction that $\deg(\mathcal{L})=-2$. Then $\deg(\mathcal{L}|_F)=-1$ for every proper subchain $F\subsetneq E$ containing E_1 or E_n. It follows that

$$\deg(\mathcal{L}|_{E_i})=\begin{cases}0 & \text{if } 1<i<n,\\ -1 & \text{otherwise.}\end{cases}$$

But then $\mathcal{L}$ is the dualizing sheaf of E, and thus $h^1(E,\mathcal{L})=1$, reaching a contradiction. The proof of Statement 1 is complete.

Statement 2 is proved in a similar way. Alternatively, it is enough to observe that $\mathcal{O}_E(-P-Q)$ is the dualizing sheaf of E, and thus, by Serre Duality,

$$h^0(E,\mathcal{L}(-P-Q))=h^1(E,\mathcal{L}^{-1}).$$

So Statement 2 follows from 1. □

3 Admissibility

Let $f\colon X\to S$ be a family of curves. Let $\psi\colon Y\to X$ be a proper morphism such that the composition $f\psi$ is another family of curves. We say that ψ is a *semistable modification* of f if for each geometric point s of S there are a collection of nodes $\mathcal{N}_s$ of X_s and a map $\eta_s\colon\mathcal{N}_s\to\mathbb{N}$ such that the induced map $\psi_s\colon Y_s\to X_s$ is X_s-isomorphic to $\mu_{\eta_s}\colon(X_s)_{\eta_s}\to X_s$. If η_s is constant and equal to 1 for every s, we say that ψ is a *small semistable modification* of f. **New definition**

Assume ψ is a semistable modification of f. Let $\mathcal{L}$ be an invertible sheaf on Y. We say that $\mathcal{L}$ is *ψ-admissible* (resp. *negatively ψ-admissible*, resp. *positively ψ-admissible*, resp. *ψ-invertible*) at a given geometric point s of S if the restriction of $\mathcal{L}$ to every chain of rational curves of Y_s over a node of X_s has degree -1, 0 or 1 (resp. -1 or 0, resp. 0 or 1, resp. 0). We say that $\mathcal{L}$ is *ψ-admissible* (resp. *negatively ψ-admissible*, resp. *positively ψ-admissible*, resp. *ψ-invertible*) if $\mathcal{L}$ is so at every s. Notice that, if $\mathcal{L}$ is negatively (resp. positively) ψ-admissible, for every chain of rational curves of Y_s over a node of X_s, the degree of $\mathcal{L}$ on each component of the chain is 0 but for at most one component where the degree is -1 (resp. 1).

Theorem 3.1. *Let $f\colon X \to S$ be a family of curves and $\psi\colon Y \to X$ a semistable modification of f. Let $\mathcal{L}$ be an invertible sheaf on Y of relative degree d over S. Then the following statements hold:*

1. *The points s of S at which $\mathcal{L}$ is ψ-admissible (resp. negatively ψ-admissible, resp. positively ψ-admissible, resp. ψ-invertible) form an open subset of S.*
2. *$\mathcal{L}$ is ψ-admissible if and only if $\psi_*\mathcal{L}$ is a torsion-free, rank-1 sheaf on X/S of relative degree d, whose formation commutes with base change. In this case, $R^1\psi_*\mathcal{L} = 0$.*
3. *If $\mathcal{L}$ is ψ-admissible then the evaluation map $v\colon \psi^*\psi_*\mathcal{L} \to \mathcal{L}$ is surjective if and only if $\mathcal{L}$ is positively ψ-admissible. Furthermore, v is bijective if and only if $\mathcal{L}$ is ψ-invertible, if and only if $\psi_*\mathcal{L}$ is invertible.*

Proof. All of the statements and hypotheses are local with respect to the étale topology of S. So we may assume S is Noetherian and that there is an invertible sheaf $\mathcal{A}$ on X that is relatively ample over S. Let $\widehat{\mathcal{A}} := \psi^*\mathcal{A}$.

We prove Statement 1 first. For each geometric point s of S, let E_s be the subcurve of Y_s which is the union of all the components contracted by ψ_s, and let $\widetilde{X}_s$ be the partial normalization of X_s obtained as the union of the remaining components. Since $\psi|_{\widetilde{X}_s} : \widetilde{X}_s \to X_s$ is a finite map, it follows that $\widehat{\mathcal{A}}|_{\widetilde{X}_s}$ is ample, and thus

$$h^1(\widetilde{X}_s, (\mathcal{L} \otimes \widehat{\mathcal{A}}^{\otimes m_s})|_{\widetilde{X}_s}(-\sum P_i)) = 0$$

for every large enough integer m_s, where the sum runs over all the branch points of $\widetilde{X}_s$ above X_s. Since S is Noetherian, a large enough integer works for all s, that is, for every $m >> 0$,

$$h^1(\widetilde{X}_s, (\mathcal{L} \otimes \widehat{\mathcal{A}}^{\otimes m})|_{\widetilde{X}_s}(-\sum P_i)) = 0 \quad \text{for each geometric point } s \text{ of } S. \tag{2}$$

Now, for each integer m consider the natural exact sequence

$$0 \longrightarrow (\mathcal{L} \otimes \widehat{\mathcal{A}}^{\otimes m})|_{\widetilde{X}_s}(-\sum P_i) \longrightarrow \mathcal{L}_s \otimes \widehat{\mathcal{A}}_s^{\otimes m} \longrightarrow (\mathcal{L} \otimes \widehat{\mathcal{A}}^{\otimes m})|_{E_s} \longrightarrow 0 \tag{3}$$

and its associated long exact sequence in cohomology. If m is large enough that (2) holds, then

$$h^1(Y_s, \mathcal{L}_s \otimes \widehat{\mathcal{A}}_s^{\otimes m}) = h^1(E_s, \mathcal{L} \otimes \widehat{\mathcal{A}}^{\otimes m}|_{E_s}). \tag{4}$$

On the other hand, since $\widehat{\mathcal{A}}$ is a pullback from X, it follows that

$$h^1(E_s, \mathcal{L}\otimes\widehat{\mathcal{A}}^{\otimes m}|_{E_s}) = \sum_F h^1(F, \mathcal{L}|_F) \quad \text{for every integer } m, \tag{5}$$

where the sum runs over all the maximal chains F of rational curves on Y_s contracted by ψ_s. Putting together (4) and (5), it follows now from Lemma 2.1 that

$$h^1(Y_s, \mathcal{L}_s\otimes\widehat{\mathcal{A}}_s^{\otimes m}) = 0 \tag{6}$$

if and only if $\deg(\mathcal{L}|_F) \geq -1$ for every chain F of rational curves on Y_s contracted by ψ_s. This is the case if $\mathcal{L}$ is ψ-admissible at s.

It follows from semicontinuity of cohomology that the geometric points s of S such that $\mathcal{L}_s$ has degree at least -1 on every chain of rational curves of Y_s contracted by ψ_s form an open subset S_1 of S. Likewise, for each integer n, the geometric points s of S such that $\mathcal{L}_s^{\otimes n}$ has degree at least -1 on every chain of rational curves of Y_s contracted by ψ_s form an open subset S_n of S. Then $S_1\cap S_{-1}$ parameterizes those s for which $\mathcal{L}_s$ is ψ_s-admissible, $S_1\cap S_{-2}$ parameterizes those s for which $\mathcal{L}_s$ is negatively ψ_s-admissible, $S_2\cap S_{-1}$ parameterizes those s for which $\mathcal{L}_s$ is positively ψ_s-admissible, and $S_2\cap S_{-2}$ parameterizes those s for which $\mathcal{L}_s$ is ψ_s-invertible.

We prove Statement 2 now. Assume for the moment that $\mathcal{L}$ is ψ-admissible. To show that $\psi_*\mathcal{L}$ is flat over S, we need only show that $f_*(\psi_*\mathcal{L}\otimes\mathcal{A}^{\otimes m})$ is locally free for each $m >> 0$. By the projection formula, we need only show that $g_*(\mathcal{L}\otimes\widehat{\mathcal{A}}^{\otimes m})$ is locally free for each $m >> 0$, where $g := f\psi$. This follows from what we have already proved: For each large enough integer m such that (2) holds, also (6) holds for each geometric point s of S, because $\mathcal{L}$ is ψ-admissible.

Furthermore, taking the long exact sequence in higher direct images of ψ_s for the exact sequence (3) with $m = 0$, using (5) and that $\psi_s|_{\widetilde{X}_s}\colon \widetilde{X}_s \to X_s$ is a finite map, it follows that $R^1\psi_{s*}(\mathcal{L}_s) = 0$ for every geometric point s of S. Since the fibers of ψ have at most dimension 1, the formation of $R^1\psi_*(\mathcal{L})$ commutes with base change, and thus $R^1\psi_*(\mathcal{L}) = 0$.

Another consequence of (6) holding for each geometric point s of S is that the formation of $g_*(\mathcal{L}\otimes\widehat{\mathcal{A}}^{\otimes m})$ commutes with base change for $m >> 0$. We claim now that the base-change map $\lambda_X^*\psi_*\mathcal{L} \to \psi_{T*}\lambda_Y^*\mathcal{L}$ is an isomorphism for each Cartesian diagram of maps

$$\begin{tikzcd}
Y_T \arrow[r, "\lambda_Y"] \arrow[d, "\psi_T"'] & Y \arrow[d, "\psi"'] \\
X_T \arrow[r, "\lambda_X"] \arrow[d, "f_T"'] & X \arrow[d, "f"'] \\
T \arrow[r, "\lambda"] & S.
\end{tikzcd}$$

Indeed, since $\mathcal{A}$ is relatively ample over S, it is enough to check that the induced map

$$f_{T*}(\lambda_X^*\psi_*\mathcal{L}\otimes\lambda_X^*\mathcal{A}^{\otimes m}) \longrightarrow f_{T*}(\psi_{T*}\lambda_Y^*\mathcal{L}\otimes\lambda_X^*\mathcal{A}^{\otimes m}) \tag{7}$$

is an isomorphism for $m >> 0$. But, by the projection formula, the right-hand side is simply $f_{T*}\psi_{T*}\lambda_Y^*(\mathcal{L}\otimes\widehat{\mathcal{A}}^{\otimes m})$. Also, since $\psi_*\mathcal{L}$ is S-flat, the left-hand side is $\lambda^* f_*(\psi_*(\mathcal{L})\otimes\mathcal{A}^{\otimes m})$ for $m >> 0$, whence equal to $\lambda^* f_*\psi_*(\mathcal{L}\otimes\widehat{\mathcal{A}}^{\otimes m})$ by the projection formula. So, since the formation of $g_*(\mathcal{L}\otimes\widehat{\mathcal{A}}^{\otimes m})$ commutes with base change for $m >> 0$, it follows that (7) is an isomorphism for $m >> 0$, as asserted.

To prove the remainder of Statement 2 and Statement 3 we may now assume that S is a geometric point. For Statement 2, we need only show now that $\psi_*\mathcal{L}$ is a torsion-free, rank-1 sheaf of degree d on X if and only if $\mathcal{L}$ is ψ-admissible. Let $F_1,\dots,F_e$ be the maximal chains of rational curves of Y contracted by ψ, to $P_1,\dots,P_e\in X$. Let E be the union of the F_i and $\widetilde{X}$ the union of the remaining components. For each $i=1,\dots,e$, let $P_{i,1},P_{i,2}\in Y$ be the points of intersection between F_i and $\widetilde{X}$. Taking higher direct images under ψ in the natural exact sequences

$$\begin{aligned}&0\to\mathcal{L}|_{\widetilde{X}}(-\textstyle\sum P_{i,j})\to\mathcal{L}\to\mathcal{L}|_E\to 0,\\ &0\to\mathcal{L}|_E(-\textstyle\sum P_{i,j})\to\mathcal{L}\to\mathcal{L}|_{\widetilde{X}}\to 0,\end{aligned}\tag{8}$$

and using that $\psi|_{\widetilde{X}}$ is a finite map, we get

$$R^1\psi_*\mathcal{L}=R^1\psi_*\mathcal{L}|_E\tag{9}$$

and the exact sequence

$$0\to\psi_*\mathcal{L}|_E(-\textstyle\sum P_{i,j})\to\psi_*\mathcal{L}\to\psi_*\mathcal{L}|_{\widetilde{X}}\to R^1\psi_*\mathcal{L}|_E(-\sum P_{i,j})\to R^1\psi_*\mathcal{L}\to 0.$$

Since $\psi|_{\widetilde{X}}$ is also birational, $\psi_*\mathcal{L}|_{\widetilde{X}}$ is a torsion-free, rank-1 sheaf of degree $\deg\mathcal{L}|_{\widetilde{X}}+e$. Since $\psi_*\mathcal{L}|_E(-\sum P_i)$ is supported at finitely many points, it follows that $\psi_*\mathcal{L}$ is torsion-free if and only if $h^0(E,\mathcal{L}|_E(-\sum P_i)=0$. The latter holds if and only if the degree of $\mathcal{L}$ on each chain of rational curves in E is at most 1, by Lemma 2.1. Furthermore, if it holds, then $R^1\psi_*\mathcal{L}|_E(-\sum P_i)$ has length $1-\deg\mathcal{L}|_{F_i}$ at each P_i by the Riemann–Roch Theorem. Since $\deg\mathcal{L}|_{\widetilde{X}}+\deg\mathcal{L}|_E=d$, it follows that $\deg\psi_*\mathcal{L}=d$ if and only if $R^1\psi_*\mathcal{L}=0$. By (9), the latter holds if and only if $h^1(E,\mathcal{L}|_E)=0$, thus if and only if the degree of $\mathcal{L}$ on each chain of rational curves in E is at least -1, by Lemma 2.1. The proof of Statement 2 is complete.

Assume from now on that $\mathcal{L}$ is ψ-admissible. Then $\psi_*\mathcal{L}|_E(-\sum P_{i,j})=0$, and thus it follows from the exact sequences in (8) that

$$\psi_*(\mathcal{L}|_{\widetilde{X}}(-\textstyle\sum P_{i,j}))\subseteq\psi_*\mathcal{L}\subseteq\psi_*(\mathcal{L}|_{\widetilde{X}}).\tag{10}$$

Furthermore, since $R^1\psi_*\mathcal{L}=0$ and since $R^1\psi_*\mathcal{L}|_E(-\sum P_{i,j})$ is supported with length $1-\deg\mathcal{L}|_{F_i}$ at P_i, the rightmost inclusion is strict at P_i if $\deg\mathcal{L}|_{F_i}=0$, and an equality if $\deg\mathcal{L}|_{F_i}=1$, for each $i=1,\dots,e$. In particular, if $\psi_*\mathcal{L}$ is invertible, then $\deg\mathcal{L}|_{F_i}=0$ for every $i=1,\dots,e$.

Moreover, for each $i=1,\dots,e$, we have the following natural commutative diagram:

$$\begin{array}{ccc} \psi_*\mathcal{L}|_{P_i} & \xrightarrow{v_i'} & \psi_*(\mathcal{L}|_{F_i}) \\ \downarrow & & \downarrow{\scriptstyle(\rho_{i,1},\rho_{i,2})} \\ \psi_*(\mathcal{L}|_{\widetilde{X}})|_{P_i} & \longrightarrow & \psi_*(\mathcal{L}|_{P_{i,1}} \oplus \mathcal{L}|_{P_{i,2}}) \end{array} \tag{11}$$

where all the maps are induced by restriction. Then $\psi_*\mathcal{L}$ is invertible at P_i if and only if $\deg \mathcal{L}|_{F_i} = 0$ and the compositions

$$\psi_*\mathcal{L} \to \psi_*(\mathcal{L}|_{\widetilde{X}}) \to \psi_*(\mathcal{L}|_{P_{i,j}}) \tag{12}$$

are nonzero for $j = 1, 2$. This is the case only if the maps $\rho_{i,1}$ and $\rho_{i,2}$ are nonzero.

Now, if $\deg \mathcal{L}|_{F_i} = 0$ then $\rho_{i,1}$ and $\rho_{i,2}$ are nonzero if and only if $\mathcal{L}|_{F_i} = \mathcal{O}_{F_i}$. Indeed, this is clear if $\mathcal{L}|_{F_i} = \mathcal{O}_{F_i}$. On the other hand, suppose $\mathcal{L}|_{F_i} \neq \mathcal{O}_{F_i}$. Let $F_{i,1}, \dots, F_{i,\ell_i}$ be the ordered sequence of components of F_i such that $P_{i,1} \in F_{i,1}$ and $P_{i,2} \in F_{i,\ell_i}$. Since $\mathcal{L}|_{F_i} \neq \mathcal{O}_{F_i}$ there is a smallest (resp. largest) integer j such that $\deg \mathcal{L}|_{F_{i,j}} \neq 0$; if $\rho_{i,1} \neq 0$ (resp. $\rho_{i,2} \neq 0$) then $\deg \mathcal{L}|_{F_{i,j}} > 0$. However, since $\mathcal{L}$ is ψ-admissible, both maps cannot be simultaneously nonzero.

To summarize, if $\psi_*\mathcal{L}$ is invertible then $\mathcal{L}$ is ψ-admissible. On the other hand, observe that v_i' is surjective for each $i = 1, \dots, e$. Indeed, it follows from applying ψ_* to the first exact sequence in (8) that the map $\psi_*\mathcal{L} \to \psi_*(\mathcal{L}|_{F_i})$ induced by restriction is surjective, and thus so is v_i'. Thus, if $\mathcal{L}|_{F_i} = \mathcal{O}_{F_i}$, the maps $\rho_{i,1}$ and $\rho_{1,2}$ are nonzero, and thus, from Diagram (11), the composition (12) is nonzero for $j = 1, 2$, whence $\psi_*\mathcal{L}$ is invertible at P_i. So, the converse holds: If $\mathcal{L}$ is ψ-admissible then $\psi_*\mathcal{L}$ is invertible.

Observe now that, for each $i = 1, \dots, e$, the restriction of the evaluation map $v\colon \psi^*\psi_*\mathcal{L} \to \mathcal{L}$ to F_i is a map $v_i\colon H^0(P_i, \psi_*\mathcal{L}|_{P_i}) \otimes \mathcal{O}_{F_i} \to \mathcal{L}|_{F_i}$. Thus, if v is surjective then $\mathcal{L}$ is positively ψ-admissible, and if v is an isomorphism then $\psi_*\mathcal{L}$ is invertible and $\mathcal{L}$ is ψ-invertible.

Assume from now on that $\mathcal{L}$ is positively ψ-admissible. Note that each v_i is obtained by composing the base-change map $v_i'\colon \psi_*\mathcal{L}|_{P_i} \to \psi_*(\mathcal{L}|_{F_i})$ with the evaluation map $v_i''\colon H^0(F_i, \mathcal{L}|_{F_i}) \otimes \mathcal{O}_{F_i} \to \mathcal{L}|_{F_i}$. Since $\mathcal{L}$ is positively ψ-admissible, it follows from Lemma 2.1 that

$$h^1(F_i, \mathcal{L}|_{F_i}) = h^1(F_i, \mathcal{L}|_{F_i}(-Q)) = 0,$$

and thus, by the Riemann–Roch Theorem, $h^0(F_i, \mathcal{L}|_{F_i}(-Q)) < h^0(F_i, \mathcal{L}|_{F_i})$ for every Q on the nonsingular locus of F_i. So v_i'' is surjective. Since the v_i' was already shown to be surjective, so is v_i for each $i = 1, \dots, e$, whence v is surjective.

Moreover, if $\psi_*\mathcal{L}$ is invertible then v is a surjective map between invertible sheaves, whence an isomorphism. □

Theorem 3.2. *Let X be a curve and $\psi\colon Y \to X$ a semistable modification of X. Let $\mathcal{L}$ and $\mathcal{M}$ be ψ-admissible invertible sheaves on Y. Assume that $\mathcal{M} \otimes \mathcal{L}^{-1}$*

is a twister of Y of the form

$$\mathcal{O}_Y\left(\sum c_E E\right), \; c_E \in \mathbb{Z},$$

where the sum runs over the components E of Y contracted by ψ. Then $\psi_*\mathcal{L} \simeq \psi_*\mathcal{M}$.

Proof. Set $\mathcal{T} := \mathcal{M} \otimes \mathcal{L}^{-1}$. Let $\mathcal{R}$ be the set of smooth, rational curves contained in Y and contracted by ψ. If $\mathcal{R} = \emptyset$, then $\mathcal{T} = \mathcal{O}_Y$ and thus $\mathcal{L} \cong \mathcal{M}$. Suppose $\mathcal{R} \neq \emptyset$. Let $\mathcal{K}$ be the set of maximal chains of rational curves contained in $\mathcal{R}$.

Claim: For every $F \in \mathcal{K}$ and every two components $E_1, E_2 \subseteq F$ such that $E_1 \cap E_2 \neq \emptyset$, we have $|c_{E_1} - c_{E_2}| \leq 1$. In addition, if E is an extreme component of F, then $|c_E| \leq 1$.

Indeed, let $E_1, \dots E_n$ be the components of F, ordered in such a way that $\# E_i \cap E_{i+1} = 1$ for $i = 1, \dots, n-1$. Since $\mathcal{L}$ and $\mathcal{M}$ are admissible, $|\deg_G \mathcal{T}| \leq 2$ for every subchain G of F. Set $c_{E_0} := c_{E_{n+1}} := 0$. We will reason by contradiction. Thus, up to reversing the order of the E_i, we may assume that $c_{E_i} - c_{E_{i+1}} \geq 2$ for some $i \in \{0, \dots, n\}$. Then

$$c_{E_i} \leq c_{E_{i-1}} \leq \dots \leq c_{E_1} \leq c_{E_0} = 0,$$

because, if $c_{E_j} > c_{E_{j-1}}$ for some $j \in \{1, \dots, i\}$, then

$$\deg_{E_j \cup \dots \cup E_i} \mathcal{T} = c_{E_{j-1}} - c_{E_j} + c_{E_{i+1}} - c_{E_i} < -2.$$

Similarly, $c_{E_{i+1}} \geq c_{E_{i+2}} \geq \dots \geq c_{E_n} \geq c_{E_{n+1}} = 0$. But then

$$0 \leq c_{E_{i+1}} < c_{E_i} \leq 0,$$

a contradiction that proves the claim.

Now, for each $F \in \mathcal{K}$, let $F^\dagger$ be the (possibly empty) union of components $E \subseteq F$ such that $c_E = 0$. For each connected component G of $\overline{F - F^\dagger}$ and irreducible components $E_1, E_2 \subseteq G$, it follows from the claim that $c_{E_1} \cdot c_{E_2} > 0$. Let $\mathcal{K}^+$ (resp. $\mathcal{K}^-$) be the collection of connected components G of $\overline{F - F^\dagger}$ for $F \in \mathcal{K}$ such that $c_E > 0$ (resp. $c_E < 0$) for every irreducible component $E \subseteq G$.

Notice that, again by the claim,

$$c_E = \begin{cases} 1 & \text{if } E \text{ is an extreme component of some } G \in \mathcal{K}_F^+ \\ -1 & \text{if } E \text{ is an extreme component of some } G \in \mathcal{K}_F^-. \end{cases} \tag{13}$$

So, being $\mathcal{L}$ and $\mathcal{M}$ admissible,

$$\deg_G \mathcal{L} = -\deg_G \mathcal{M} = \begin{cases} 1 & \text{if } G \in \mathcal{K}^+ \\ -1 & \text{if } G \in \mathcal{K}^-. \end{cases} \tag{14}$$

Define

$$W^+ := \overline{Y - \cup_{G \in \mathcal{K}^+} G}, \qquad W^- := \overline{Y - \cup_{G \in \mathcal{K}^-} G}, \qquad W := \overline{Y - \cup_{G \in \mathcal{K}^- \cup \mathcal{K}^+} G}.$$

For each $G \in \mathcal{K}^+ \cup \mathcal{K}^-$, let N_G and N'_G denote the points of $G \cap \overline{Y - G}$, and put

$$D^+ := \sum_{G \in \mathcal{K}^+} (N_G + N'_G) \quad \text{and} \quad D^- := \sum_{G \in \mathcal{K}^-} (N_G + N'_G).$$

We may view D^+ and D^- as divisors of W. Thus, by (13),

$$\mathcal{M}|_W \simeq \mathcal{L}|_W(D^+ - D^-). \tag{15}$$

Consider the natural diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \mathcal{L}|_W(-D^-) & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{G \in \mathcal{K}^+} \mathcal{L}|_G(-N_G - N'_G) & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{L}|_{W^+} & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & \bigoplus_{G \in \mathcal{K}^-} \mathcal{L}|_G & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

where the horizontal and vertical sequences are exact. By (14) and Lemma 2.1, and using the Riemann–Roch Theorem,

$$R^i\psi_*\mathcal{L}|_G(-N_G - N'_G) = H^i(G, \mathcal{L}|_G(-N_G - N'_G)) \otimes \mathcal{O}_{\psi(G)} = 0$$

for $G \in \mathcal{K}^+$ and $i = 0, 1$, whereas

$$\psi_*\mathcal{L}|_G = H^0(G, \mathcal{L}|_G) \otimes \mathcal{O}_{\psi(G)} = 0 \quad \text{for } G \in \mathcal{K}^-.$$

Hence, it follows from the above diagram, by considering the associated long exact sequences in higher direct images of ψ, that

$$\psi_*\mathcal{L} \simeq (\psi|_W)_*\mathcal{L}|_W(-D^-). \tag{16}$$

Consider a second diagram, similar to the above, but with the roles of $\mathcal{K}^+$ and $\mathcal{K}^-$, and thus of D^+ and D^-, reversed, and $\mathcal{M}$ substituted for $\mathcal{L}$. As before,

$$R^i\psi_*\mathcal{M}|_G(-N_G - N'_G)) = H^i(G, \mathcal{M}|_G(-N_G - N'_G)) \otimes \mathcal{O}_{\psi(G)} = 0$$

for $G \in \mathcal{K}^-$ and $i = 0, 1$, whereas

$$\psi_*\mathcal{M}|_G \simeq H^0(G, \mathcal{M}|_G) \otimes \mathcal{O}_{\psi(G)} = 0 \text{ for } G \in \mathcal{K}^+.$$

Hence, taking the associated long exact sequences,

$$\psi_*\mathcal{M} \simeq (\psi|_W)_*\mathcal{M}|_W(-D^+). \tag{17}$$

Combining (15), (16) and (17), we get $\psi_*\mathcal{L} \simeq \psi_*\mathcal{M}$. $\square$

4 Stability

Theorem 4.1. *Let X be a curve and $\psi\colon Y \to X$ a semistable modification of X. Let P be a simple point of Y not lying on any component contracted by ψ. Let $\mathcal{E}$ be a locally free sheaf on X and $\mathcal{L}$ an invertible sheaf on Y. Then $\mathcal{L}$ is semistable (resp. P-quasistable, resp. stable) with respect to $\psi^*\mathcal{E}$ if and only if $\mathcal{L}$ is ψ-admissible (resp. negatively ψ-admissible, resp. ψ-invertible) and $\psi_*\mathcal{L}$ is semistable (resp. $\psi(P)$-quasistable, resp. stable) with respect to $\mathcal{E}$.*

Proof. Since $\psi^*\mathcal{E}$ has degree 0 on every component of Y contracted by ψ, and P does not lie on any of these components, it follows from the definitions that a semistable (resp. P-quasistable, resp. stable) sheaf has degree -1, 0 or 1 (resp. -1 or 0, resp. 0) on every chain of rational curves of Y contracted by ψ.

We may thus assume that $\mathcal{L}$ is ψ-admissible. Let W be any connected subcurve of X. Set $W' := \overline{X - W}$ and $\Delta_W := W \cap W'$. Set $\delta := \#\Delta_W$. Let $V_1 := \overline{Y - \psi^{-1}(W')}$ and $V_2 := \overline{Y - \psi^{-1}(W)}$. Let $F_1, \dots, F_r$ be the maximal chains of rational curves contained in $\psi^{-1}(\Delta_W)$. Then $0 \leq r \leq \delta$.

Claim: $(\psi_*\mathcal{L})_W \cong \psi_*(\mathcal{L}|_Z)$ for a certain connected subcurve $Z \subseteq Y$ such that:

1. $V_1 \subseteq Z \subseteq \psi^{-1}(W)$.

2. For each connected subcurve $U \subseteq Y$ such that $V_1 \subseteq U \subseteq \psi^{-1}(W)$,
$$\deg(\mathcal{L}|_U) \geq \deg(\mathcal{L}|_Z).$$

(Notice that Property 1 implies that $P \in Z$ if and only if $\psi(P) \in W$.)

Indeed, if $W = X$, let $Z := \psi^{-1}(W)$. Suppose $W \neq X$. Then $\delta > 0$. Let $M_1, \dots, M_\delta$ be the points of intersection of V_1 with $V_1' := \overline{Y - V_1}$ and $N_1, \dots, N_\delta$ those of V_2 with $V_2' := \overline{Y - V_2}$.

Write $F_i = F_{i,1} \cup \cdots \cup F_{i,e_i}$, where $F_{i,j} \cap F_{i,j+1} \neq \emptyset$ for $j = 1, \dots, e_i - 1$ and $F_{i,1}$ intersects V_1. Up to reordering the M_i and N_i, we may assume that $F_{i,1}$ intersects V_1 at M_i and F_{i,e_i} intersects V_2 at N_i for $i = 1, \dots, r$. (Thus $M_i = N_i$ for $i = r+1, \dots, \delta$.) Up to reordering the F_i, we may also assume that there are nonnegative integers u and t with $u \leq t$ such that

$$\deg(\mathcal{L}|_{F_i}) = \begin{cases} 1 & \text{for } i = 1, \dots, u \\ 0 & \text{for } i = u+1, \dots, t \\ -1 & \text{for } i = t+1, \dots, r. \end{cases}$$

Up to reordering the F_i, we may assume there is an integer b with $u \leq b \leq t$ such that, for each $i = u+1, \dots, t$, we have that $i > b$ if and only if $\deg(\mathcal{L}|_{F_{i,j}}) = 0$ for every j or the largest integer j such that $\deg(\mathcal{L}|_{F_{i,j}}) \neq 0$ is such that $\deg(\mathcal{L}|_{F_{i,j}}) = -1$. Set $G_i := F_i$ for $i = b+1, \dots, r$. For each $i = u+1, \dots, b$, let $G_i := F_{i,1} \cup \cdots \cup F_{i,j-1}$, where j is the largest integer such that $\deg(\mathcal{L}|_{F_{i,j}}) = 1$, let $\widehat{G}_i := \overline{F_i - G_i}$ and denote by B_i the point of intersection of G_i and $\widehat{G}_i$. (Notice that $1 < j \leq e_i$.) Let $B_i := M_i$ and $\widehat{G}_i := F_i$ for $i = 1, \dots, u$, and $B_i := N_i$ for $i = b+1, \dots, \delta$.

For $i = u+1, \dots, r$, since the degree of $\mathcal{L}|_{G_i}(B_i)$ on each subchain of G_i is at most 1, it follows from Lemma 2.1 that

$$h^0(G_i, \mathcal{L}|_{G_i}(-M_i)) = 0 \quad \text{for } i = u+1, \dots, r. \tag{18}$$

Furthermore, for $i = 1, \dots, b$, the total degree of $\mathcal{L}|_{\widehat{G}_i}$ is 1; thus, by Lemma 2.1 and the Riemann–Roch Theorem,

$$h^1(\widehat{G}_i, \mathcal{L}|_{\widehat{G}_i}(-B_i - N_i)) = 0 \quad \text{for } i = 1, \dots, b. \tag{19}$$

Set

$$Z := V_1 \cup G_{u+1} \cup \cdots \cup G_r$$

and $Z' := \overline{Y - Z}$. Put $\Delta_Z := Z \cap Z'$. Notice that $\Delta_Z = \{B_1, \dots, B_\delta\}$. Also, notice that Z is connected, and

$$\deg(\mathcal{L}|_U) \geq \deg(\mathcal{L}|_Z) = \deg(\mathcal{L}|_{V_1}) - (b-u) - (r-t)$$

for each connected subcurve $U \subseteq Y$ such that $V_1 \subseteq U \subseteq \psi^{-1}(W)$.

We have three natural exact sequences:

$$0 \to \mathcal{L}|_{Z'}\Big(-\sum_{i=1}^{\delta} B_i\Big) \to \mathcal{L} \to \mathcal{L}|_Z \to 0, \tag{20}$$

$$0 \to \bigoplus_{i=1}^{b} \mathcal{L}|_{\widehat{G}_i}(-B_i - N_i) \to \mathcal{L}|_{Z'}\Big(-\sum_{i=1}^{\delta} B_i\Big) \to \mathcal{L}|_{V_2}\Big(-\sum_{i=b+1}^{\delta} B_i\Big) \to 0, \tag{21}$$

$$0 \to \bigoplus_{i=u+1}^{r} \mathcal{L}|_{G_i}(-M_i) \to \mathcal{L}|_Z \to \mathcal{L}|_{V_1} \to 0. \tag{22}$$

Since $\mathcal{L}$ is ψ-admissible, so are $\mathcal{L}|_{V_1}$ with respect to $\psi|_{V_1} \colon V_1 \to W$ and $\mathcal{L}|_{V_2}$ with respect to $\psi|_{V_2} \colon V_2 \to W'$. Then $\psi_*(\mathcal{L}|_{V_1})$ is a torsion-free, rank-1 sheaf on W and $R^1\psi_*(\mathcal{L}|_{V_2}(-\sum B_i)) = 0$ by Theorem 3.1.

Since $R^1\psi_*(\mathcal{L}|_{V_2}(-\sum B_i)) = 0$, from (19) and the long exact sequence of higher direct images under ψ of (21) and (20) we get that $R^1\psi_*(\mathcal{L}|_{Z'}(-\sum B_i)) = 0$ and the natural map $\psi_*\mathcal{L} \to \psi_*(\mathcal{L}|_Z)$ is surjective. Also, it follows from (18) and the long exact sequence of higher direct images under ψ of (22) that the natural map $\psi_*(\mathcal{L}|_Z) \to \psi_*(\mathcal{L}|_{V_1})$ is injective. Thus, since $\psi_*(\mathcal{L}|_{V_1})$ is a torsion-free, rank-1 sheaf on W, so is $\psi_*(\mathcal{L}|_Z)$. And, since $\psi_*\mathcal{L} \to \psi_*(\mathcal{L}|_Z)$ is surjective, we get an isomorphism $(\psi_*\mathcal{L})_W \cong \psi_*(\mathcal{L}|_Z)$, finishing the proof of the claim.

To prove the "only if" part, let W be any connected subcurve of X. Let Z be as in the claim. Since $\mathcal{L}$ is admissible with respect to ψ, Theorem 3.1 yields $R^1\psi_*\mathcal{L} = 0$, and hence $R^1\psi_*(\mathcal{L}|_Z) = 0$ from the long exact sequence of higher direct images under ψ of (20). Thus, by the claim and the projection formula,

$$\chi((\psi_*\mathcal{L})_W \otimes \mathcal{E}|_W) = \chi(\psi_*(\mathcal{L}|_Z) \otimes \mathcal{E}|_W) = \chi(\mathcal{L}|_Z \otimes (\psi^*\mathcal{E})|_Z). \tag{23}$$

If $\mathcal{L}$ is semistable (resp. P-quasistable, resp. stable) then $\chi(\mathcal{L}|_Z \otimes (\psi^*\mathcal{E})|_Z) \geq 0$ (resp. with equality only if $Z = Y$ or $Z \not\ni P$, resp. with equality only if $Z = Y$). Now, if $Z = Y$ then $W = X$. Also, $P \in Z$ if and only if $\psi(P) \in W$. So (23) yields $\chi((\psi_*\mathcal{L})_W \otimes \mathcal{E}|_W) \geq 0$ (resp. with equality only if $W = X$ or $W \not\ni \psi(P)$, resp. with equality only if $W = X$).

As for the "if" part, let U be a connected subcurve of Y. If U is a union of components of Y contracted by ψ, then U is a chain of rational curves of Y collapsing to a node of X, and hence $\mathcal{L}|_U$ has degree at least -1 (exactly 0 if $\mathcal{L}$ is ψ-invertible). Thus

$$\chi(\mathcal{L}|_U \otimes \psi^*\mathcal{E}|_U) = \mathrm{rk}(\mathcal{E})\chi(\mathcal{L}|_U) \geq 0,$$

with equality only if $\mathcal{L}$ is not ψ-invertible.

Suppose now that U contains a component of Y not contracted by ψ. Then $W := \psi(U)$ is a connected subcurve of X. Let $\widehat{U}$ be the smallest subcurve of Y containing U and $\overline{Y - \psi^{-1}(W')}$, where $W' := \overline{X - W}$. Then $\widehat{U}$ is connected and contained in $\psi^{-1}(W)$. Furthermore, $\chi(\mathcal{O}_U) - \chi(\mathcal{O}_{\widehat{U}})$ is the number of connected components of $\overline{\widehat{U} - U}$. Thus

$$\deg(\mathcal{L}|_U) + \chi(\mathcal{O}_U) \geq \deg(\mathcal{L}|_{\widehat{U}}) + \chi(\mathcal{O}_{\widehat{U}}), \tag{24}$$

with equality only if $\mathcal{L}$ has degree 1 on every connected component of $\overline{\widehat{U} - U}$. Let Z be as in the claim. Notice that $\chi(\mathcal{O}_{\widehat{U}}) = \chi(\mathcal{O}_Z)$. Since $\deg(\mathcal{L}|_{\widehat{U}}) \geq \deg(\mathcal{L}|_Z)$ by the claim, using (23) and (24) we get

$$\begin{aligned}
\chi(\mathcal{L}|_U \otimes \psi^*\mathcal{E}|_U) &= \mathrm{rk}(\mathcal{E})(\deg(\mathcal{L}|_U) + \chi(\mathcal{O}_U)) + \deg(\psi^*\mathcal{E}|_U) \\
&\geq \mathrm{rk}(\mathcal{E})(\deg(\mathcal{L}|_{\widehat{U}}) + \chi(\mathcal{O}_{\widehat{U}})) + \deg(\psi^*\mathcal{E}|_{\widehat{U}}) \\
&= \mathrm{rk}(\mathcal{E})(\deg(\mathcal{L}|_{\widehat{U}}) + \chi(\mathcal{O}_Z)) + \deg(\psi^*\mathcal{E}|_Z) \\
&\geq \mathrm{rk}(\mathcal{E})(\deg(\mathcal{L}|_Z) + \chi(\mathcal{O}_Z)) + \deg(\psi^*\mathcal{E}|_Z) \\
&= \chi(\mathcal{L}|_Z \otimes (\psi^*\mathcal{E})|_Z) \\
&= \chi((\psi_*\mathcal{L})_W \otimes \mathcal{E}|_W).
\end{aligned}$$

Assume that $\psi_*\mathcal{L}$ is semistable (resp. $\psi(P)$-quasistable, resp. stable) with respect to $\mathcal{E}$. Then $\chi((\psi_*\mathcal{L})_W \otimes \mathcal{E}|_W) \geq 0$ (resp. with equality only if $W = X$ or $W \not\ni \psi(P)$, resp. with equality only if $W = X$). So $\chi(\mathcal{L}|_U \otimes \psi^*\mathcal{E}|_U) \geq 0$. Suppose $\chi(\mathcal{L}|_U \otimes \psi^*\mathcal{E}|_U) = 0$. Then $\chi((\psi_*\mathcal{L})_W \otimes \mathcal{E}|_W) = 0$ and equality holds in (24). If $W \not\ni \psi(P)$ then $U \not\ni P$. Suppose $W = X$. Then $\widehat{U} = Y$. If $U \neq Y$ then $\mathcal{L}$ has degree 1 on each connected component of $\overline{Y - U}$, and thus $\mathcal{L}$ is not negatively ψ-admissible. ∎

5 Sheaves on quasistable curves

If X is a semistable curve, a stable curve $\check{X}$ may be obtained from X by contracting all exceptional components. We say that $\check{X}$ is the *stable model* of X.

Let $f\colon Y \to S$ be a family of semistable curves. We call the pair $(\check{f}, \psi)$, consisting of a family of stable curves $\check{f}\colon X \to S$ and a S-map $\psi\colon Y \to X$, a *stable model* of f if ψ is a semistable modification of $\check{f}$. So, for every geometric point s of S the induced map $\psi_s\colon Y_s \to X_s$ is the map contracting all exceptional components of Y_s. We will also call $\check{f}$ the stable model of f and ψ the *contraction map*.

Stable models always exist, and are unique up to unique isomorphism by the following proposition.

Proposition 5.1. *Let $f\colon Y \to S$ be a family of semistable curves. The following statements hold:*

1. *The family f has a stable model.*
2. *If $\check{f}\colon X \to S$ and $\check{f}'\colon X' \to S$ are stable models of f, with contraction maps $\psi\colon Y \to X$ and $\psi'\colon Y \to X'$, then there is a unique isomorphism $u\colon X' \to X$ such that $\check{f}' = \check{f}u$ and $\psi = u\psi'$.*
3. *For each stable model $\check{f}$ with contraction map ψ, the comorphism $\mathcal{O}_X \to \psi_*\mathcal{O}_Y$ is an isomorphism, $R^1\psi_*\mathcal{L} = 0$, and the pullback of the relative dualizing sheaf of $\check{f}$ under ψ is the relative dualizing sheaf of f.*

Proof. We will prove Statement 3 first. So, let $\check{f}\colon X \to S$ be a stable model of f with contraction map $\psi\colon Y \to X$. Then $R^1\psi_*\mathcal{O}_Y = 0$ by Theorem 3.1. Furthermore, $\psi_*\mathcal{O}_Y$ is invertible and the evaluation map $v\colon \psi^*\psi_*\mathcal{L} \to \mathcal{L}$ is an isomorphism. If $\psi^{\#}\colon \mathcal{O}_X \to \psi_*\mathcal{O}_Y$ is the comorphism, since $v\psi^*(\psi^{\#})$ is a natural isomorphism, it follows that $\psi^*(\psi^{\#})$ is an isomorphism, and thus that $\psi^{\#}$ is surjective. Since $\psi^{\#}$ is a surjection between invertible sheaves, it is an isomorphism.

Let $\check{\omega}$ be the relative dualizing sheaf of $\check{f}$. Then

$$
\begin{aligned}
R^1f_*(\psi^*\check{\omega}) =& R^1\check{f}_*\psi_*(\psi^*\check{\omega}) = R^1\check{f}_*(\check{\omega} \otimes \psi_*\mathcal{O}_Y)\\
=& R^1\check{f}_*(\check{\omega}) = \mathcal{O}_S.
\end{aligned} \tag{25}
$$

Indeed, the fourth equality in (25) is given by the trace map, an isomorphism because the fibers of $\check{f}$ are connected. The third equality follows from $\mathcal{O}_X = \psi_*\mathcal{O}_Y$. The projection formula, which holds because $\check{\omega}$ is invertible, yields the second equality. Finally, the first equality holds because of the degeneration of the spectral sequence associated to the composition $\check{f}\psi$, since

$$R^1\psi_*(\psi^*\check{\omega}) = \check{\omega} \otimes R^1\psi_*\mathcal{O}_Y = 0.$$

Let ω be the relative dualizing sheaf of f. By [19], Thm. 21, p. 55, for each coherent sheaf $\mathcal{N}$ on S, there is a functorial (on $\mathcal{N}$) isomorphism

$$f_*Hom(\psi^*\check{\omega}, \omega \otimes f^*\mathcal{N}) \to Hom(R^1f_*(\psi^*\check{\omega}), \mathcal{N}). \tag{26}$$

Putting the isomorphisms (25) and (26) together, we get a functorial (on $\mathcal{N}$) isomorphism

$$f_*Hom(\psi^*\check{\omega}, \omega \otimes f^*\mathcal{N}) \to \mathcal{N}. \tag{27}$$

In particular, replacing $\mathcal{N}$ by $\mathcal{O}_S$, we get a natural map $h\colon \psi^*\breve{\omega} \to \omega$, corresponding to the constant function 1_S. This map is fiberwise (over S) nonzero, a fact that can be shown by replacing $\mathcal{N}$ by skyscraper sheaves and using the functoriality of (27).

Since both $\breve{\omega}$ and ω are invertible, we need only show that h is surjective, and thus we may assume that S is the spectrum of an algebraically closed field. Now, ω and $\psi^*\breve{\omega}$ restrict to isomorphic sheaves on each component Z of Y. In fact, it follows from adjunction that

$$\omega|_Z \cong \mathcal{L} \otimes \mathcal{O}_Z\left(\sum_{P\in Z\cap Z'} P\right) \cong \psi^*\breve{\omega}|_Z,$$

where $\mathcal{L}$ is the dualizing sheaf of Z. In particular, ω and $\psi^*\breve{\omega}$ have the same multidegree. Since h is nonzero, it follows that h is an isomorphism.

We will now prove Statement 1. Let ω be the relative dualizing sheaf of f and consider the S-scheme:

$$X := \mathrm{Proj}_S\big(\mathcal{O}_S \oplus f_*\omega \oplus f_*(\omega^{\otimes 2}) \oplus \cdots\big).$$

Let $\check{f}\colon X \to S$ denote the structure map.

For each geometric point s of S, by adjunction, ω_s has positive degree on each nonexceptional component of Y_s, and thus, by duality,

$$H^1(Y_s, \omega_s^{\otimes n}) = H^0(Y_s, \omega_s^{\otimes 1-n})^* = 0 \quad \text{for each } n \geq 2.$$

It follows that the direct image $f_*(\omega^{\otimes n})$ is locally free, with formation commuting with base change, for each $n \geq 2$. Also, $f_*\omega$ is locally free, with formation commuting with base change, because $R^1 f_*\omega \cong \mathcal{O}_S$, the trace map being an isomorphism. So, $\check{f}$ is flat, and its formation commutes with base change, so

$$X_s = \mathrm{Proj}\big(H^0(Y_s, \mathcal{O}_{Y_s}) \oplus H^0(Y_s, \omega_s) \oplus H^0(Y_s, \omega_s^{\otimes 2}) \oplus \cdots\big) \tag{28}$$

for each geometric point s of S.

By [9], Thm. A, p. 68, the sheaf $\omega_s^{\otimes n}$ is globally generated for each integer $n \geq 2$ and each geometric points s of S. Thus, the natural maps $f^*f_*(\omega^{\otimes n}) \to \omega^{\otimes n}$ are surjections for $n \geq 2$, and hence induce a globally defined S-map $\psi\colon Y \to X$.

We need only show now that, for each geometric point s of S, the scheme X_s is a stable model of Y_s and ψ_s is a contraction map. Indeed, let Z be a stable model of Y_s, and let $b\colon Y_s \to Z$ be a contraction map. Let $\mathcal{L}$ be the dualizing sheaf of Z. Then $b^*\mathcal{L} \cong \omega_s$ by Statement 3. Since $b_*\mathcal{O}_{Y_s} = \mathcal{O}_Z$, it follows that

$$H^0(Z, \mathcal{L}^{\otimes n}) = H^0(Y_s, \omega_s^{\otimes n}) \quad \text{for each integer } n > 0. \tag{29}$$

On the other hand, since Z is stable, $\mathcal{L}$ is ample, and thus

$$Z = \mathrm{Proj}\big(H^0(Z, \mathcal{O}_Z) \oplus H^0(Z, \mathcal{L}) \oplus H^0(Z, \mathcal{L}^{\otimes 2}) \oplus \cdots\big).$$

It follows now from (28) and (29) that there is an isomorphism $u\colon Z \to X_s$ such that $\psi_s = ub$. $\square$

If X is a scheme and $\mathcal{F}$ is a coherent sheaf on X, let

$$Sym(\mathcal{F}) = \bigoplus_{n\geq 0} Sym^n(\mathcal{F}) \quad \text{and} \quad \mathbb{P}_X(\mathcal{F}) := \mathrm{Proj}(Sym(\mathcal{F})),$$

where $Sym^n(\mathcal{F})$ is the nth symmetric product of $\mathcal{F}$, for each integer $n \geq 0$.

Proposition 5.2. *Let X be a curve and $\mathcal{I}$ a torsion-free, rank-1 sheaf on X. Set $Y := \mathbb{P}_X(\mathcal{I})$, and let $\psi\colon Y \to X$ be the structure map. Then ψ is a small semistable modification of X. The exceptional components of Y contracted by ψ are the fibers of ψ over the points of X where $\mathcal{I}$ is not invertible. In particular, if X is stable, then Y is quasistable with stable model X and contraction map ψ.*

Proof. Wherever $\mathcal{I}$ is invertible, ψ is an isomorphism. So, let us analyze ψ on a neighborhood of a node P of X where $\mathcal{I}$ fails to be invertible. In fact, consider the base change of ψ to the spectrum of the completion $\widehat{\mathcal{O}}_{X,P}$. Since P is a node, where $\mathcal{I}$ fails to be invertible, $\widehat{\mathcal{I}}_P \cong \mathfrak{m}_P$, where $\mathfrak{m}_P$ is the maximal ideal of $\widehat{\mathcal{O}}_{X,P}$. Also, since P is a node,

$$\widehat{\mathcal{O}}_{X,P} \cong \frac{K[[u,v]]}{(uv)},$$

where K is the base field of X. Now, under the above identification,

$$\mathfrak{m}_P \cong \frac{\widehat{\mathcal{O}}_{X,P} \oplus \widehat{\mathcal{O}}_{X,P}}{v\widehat{\mathcal{O}}_{X,P} \oplus u\widehat{\mathcal{O}}_{X,P}}$$

as an $\widehat{\mathcal{O}}_{X,P}$-module. So, locally analytically, Y is the subscheme of $\mathbf{A}^2_K \times \mathbb{P}^1_K$ defined by the equations $uv = sv = tu = 0$, where u and v are the coordinates of $\mathbf{A}^2_K$ and s and t are homogeneous coordinates of $\mathbb{P}^1_K$. Also, ψ is the restriction to Y of the projection $\mathbf{A}^2_K \times \mathbb{P}^1_K \to \mathbf{A}^2_K$ onto the first factor. Then Y is the union of three lines, the projective line given by $u = v = 0$, and the affine lines given by $u = s = 0$ and $v = t = 0$, the latter two not meeting each other, but intersecting the former transversally.

As the above reasoning applies to any node P of X where $\mathcal{I}$ fails to be invertible, it follows that the singularities of Y are nodes, that Y is a curve, and that $\psi^{-1}(P)$ is a smooth, rational component of Y with $k_{\psi^{-1}(P)} = 2$ for any such P. ∎

Lemma 5.3. (E–Kleiman) *Let $p\colon X \to S$ be a flat map and $\mathcal{F}$ a S-flat coherent sheaf on X. Assume $\mathcal{F}$ is invertible at each associated point of X, and is everywhere locally generated by two sections. Set $W := \mathbb{P}_X(\mathcal{F})$, and let $w\colon W \to X$ be the structure map. Then W is S-flat and Serre's graded $\mathcal{O}_X$-algebra homomorphism*

$$Sym(\mathcal{F}) \longrightarrow \bigoplus_{n\geq 0} w_*\mathcal{O}_W(n)$$

is an isomorphism.

Proof. We refer to the proof of [14], Lemma 3.1, p. 491 and its notation. To complement the proof, we need only observe that W is S-flat. First, notice that $\mathcal{N}$ is S-flat, because of the first exact sequence in the proof. Second, notice that V is S-flat, being a projective bundle over X. The structure map is denoted $v\colon V \to X$. Since $\mathcal{N}$ is flat, and v is a projective bundle map, it follows from the third exact sequence in the proof that W is a subscheme of V with a S-flat sheaf of ideals. Now, the formation of this third exact sequence commutes with base change. So W is S-flat. □

Proposition 5.4. *Let $f\colon Y \to S$ be a family of quasistable curves. Let $\mathcal{L}$ be an invertible sheaf on Y of degree d on Y/S such that $\deg_E(\mathcal{L}) = 1$ for every exceptional component E of every geometric fiber of Y/S. Let $\check{f}\colon X \to S$ be a stable model of f and $\psi\colon Y \to X$ the contraction map. Let $\mathcal{I} := \psi_*\mathcal{L}$. Then the following statements hold:*

1. *The direct image $\psi_*\mathcal{L}$ is a torsion-free, rank-1 sheaf on X/S of relative degree d, whose formation commutes with base change.*

2. *For each geometric point $s \in S$ and each node P of X_s, the sheaf $\mathcal{I}_s$ is invertible at P if and only if ψ is an isomorphism over a neighborhood of P.*

3. *The evaluation map $e\colon \psi^*\mathcal{I} \to \mathcal{L}$ is surjective.*

4. *There is an isomorphism $u\colon Y \to \mathbb{P}_X(\mathcal{I})$ over X such that $u^*\mathcal{O}(1) \cong \mathcal{L}$.*

Proof. Statement 1 follows readily from Theorem 3.1, as well as Statement 3. It follows from Statement 3 that e defines a X-map $u\colon Y \to \mathbb{P}_X(\mathcal{I})$ such that $u^*\mathcal{O}(1) \cong \mathcal{L}$. Then, to prove Statement 4, since both Y and $\mathbb{P}_X(\mathcal{I})$ are S-flat, the latter by Lemma 5.3, and the formation of $\mathcal{I}$ commutes with base change by Statement 1, we need only check that u_s is an isomorphism for every geometric point s of S.

So, for the remainder of the proof, we may now assume S is the spectrum of an algebraically closed field.

The contraction map ψ factors as the composition of several maps, each contracting a single exceptional component. Thus, to prove Statement 2 we may assume that ψ contracts a single component. Then Statement 2 follows from Theorem 3.1 as well.

As for Statement 4, first observe that $\mathbb{P}_X(\mathcal{I})$ is a quasistable curve isomorphic to Y, by Proposition 5.2 and Statement 2. So, since u is an X-morphism, to check that u is an isomorphism we need only check that, for each exceptional component $F \subset Y$, the restriction $u|_F$ is an isomorphism onto the corresponding exceptional component of $\mathbb{P}(\mathcal{I})$. But this is so, because, letting $R \in X$ denote the point below F, the restriction $u|_F$ is the map to $\mathbb{P}(\mathcal{I}|_R)$ given by the surjection $e|_F$. So, $u|_F$ is an isomorphism because $e|_F$ is the evaluation map of the degree-1 sheaf $\mathcal{L}|_F$. □

Proposition 5.5. *Let $f\colon X \to S$ be a family of curves. Let $\mathcal{I}$ be a torsion-free, rank-1 sheaf of degree d on X/S. Let $Y := \mathbb{P}_X(\mathcal{I})$, with structure map $\psi\colon Y \to X$, and let $\mathcal{L}$ denote the tautological invertible sheaf on Y. Then ψ is a small semistable modification of X/S. In particular, if X/S is a family of stable curves, then Y/S is a family of quasistable curves, X/S is its stable model, and ψ is the contraction map. Furthermore, $\mathcal{L}$ has degree d on Y/S, the degree of $\mathcal{L}$ on every exceptional component contracted by ψ of every geometric fiber of Y/S is* 1*, and $\mathcal{I} = \psi_*\mathcal{L}$.*

Proof. We apply Lemma 5.3 for $\mathcal{F} := \mathcal{I}$. The hypotheses are verified because the associated points of X are generic points of certain fibers of f, where $\mathcal{I}$ is invertible, and $\mathcal{I}$ is everywhere locally generated by two sections, since X/S is a family of nodal curves. So Y is S-flat.

It follows from Lemma 5.3 as well that $\mathcal{I} = \psi_*\mathcal{L}$. Since the formation of $\mathbb{P}_X(\mathcal{I})$ commutes with base change, it follows from Proposition 5.2 that ψ is a semistable modification of X/S.

By Proposition 5.2, the exceptional components contracted by ψ of the geometric fibers of Y/S are the fibers of $\mathbb{P}_X(\mathcal{I})$ over the nodes of the geometric fibers of X/S where $\mathcal{I}$ is not invertible. Since $\mathcal{L}$ is the tautological sheaf of $\mathbb{P}_X(\mathcal{I})$, its restriction to a fiber over X is also tautological. So $\mathcal{L}$ has degree 1 on every exceptional component contracted by ψ of every geometric fiber of Y/S. Finally, that $\mathcal{L}$ has relative degree d over S follows now from Statement 1 of Proposition 5.4. □

6 Functorial isomorphisms

Let $\mathcal{P}_{d,g}$ be the contravariant functor from the category of schemes to that of sets defined in the following way: For each scheme S, let $\mathcal{P}_{d,g}(S)$ be the set of equivalence classes of pairs $(f, \mathcal{L})$, where $f\colon Y \to S$ is a family of quasistable curves of genus g over S, and $\mathcal{L}$ is an invertible sheaf on Y of relative degree d over S whose degree on every exceptional component of every geometric fiber of Y/S is 1. Two such pairs, $(f\colon Y \to S, \mathcal{L})$ and $(f'\colon Y' \to S, \mathcal{L}')$, are said to be equivalent if there are an S-isomorphism $u\colon Y \to Y'$ and an invertible sheaf $\mathcal{N}$ on S such that $u^*\mathcal{L}' \cong \mathcal{L} \otimes f^*\mathcal{N}$. We leave it to the reader to define the functor on maps.

On the other hand, let $\mathcal{J}_{d,g}$ be the contravariant functor from the category of schemes to that of sets defined in the following way: For each scheme S, let $\mathcal{J}_{d,g}(S)$ be the set of equivalence classes of pairs $(f, \mathcal{L})$, where $f\colon X \to S$ is a family of stable curves of genus g over S, and $\mathcal{I}$ is a torsion-free, rank-1 sheaf on X/S of relative degree d. Two such pairs, $(f\colon X \to S, \mathcal{I})$ and $(f'\colon X' \to S, \mathcal{I}')$, are said to be equivalent if there are an S-isomorphism $u\colon X \to X'$ and an invertible sheaf $\mathcal{N}$ on S such that $u^*\mathcal{I}' \cong \mathcal{I} \otimes f^*\mathcal{N}$. Again, we leave it to the reader to define the functor on maps.

Finally, let $\overline{\mathcal{M}}_g$ be the usual moduli functor of stable curves of genus g. There are natural "forgetful" maps of functors $\mathcal{P}_{d,g} \to \overline{\mathcal{M}}_g$, defined by taking a pair $(f\colon Y \to S, \mathcal{L})$ to the stable model X/S of Y/S, and $\mathcal{J}_{d,g} \to \overline{\mathcal{M}}_g$, defined by taking a pair $(f\colon X \to S, \mathcal{I})$ to X/S. The former forgetful map is well-defined by Proposition 5.1.

Theorem 6.1. *There is a natural isomorphism of functors*

$$\Phi\colon \mathcal{P}_{d,g} \longrightarrow \mathcal{J}_{d,g}$$

over $\overline{\mathcal{M}}_g$. The isomorphism Φ takes a pair $(f\colon Y \to S, \mathcal{L})$ of a family of quasistable curves f and an invertible sheaf $\mathcal{L}$ on Y to $(X \to S, \psi_*\mathcal{L})$, where X/S is the stable model of Y/S and $\psi\colon Y \to X$ is the contraction map. Its inverse takes a pair $(f\colon X \to S, \mathcal{I})$ of a family of stable curves f and a torsion-free, rank-1 sheaf $\mathcal{I}$ on X/S to $(\mathbb{P}_X(\mathcal{I}) \to S, \mathcal{O}(1))$.

Proof. Just combine Propositions 5.4 and 5.5. □

Let g and d be integers with $g \geq 2$. Let Y be a curve of genus g, and ω a dualizing sheaf of Y. The degree-d *canonical polarization* of Y is the sheaf

$$\mathcal{E}_d := \mathcal{O}_Y^{\oplus 2g-3} \oplus \omega^{\otimes g-1-d}.$$

Let $\mathcal{I}$ be a torsion-free, rank-1 sheaf on Y of degree d. We say that $\mathcal{I}$ is *semistable* (resp. *stable*) if $\mathcal{I}$ is semistable (resp. stable) with respect to $\mathcal{E}_d$.

Since $\chi(\mathcal{I}) = d + 1 - g$, and thus

$$\chi(\mathcal{I}_Z \otimes \mathcal{E}_d|_Z) = (2g-2)\chi(\mathcal{I}_Z) - \chi(\mathcal{I}) \deg_Z(\omega)$$

for every subcurve $Z \subseteq Y$, it follows that $\mathcal{I}$ is semistable (resp. stable) if and only if

$$\chi(\mathcal{I}_Z) \geq \frac{\deg_Z(\omega)}{2g-2}\chi(\mathcal{I}) \tag{30}$$

for every subcurve $Z \subseteq Y$ (resp. with equality only if $Z = Y$).

If Y is stable, the above condition is the same as Seshadri's in [27], Part 7, Def. 9, p. 153, when the polarization chosen (in Seshadri's sense) is the so-called canonical: If $Y_1, \dots, Y_p$ denote the components of Y, the *canonical polarization* is the p-tuple $\mathfrak{a} := (a_1, \dots, a_p)$ where

$$a_i := \frac{\deg_{Y_i}(\omega)}{2g-2}.$$

That $\mathfrak{a}$ is indeed a polarization in Seshadri's sense follows from the ampleness of ω, by the stability of Y. That the above notion of (semi)stability is Seshadri's follows from the fact that the nonzero torsion-free quotients of $\mathcal{I}$ are the sheaves $\mathcal{I}_Z$ for subcurves Z of Y.

On the other hand, $\chi(\mathcal{I}_Z) = \deg_Z(\mathcal{I}) + \chi(\mathcal{O}_Z)$ for each subcurve Z of Y. Also, it follows from adjunction and duality that

$$\deg_Z(\omega) = \deg(\mathcal{F}) + k_Z = \chi(\mathcal{F}) - \chi(\mathcal{O}_Z) + k_Z = -2\chi(\mathcal{O}_Z) + k_Z,$$

where $\mathcal{F}$ is the dualizing sheaf of Z. Thus, (30) holds for each proper subcurve $Z \subset Y$ if and only if

$$\deg_Z(\mathcal{I}) \geq d\left(\frac{\deg_Z(\omega)}{2g-2}\right) - \frac{k_Z}{2}, \tag{31}$$

with equality if and only if equality holds in (30).

Let X/S be a family of stable curves. A torsion-free, rank-1 sheaf $\mathcal{I}$ on X/S is said to be *semistable* (resp. *stable*) if $\mathcal{I}_s$ is semistable (resp. stable) for each geometric point s of S. Let $\mathcal{J}_{d,g}^{ss}$ (resp. $\mathcal{J}_{d,g}^{s}$) denote the subfunctor of $\mathcal{J}_{d,g}$ parameterizing the pairs $(X/S, \mathcal{I})$ with $\mathcal{I}$ semistable (resp. stable) on X/S.

According to [8], Def. 5.1.1, p. 3756, if Y is quasistable, a degree-d invertible sheaf $\mathcal{L}$ on Y is called *balanced* if $\deg_E(\mathcal{L}) = 1$ for each exceptional component E of Y and the "Basic Inequality" holds,

$$\left|\deg_Z(\mathcal{L}) - d\left(\frac{\deg_Z(\omega)}{2g-2}\right)\right| \leq \frac{k_Z}{2}, \tag{32}$$

for every proper subcurve $Z \subset Y$. Furthermore, $\mathcal{L}$ is called *stably balanced* if $\mathcal{L}$ is balanced and equality holds in

$$\deg_Z(\mathcal{L}) \geq d\left(\frac{\deg_Z(\omega)}{2g-2}\right) - \frac{k_Z}{2} \tag{33}$$

only if Z' is a union of exceptional components of Y.

Notice that (32) for every proper subcurve $Z \subset Y$ is equivalent to (33) for every proper subcurve $Z \subset Y$, which is in turn equivalent to

$$\deg_Z(\mathcal{L}) \leq d\left(\frac{\deg_Z(\omega)}{2g-2}\right) + \frac{k_Z}{2} \tag{34}$$

for every proper subcurve $Z \subseteq Y$. In addition, if $\deg_E(\mathcal{L}) = 1$ for each exceptional component E of Y, then equality holds in (33) (resp. (34)) if Z' (resp. Z) is a union of exceptional components of Y. So, in a formulation analogous to that of semistability and stability, $\mathcal{L}$ is balanced (resp. stably balanced) if $\deg_E(\mathcal{L}) = 1$ for each exceptional component E of Y and (33) holds for every proper subcurve $Z \subset Y$ (resp. with equality only if Z' is a union of exceptional components of Y).

Let Y/S be a family of quasistable curves. An invertible sheaf $\mathcal{L}$ on Y is said to be balanced (resp. stably balanced) on Y/S if $\mathcal{L}_s$ is balanced (resp. stably balanced) on Y_s for each geometric point s of S. Let $\mathcal{P}_{d,g}^{b}$ (resp. $\mathcal{P}_{d,g}^{sb}$) denote the subfunctor of $\mathcal{P}_{d,g}$ parameterizing the pairs $(Y/S, \mathcal{L})$ with $\mathcal{L}$ balanced (resp. stably balanced) on Y/S.

Proposition 6.2. *Let Y be a quasistable curve. Let X be its stable model and $\psi\colon Y \to X$ the contraction map. Let $\mathcal{L}$ be an invertible sheaf on Y such that $\deg_E(\mathcal{L}) = 1$ for every exceptional component $E \subset Y$. Then $\mathcal{L}$ is balanced (resp. stably balanced) if and only if $\psi_*\mathcal{L}$ is semistable (resp. stable).*

Proof. Let d be the degree of $\mathcal{L}$ and $\mathcal{E}_d$ the degree-d canonical polarization on X. Let $\check{\omega}$ be a dualizing sheaf of X. It follows from Proposition 5.1 that $\omega := \psi^*\check{\omega}$ is a dualizing sheaf of Y. Thus $\psi^*\mathcal{E}_d$ is the degree-d canonical polarization of Y.

Since $\mathcal{L}$ is ψ-admissible, it follows from Theorem 3.1 that $\psi_*\mathcal{L}$ is torsion-free, rank-1 and of degree d. Define the invertible sheaf

$$\mathcal{I} := \mathcal{L} \otimes \mathcal{O}_Y\left(\sum E\right) \tag{35}$$

on Y, where E runs over the set of components of Y contracted by ψ. Then $\mathcal{I}$ is negatively ψ-admissible. Furthermore, $\psi_*\mathcal{I} = \psi_*\mathcal{L}$ by Theorem 3.2. We claim first that $\mathcal{L}$ is balanced if and only if $\mathcal{I}$ is semistable. Furthermore, let $X_1, \dots, X_p$ be all the components of X, and $Y_1, \dots, Y_p$ those of Y such that $\psi(Y_i) = X_i$ for $i = 1, \dots, p$. We have that $\psi_*\mathcal{I}$ is stable if and only if $\psi_*\mathcal{I}$ is X_i-quasistable with respect to $\mathcal{E}_d$ for every $i = 1, \dots, p$. We claim as well that $\mathcal{L}$ is stably balanced if and only if $\mathcal{I}$ is Y_i-quasistable with respect to $\psi^*\mathcal{E}_d$ for every $i = 1, \dots, p$. Once the claims are proved, an application of Theorem 4.1 finishes the proof of the proposition.

Let Z be a proper subcurve of Y. If Z is a union of exceptional components of Y, then

$$\deg_Z(\mathcal{L}) = -\deg_Z(\mathcal{I}) = k_Z/2,$$

whence equality holds in (31) whereas strict inequality holds in (33). On the other hand, if Z' is a union of exceptional components of Y, then strict inequality holds in (31) whereas equality holds in (33).

Assume now that neither Z nor Z' is a union of exceptional components of Y. Let n (resp. n') be the number of connected components of Z' (resp. Z) which are exceptional components of Y. Let Z_1 (resp. Z_2) be the subcurve of Y obtaining by removing from (resp. adding to) Z all the exceptional components E of Y intersecting Z' (resp. Z) at exactly 1 or 2 points. Then Z_1 and Z_2 are proper subcurves of Y such that

$$k_{Z_1} + 2n' = k_{Z_2} + 2n = k_Z$$

and

$$\deg_{Z_1}(\omega) = \deg_Z(\omega) = \deg_{Z_2}(\omega).$$

Furthermore,

$$\deg_{Z_1}(\mathcal{L}) - n' \leq \deg_Z(\mathcal{I}) \text{ and } \deg_{Z_2}(\mathcal{I}) - n \leq \deg_Z(\mathcal{L}).$$

So (31) holds for Z replaced by Z_2 only if (33) holds, whereas (33) holds for Z replaced by Z_1 only if (31) holds. Furthermore, equality holds in (33) only if equality holds in (31) for Z replaced by Z_2. Since Z_2 contains some Y_i, this is not possible if $\mathcal{I}$ is Y_i-quasistable for every $i = 1, \dots, p$. Also, equality holds in (31) only if it holds in (33) for Z replaced by Z_1. Since Z_1' is not a union of exceptional components of Y, this is not possible if $\mathcal{L}$ is stably balanced. $\square$

Theorem 6.3. *The isomorphism of functors Φ of Theorem 6.1 restricts to isomorphisms of functors*

$$\Phi^b : \mathcal{P}^b_{d,g} \longrightarrow \mathcal{J}^{ss}_{d,g} \quad \textit{and} \quad \Phi^{sb} : \mathcal{P}^{sb}_{d,g} \longrightarrow \mathcal{J}^{s}_{d,g}.$$

Proof. Just combine Theorem 6.1 with Proposition 6.2. □

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